NICE INFINITARY LOGICS
SH797

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ABSTRACT. We deal with soft model theory of infinitary logics. We find a logic between $\mathbb{L}_{\infty,\aleph_0}$ and $\mathbb{L}_{\infty,\infty}$ which has some striking properties. First, it has interpolations (it was known that each of those logics fail interpolation though the pair has). Second, well ordering is not characterized in a strong way. Third, it can be characterized as the maximal such nice logic (in fact, is the maximal logic stronger than $\mathbb{L}_{\infty,\aleph_0}$ and which satisfies "well ordering is not characterized in a strong way").

Anotated Content

§ 0. Introduction for non-logicians

The first part of the introduction urged to try to explain the aim to general mathematical audience, so may be skipped by a knowledgeable reader; naturally we should start by explaining what is first order logic and a general (abstract) logic from model theoretic perspective.

We may consider classes of rings, and classes of groups but usually we do not consider a class containing structures of both kinds. Formally a ring is a structure (or model) M consistent of its universe, set of elements called $|M|$ (but we may write $a \in M$) and interpretations $+^M, \times^M$ and 0^M of the binary function symbols $+, \times$ and the individual constant symbol (= zero place function symbol) 0. We also write $^n M$ for $\{(a_0, \ldots, a_{n-1}) : a_0, \ldots, a_{n-1} \text{ are elements of } M\}$.

Generally we have a so-called vocabulary τ consisting of relation symbols (= predicates) and function symbols, each with a given number of places (= arity).

For a ring M we many times consider the set of n-tuples satisfying some equations. Model theorist usually look at a wider class of such sets, which start with the family $\{\{\bar{a} \in {}^n M : \bar{a} \text{ satisfies an equation } \varphi\} : n \in \mathbb{N} \text{ and } \varphi \text{ an equation}\}$ and close it under intersection of two (with the same n), compliment inside the relevant $^n(M)$ and projection (from $^{n+1} M$ to $^n(M)$). So a first order formula for the vocabulary $\tau, \varphi = \varphi(x_0, \ldots, x_{n-1})$ is a scheme giving for a τ-structure M a subset $\varphi(M)$ of $^n M$ as above. If $n = 0, \varphi(M) \in \{\{\langle\rangle\}, \emptyset\}$ then we call φ a sentence and say M satisfifes it, $M \models \varphi$ iff $\varphi(M) \neq \emptyset$; let $\mathbb{L}(\tau)$ be the set of first order sentences or formulas in the vocabulary τ; as we can add additional individual constants the difference is minor. We may consider sets definable with parameters by, i.e. $\varphi(M, \bar{b}) = \{\bar{a} : M \models \varphi[\bar{a}, \bar{b}]\}$.

There is much to be said for first order logic; e.g. for this family of subsets of $^n M$ for $n \in \mathbb{N}$ has better closure properties than the set "solutions of finitely many equations", i.e. varieties; however we shall not say it here. But in first order logic we cannot express, e.g. "a group G is locally finite, i.e. every finitely generated subgroup is finite". To express this we may allow: if $\varphi_k(x_0, \ldots, x_{n-1})$ is a formulas for $k \in \mathbb{N}$ then so is $\varphi(x_0, \ldots, x_{n-1}) = \bigwedge_k \varphi_k(x_0, \ldots, x_k)$, i.e. $\varphi(M) = \cap\{\varphi_k(M) : k\}$.

Allowing this we get the logic $\mathbb{L}_{\aleph_1, \aleph_0}$, more generally

⊞$_0$ the logic $\mathbb{L}_{\lambda, \kappa}$ is defined similarly but the formulas have the form $\varphi = \varphi(\langle x_i : i < \gamma \rangle), \gamma < \kappa$ and we allow $\bigwedge_{\alpha < \beta} \varphi_\alpha(\bar{x})$ for $\beta < \lambda$ and $(\exists y_0, \ldots, y_\alpha, \ldots)_{\alpha < \ell g(\bar{y})} \varphi(\bar{y}, \bar{x})$

where we allow a formula to have $< \kappa$ free variables, i.e. we consider subsets of $^\alpha M$ for $\alpha < \kappa$ and allow intersection of $< \lambda$ (instead of two) and projection "forgetting" $< \chi$ variables (instead of one).

Another strengthening of first order logic is allowing $\psi(\bar{y}) = (\exists^{\geq \aleph_1} x) \varphi(x, \bar{y})$, i.e. $\psi(\bar{M}) = \{\bar{b}: \text{ for uncountably many } a \in M \text{ we have } M \models \varphi[b, \bar{a}]\}$. There are many other logics.

Now first order logic has many good properties, including (recall, the cardinality of a set A is the number of elements, which may be infinite, the cardinality of a model is the number of its element, i.e. the cardinality of its universe, $\|M\|$)

⊞$_1$ the LST (Löwenheim-Skolem-Tarski) property:

(a) if a sentence $\psi \in \mathbb{L}(\tau)$ has a model, i.e. $M \models \psi$ then it has a countable model (the downward LST to $\aleph_0$)

(b) if τ is a vocabulary, M is a τ-model, $A \subseteq |M|, |A|+|\tau|+\aleph_0 \leq \lambda < \|M\|$ <u>then</u> there is a τ-model N of cardinality λ, a sub-model of M such that $A \subseteq |N|$ and $\mathrm{Th}_{\mathbb{L}}(N) = \mathrm{Th}_{\mathbb{L}}(M)$ where
- $\mathrm{Th}_{\mathbb{L}}(M) = \{\psi \in \mathbb{L}(\tau_M) : \psi$ a sentence $M \models \psi\}$.

This (together with the upward LST property) means that first order logic does not distinguish infinite cardinals.

$\boxplus_2$ compactness: if T is a set of sentences in $\mathbb{L}(\tau)$ and every finite $T' \subseteq T$ has a model, i.e. for some τ-model M we have $\varphi \in T' \Rightarrow M \models \varphi$ <u>then</u> T has a model.

The desirability of this should be obvious.

$\boxplus_3$ interpolation: <u>if</u> $\tau_0 = \tau_1 \cap \tau_2$ are vocabularies, $\psi_1 \in \mathbb{L}(\tau_1), \psi_2 \in \mathbb{L}(\tau_2)$ and $\psi_1 \vdash \psi_2$, i.e. there is no model of $\psi_1 \wedge \neg\psi_2$, (equivalently if M is a $(\tau_1 \cup \tau_2)$-model and $M \models \psi_1$ then $M \models \psi_2$), <u>then</u> there is $\varphi \in \mathbb{L}(\tau_0)$ such that $\psi_1 \vdash \varphi$ and $\varphi \vdash \psi_2$.

First order logic satisfies interpolation: this is Craig theorem. Lindström set out to show that first order logic is the natural choice, recalling there are many logics; for this he has first to define a logic, essentially (see more in Definition 1.9)

$\boxplus_4$ a logic $\mathscr{L}$ consists of the following

(a) a set of sentences $\mathscr{L}(\tau)$ for any vocabulary τ, we can define formulas $\varphi(x_0, \ldots, x_{n-1})$ by adding to τ individual constants

(b) satisfaction relation $\models_{\mathscr{L}}$, i.e. $M \models_{\mathscr{L}} \psi$ where M a model, $\psi \in \mathscr{L}(\tau_M)$

(c) natural properties like preservation under isomorphisms and monotonicity (i.e. $\tau_1 \subseteq \tau_2 \Rightarrow \mathscr{L}(\tau_1) \subseteq \mathscr{L}(\tau_2)$).

This seems too wide so, (see more in Definition 1.12).

$\boxplus_5$ $\mathscr{L}$ is a nice logic <u>if</u> the sets of sentences $\mathscr{L}(\tau)$ has some natural closure properties like:
- if $\psi_1, \psi_2 \in \mathscr{L}(\tau)$ then for some $\psi \in \mathscr{L}(\tau)$ we have:
  $M \models \psi$ iff $M \models \psi_1$ and $M \models \psi_2$.

There is a natural order on the class of logics:

$\boxplus_6$ $\mathscr{L}_1 \leq \mathscr{L}_2$ <u>iff</u> for every vocabulary τ and $\psi_1 \in \mathscr{L}_1(\tau)$ there is $\psi_2 \in \mathscr{L}_2(\tau)$ such that: if M is a τ-model then $M \models_{\mathscr{L}_1} \psi_1$ <u>iff</u> $M \models_{\mathscr{L}_2} \psi_2$

$\boxplus_7$ $\mathscr{L}_1, \mathscr{L}_2$ are equivalent if $\mathscr{L}_1 \leq \mathscr{L}_2$ and $\mathscr{L}_2 \leq \mathscr{L}_1$.

Now we can phrase (really just one of the versions[1] of)

$\boxtimes$ (Lindström theorem) The logic $\mathscr{L}$ is equivalent to $\mathbb{L}$, first order logic <u>when</u>

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[1]e.g. compactness just for countable theorems but then we have to add the occurance number is $\aleph_0$ or just $\aleph_0$, see Definition 1.11.

(a) $\mathscr{L}$ is a nice logic

(b) $\mathscr{L}$ satisfies LST to $\aleph_0$: i.e. $\boxplus_1(a)$

(c) $\mathscr{L}$ satisfies compactness, see $\boxplus_2$

This indicates that the family of nice logics not equivalent to $\mathbb{L}$ is the union of:

- the infinitary ones, usually above $\mathbb{L}_{\aleph_1, \aleph_0}$
- the somewhat compact, usually $\aleph_0$-compact ones.

We here deal with the first.

Lindström theorem founded "abstract model theory" where we have variables over logics. In the seventies and eighties this area flourished but a reason for its almost dying out is the lack of similar theorems for other logics, i.e. discovering (or pointing out) "interesting" logic which can be characterized in a reasonable way.

The aim of this work is to present such an infinitary logic and prove that it has some desirable properties. In particular it satifies interpolation which holds only in "few" cases. This solves some more specific old problems and we hope it will reopen the case of "abstract model theory".

In more details, we define the logic is $\mathbb{L}^1_\kappa$ for any suitable cardinal κ playing the role of $\aleph_0$ in first order logic

$\boxplus$ $\mathbb{L}^1_\kappa$ satisfies

(a) a downward LST; any sentence which has a model, has one of cardinality $< \kappa$

(b) if the vocabulary has cardinality $< \kappa$ then the number of sentences is κ

(c) a weak substitute of compactness: well ordering is not definable;

(d) the $\mathbb{L}^1_\kappa$-theory of a product of two τ-models $M_1 \times M_2$ depend just on the $\mathbb{L}^1_\kappa$-theories of M_1 and M_2

(e) interpolation, see $\boxplus_3$.

§ 1. Introduction and preliminaries

§ 1(A). **Aims.**

We feel that this is an important one among my works and will attract little attention. Is this an oxymoron? We do not think so. See below.

The investigation of model theoretic logics and soft model theory has started with Lindström theorems and it was a central topic of model theory in the seventies. Major aims were to find characterization theorems, new important logics and non-trivial implications. The achievements were to a large extent summed up in the handbook Barwise-Feferman [**?**] but then the subject become quite muted. There were some external reasons: stability theory and theoretical computer science draw people away and there were also some incidental personal reasons. But probably the profound reason was a disappointment. The impression was that there were just too many examples and counterexamples but not enough deep results and, particularly, too many logics and too few characterization theorems (saying a logic $\mathbb{L}$ is the unique logic such that ...). Recall that Lindström characterize first order logic; e.g. as the only "reasonable" logic satisfying compactness (for $\aleph_0$ sentences) and the downward LST theorem (for one sentence, to $\aleph_0$), (but see §(1D) below). Still there was some activity later, particularly of Väänänen.

Here we try to reopen the case. A property which remains mysterious was interpolation, see Makowsky [**?**] in the handbook. It was known that $\mathbb{L}_{\aleph_1,\aleph_0}$ has interpolation, (Lopez-Escobar) but not $\mathbb{L}_{\lambda,\kappa}$ when $(\lambda,\kappa) \neq (\aleph_0,\aleph_0),(\aleph_1,\aleph_0)$,(Malitz). On $\mathbb{L}_{\kappa,\theta}$ see Dickman [**?**]. However, the pair $(\mathbb{L}_{<\infty,\omega},\mathbb{L}_{\infty,\infty})$ and even $(\mathbb{L}_{\lambda^+,\omega},\mathbb{L}_{(2^\lambda)^+,\lambda^+})$ has interpolation, a puzzling result where

> ⊕ the pair of logics $(\mathscr{L}_1,\mathscr{L}_2)$ has interpolation <u>when</u> : if $\psi_\ell \in \mathscr{L}_1(\tau_\ell)$ for $\ell = 1,2$ and $\psi_1 \vdash \psi_2$ <u>then</u> for some $\varphi \in \mathscr{L}_2(\tau_1 \cap \tau_2)$ we have $\psi_1 \vdash \varphi$ and $\varphi \vdash \psi_2$.

This leads naturally to a question: does this interpolation come from the existence of an intermediate logic which has interpolation? See more on the history of those questions and on interpolation and related subjects, [**?**];

Let us recall some old questions on which we do not advance here. Feferman raises the question

Question 1.1. Is there an $\aleph_0$-compact logic strenghtening $\mathbb{L}(\exists^{\geq\aleph_1})$ with interpolation.

Note the plethora of extensions of $\mathbb{L}(\exists^{\geq\aleph_1})$. For my taste, preferable is

Question 1.2. 1) Is there a λ-compact logic stronger than first order satisfying interpolation, for any λ?
2) Moreover, fully compact one?

Of course, part (2) becomes a question only after fully compact logics $> \mathbb{L}$ were discovered ([**?**]).

The introduction of [**?**] mentions the (then latest advance): some compact logic strengthening first order logic satisfies the Beth definability theorem, ([**?**]) a puzzling result. Also the pair $(\mathbb{L}(Q^{\mathrm{cf}}_{\aleph_0}),\mathbb{L}(aa))$ of logics satisfies interpolation. Again a puzzling result. Those cases give place to hope of better results using related new

logics. Returning to infinitary logics, the following old problems are (and will be our main concern):

Problem 1.3. Is there a logic $\mathbb{L}$ satisfying interpolation such that $\mathbb{L}_{<\infty,\aleph_0} \subseteq \mathbb{L} \subseteq \mathbb{L}_{<\infty,<\infty}$?

Problem 1.4. Assume κ is strong limit singular of cofinality $\aleph_0$. Is there a logic $\mathbb{L}$ satisfying interpolation such that $\mathbb{L}_{\kappa^+,\aleph_0} \subseteq \mathbb{L} \subseteq \mathbb{L}_{\kappa^+,\kappa}$?

Later we have asked ourselves:

Problem 1.5. Is there, for arbitrarily large cardinal κ, a logic $\mathbb{L}$ such that:

- (a) $\mathbb{L}_{\kappa,\omega} \subseteq \mathbb{L}$ and has reasonable closure properties
- (b) $\mathbb{L}$ has the downward LST property in the sense that every sentence which has a model N, has a model N of "small" cardinality, moreover $M \subseteq N$
- (c) $\mathbb{L}$ has interpolation
- (d) undefinability of well ordering (in a strong sense) which means: if M expands $(\mathscr{H}(\lambda), \in)$ and $M \models \psi$ <u>then</u> for some N we have
  - (α) $N \models \psi$
  - (β) ord^N is not well founded
    a posteriori we add
  - (γ) $|N|$ is the union of $\aleph_0$, an internal set of bounded cardinality, i.e. for some $\langle a_n : n < \omega \rangle, \theta, N \models$ "θ a cardinal such that $|a_n| \leq \theta$ and $(\forall b \in N)(\bigvee_{n<\omega} N \models$ "$b \in a_n$")
- (e) $\mathbb{L} \subseteq \mathbb{L}_{\theta,\theta}$ for a suitable θ.

Problem 1.6. Is there a maximal such logic?

There is a feeling that Lindström theorem, EF-games and interpolation are inherently connected, though I do not know of a formlization of it, the present work gives evidence strengthening this feeling.

§ 1(B). **What is achieved.**

We feel that here we reasonably fulfill those old hopes mentioned above, in the direction of non-compact logics; (recall that by Lindstrom theorem, any (nice) logic stronger than first order logic, fail downward LST to $\aleph_0$ or fail $\aleph_0$-compactness).

Assume for transparency $\kappa = \beth_\kappa$, we find an interesting logic, $\mathbb{L}^1_\kappa$ such that:

$\circledast_1$ (A) $\mathbb{L}^1_\kappa$ is a nice logic

  (B) it has a reasonable characterization: it is the maximal nice logic, see Definition 1.9, 1.12 such that $(\alpha, <)$ can be characterized up to isomorphism by some $\psi_\alpha \in \mathbb{L}^1_\kappa$ for $\alpha < \kappa$, has occurance number $\leq \kappa$, see Definition 1.11 and well ordering is not definable in a strong way

  (C) it satisfies interpolation[2] (see $\boxplus_3$ above or Definition 1.14 below; answering an old question on the existence of such logic)

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[2]For consequences of interpolation, see [?].

(D)   it is between $\mathbb{L}_{<\kappa}^{-1} = \cup\{\mathbb{L}_{\lambda^+,\aleph_0} : \lambda < \kappa\}$ and
      $\mathbb{L}_{<\kappa}^0 = \cup\{\mathbb{L}_{\lambda^+,\lambda^+} : \lambda < \kappa\}$, see $\boxplus_0$ above or Definition 1.21 below

(E)   has many of the good properties of $\mathbb{L}_\kappa^0 := \mathbb{L}_{<\kappa}^0$:

   (α)   downward LST, see $\boxplus_1$ above, specifically every $\psi \in \mathbb{L}_\kappa^1(\tau)$
         has a model of cardinality $< \kappa$
         (see some variants in Definition 2.14 below)

   (β)   well ordering is not an expressible[3]

   (γ)   addition of theories, see 3.14

   (δ)   product of (two) theories, see 3.14

(F)   alternative characterization: $\mathbb{L}_\kappa^1$ is a minimal nice logic $\mathscr{L}$ for which
   •   for any ordinal $\alpha < \kappa$ we can characterize $(\alpha, <)$ up to
       isomorphism by some $\psi_\alpha \in \mathscr{L}$
   •   for each $\mu < \kappa$ we can characterize the class of $(A \cup \mathscr{P}, A, \in)$
       where $\mathscr{P} \subseteq [A]^{\leq\mu}$ is an $\aleph_0$-cover (and $A \cap \mathscr{P} = \emptyset$)
       by some sentence from $\mathscr{L}$ (see Definition 3.6)
   • $\mathscr{L}$ is Δ-closed, see Definition 3.8
   • $\mathscr{L}$ has occurance number $\leq \kappa$, see Definition 1.11.

We do not have a generalization of the Feferman-Vaught theorem [?] on general operations and even not the Mostowski one, [?], on reduced products, even for the product of countably many models, see Theorem 4.10.

   Here in §2 we choose a definition of the logic closest to the way we arrive to it and to the proof. For $\alpha, \theta < \kappa$ we generalize the Ehrenfeucht-Fraisse game allowing "rescheduling of debts". This does not give an equivalence relation so we close the induced relation to an equivalence relation and a sentence, i.e. the class of models of a sentence is the union of some such equivalence classes; then we prove the basic properties.

   In §3 we deal with the deeper properties as promised in $\circledast_1$ above: non-definability of well ordering, characterization and interpolation.

   In §4 we show how close is our logic to $\mathbb{L}_\kappa^0$ and deal with sums and products. We intend to continue in [?], in particular concerning to 1.4.

   More than once, lecturing on this some in the audience "complain" that this definition does not sound like a definition of logic. So in [?] we intend to give presentation close to the ways logic are traditionally defined (we could have done it for $\mathbb{L}_\kappa^1$, too) <u>but</u> our characterization theorem shows that we shall get the same logic.

   The logic $\mathbb{L}_\kappa^1$ from §1 is quite satisfactory; many of the good properties of $\mathbb{L}_{<\kappa,\aleph_0}$ and interpolation, and a characterization (parallel to Lindström theorem). <u>But</u> compared to $\mathbb{L}_{<\kappa,\aleph_0}$ we lose the upward LST.

Question 1.7. Let $\kappa = \beth_\kappa$, is there $\mathbb{L}$ such that $\mathbb{L}_\kappa^{-1} \leq \mathscr{L} \leq \mathbb{L}_\kappa^0$ satisfying interpolation and the upward LST theorem?
   We intend to deal with this in [?].

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[3]That is, if ψ is a sentence and for every ordinal α for some $M, M \models \psi$ and $(P^M, <^M)$ is a well ordering of order type $\geq \alpha$ <u>then</u> for some model M of $\psi, (P^M, <^M)$ is not a well ordering.

§ 1(C). **Why characterizations?** Note that characterization theorems are central for several reasons:

$\circledast_2$ (a) per se, uniqueness results are nice, of course

     (b) historically - Lindström theorem has this form

     (c) they prove a logic is natural logic

     (d) for a logic which lacks such theorem we may well suspect that

       •   there are many relatives of similar good properties, without a special reason to prefer one or another.

How good is a characterization theorem? Of course, it all depends on the properties appearing in the characterization being natural and preferably well established.

The situation of having reasonable logics which we can strengthen preseving their main positive properties but neither seeing a maximal one nor proving such extension do not exist has been prominent in the area; e.g. the most well established ones like $\mathbb{L}(Q) = \mathbb{L}(\exists^{\geq \aleph_1})$ and also $\mathbb{L}_{\kappa^+,\aleph_0}$.

Why here we tend to look at strong limit cardinals, in particular, $\kappa = \beth_\kappa$? Note that in first order formula with a fix finite vocabulary, with predicates only (for transparency), we use the q-depth theory, the set of sentences of quantifier depth $\leq q$ a model satisfies. Now the set of such sentences is finite;; in fact the number of sentences of quantifier depth q has order of magnitude $\beth_q$, iterating $x \mapsto 2^x$ (power) q times. In infinitary logic, if we like that still there is a sentence expressing the "quantifier depth $\leq \alpha$ theory" we need $\kappa = \beth_\kappa$. A price is that for κ singular, we lose full substitution, and moreover, full closure under conjunctions of $< \kappa$. This is resolved if we demand κ is strong limit regular, i.e. (strongly) inaccessible; this is fine but the existence of such cardinals is unprovable in ZFC.

§ 1(D). **Directions ignored here.** We do not deal here with some other major directions:

$\circledast_3$ (a) $\aleph_0$-compact (nice logic)

     (b) logics without negation and continuous logic

     (c) almost isomorphism (and absolute logic).

We may look at model theory essentially replacing "isomorphic" by "almost isomorphic", that is isomorphisms by potential isomorphisms, i.e. isomorphism in some forcing extension (= generic extension). In [**?**] we have suggested to reconsider a major theme in model theory, counting the number of isomorphism types. Recall that M, N are almost-isomorphic $\underline{\text{iff}}$ M, N have (the same vocabulary and) the same $\mathbb{L}_{\infty,\aleph_0}$-theory. For a theory T let $\mathbb{I}_{\text{ai}}(\lambda, T)$ be the number of models of T of cardinality λ up to almost isomorphism, i.e. $|\{M/\equiv_{\mathbb{L}_{\infty,\aleph_0}}: M$ a model of T of cardinality $\lambda\}|$. This behaves nicely ([**?**]): if T has cardinality $\leq \lambda$, is first order or just $\subseteq \mathbb{L}_{\lambda^+,\aleph_0}$ then $\dot{\mathbb{I}}_{\text{ai}}(\lambda, T) \leq \lambda < \mu \Rightarrow \dot{\mathbb{I}}_{\text{ai}}(\mu, T) \leq \dot{\mathbb{I}}_{\text{ai}}(\lambda, T)$, (on $\dot{\mathbb{I}}_{\text{ai}}(-, T)$ for $\aleph_0$-stable T, see a work of Laskowski-Shelah in preparation). In [**?**] we also define "M is ai-rigid, i.e. $a \neq b \in M \Rightarrow (M, a) \not\equiv_{\mathbb{L}_{\infty,\aleph_0}} (N, a)$" and have downward LST theorem for it. Later Nadel suggested further to consider homomorphisms, in particular for abelian groups, see [**?**, Ch.IV,§3,pg.487], more Göbel-Shelah [**?**], Göbel-Herden-Shelah [**?**]. Barwise characterized the relevant logic, $\mathbb{L}_{\infty,\aleph_0}$ by absoluteness: (among logics with satisfaction being absolute under forcing it is maximal).

§ 1(E). **Can soft model theory be applied?**

What about applications of soft model theory? There are some applications using compact logics on other directions, (concerning §1(D), §1(E)) see [?]. See [?] on extending first order logic by second order quantifiers restricted in some ways. Of course, the expressive power of the logic depend on the restriction, see the examples below. In cofinality logic, $\mathbb{L}(\mathbf{Q}_C^{\mathrm{cf}})$ with $\mathbf{C}$ a class of regular cardinals, we are allowed to say: the formula $\varphi(x,y)$, possibly with parameters, define a linear order with no last element of cofinality from $\mathbf{C}$, recalling that the cofinality of a linear oder I is the minimal cardinality of an unbounded (equivalently cofinal) subset. If $\mathbf{C}$ is non-trivial, this logic is a very interesting logic (e.g. fully compact), in particular showing what we cannot prove so full compactness is not sufficient to characterize first order logic. But its expressive power is weak so we do not expect it to have applications.

In [?] (where you can find something about the history of the topic) we prove the compactness of the quantifier (that is for first order logic extended by it) $\mathscr{L}^{\mathrm{ceab}} = \mathbb{L}(\mathbb{Q}^{\mathrm{ceab}})$ — quantifying over complete embeddings of one atomless Boolean ring into another. Moreover, for this logic we prove completeness for a natural set of axioms. Now consider the problem "can the automorphism groups of a 1-homogeneous[4] Boolean algebra be non-simple[5]"? Much is known on this group and, in particular, that it is "almost" simple — see Rubin-Štěpánek [?]. It was known that there may exist such Boolean Algebras as by [?, IV] in some generic extension, all automorphisms of $\mathscr{P}(\omega)/\text{finite}$ are trivial (i.e. induced by permutations π of $\mathbb{Z}$ such that $\{n \in \mathbb{Z} : n \geq 0$ but $\pi(n) < 0\}$ is finite) and van Dowen note that the group of trivial automorphisms of $\mathscr{P}(\omega)/\text{finite}$ is not simple (as the subgoup of the automorphisms induced by permutations of ω is a normal subgroup) and the quotient is isomorphic to $(\mathbb{Z}, +)$. Alternatively, Koppelberg [?] has directly constructed such Boolean Algebras of cardinality $\aleph_1$ assuming (the more natural assumption) CH. So by the completeness theorem (recalling that the set of axioms is absolute), as the relevant facts are expressible in $\mathbb{L}(\mathbb{Q}^{\mathrm{ceab}})$, the existence is proved in ZFC. Some may want to consider a direct proof. It almost certainly will give more specific desirable information.

Another helpful quantifier is on branches of trees (see [?]). In Fuchs-Shelah [?] it is used to eliminate the use of diamond, i.e. to prove in ZFC the existence of valuation domains R such that there are R-modules which are univerasl (i.e. the family of sub-modules is linearly ordered by inclusion) but not standard. Note the obvious examples (which are called standard): R itself or appropriate quotients. Actually the completeness theorem for this logic gives an absoluteness result which is used.

We believe that quantifiers with completeness and compactness will be useful so it is worthwhile to find such quantifiers. Hopefully see more in [?].

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[4]A Boolean algebra B is 1-homogeneous if it is atomless for every $a, b \in B \setminus \{0_{\mathbb{B}}\}$ we have $B \cong B \restriction b$ (equivalently for $a, b \in \mathbb{B} \setminus \{0_B, 1_B\}$ for some automorphism f of B, $f(a) = b$)

[5]That is has no normal subgroup which is neither the full group nor the one-element sub-group

§ 1(F). **Preliminaries.**

Notation 1.8. 1) τ denotes a vocabulary, i.e. a set of predicates so each $P \in \tau$ has arity$_\tau(P) < \omega$ places and each function symbol $F \in \tau$ has arity$_\tau(F) < \omega$ places; of course, individual constants are zero-place function symbols and we may write arity(P), arity(F) when τ is clear from the context.

1A) For a structure M let τ_M be the vocabulary of M, for a predicate P from τ_M, P^M is the interpretation of P so an arity$_\tau(P)$-place relation on $|M|$, the universe of M; similarly for a function symbol F from τ and in particular for an individual constant c from τ.

1B) Let $\tau_1 \subseteq \tau_2$ means that for every predicate P from τ_1 is a predicate of τ_2 and arity$_{\tau_2}(P) = $ arity$_{\tau_1}(P)$ and similarly for every function symbol F.

1C) Let $\tau_0 = \tau_1 \cap \tau_2$ means that $\tau_0 \subseteq \tau_1, \tau_0 \subseteq \tau_2$ and $P \in \tau_1 \wedge P \in \tau_2 \Rightarrow P \in \tau_0$ and similarly for function symbols.

2) $\mathscr{L}$ denotes a logic, see Definition 1.9.

3) $\bar{x}, \bar{y}, \bar{z}$ denote sequence of variables (with no repetition). Usually $\bar{x} = \langle x_i : i < \alpha \rangle$ so $\alpha = \ell g(\bar{\alpha})$ but even possibly $\bar{x} = \langle x_s : s \in S \rangle$ and then we let $\ell g(\bar{x}) = \text{Dom}(S) := S$.

4) We say τ is a relational vocabulary <u>when</u> it has no function symbol.

Recall

Definition 1.9. 1) A logic $\mathscr{L}$ consists of

(a) function $\tau \mapsto \mathscr{L}(\tau)$ giving a set of sentences φ (or formulas $\varphi(\bar{x})$, see 1.10 below) for any vocabulary τ; the function is a class function that is a definition

(b) $\models_{\mathscr{L}}$, satisfaction, i.e. the relation $M \models_{\mathscr{L}} \varphi$ for M a model, $\varphi \in \mathscr{L}(\tau_M)$

(c) renaming: the function $\hat{\pi}$, depending on $(\tau_1, \tau_2, \pi, \mathscr{L})$, is a one-to-one function from $\mathscr{L}(\tau_1)$ onto $\mathscr{L}(\tau_2)$ <u>when</u> π is an isomorphism from the vocabulary τ_1 onto the vocabulary τ_2 (i.e. if $P \in \tau_1 \Rightarrow$ then $\pi(P) \in \tau_2$ is a predicate and arity$_{\tau_1}(P) = $ arity$_{\tau_2}(\pi(P))$ and similarly for $F \in \tau_1$)

(d) if π is an isomorphism from the vocabulary τ_1 onto the vocabulary τ_2 and M_1 is a τ_1-model and $M_2 = \pi(M_1)$ naturally defined <u>then</u> $\varphi \in \mathscr{L}(\tau_1(M_1)) \Rightarrow [M_1 \models \varphi \Leftrightarrow M_2 \models \hat{\pi}(\varphi)]$

(e) (isomorphism): if M_1, M_2 are isomorphic τ-models and $\varphi \in \mathscr{L}(\tau)$ then $M_1 \models_{\mathscr{L}} \varphi \Leftrightarrow M_2 \models_{\mathscr{L}} \varphi$

(f) (monotonicity): if $\tau_1 \subseteq \tau_2$ <u>then</u> $\mathscr{L}(\tau_1) \subseteq \mathscr{L}(\tau_2)$ and for any τ_2-model M_2 and $\varphi \in \mathscr{L}(\tau_1)$ we have $M_2 \models \varphi \Leftrightarrow (M_2 {\restriction} \tau_1) \models \varphi$.

Convention 1.10. We define a formula $\varphi = \varphi(\bar{x})$ in $\mathscr{L}(\tau)$ as a sentence in $\mathscr{L}(\tau \cup \{c_i : i < \ell g(\bar{x})\})$ with $c_i(i < \alpha)$ pairwise distinct individual constants not from τ and if $\varphi(\bar{x}) \in \mathscr{L}(\tau)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ then $M \models_{\mathscr{L}} \varphi[\bar{a}]$ means $M^+ \models_{\mathscr{L}} \varphi$ where M^+ is the expansion of M by $c_i^{M^+} = a_i$ for $i < \ell g(\bar{x})$, in fact, $\ell g(\bar{x})$ can be any index set.

Definition 1.11. For a logic $\mathscr{L}$, the occurance number oc$(\mathscr{L})$ of $\mathscr{L}$ is the minimal cardinal κ (or ∞) such that $\mathscr{L}(\tau) = \cup\{\mathscr{L}(\tau') : \tau' \subseteq \tau$ is of cardinality $< \kappa\}$ for any vocabulary τ.

Definition 1.12. We say a logic $\mathscr{L}$ is nice <u>when</u>:

(a) (α) applying predicates: if $P \in \tau$ is an n-place predicate then
$P(x_0, \ldots, x_{n-1}) \in \mathscr{L}(\tau)$ is a formula of $\mathscr{L}(\tau)$

(β) equality: $x_0 = x_1 \in \mathscr{L}(\tau)$, i.e. for any individual constants $c_0, c_1 \in \tau$
there is $\varphi \in \mathscr{L}(\tau)$ such that $M \models \varphi$ iff $c_0^M = c_1^M$

(γ) applying[6] functions: for an n-place function symbol $F \in \tau$,
$x_0 = F(x_1, \ldots, x_{n-1})$ is a formula of $\mathscr{L}(\tau)$

(b) (α) $\mathscr{L}$ is closed under conjunction, i.e. for every $\varphi_1, \varphi_2 \in \mathscr{L}(\tau)$ there is
$\varphi_3 \in \mathscr{L}(\tau)$ such that for every τ-model M we have:
$M \models \varphi_3$ <u>iff</u> $M \models \varphi_1$ and $M \models \varphi_2$

(β) $\mathscr{L}$ is closed under existential quantifier, $(\exists x)$, i.e. for any $\varphi \in \mathscr{L}(\tau \cup \{c\})$, c an individual constant not in τ, there is $\psi \in \mathscr{L}(\tau)$ such
that for any τ-model M we have $M \models \varphi$ iff $M^+ \models \varphi$ for some
$(\tau \cup \{c\})$-expansion M^+ of μ

(γ) $\mathscr{L}$ is closed under negation, i.e. for any $\varphi \in \mathscr{L}(\tau)$ there is $\psi \in \mathscr{L}(\tau)$
such that for any τ-model $M, M \nVDash_{\mathscr{L}} \varphi \Leftrightarrow M \models_{\mathscr{L}} \psi$

(c) restricting a sentence ψ to a predicate $P, \psi{\restriction}P \in \mathscr{L}(\tau)$ when $\psi \in \mathscr{L}(\tau)$,
where $P \in \tau$ is a unary predicate, and τ is a relational vocabulary, see
Definition 1.15, 1.8(4)

(d) weak substitution, that is, has substitution for very simple schemes, see
1.18 below.

Remark 1.13. 1) Above we prefer $(a)(\gamma)$ on using $R'(\sigma_0, \ldots, \sigma_{m-1})$ where each
$\sigma_\ell = \sigma_\ell(\bar{x})$ is a term; it does not really matter.
2) Below we can define the multi-sort vesrion.

Definition 1.14. 1) A logic $\mathscr{L}$ satisfies interpolation <u>when</u> for any sentence $\psi_1 \in \mathscr{L}(\tau_1), \varphi_2 \in \mathscr{L}(\tau_2)$ and $\tau = \tau_1 \cap \tau$, we have: if $\varphi_1 \vdash \varphi_2$, i.e. for any $(\tau_1 \cup \tau_2)$-model
$M, M \models \varphi_1 \Rightarrow M \neq \varphi_2$, <u>then</u> for some sentence $\psi \in \mathscr{L}(\tau)$ we have $\varphi_1 \vdash \psi$ and
$\psi \vdash \varphi_2$.
2) We define "$\mathscr{L}$ satisfies interpolation with finitely many sorts" defined.

Definition 1.15. 1) We say[7] $\varphi \equiv \psi{\restriction}P$ where $\varphi, \psi \in \mathscr{L}(\tau)$ or pedantically[8] $\varphi = \psi{\restriction}_\tau P$ <u>where</u> P is a unary predicate in the vocabulary τ, τ is of minimal cardinality
such that $\psi \in \mathscr{L}(\tau)$ <u>when</u>:

for any τ-models $M, M \models (\psi{\restriction}_\tau P)$ <u>iff</u> P^M is non-empty, closed under F^M for
every function symbol from τ and $M{\restriction}_\tau P^M \models \psi$, see below.
2) For a τ-model M and unary predicate P let[9] $N = M{\restriction}_\tau P^M$ be the τ-model with
universe P^M, for n-place predicate $Q \in \tau$ we have $Q^N = Q^M \cap {}^n|N|$ and for n-place
function symbol $F \in \tau, F^N(\bar{a}) = b \Leftrightarrow \bar{a} \in {}^n|N| \wedge b \in N \wedge F^M(\bar{a}) = b$.

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[6]We may put together clauses $(a)(\alpha)$ and $(a)(\gamma)$ allowing $P(\sigma_0(\bar{x}), \ldots, \sigma_{n-1}(\bar{x})), \sigma_\ell$ a term;
the choice is immaterial.

[7]so this is a relation, as pedantically φ is not uniquely determined

[8]of course $\psi{\restriction}\tau$ is not uniquely determined. We may like to be more liberal in restricting a
model, in 1.8(1) allow F^M to be a partial function for F a function symbol from $\tau(M)$. We may
combine this with restriction (see Definition 1.15), then we still have to demand $P^M \neq \emptyset$ (except
if we go further and allow empty models). Note $M \models$ "$\neg F(\bar{a}) = b$" when $F^M(\bar{a})$ is not well
defined.

[9]again, not always exist

Definition 1.16. Let $\mathscr{L}$ be a logic.

0) $\mathrm{at}_{\mathscr{L}}(\tau)$ is the set of atomic formulas $\varphi(\bar{x})$ from 1.12(a); $\mathrm{bs}_{\mathscr{L}}(\tau) = \{\varphi, \neg\varphi : \varphi \in \mathrm{at}_{\mathscr{L}}(\tau)\}$; we may omit $\mathscr{L}$ if clear from the context.

1) Let $\varphi \vdash_{\mathscr{L}} \psi$ where $\varphi, \psi \in \mathscr{L}(\tau)$ mean that $M \models \varphi \Rightarrow M \models \psi$ for any τ-model (this does not depend on τ by Definition 1.9).

2) We say $\bar{\vartheta}$ is an $(\mathscr{L}, \tau_1, \tau_2)$-interpretation scheme <u>when</u>:

    (a) $\mathscr{L}$ a logic

    (b) τ_1, τ_2 vocabularies

    (c) $\bar{\vartheta} = \langle \vartheta_{\varphi(\bar{x})}(\bar{x}) : \varphi(\bar{x}) \in \mathrm{at}_{\mathscr{L}}(\tau_2) \rangle$

    (d) $\vartheta_{\varphi(\bar{x})}(\bar{x})$ is a formula in $\mathscr{L}(\tau_1)$ so $\varphi = P(x_0, \ldots, x_{n-1}), \bar{x} = \langle x_\ell : \ell < n \rangle$ or $\varphi = (x_0 = F(x_1, \ldots, x_n)), \bar{x} = \langle x_\ell : \ell \leq n \rangle$.

2A) Above we say the $((\mathscr{L}_1, \tau_1, \tau_2)$-interpretation) scheme $\bar{\vartheta}$ is <u>simple</u> when $\vartheta_{x_0=x_1}$ has the form $(x_0 = x_1)$.

3) For $\bar{\vartheta}$ as above we say $N = N_{\bar{\vartheta}}[M] = M_1[\bar{\vartheta}]$ <u>when</u>

    (a) $|N| = |M|/\varphi_=(x_0, x_1)$ which means

        • $E^M_{\varphi_=} = \{(a, b) : M \models \varphi(a, b)\}$ is an equivalence relation on $|M|$ or just some non-empty subset

        • $|N| = \{a/E^M_{\varphi_=} : a \in M \text{ and } M \models \varphi_=[a, a]\}$

    (b) $N \models \varphi[\bar{a}] \Leftrightarrow M \models \vartheta_{\varphi(\bar{x})}[\bar{a}]$ for $\varphi(\bar{x}) \in \mathrm{at}(\tau_2)$ and $\bar{a} \in {}^{\omega>}M$ of length $\ell g(\bar{x})$

(note: not for every such $\bar{\vartheta}$ and τ_1-model M_1 is $N_{\bar{\vartheta}}[M_2]$ well defined, we need that $\varphi_=(-, -)$ defines an equivalence relation on $|M_1|$, which is a congruence relation for the τ_2-relations and functions we define and, of course, the definition of a function gives a function, similarly below; but if $\bar{\vartheta}$ is simple this problem does not arise).

4) We say the logic $\mathscr{L}$ satisfies full substitution[10] <u>when</u>: if τ_1, τ_2 are vocabularies, $\bar{\vartheta} = \langle \vartheta_{\varphi(\bar{x})}(\bar{x}) : \varphi(\bar{x}) \in \mathrm{at}_{\mathscr{L}}(\tau_2) \rangle$ is a $(\mathscr{L}, \tau_1, \tau_2)$-interpretation scheme, and $\psi_2 \in \mathscr{L}(\tau_2)$, <u>then</u> there is $\psi_1 \in \mathscr{L}(\tau_1)$ such that: if $M_1 = N_{\bar{\vartheta}}[M_2]$, see below, so M_ℓ is a τ_ℓ-models for $\ell = 1, 2$ then $M_1 \models \psi_1 \Leftrightarrow M_2 \models \psi_2$.

5) We say that the logic $\mathscr{L}$ satisfies substitution <u>when</u> we require $\bar{\vartheta}$ to be simple.

Remark 1.17. On substitutions, what are the problems:

    (a) for $\mathbb{L}^1_\kappa$ if $\mathrm{cf}(\kappa) < \kappa$ then we cannot use a scheme $\bar{\vartheta}$ such that for every $\lambda < \kappa$, some $\vartheta_{\varphi(\bar{x})}$ is not in $\mathbb{L}^1_\lambda$

    (b) note that atomic formulas have finitely many variables otherwise one may have problems: increasing union of ω-chains of models is not well defined

    (c) but substitution in some form is needed in e.g. characterization

    (d) can use $\kappa, \mathbb{L}^1_\kappa, \kappa$ inaccessible and have full substitution, but then not in ZFC.

Definition 1.18. 1) We say $\bar{\vartheta}$ is a weak $(\mathscr{L}, \tau_1, \tau_2)$-scheme (but $\mathscr{L}$ is immaterial so can be omitted) <u>when</u>:

    (a) τ_1, τ_2 are vocabularies

    (b) $\bar{\vartheta} = \langle \vartheta_{\varphi(\bar{x})}(\bar{x}) : \varphi(\bar{x}) \in \mathrm{at}(\tau_2) \rangle$

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[10]We do not use full substitution; as for κ singular $\mathbb{L}^0_\kappa, \mathbb{L}^1_\kappa$ are not closed under full substitution

(c) the formulas $\vartheta_{\varphi(\bar{x})}$ are atomic (or conjunction of two atomic for equality) formulas in which we substitute some variables by individual constants, moreover:

    (α) if $\varphi(\bar{x})$ is equal to $(x_0 = x_1)$ so $\ell g(\bar{x}) = 2$ then $\vartheta_{\varphi(\bar{x})} = (x_0 = x_1 \wedge P_{\varphi(\bar{x})}(x_0, \bar{c}_{\varphi(\bar{x})})$

    (β) if $\varphi(\bar{x}) = P(\bar{x})$, so $P \in \tau_2$ a predicate <u>then</u>

        $\bullet_1$ $\vartheta(\bar{x}) = Q_{\varphi(\bar{x})}(\bar{x}, \bar{c})$ where

        $\bullet_2$ $Q_{\varphi(\bar{x})}$ a predicate from τ_1

        $\bullet_3$ $\bar{c}$ a sequence of individual constants from τ_1

        $\bullet_4$ so $\text{arity}_{\tau_2}(P) = \ell g(\bar{x})$, $\text{arity}_{\tau_1}(Q_p) = \ell g(\bar{x}) + \ell g(\bar{c})$

    (γ) if $\varphi(\bar{x}) = (x_0 = F(x_1, \ldots, x_n))$ so $F \in \tau_2$ a function symbol <u>then</u>

        $\bullet_1$ $\vartheta_{\varphi(\bar{x})}(\bar{x})$ is $(x_0 = H_F(x_1, \ldots, x_n, \bar{c}_F))$ or $Q_F(x_0, \ldots, x_n, \bar{c}_F)$

        $\bullet_2, \bullet_3$ as above.

2) We say $\mathscr{L}$ has weak substitution <u>when</u> satisfies Definition 1.16(4), but restricting ourselves to weak schemes.

Remark 1.19. 1) The meaning of 1.18(1)(c)(γ) is

    $(*)_1$ $\bullet$ $\vartheta_{\varphi(\bar{x})}(\bar{x}) = (x_0 = H_F(x_1, \ldots, x_n, \bar{c}_F))$

         $\bullet$ $H_F \in \tau_1$ a function symbol of arity $n + \ell g(\bar{c}_F)$

         $\bullet$ $\bar{c}_F$ a sequence of individual constants from

        <u>or</u>

    $(*)_2$ similarly using $Q_F(x_0, \ldots, x_n, \bar{c}_F)$.

2) Why in 1.18(1)(c) the "moreover", i.e. why is the second version stronger (so have fewer substitutions)? As we demand the $\bar{c}_\varphi$'s to be as in the end. Does not matter whether we ask it or not. We could further demand $\bar{c}_\varphi = \langle c_* \rangle$ for all $\bar{\varphi} \in \text{at}(\tau_2)$.

3) In "weak $(\mathscr{L}, \tau_1, \tau_2)$-scheme" (hence in "weak substitution") we may use first order formulas (instead of atomic), i.e. that is our results will not be affected by this change in the definition.

4) Also, in "weak $(\mathscr{L}, \tau_1, \tau_2)$-scheme" we may add $\bar{c}_\varphi = \langle c_* \rangle$, for one c_*, and/or demand $\bar{\vartheta}$ is simple. For the later change in the proof of 3.4, we have to say that "without loss of generality all $\|M_{\alpha,n}\|$ have cardinality $\geq \mu_1$ and in fact $= \mu_1$", for this we need a suitable version of LST, see so have to add the assumption "$\mathscr{L}$ satisfies suitable version of LST.

Definition 1.20. 1) For logics $\mathscr{L}_1, \mathscr{L}_2$ let $\mathscr{L}_1 \subseteq \mathscr{L}_2$ mean that: $\mathscr{L}_1(\tau) \subseteq \mathscr{L}_2(\tau)$ for any vocabulary τ and $M \models_{\mathscr{L}_1} \varphi$ is equivalent to $M \models_{\mathscr{L}_2} \varphi$ when $\varphi \in \mathscr{L}(\tau)$ and M is a τ-model.

2) For logics $\mathscr{L}_1, \mathscr{L}_2$ let $\mathscr{L}_1 \leq \mathscr{L}_2$ or $\mathscr{L}_2$ is stronger than $\mathscr{L}_1$ means that for every τ and $\varphi \in \mathscr{L}_1(\tau)$ there is $\psi \in \mathscr{L}_2(\tau)$ such that φ, ψ are equivalent, i.e. $M \models_{\mathscr{L}_1} \varphi \Leftrightarrow M \models_{\mathscr{L}_2} \psi$ for any τ-model M.

3) We say that the logics $\mathscr{L}_1, \mathscr{L}_2$ are equivalent, $\mathscr{L}_1 \equiv \mathscr{L}_2$ when $\mathscr{L}_1 \leq \mathscr{L}_2$ and $\mathscr{L}_2 \leq \mathscr{L}_1$.

Definition 1.21. 1) The logic $\mathbb{L}_{\kappa,\theta}$ for $\kappa \geq \theta \geq \aleph_0$ is defined like first order logic, but $\mathbb{L}_{\kappa,\theta}(\tau)$ is the closure of the set of atomic formulas under $\neg\varphi$, $\bigwedge\limits_{i<\alpha} \varphi_i$ where $\alpha < \kappa$ and $(\forall x_0, \ldots, x_i, \ldots)_{i<\alpha}\varphi$ where $\alpha < \theta$; a sentence is a formula with no free variables and satisfaction is defined naturally.

1A) First order logic is $\mathbb{L} = \mathbb{L}_{\aleph_0,\aleph_0}$.

2) Let $\mathbb{L}_{\kappa,\theta,\gamma}(\tau)$ be the set of formulas $\varphi(\bar{x}) \in \mathbb{L}_{\kappa,\theta}(\tau)$ of quantifier depth $\leq \gamma$; similarly in the other cases.

3) Let $\mathbb{L}^0_{\leq\kappa,\leq\theta} = \mathbb{L}_{\kappa^+,\theta^+}, \mathbb{L}^0_{<\kappa,<\theta} = \cup\{\mathbb{L}^0_{\leq\kappa_1,\leq\theta_1} : \kappa_1 < \kappa, \theta_1 < \theta\}$; similarly $\mathbb{L}^0_{\leq\kappa,\leq\theta,<\gamma}$, etc.

4) Let $\mathbb{L}^{-1}_\kappa = \mathbb{L}^0_{<\kappa,\aleph_0}, \mathbb{L}^0_\kappa = \mathbb{L}^0_{<\kappa,<\kappa}$.

5) We define the logic $\mathbb{L}_{\infty,\theta,\gamma}$ by induction on the ordinal γ such that $\mathbb{L}_{\infty,\theta,\gamma}(\tau)$ is a set of cardinality $\leq \beth_\gamma(|\tau| + \aleph_0)$, increasing with γ as follows:

- for $\gamma = 0$, the set of basic sentences
- for γ limit, it is $\cup\{\mathbb{L}_{\infty,\theta,\beta}(\tau) : \beta < \gamma\}$
- for $\gamma = \beta + 1$ it is the set of sentences of the form $(\exists\bar{x}) \bigwedge\limits_{i<\alpha} \varphi_i(\bar{x})$ or its negation where $\ell g(\bar{x}) < \theta, \varphi_i(\bar{x})$ a formula from $\mathbb{L}_{\infty,\theta,\beta}(\tau)$ with $\alpha < \beth_\beta(|\tau| + \aleph_0)$.

Observation 1.22. *If κ is strong limit singular, <u>then</u> there is no logic $\mathscr{L}$ such that:*

(a) $\mathbb{L}^{-1}_\kappa \leq \mathscr{L} \leq \mathbb{L}^0_\kappa$

(b) $\mathscr{L}$ *satisfies full substitution.*

Note that

Claim 1.23. *1) In Definition 1.21 for κ singular, $\mathbb{L}_{\kappa^+,\theta} \leq \mathbb{L}_{\kappa,\theta} \subseteq \mathbb{L}_{\kappa^+,\theta}$, i.e. every sentence in $\mathbb{L}_{\kappa^+,\theta}(\tau)$ is equivalent to one in $\mathbb{L}_{\kappa,\theta}$ (and $\mathbb{L}_{\kappa,\theta}(\tau) \subseteq \mathbb{L}_{\kappa^+,\theta}(\tau)$), so it seems redundant to allow κ to be singular as was originally done. Still this is the tradition and we use it.*

2) If κ is a regular cardinal <u>then</u> the logics $\mathbb{L}^0_{\kappa,\theta}$ and $\mathbb{L}^0_{<\kappa,\theta}$ are equal.

3) If $\kappa = \beth_\kappa$ then $\mathbb{L}^0_{<\kappa,<\kappa,<\kappa} = \mathbb{L}^0_{<\kappa,<\kappa}$.

§ 2. The logic $\mathbb{L}^1_\kappa$

Our main definition is

Definition 2.1. For a vocabulary τ, τ-models M_1, M_2, set Γ formulas in the vocabulary τ in any logic (each with finitely many free variables if not said otherwise; see 2.3(4)), cardinal θ and ordinal α we define a game $\partial = \partial_{\Gamma, \theta, \alpha}[M_1, M_2]$ as follows, and using $(M_1, \bar{b}_1), (M_2, \bar{b}_2)$ with their natural meaning when $\mathrm{Dom}(\bar{b}_1) = \mathrm{Dom}(\bar{b}_2)$.

(A) The moves are indexed by $n < \omega$ (but every actual play is finite), just before the n-th move we have a state $\mathbf{s}_n = (A_n^1, A_n^2, h_n^1, h_n^2, g_n, \beta_n, n)$

(B) $\mathbf{s} = (A^1, A^2, h^1, h^2, g, \beta, n) = (A_{\mathbf{s}}^1, A_{\mathbf{s}}^2, h_{\mathbf{s}}^1, h_{\mathbf{s}}^2, g_{\mathbf{s}}, \beta_{\mathbf{s}}, n_{\mathbf{s}})$ is a state (or n-state or (θ, n)-state or $(\theta, < \omega)$-state) <u>when</u>:

    (a) $A^\ell \in [M_\ell]^{\leq \theta}$ for $\ell = 1, 2$

    (b) $\beta \leq \alpha$ so an ordinal

    (c) h^ℓ is a function from A^ℓ into ω

    (d) g is a partial one-to-one function from M_1 to M_2 and let $g_s^1 = g^1 = g_s = g$ and let $g_s^2 = g^2 = (g_s^1)^{-1}$,

    (e) $\mathrm{Dom}(g^\ell) \subseteq A^\ell$ for $\ell = 1, 2$

    (f) g preserves satisfaction of the formulas in Γ and their negations, i.e. for $\varphi(\bar{x}) \in \Gamma$ and $\bar{a} \in {}^{\ell g(\bar{x})}\mathrm{Dom}(g)$ we have $M_1 \models \varphi[\bar{a}] \Leftrightarrow M_2 \models \varphi[g(\bar{a})]$

    (g) if $a \in \mathrm{Dom}(g^\ell)$ then $h^\ell(a) < n$

(C) we define the state $\mathbf{s} = \mathbf{s}_0 = \mathbf{s}_\alpha^0$ by letting $n_{\mathbf{s}} = 0, A_{\mathbf{s}}^1 = \emptyset = A_{\mathbf{s}}^1, \beta_{\mathbf{s}} = \alpha, h_{\mathbf{s}}^1 = \emptyset = h_{\mathbf{s}}^2, g_s = \emptyset$; so really $\mathbf{s}$ depend only on α (but in general, this may not be a state for our game as possibly for some sentence $\psi \in \Gamma$ we have $M_1 \models \psi \Leftrightarrow M_2 \models \neg\psi$)

(D) we say that a state $\mathbf{t}$ extends a state $\mathbf{s}$ when $A_{\mathbf{s}}^\ell \subseteq A_{\mathbf{t}}^\ell, h_{\mathbf{s}}^\ell \subseteq h_{\mathbf{t}}^\ell$ for $\ell = 1, 2$ and $g_{\mathbf{s}} \subseteq g_{\mathbf{t}}, \beta_{\mathbf{s}} > \beta_{\mathbf{t}}, n_{\mathbf{s}} < n_{\mathbf{t}}$; we say $\mathbf{t}$ is a successor of $\mathbf{s}$ if in addition $n_{\mathbf{t}} = n_{\mathbf{s}} + 1$

(E) in the n-th move

    <u>the anti-isomorphism</u> player (AIS) chooses $(\beta_{n+1}, \iota_n, A_n')$ such that:

    $\iota_n \in \{1, 2\}, \beta_{n+1} < \beta_n$ and $A_n^{\iota_n} \subseteq A_n' \in [M_{\iota_n}]^{\leq \theta}$,

    <u>the isomorphism</u> player (ISO) chooses a state $\mathbf{s}_{n+1}$ such that

    • $\mathbf{s}_{n+1}$ is a successor of $\mathbf{s}_n$

    • $A_{\mathbf{s}_{n+1}}^{\iota_n} = A_n'$

    • $A_{\mathbf{s}_{n+1}}^{3-\iota_n} = A_{\mathbf{s}_n}^{3-\iota_n} \cup \mathrm{Dom}(g_{\mathbf{s}_{n+1}}^{3-\iota_n})$

    • if $a \in A_n' \backslash A_{\mathbf{s}_n}^{\iota_n}$ then $h_{\mathbf{s}_{n+1}}^{\iota_n}(a) \geq n + 1$

    • $\mathrm{Dom}(g_{\mathbf{s}_{n+1}}^{\iota_n}) = \{a \in A_{\mathbf{s}_n}^{\iota_n} : h_{\mathbf{s}_n}^{\iota_n}(a) < n + 1\}$ so it includes $\mathrm{Dom}(g_{\mathbf{s}_n}^{\iota_n})$

    • $\beta_{\mathbf{s}_{n+1}} = \beta_{n+1}$.

(F)   • the play ends when one of the player has no legal moves (always occur as $\beta_n < \beta_{n-1}$) and then this player loses, this may occur for $n = 0$

    • for $\alpha = 0$ we stipulate that ISO wins iff $\mathbf{s}_\alpha^0$ is a state.

Discussion 2.2. 1) This is a parallel to EF-games. Note that we like on the one hand the game to have $\leq \omega$ moves, really each play has $< \omega$ moves and deal with

sets of cardinality $\leq \theta$ and on the other hand we do not like well ordering to be definable, i.e. allow M_1 to be well ordered while M_2 to be non-well ordered but still the ISO player wins. We do this by "rescheduling our debts", i.e. using the h_n's.

Definition 2.3. 1) Let $\mathscr{E}^{0,\tau}_{\Gamma,\theta,\alpha}$ be the class $\{(M_1, M_2) : M_1, M_2 \text{ are } \tau\text{-models and}$ in the game $\partial_{\Gamma,\theta,\alpha}[M_1, M_2]$ the ISO player has a winning strategy$\}$ where Γ is a set of formulas in the vocabulary τ, each with finitely many free variables.
2) $\mathscr{E}^{1,\tau}_{\Gamma,\theta,\alpha}$ is the closure of $\mathscr{E}^{0,\tau}_{\Gamma,\theta,\alpha}$ to an equivalence relation (on the class of τ-models).
3) Above we may replace Γ by qf(τ) which means $\Gamma = $ the set at(τ) or bs(τ) formulas in the vocabulary τ.
4) Above if we omit τ we mean $\tau = \tau_\Gamma$ and if we omit Γ we mean bs(τ). Abusing notation we may say M_1, M_2 are $\mathscr{E}^{0,\tau}_{\Gamma,\theta,\alpha}$-equivalent.

Fact 2.4. Assume $\partial_{\Gamma,\theta,\alpha}[M_1, M_2]$ is well defined and M_1, M_2 are τ-models.
1) The game $\partial_{\Gamma,\theta,\alpha}[M_1, M_2]$ is a determined game and is without memory, i.e. during a play, being a winning situation does not depend on the history, just on the current state, also only $M_\ell{\restriction}\tau_\Gamma$ for $\ell = 1, 2$ are relevant.
2a) The relation $\mathscr{E}^0_{\Gamma,\theta,\alpha}$ holds for (M_1, M_2) when M_1, M_2 are isomorphic τ-models.
2b) If $M_1 \cong M_1', M_2 \cong M_2'$ then $M_1 \mathscr{E}^0_{\Gamma,\theta,\alpha} M_2 \Leftrightarrow M_1' \mathscr{E}^0_{\Gamma,\theta,\alpha} M_2'$.
2c) $\mathscr{E}^0_{\Gamma,\theta,\alpha}$ is reflexive and symmetric.
3) The relation $\mathscr{E}^1_{\Gamma,\theta,\alpha}$ is an equivalence relation on the class of τ-models.
4) If α is a limit ordinal <u>then</u> $M_1 \mathscr{E}^0_{\Gamma,\theta,\alpha} M_2$ iff $[\beta < \alpha \Rightarrow M_1 \mathscr{E}^0_{\Gamma,\theta,\beta} M_2]$.
5) $\mathscr{E}^1_{\Gamma,\theta,\alpha}$ has $\leq \beth_{\alpha+1}(|\Gamma| + \theta)$ equivalence classes.
6) If $\tau^1 \subseteq \tau^2, \Gamma_1 \subseteq \Gamma_2, \theta_1 \leq \theta_2$ and $\alpha_1 \leq \alpha_2$ and $M_1 \mathscr{E}^{0,\tau^2}_{\Gamma_2,\theta_2,\alpha_2} M_2$ <u>then</u> $M_1 \mathscr{E}^{0,\tau^1}_{\Gamma_1,\theta_1,\alpha_1} M_2$.

Proof. 1) Obvious.
2a) Let g_* be an isomorphism from M_1 onto M_2. Now a winning strategy for the player ISO in $\partial_{\Gamma,\theta,\alpha}[M_1, M_2]$ is to preserve "$g_{\mathbf{s}_n} \subseteq g_*$".
2b) Should be clear.
2c) Let us check.
    <u>Reflexivity</u>:
    Follows from part (2a) as $M \cong M$.
    <u>Symmetry</u>:
    Reading the definition carefully it should be clear.
3),4) Easy, too.
5) We prove by induction on α that there is an equivalence relation $\mathscr{E}^2_{\Gamma,\theta,\alpha}$ on the class of $\tau(\Gamma)$-models such that $M_1 \mathscr{E}^2_{\Gamma,\theta,\alpha} M_2 \Rightarrow M_1 \mathscr{E}^0_{\Gamma,\theta,\alpha} M_2$ and $\mathscr{E}^2_{\Gamma,\theta,\alpha}$ has $\leq \beth_{\alpha+1}(|\Gamma| + \theta)$ equivalence classes. For $\alpha = 0$, recall clause (F) of Definition 2.1 so easily $M_1 \mathscr{E}^0_{\Gamma,\theta,\alpha} M_2$ means that $\Gamma_\ell = \{\varphi : M_\ell \models \varphi \text{ and } \varphi \in \Gamma, \varphi \text{ is a sentence}\}$ does not depend on ℓ; clearly $\mathscr{E}^0_{\Gamma,\theta,\alpha}$ is an equivalence relation with $\leq 2^{|\Gamma|}$ equivalence classes and $\mathscr{E}^1_{\Gamma,\theta,\alpha} = \mathscr{E}^0_{\Gamma,\theta,\alpha}$; let $\mathscr{E}^2_{\Gamma,\theta,\alpha} = \mathscr{E}^0_{\Gamma,\theta,\alpha}$. For α a limit ordinal use part (4) and choose $\mathscr{E}^2_{\Gamma,\theta,\alpha} = \cap\{\mathscr{E}^2_{\Gamma,\theta,\beta} : \beta < \alpha\}$ recalling $\beth_\alpha(|\Gamma| + \theta) = \Sigma\{\beth_\beta(|\Gamma| + \theta) : \beta < \alpha\}$.

    Lastly, for $\alpha = \beta + 1$ use the induction hypothesis and $\beth_\alpha(|\Gamma| + \theta) = 2^{\beth_\beta(|\Gamma|+\theta)}$. Alternatively use $\mathbb{L}_{\theta^+,\theta^+,\alpha}$, i.e. use the proof of 2.8, when we replace $\beth_\alpha(|\Gamma| + \theta)$ by $\beth_{(|\Gamma|+\theta)^+}$.
6) Easy. $\square_{2.4}$

Definition 2.5. We define the logic $\mathbb{L}^1_{\leq\theta}$ as follows: a sentence $\psi \in \mathbb{L}_{\leq\theta}(\tau)$ iff the sentence is defined using (or by) a triple $(\mathrm{qf}(\tau_1), \theta, \alpha)$ which means: τ_1 a sub-vocabulary of τ of cardinality $\leq \theta$ and $\alpha < \theta^+$ and for some sequence $\langle M_\beta : \beta < \beta(*)\rangle$ of τ_1-models of length $\beta(*) \leq \beth_{\alpha+1}(\theta)$ we have: $M \models \psi$ iff M is $\mathscr{E}^1_{\mathrm{qf}(\tau_1),\theta,\alpha}$-equivalent to M_α for some $\beta < \beta(*)$.

2) Let $\mathbb{L}^1_\kappa = \cup\{\mathbb{L}^1_{\leq\theta} : \theta < \kappa\}$ so $\mathbb{L}^1_{\theta^+} = \mathbb{L}^1_{\leq\theta}$.

Remark 2.6. 1) The present definition of the logic (2.5) is interesting mainly for κ strong limit such that $\kappa = \beth_\kappa$ and it seems to me that it makes the presentation transparent. Note that $\mathbb{L}^1_{\leq\theta}$ is similar to the set of formulas of quantifier depth $< \theta^+$.

2) Why? Just note that for vocabulary τ of cardinality $\leq \theta, \mathbb{L}^1_{\leq\theta}(\tau)$ has cardinality $\beth_{\theta^+}$, and it helps to have arbitrary Boolean combinations of formulas of a fix quantifier depth $[< \theta^+]$. Note that if $\varphi(\bar{x}) \in \mathbb{L}^1_\kappa(\tau)$ has infinitely many free variables, we cannot "close" it to a sentence.

3) We may instead define $\mathbb{L}^{1,*}_{\leq\theta}$ by: $\psi \in \mathbb{L}^1_{\leq\theta}(\tau)$ iff for some $\tau' \subseteq \tau$ of cardinality $\leq \theta$ and some sequence $\langle M_i : i < i(*)\rangle$ of τ'-models of cardinality $\leq 2^\theta$ and some $\gamma < \theta^+$ we have: a τ-model M satisfies ψ iff $M \mathscr{E}^1_{\mathrm{qf}(\tau'),\theta,\gamma} M_i$ for some i.

Claim 2.7. *1)* $\mathbb{L}^1_{\leq\theta}$ *is a nice logic, see Definition 1.9.*

2) The logic $\mathbb{L}^1_{\leq\theta}$ *has full substitution.*

3) $\mathbb{L}^1_{\leq\theta}(\tau)$ *has cardinality* $\leq |\tau|^\theta + \beth_{\theta^+}$ *for any vocabulary* τ.

4) If $\kappa = \beth_\kappa$ *then* $\mathbb{L}^1_\kappa$ *is a nice logic,* $\mathbb{L}^1_\kappa(\tau)$ *has cardinality* κ *whenever* τ *is a vocabulary of cardinality* $< \kappa$ *and* $\mathbb{L}^1_\kappa(\tau)$ *has cardinality* $|\tau|^{<\kappa} + \kappa$ *for any vocabulary* τ.

5) In part (4), if κ *is regular then the logic* $\mathbb{L}^1_\kappa$ *has full substitution.*

6) If $\theta, \alpha < \kappa$ *and* τ *is a vocabulary of cardinality* $< \kappa$ *then for any set* $\mathscr{U}$ *of* $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$-*equivalence classes for some* $\psi \in \mathbb{L}^1_\kappa(\tau)$ *we have* $\{M : M/\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha} \in \mathscr{U}\} = \{M : M$ *a* τ-*model of* $\psi\}$.

Proof. 1),2) Just check definition 1.9, 1.12, 1.16(5) but still we elaborate the relatively more substantial 1.12, 1.16(5).

Clause (a): <u>Atomic formulas</u>

Just note that if $M_1 \mathscr{E}^0_{\mathrm{qf}(\tau),\theta,\alpha} M_2$ for $\alpha = 0$ (any θ) <u>then</u> M_1, M_2 satisfies the same τ-atomic sentences.

Clause $(b)(\alpha)$: <u>Conjunction, (similarly Disjunction)</u>

Just note that if φ_ℓ is defined using $(\mathrm{qf}(\tau_\ell), \theta_\ell, \alpha_\ell)$ and $\tau = \tau_1 \cup \tau_2, \theta = \max\{\theta_1, \theta_2\}, \alpha = \max\{\alpha_1, \alpha_2\}$ then the equivalence relation $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$ refine the equivalence relation $\mathscr{E}^1_{\mathrm{qf}(\tau_\ell),\theta_\ell,\alpha_\ell}$ for $\ell = 1, 2$.

Clause $(b)(\beta)$: <u>Existential Quantifier</u>

Assume $\varphi \in \mathbb{L}^1_{\leq\theta}(\tau \cup \{c\})$ is defined by the triple $(\mathrm{qf}(\tau \cup \{c\}), \theta, \alpha)$ and $\varphi(x)$ is the corresponding formula from $\mathbb{L}^1_{\leq\theta}(\tau)$ and $\exists x \varphi(x)$ is the naturally defined sentence.

Now if M_1, M_2 are $\mathscr{E}^0_{\mathrm{qf}(\tau),\theta,\alpha+1}$-equivalent τ-models then $M_1 \models \exists x \varphi(x)$ iff $M_2 \models (\exists x)\varphi(x)$ by the definition of the game. Hence this holds for "M_1, M_2 are $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha+1}$-equivalent τ-models".

Clause $(b)(\gamma)$: <u>Negation</u>

Obvious by the definition because for any vocabulary τ of cardinality $\leq \theta$ the equivalence relation $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$ has $\leq \beth_{\alpha+1}(\theta)$ equivalence classes by 2.4(5).

Clause (c): Restricting to a unary predicate P

Easily if ψ is defined using $(\mathrm{qf}(\tau), \theta, \alpha)$ and without loss of generality $P \in \tau$ then so is $\psi \restriction P$.

Clause (d): We prove more: Full substitution:

Assume we are given vocabularies τ_1, τ_2 and consider substituting $\varphi_P(x_0, \ldots, x_{\mathrm{arity}_{\tau_2}(P)-1}) \in \mathbb{L}^1_{\leq\theta}(\tau_0)$ for $P \in \tau_2$ treating $F(x_0, \ldots, x_{\mathrm{arity}(F)-1}) = x_{\mathrm{arity}(F)}$ as an $(\mathrm{arity}_{\tau_2(F)+1})$-place predicate. Let φ_P be defined by $(\mathrm{qf}(\tau^{+\mathrm{arity}(P)}), \theta_P, \alpha_P)$ and let $\alpha_0 = \sup\{\alpha_P : P \in \tau_2\}$.

We are given $\psi \in \mathbb{L}^1_{\leq\theta}(\tau_1)$ and we shall find $\psi' \in \mathbb{L}^1_{\leq\theta}(\tau_2)$ which says that if we substitute $\varphi_P(x_0, \ldots, x_{\mathrm{arity}_{\tau_2}(P)})$ instead $P(x_0, \ldots)$ in ψ for every $P \in \tau_1$, we get (up to equivalence) ψ'. Let ψ be defined by $(\mathrm{qf}(\tau_1), \theta_1, \alpha_1)$. Let $\alpha_2 = \alpha_1 + \alpha_0$ and easily there is ψ' as required defined by $(\mathrm{qf}(\tau_2), \theta_1, \alpha_2)$.

3)-6) Easy. $\hfill\square_{2.7}$

Claim 2.8. *1) Let $\kappa = \beth_{\theta^+}$.*
$\mathbb{L}^1_{\leq\theta} \leq \mathbb{L}^0_\kappa$, *i.e. every formula of $\mathbb{L}^1_{\leq\theta}$ is equivalent to, hence can be looked at, as a formula of $\mathbb{L}^0_\kappa$.*
2) $\mathbb{L}^{-1}_{\leq\theta} \leq \mathbb{L}^1_{\leq\theta}$.

Remark 2.9. For many purposes we identify them, i.e. say $\mathbb{L}^1_{\leq\theta} \subseteq \mathbb{L}^0_\kappa$; this gives another reasonable version of subformulas.

Proof. 1) We first prove:

$\boxplus_1$ if $\mathbf{s}$ is a state in the game $\partial_{\mathrm{qf}(\tau),\theta,\alpha}[M_1, M_2]$ and $\beta = \beta_{\mathbf{s}}, \tau = \tau(M_\ell), \bar{a}_1 = \langle a^1_\varepsilon : \varepsilon < \varepsilon(*)\rangle \in {}^{\theta^+>}(M_1)$ list $\mathrm{Dom}(g_{\mathbf{s}})$ and $\bar{a}_2 = \langle a^2_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ where $a^2_\varepsilon = g_{\mathbf{s}}(a^1_\varepsilon)$ and $M_2 \models \varphi[\bar{a}_1] \Leftrightarrow M_2 \models \varphi[\bar{a}_2]$ for every $\varphi = \varphi(\langle x_\varepsilon : \varepsilon < \varepsilon(*)\rangle) \in L^*_\beta := \bigcup_{\zeta<\beta} \mathbb{L}_{(\beth_\zeta(\theta+|\tau|))^+,\theta^+}(\tau)$, or just $\varphi \in L^*_\beta = \mathbb{L}_{\infty,\theta^+,\beta}(\tau)$, see 1.21(5), then $\mathbf{s}$ is a winning state for the player ISO in the game.

[Why? We prove this by induction on β. First, for $\beta = 0$ this is trivial as $\mathbf{s}$ is a state. Second, for β limit, any choice of the AIS player includes an ordinal $\gamma = \beta_{n_{\mathbf{s}}+1} < \beta$, so the ISO player may "pretend" that the given state $\mathbf{s}$ has $\beta_{\mathbf{s}} = \gamma + 1$ and use the induction hypothesis. Third, if $\beta = \gamma + 1$, let the AIS player make his choice $(\beta_{n_{\mathbf{s}}+1}, \iota, A)$. Now ISO has to extend $g^\iota_{\mathbf{s}}$ adding some $\leq \theta$ elements of M_ι to its domain, the elements in $\{a \in \mathrm{Dom}(g^\iota_{\mathbf{s}}) : h^\iota_{\mathbf{s}}(a) < n_{\mathbf{s}} + 1\} \cup A$, let $\bar{b}_\iota$ list them. Let $\bar{x} = \langle x_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ and let $\bar{y} = \langle y_\varepsilon : \varepsilon < \ell g(\bar{b}_\iota)\rangle$ and define $\varphi_*(\bar{y}, \bar{x}) = \bigwedge\{\varphi(\bar{y}, \bar{x}) \in L^*_\gamma : M_\iota \models \varphi[\bar{b}, \bar{a}_\iota]\}$.

So $M_\iota \models \varphi_*[\bar{b}, \bar{a}_\iota]$ hence $M_\iota \models (\exists\bar{y})\varphi_*(\bar{y}, \bar{a}_\iota)$ hence by the assumption of $\boxplus_1$ we have $M_{3-\iota} \models ``(\exists\bar{y})\varphi_*(\bar{y}, \bar{a}_{3-\iota})"$ hence there is $\bar{b}_{3-\iota} \in {}^{\ell g(\bar{b}_\iota)}(M_{3-\iota})$ such that $M_{3-\iota} \models \varphi_*[\bar{b}_{3-\iota}, \bar{a}_{3-\iota}]$.

Now the player ISO can make its move getting the state $\mathbf{t}$, a successor of $\mathbf{s}$ such that $g_{\mathbf{t}} = g_{\mathbf{s}} \cup \{\langle b^1_\varepsilon, b^2_\varepsilon\rangle : \varepsilon < \ell g(\bar{y})\rangle, g_{\mathbf{t}}$ maps $\bar{b}_1$ to $\bar{b}_2$ and $n_{\mathbf{t}} = n_{\mathbf{s}} + 1$ and $\beta_{\mathbf{t}} = \gamma$; clearly possible. As the ordinal $\beta_{n_{\mathbf{s}}+1}$ chosen by the AIS is $\leq \gamma$ also $\mathbf{t}$ is as required in $\boxplus_1$ but so, by the induction hypothesis also it is a winning state.]

Let

$\boxplus_2$ $\mathscr{E}_{\tau,\theta,\alpha} = \{(M_1, M_2) : M_1, M_2$ are τ-models satisfying the same sentences from $L^*_\alpha\}$.

Now we shall prove $\boxplus_3$ and by $\boxplus_3(d)$ we are done.

$\boxplus_3$ (a) $\mathscr{E}_{\tau,\theta,\alpha}$ is an equivalence relation with $\leq (\beth_{\alpha+1}(\theta + |\tau|))$ equivalence classes

(b) $\mathscr{E}_{\tau,\theta,\alpha}$ is an equivalence relation included in $\{(M_1, M_2) : M_1, M_2 \text{ are } \tau\text{-models such that the player ISO wins in } \partial_{\mathrm{qf}(\tau),\theta,\alpha}[M_1, M_2]\}$

(c) $\mathscr{E}_{\tau,\alpha,\theta}$ refines the equivalence relation $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$

(d) $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$ has $\leq \beth_{\alpha+1}(\theta + |\tau|)$-equivalence classes.

[Why? Clause (a) follows from the number of such sentences being $\leq \beth_\alpha(\theta + |\tau|)$, clause (b) follows by $\boxplus_1$, clause (c) follows from (b) and the definitions of $\mathscr{E}^0_{\mathrm{qf}(\tau),\theta,\alpha}$ and $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$. Lastly, clause (d) follows from clauses (a) + (b) + (c).
2) Easy. $\square_{2.8}$

Remark 2.10. 1) The proof above may seem wasteful <u>but</u> for our purposes this is immaterial.
2) We can do better as follows, i.e. another proof of 2.8 runs as follows:
By induction on the formula. Without loss of generality we deal with a formula $\Theta_{N,\bar{d},\theta,\alpha,\Gamma}(\bar{x})$ by induction on $\beta \leq \alpha$ we prove:

$\boxplus_1$ if $n \in [-1, \omega)$ and $\gamma(1), \gamma(2) < \theta^+, \bar{b} \in {}^{\gamma(2)}N, \mathbf{h}_\ell : \gamma(\ell) \to \omega$ letting $\bar{x} = \langle x_i : i < \gamma(2)\rangle; u_\ell \subseteq \gamma(\ell)$ for $\ell = 1, 2, \mathbf{g}$ a one-to-one mapping from u_1 onto u_2 (so $n = -1 \Rightarrow \gamma(1), \gamma(2) = 0$) <u>then</u> for some formula $\varphi(\bar{z}, \bar{x}) = \varphi^0_{N,\bar{d},\theta,\alpha,\Gamma,\mathbf{h}_1,\mathbf{h}_2,g,\beta}(\bar{x}) \in \mathbb{L}^0_\kappa$ we have: for any model M of the vocabulary of N and $\bar{c} \in {}^{(\ell g(\bar{d}))}M$ and $\bar{a} \in {}^{\ell g(\bar{b})}M$ letting
  • $A^1_n = \mathrm{Rang}(\bar{c}), A^2_n = \mathrm{Rang}(\bar{d})$
  • g_n has domain $\{c_i : i \in u\}$ and $g_n(c_i) = d_{\mathbf{g}(i)}$ for $i \in u$
  • $h^\ell_n(c_i) = d_{\mathbf{h}_\ell}(i)$
the following are equivalent:
  (i) $M \models \varphi[\bar{c}, \bar{a}]$
  (ii) $(A_n, B_n, h^1_n, h^2_n, g_n, \beta)$ is a winning position for the equivalence player in the game $\partial_{N,\theta,\alpha,\Gamma}[(M, \bar{c}), (N, \bar{d})]$
$\boxplus_2$ similarly a formula $\varphi^1_{N,\bar{c},\theta,\alpha,\Gamma,h,h',\beta}(\bar{z}, \bar{x})$ expressing situation after the AIS player moves.

The proof is straightforward.

The Crucial Claim 2.11. *We have $M_n \equiv_{\mathbb{L}^1_\theta} M_\omega$ for $n < \omega$ and even $M_n \models \psi[\bar{a}] \Leftrightarrow M_\omega \models \psi[\bar{a}]$ <u>when</u>:*

(a) $\psi(\bar{z}) \in \mathbb{L}^1_{\leq\theta}(\tau)$ *a formula*

(b) $M_n \prec_{\mathbb{L}_{<\partial,\theta^+}} M_{n+1}$ *where $\partial = \beth_{\theta^+}$*

(c) $M_\omega := \bigcup_{n<\omega} M_n$

(d) $\bar{a} \in {}^{\ell g(\bar{z})}(M_0)$

(e) $\tau = \tau(M_n)$ *for $n < \omega$.*

Remark 2.12. In fact the logic $\mathbb{L}^0_{\partial^+,\theta^+,\theta^+}$ suffice, see Definition 1.21(2).

Proof. Without loss of generality $|\tau| \leq \theta$. We shall prove by induction on $\beta < \theta^+$ the following

$\boxplus_\beta$ **s** is a winning state for the ISO player in the game $\partial_{\mathrm{qf}(\tau),\theta,\alpha}[M_n, M_\omega]$ <u>when</u>
    for some k we have
    (a) **s** is a state in this game
    (b) $n \leq k < \omega$
    (c) $\beta_{\mathbf{s}} = \beta$
    (d) $\mathrm{Rang}(g_{\mathbf{s}}^2) \subseteq M_k$
    (e) if $b \in A_{\mathbf{s}}^2$ then $b \in M_{h_{\mathbf{s}}^2(b)}$
    (f) $A_{\mathbf{s}}^1 \subseteq \mathrm{Dom}(f_{\mathbf{s}}^1)$
    (g) letting $\bar{a}^1$ list $\mathrm{Dom}(g_{\mathbf{s}})$ and $\bar{a}^2$ be $\langle g_{\mathbf{s}}(a_i^1) : i < \ell g(\bar{a}^1)\rangle$ and $\lambda_\beta = \beth_\beta(\theta)$
        for every $\varphi(\bar{x}) \in \mathbb{L}_{\lambda_\beta^+,\theta^+}$ we have $M_n \models \varphi[\bar{a}^1] \Leftrightarrow M_k \models \varphi[\bar{a}^2]$.

The case $\mathbf{s} = \mathbf{s}_\beta^0$, the initial state, suffice to prove the desired result.

<u>First Case</u>: $\beta = 0$.
    Trivial.

<u>Second Case</u>: β a limit ordinal.
    If the AIS makes its move choosing (A, ι, β_1), the ISO player may pretend $\beta_{\mathbf{s}} = \beta + 1$ and use the induction hypothesis, this is O.K. as $\beta < \beta + 1$, as in proving 2.8.

<u>Third Case</u>: $\beta = \gamma + 1$.
    As in the proof of 2.8. $\square_{2.11}$

We can sum up the easy properties but first we present two definitions.

Definition 2.13. 1) We say that N code $[N]^{\leq\lambda}$ <u>when</u> for some two-place predicate R we have $\{\mathrm{set}(b, N) : b \in N\}$ is a cofinal subset of $([N]^{\leq\lambda}, \subseteq)$, where:
2) In this case for $b \in N$ let $\mathrm{set}(b, N) = \mathrm{set}_R(b, N) = \{a : aR^N b\}$ and if R is clear from the context we may omit it.

For our results it seems helpful to define some variants of "$\mathscr{L}$ satisfies the downward LST".

Definition 2.14. Let $\mathscr{L}$ be a logic, τ_0 a vocabulary $\psi_0 \in \mathscr{L}(\tau_0)$ a sentence with τ_0 of minimal cardinality <u>or</u> $T \subseteq \mathscr{L}(\tau_0)$, i.e. a set of sentences with $|\tau_0|$ minimal.
1) $\mathscr{L}$ satisfies $\mathrm{LST}_{<\kappa}^\iota$ <u>when</u> for every vocabulary τ of cardinality $< \kappa$ and $\varphi \in \mathscr{L}(\tau)$ we have $\mathrm{LST}_{<\kappa}^\iota(\varphi)$, see below; if $\iota = 0$ we may omit it.
    If $\kappa = \lambda^+$ we may write "λ" instead "$< \kappa$". $\mathrm{LST}_{<\kappa}^\iota(\psi)$ means $\mathrm{LST}_{<\kappa}^\iota(\{\psi\})$ similarly below and $\mathrm{LST}_{<\kappa}^\iota(\leq \theta)$ means $\mathrm{LST}_{<\kappa}^\iota(T)$ when $|T| \leq \theta$; the reader may concentrate on the case $\kappa = \lambda^+$.
2) For $\iota = 0$, let $\mathrm{LST}_{<\kappa}^\iota(T)$ means: if T has a model N of cardinality $\geq \kappa$ <u>then</u> it has a model M of cardinality λ for arbitrary large cardinals $\lambda < \kappa$.
3) If $\iota = 1$, similarly but $M \subseteq N$ (moreover can demand $A \subseteq M$ for a given $A \in [N]^{<\kappa}$.
4) For $\iota = 2$, let $\mathrm{LST}_{<\kappa}^\iota(T)$ means: if N is a model of T of cardinality $\geq \kappa$ <u>then</u> for arbitrarily large $\lambda < \kappa$ we have: if N^+ is an expansion of N and $\tau(N^+)$ has cardinality $\leq \lambda$ <u>then</u> some $M^+ \subseteq N^+$ of cardinality λ satisfies $M = M^+{\upharpoonright}\tau_T$ is a model of T.

5) For $\iota = 3$, as in $\iota = 2$, but we further assume that N^+ code $[N]^{\leq \lambda}$, see 2.13 below and we further conclude that for some $\langle d_n : n < \omega \rangle \in {}^\omega(M^+)$ we have $|M^+| = \cup \{\mathrm{set}(d_n, N^+) : n < \omega\}$.

6) We define $\mathrm{LST}^{\iota,*}_{<\kappa}(T)$ similarly but for a pregiven $\mu < \kappa$ can demand $\|M\| = \|M\|^\mu$ and similarly $\mathrm{LST}^{\iota,*}_{<\kappa}$.

Now we can sum up what we have learned on this logic.

Conclusion 2.15. Assume $\theta < \kappa = \beth_\kappa$.

1) $\mathbb{L}^1_{\leq \theta}$ is a nice logic with full substitution, see Definition 1.12.

2) $\mathbb{L}^1_\kappa$ is a nice logic and if κ is regular (equivalently strongly inaccessible) then it has full substitution.

3) $\mathbb{L}^{-1}_\kappa \leq \mathbb{L}^1_\kappa \leq \mathbb{L}^0_\kappa$, in fact $\mathbb{L}_{\theta^+,\aleph_0} \leq \mathbb{L}^1_{\leq \theta} \leq \mathbb{L}_{\beth_{\theta^+},\theta^+}$ (really $\mathbb{L}^1_{\leq \theta} \leq \mathbb{L}_{\beth_{\theta^+},\theta^+,\theta^+}$ and even $\leq \mathbb{L}_{\infty,\theta^+,\theta^+}$).

4) $\mathbb{L}^1_\kappa$ satisfies the following versions of the downward LST:

- (a) every sentence $\psi \in \mathbb{L}^1_\kappa$ which has a model has a model of cardinality $< \kappa$
- (b) $\mathbb{L}^1_\kappa$ has $\mathrm{LST}^\iota_{<\kappa}$ for $\iota \leq 3$ (see 2.14 above)
- (c) for every $\psi \in \mathbb{L}^1_\kappa$ there is $\partial < \kappa$ such that: if N is a model of ψ of cardinality λ and $\mu = \mu^{<\partial} \leq \lambda$ or at least $\mu = \sum_{n<\omega} \mu_n \leq \lambda$ and $(\mu_n)^{<\partial} = \mu_n$ for $n < \omega$ <u>then</u> some submodel M of N of cardinality μ is a model of ψ.

Proof. 1) By 2.7.

2) Follows as $\mathbb{L}^1_{\leq \theta}$ is $\subseteq$-increasing with θ and the defintion of $\mathbb{L}^1_\kappa$ in Definition 2.5(2).

3) We have $\mathbb{L}^1_{\leq \theta} \leq \mathbb{L}_{\beth_{\theta^+},\theta^+}$ (and moreover $\mathbb{L}^1_{\leq \theta} \leq \mathbb{L}_{\beth_{\theta^+},\theta^+,\theta^+}$) by 2.8. So by the definition, $\mathbb{L}^1_\kappa \leq \mathbb{L}^0_\kappa$. Note that if M_1, M_2 are $\mathscr{E}^1_{\mathrm{qf}(\tau),\aleph_0,\alpha}$-equivalent then $M_1 \models \psi \Leftrightarrow M_2 \models \psi$ for every $\psi \in \mathbb{L}_{\beth^+_\alpha,\aleph_0,\alpha}$ hence $\mathbb{L}_{\theta^+,\aleph_0} \leq \mathbb{L}^1_{\leq \theta}$ and so by the definitions, $\mathbb{L}^{-1}_\kappa \leq \mathbb{L}^1_\kappa$.

4) Clause (a) follows by part (3) and the parallel results on $\mathbb{L}_{\lambda^+,\theta^+}$ of Hanf (see, e.g. [?]). Clauses (b),(c) follows by 2.11. $\qquad \square_{2.15}$

§ 3. Serious properties of $\mathbb{L}^1_\kappa$

First we prove a strong form of non-definability of well ordering.

Claim 3.1. *Let $\kappa = \beth_\kappa$ so a strong limit cardinal.*
1) Property (d) of Problem 1.5 holds.
2) Moreover, if (A) then (B) <u>where</u> τ is a vocabulary to which the predicates $P, <, R$ (unary, binary, binary) belongs and:

- (A) (a) $\psi \in \mathbb{L}^1_{\leq\theta}(\tau)$ *is defined using* $(\mathrm{qf}(\tau), \theta, \alpha)$
    - (b) $\theta \geq |\tau|$
    - (c) $\partial = \beth_{\alpha+1}(\theta)$
    - (d) $\mu = 2^\partial$
    - (e) $M \models \psi$
    - (f) $(P^M, <^M) \cong ((2^\mu)^+, <)$
    - (g) M *is a τ-model satisfying:*
        $\{\{a : aR^M b\} : b \in M\} = [M]^{\leq\mu}$
        or just the former family is cofinal in the latter family (both ordered by inclusion)
- (B) *there are a τ-model N and a sequence $\langle b_n : n < \omega \rangle$ such that:*
    - (α) $N \models \psi$
    - (β) $(P^N, <^N)$ *is not well ordered*
    - (γ) $b_n \in N$ *and* $N \models (\forall x)(xRb_n \to xRb_{n+1})$, *may add:* $(\forall x)(xRb_n \to (\forall y)(yRx \to yRb_{n+1}))$
    - (δ) $N \models (\forall b) \bigvee_{n<\omega} [bRb_n]$.

Proof. 1) Follows by (2).
2) Without loss of generality $|M|$ is an ordinal, $P^M = (2^\mu)^+$ and $<^M$ is the usual order of the ordinals on $|M|$. By induction on n we choose a sequence $\langle (M_{n,\gamma}, \bar{\beta}_{n,\gamma}, \bar{b}_{n,\gamma}) : \gamma < (2^\mu)^+ \rangle$ such that

$\boxplus_1$ (a) $\bar{\beta}_{n,\gamma} = \langle \beta_{n,\gamma,\ell} : \ell < n \rangle$ is a decreasing sequence of ordinals from P^M, which are $> \gamma$ and $\bar{b}_{n,\gamma} = \langle b_{n,\gamma,\ell} : \ell < n \rangle$ is a sequence of members of M

- (b) $M_{n,\gamma} \prec_{\mathbb{L}_{\partial^+,\theta^+}} M, \|M_{n,\gamma}\| = \mu$ and $b_{n,\gamma,\ell}, \beta_{n,\gamma,\ell} \in M_{n,\gamma}$ for $\ell < n$ (recall $\mu = 2^\partial$)

- (c) for $\gamma_1 < \gamma_2 < (2^\mu)^+$ we have $(M_{n,\gamma_1}, \bar{b}_{n,\gamma_1}, \bar{\beta}_{n,\gamma_1}) \cong (M_{n,\gamma_2}, \bar{b}_{n,\gamma_2}, \bar{\beta}_{n,\gamma_2})$, note that the isomorphism is unique as $<^M$ is a well ordering and necessarily the isomorphism maps $b_{n,\gamma_1,\ell}$ to $b_{n,\gamma_2,\ell}$ and $\beta_{n,\gamma_1,\ell}$ to $\beta_{n,\gamma_2,\ell}$ for $\ell < n$. Let $\pi_{n,\gamma_2,\gamma_1}$ be this isomorphism from M_{n,γ_1} onto M_{n,γ_2}

- (d) if $n = m+1$ then
    - (i) $\bar{\beta}_{n,\gamma} \restriction m = \bar{\beta}_{m,\zeta}$ and $\bar{b}_{n,\gamma} \restriction m = \bar{b}_{m,\zeta}$ when $\zeta = \beta_{n,\gamma,m}$
    - (ii) $M_{m,\beta_{n,\gamma,m}} \prec_{\mathbb{L}_{\partial^+,\theta^+}} M_{n,\gamma}$
    - (iii) $a \in M_{m,\beta_{n,\gamma,m}} \Rightarrow M \models aRb_{n,\gamma}$
    - (iv) if $b \in M_{m,\beta_{n,\gamma,m}}$ then $\{a : M \models aRb\} \subseteq \{a : M \models aRb_{n,\gamma}\}$
    - (v) $\pi_{m,\zeta_2,\zeta_2} \subseteq \pi_{n,\gamma_2,\gamma_1}$ when $\zeta_\ell = \beta_{n,\gamma_\ell,m}$ for $\ell = 1, 2$

(e) if $n > m$ then $(i) - (v)$ of (d) holds (follows).

For $n = 0$ just choose $M_{0,\gamma} = M_0 \prec_{\mathbb{L}_{\partial^+,\theta^+}} M, \|M_0\| = \mu$, (OK as $\mu^\partial = \mu$, in fact an overkill).

For $n = m + 1$, choose $M'_{m,\gamma}, b'_{m,\gamma}$ such that:

$\boxplus_2$ (i) if $a \in M_{m,\gamma}$ then $aR^M b'_{m,\gamma}$

(ii) moreover if $b \in M_{m,\gamma}$ then $\{a \in M : M \models aRb\} \subseteq$
$\{a \in M : M \models aRb'_{m,\gamma}\}$

(iii) $M_{m,\gamma} \cup \{b'_{m,\gamma}\} \cup \{\gamma\} \subseteq M'_{m,\gamma}$

(iv) $M'_{m,\gamma} \prec_{\mathbb{L}_{\partial^+,\theta^+}} M$

(v) $M'_{m,\gamma}$ has cardinality μ.

The number of $(M'_{m,\gamma}, M_{m,\gamma}, \bar{\beta}_{m,\gamma+1}, \bar{b}_{n,\gamma+1}, \gamma, b'_{m,\gamma})/ \cong$ is $\leq 2^\mu$ so for some unbounded $Y_n \subseteq (2^\mu)^+$, the models $\langle (M'_{m,\gamma}, M_{m,\gamma+1}, \bar{\beta}_{m,\gamma+1}, \bar{b}_{n,\gamma+1}, \gamma, b'_{m,\gamma}) : \gamma \in Y_n \rangle$ are pairwise isomorphic.

For any $\zeta < (2^\mu)^+$ let $\gamma_\zeta = \text{Min}(Y \backslash (\zeta+1)), M_{n,\zeta} = M'_{m,\gamma_\zeta}, \bar{b}_{\zeta,n} = \bar{b}_{m,\gamma_\zeta} \hat{\ } \langle b'_{m,\gamma_\zeta} \rangle, \bar{\beta}_{n,\zeta} = \bar{\beta}_{m,\gamma_\zeta} \hat{\ } \langle \gamma_\zeta \rangle$. Clearly $\langle (M_{n,\zeta}, \bar{b}_{n,\zeta}, \bar{\beta}_{n,\zeta}) : \zeta < (2^\mu)^+ \rangle$ is as required.

Having carried the induction, it is easy to find models N_n and elements a_ℓ, b_ℓ for $\ell < n$ by induction on $n < \omega$ such that:

$\boxplus_3$ (a) $(N_n, a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ is isomorphic to $(M_{\gamma,n}, \bar{\beta}_{\gamma,n}, \bar{b}_{\gamma,n})$
for every $\gamma < (2^\mu)^+$

(b) $N_m \prec_{\mathbb{L}_{\partial^+,\theta^+}} N_n$ if $m < n$.

Now easily by the crucial claim 2.11 the model $N = \bigcup_n N_n$ and the sequence $\bar{b} = \langle b_n : n < \omega \rangle$ are as required in clause (B) with $\langle a_n : n < \omega \rangle$ witnessing clause (β) of clause (B). $\square_{3.1}$

Conclusion 3.2. For $\kappa = \beth_\kappa$, the logic $\mathbb{L}^1_\kappa$ satisfies:
$\underline{\text{SUDWO}^1_\kappa}$ (strong undefinability of well ordering) which means:
if $\psi \in \mathbb{L}^1_\kappa(\tau), |\tau| < \kappa$ and $<, R$ are two place predicates and c_1, c_2 individual constants from τ <u>then</u> for every large enough $\mu_1 < \kappa$ for arbitrarily large enough $\mu_2 < \kappa$ we have:

⊛ <u>if</u> $\lambda > \mu_2$ and $\mathfrak{A}$ is a τ-expansion of $(\mathscr{H}(\lambda), \in, \mu_1, \mu_2, <)$ with $<$ the order on the ordinals and $R^\mathfrak{A}$ being $\in$, that is $\in \restriction \mathscr{H}(\lambda), c_1^\mathfrak{A} = \mu_2$ and $c_2^\mathfrak{A} = \mu_2$
<u>then</u> we can find $\mathfrak{B}, a_n, d_n$ (for $n < \omega$) such that

(a) $\mathfrak{B} \models \psi \Leftrightarrow \mathfrak{A} \models \psi$

(b) $\mathfrak{B} \models$ "$d_{n+1} < d_n < \mu_2$" for $n < \omega$

(c) $\mathfrak{B} \models$ "$a_n \subseteq a_{n+1}$ has cardinality $\leq \mu_1$"

(d) if $e \in \mathfrak{B}$ then $\mathfrak{B} \models$ "$e \in a_n$" for some n
we may add

$(d)^+$ if $e \in \mathfrak{B}$ and $\mathfrak{B} \models$ "$|e| \leq \mu_1$" <u>then</u> $\mathfrak{B} \models$ "$e \subseteq a_n$" for some n.

Remark 3.3. There are other variants. At the moment the distinction is not crucial.

Proof. By 3.1. $\square_{3.2}$

Theorem 3.4. _First Characterization Theorem_

We have $\mathscr{L} \leq \mathbb{L}^1_\kappa$, i.e. every $\psi \in \mathscr{L}$ is equivalent to some $\psi' \in \mathbb{L}^1_\kappa$ (i.e. for any vocabulary $\tau, \ldots$) _when_ :

$\boxplus$ (a) $\kappa = \beth_\kappa$ hence is a strong limit uncountable cardinal

 (b) $\mathscr{L}$ is a nice logic

 (c) Υ, the occurance number of $\mathscr{L}$, is $\leq \kappa$, see Definition 1.11

 (d) $\mathbb{L}^{-1}_\kappa \leq \mathscr{L}$, i.e. $\mathbb{L}_{\theta^+,\aleph_0} \leq \mathscr{L}$ for $\theta < \kappa$ or just

 $(d)^-$ for every ordinal $\alpha < \kappa$, the model M_α expanding $(\alpha, <)$ is characterized up to isomorphism by some sentence from $\mathbb{L}^1_\kappa(\tau(M_\alpha))$, so $|\tau(M_\alpha)| < \kappa$

 (e) $\mathbb{L}$ satisfies SUDWO^1_κ.

Remark 3.5. 1) We could add the downward LST and restrict somewhat SUDWO_κ.
2) As $\mathbb{L}^1_\kappa$ satisfies the demands (by 2.15) we can rephrase Theorem 3.4 as: $\mathbb{L}$ is $\leq$-maximal such that (a)-(e) holds.

Proof. Assume toward contradiction that $\psi^* \in \mathscr{L}(\tau)$ is a sentence which is not equivalent to any sentence of $\mathbb{L}^1_\kappa(\tau)$. As the occurance number of $\mathscr{L}$ is $\leq \kappa$ by clause (c) of the assumption, without loss of generality the vocabulary τ has cardinality $< \Upsilon \leq \kappa$. Similarly by $\mathscr{L}$ being a logic, using clauses (c),(d) of Definition 1.9, without loss of generality the symbols we add to τ in $(*)_3(a)$ below do not belong to τ. We shall derive another sentence ψ^{**} in a somewhat bigger vocabulary, and let (μ_1, μ_2) be as in Definition 3.2 for ψ^{**} and $\mu_1 \geq \theta_*$ where $\theta_* = |\tau| + \aleph_0$; we may add ψ^* is defined by $(\mathrm{qf}(\tau^*), \theta^*, \alpha^*)$ and let $\theta_* = |\tau| + \theta^* + |\alpha^*| + \aleph_0$.

Let

$(*)_0$ (a) $\{P_i : i < i(*)\}$ list the predicates in τ

 (b) $\{H_j : j < j(*)\}$ list the function symbols in τ

 (c)   let the vocabulary τ_1 be

 •   the set of predicates $\{P'_i : i \leq i(*)\}$

 •   the set of function symbols $\{H'_j : j < j(*)\}$

 • $\mathrm{arity}_{\tau_1}(P'_{i(*)}) = 2$

 • $\mathrm{arity}_{\tau_1}(P'_i) = \mathrm{arity}_\tau(P_i) + 1$ for $i < i(*)$

 • $\mathrm{arity}_{\tau_1}(H'_j) = \mathrm{arity}_\tau(H_j) + 1$ for $j < j(*)$

 (d)   for a τ_1-structure $\mathfrak{B}$ and $c \in \mathfrak{B}$ let $N^{\mathfrak{B}}_c$ be the τ-model (if exists) such that

 •   it has universe $\{d : \mathfrak{B} \models P'_{i(*)}(d, d)\}$

 • $(P_i)^{N^{\mathfrak{B}}_c} = \{\bar{a} : \langle c \rangle ^\frown \bar{a} \in (P'_i)^{\mathfrak{B}}\}$ for $i < i(*), j < j(*)$

 • $H^{N^{\mathfrak{B}}_c}_j(\bar{a}) = (H'_j)^{\mathfrak{B}}(\langle c \rangle ^\frown \bar{a})$ for $i < i(*)$.

For each $\alpha < \kappa$, let $\theta(\alpha) = \theta_\alpha = |\alpha| + \aleph_0$, so the equivalence relation $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta(\alpha),\alpha}$ is well defined, and has $< \beth_{\theta(\alpha)^+}$ equivalence classes say $\langle N_{\alpha,\varepsilon}/\mathscr{E}^1_{\mathrm{qf}(\tau),\theta(\alpha),\alpha} : \varepsilon < \varepsilon_\alpha < \beth_{\theta(\alpha)^+}(|\tau|) \rangle$.

By the definition of $\mathbb{L}^1_{\leq\theta}(\tau)$ which is $\leq \mathbb{L}^1_\kappa(\tau)$, for each such pair (α, ε) there is a sentence $\vartheta_{\alpha,\varepsilon} \in \mathbb{L}^1_{\leq\theta(\alpha)}(\tau)$ which define $N_{\alpha,\varepsilon}/\mathscr{E}^1_{\mathrm{qf}(\tau),\theta(\alpha),\alpha}$ and moreover for every

$u \subseteq \varepsilon_\alpha$ the sentence $\vartheta_{\alpha,u} = \vee\{\vartheta_{\alpha,\varepsilon} : \varepsilon \in u\}$ belongs to $\mathbb{L}^1_\kappa(\tau)$, i.e. up to equivalence. Hence by our assumption toward contradiction for some $\zeta(\alpha) < \varepsilon_\alpha$ there are

$(*)_1$ (a) $M_\alpha, N_\alpha \in N_{\alpha,\zeta(\alpha)}/\mathscr{E}^1_{\mathrm{qf}(\tau),\theta(\alpha),\alpha}$

       (b) $N_\alpha \models \psi^*$ but $M_\alpha \models \neg\psi^*$.

By the definition of $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta(\alpha),\theta}$ there is a sequence $\bar{M}_\alpha$ such that

$(*)_2$ (a) $\bar{M}_\alpha = \langle M_{\alpha,k} : k \leq k(\alpha)\rangle$ so $k(\alpha) < \omega$

       (b) $M_{\alpha,0} = M_\alpha, M_{\alpha,k(\alpha)} = N_\alpha$

       (c) $M_{\alpha,k}, M_{\alpha,k+1}$ are $\mathscr{E}^0_{\mathrm{qf}(\tau),\theta(\alpha),\alpha}$-equivalent for $k < k(\alpha)$

       (d) let $M_{\alpha,k} \cong M_{\alpha,k(\alpha)}$ for $k \in (k(\alpha),\omega)$.

Without loss of generality the universes of the models $M_{\alpha,k}(\alpha < \kappa, k < \omega)$ are pairwise disjoint and $\tau \subseteq \mathscr{H}_{<\aleph_0}(|\tau|)$. For large enough λ we have $\langle\langle M_{\alpha,k} : k \leq k(\alpha)\rangle : \alpha < \kappa\rangle \in \mathscr{H}(\lambda)$ for every $\alpha < \kappa$ and so $\lambda > \kappa$; let $\mathfrak{A}_\lambda = \mathfrak{A}(\lambda)$ be a model such that:

$(*)_4$ (a) the vocabulary τ_2 of $\mathfrak{A}_\lambda$ is $\tau_1 \cup \{<, E, R, R_1, F_1, F_2\} \cup \{c_\alpha : \alpha \leq \theta^+_*\}$, with c_α individual constants, F_1, F_2 unary function symbols, $<, E$ binary predicates

       (b) universe $\mathscr{H}(\lambda)$

       (c) among the elements $c_\alpha^{\mathfrak{A}(\lambda)}, (\alpha < \theta_*)$ will be $\kappa, \theta_*, \theta^+_*, \mu_1, \mu_2, \tau, \tau_1$ and every symbol in τ and $\alpha \leq \theta_* \Rightarrow c_{\theta^*+\alpha}(\mathfrak{A}) = \alpha$; so $c_0^{\mathfrak{A}(\lambda)} = \kappa, c_1^{\mathfrak{A}(\lambda)} = \theta_*$, etc.

       (d) the functions $\omega\alpha + k \mapsto M_{\alpha,k}$ and $\alpha \mapsto k(\alpha)$ are $F_1^{\mathfrak{A}(\lambda)}, F_2^{\mathfrak{A}(\lambda)}$ respectively

       (e) $<^{\mathfrak{A}(\lambda)}$ is the order on the ordinals

       (f) $E^{\mathfrak{A}(\lambda)}$ is an equivalence relation such that the equivalence classes of $E^{\mathfrak{A}}$ are $\{|M_{\alpha,k}| : \alpha < \kappa$ and $k < \omega\}$

   $(g)(\alpha)$ $R^{\mathfrak{A}(\lambda)} = \{(a,b) : a \in b \in \mathscr{H}(\lambda)$ and $|b| \leq \mu_1\}$

       (β) $R_1^{\mathfrak{A}(\lambda)} = \in \restriction \mathscr{H}(\lambda)$

       (h) $(P'_{i(*)})^{\mathfrak{A}(\lambda)} = \{(a, \omega\alpha + k) : k < \omega, \alpha < \kappa$ and $a \in M_{\alpha,k}\}$

       (i) $(P'_i)^{\mathfrak{A}(\lambda)} = \{\bar{a}^\frown\langle\omega\alpha + k\rangle : \alpha < \kappa, \bar{a} \in P_i^{M_{\alpha,k}}$ and $k < \omega\}$ for $i < i(*)$

       (j) $H_j^{\mathfrak{A}(\lambda)}$ is an $(\mathrm{arity}_\tau(H_j)+1)$-place function satisfying $H_j^{\mathfrak{A}}(\bar{a}^\frown\langle\omega\alpha + k\rangle) = H_j^{M_{\alpha,k}}(\bar{a})$ for $j < j(*), \alpha < \kappa$ and $k < \omega$.

Let ψ^{**} says:

$(*)_5$ (a) one first order sentence saying all relevant (set-theoretic) properties

       (b) description of τ, τ_1

       (c) $(\forall\alpha < \kappa)(M_{\alpha,0} \models \psi^*$ and $M_{\alpha,k(\alpha)} \models \neg\psi^*)$.

Why possible? For clauses (a),(b) as $\mathbb{L}^{-1}_\kappa \leq \mathscr{L}$, by clause (d) of the assumption or just ensuring $\{a : a < c_1\}$ isomorphic to θ^+_* if we just assume clause (d)$^-$ of the assumption and then the rest is said by a first order sentence. For clause (c) as the

logic has restriction and weak substitution (see clauses (c),(d) of Definition 1.12) recalling the logic is closed under the first order operations.

Now apply SUDWO^1_κ to $\psi^{**}, (\mathfrak{A}_\lambda, \mu_1, \mu_2)$, and we get $\mathfrak{B}, a_n, d_n (n < \omega)$ as there. Now

$(*)_6$ without loss of generality τ, τ_1 are interpreted in $\mathfrak{B}$ as τ, τ_1 respectively; similarly the individual constant c_α for $\alpha < \theta^+_*$ is interpreted as $c_\alpha^{\mathfrak{A}(\lambda)}$, hence if $\mathfrak{B} \models$ "c is an ordinal $< \kappa$ and $k(c) = F_2(c)$" then for $k < k(c), m \le k(c)$ we have:

(a) $M^{\mathfrak{B}}_{c,m}$ defined naturally as $N^{\mathfrak{B}}_{F^{\mathfrak{B}}_1(c,m)}$ is a τ-model (i.e. no non-standard predicates!)

(b) $M^{\mathfrak{B}}_{c,0} \models$ "ψ" and $M^{\mathfrak{B}}_{c,k(c)} \models$ "$\neg\psi$"

(c) $\mathfrak{B} \models$ "in the game $\beth_{\mathrm{qf}(\tau),\theta_*,c}[M^{\mathfrak{B}}_{c,k}, M^{\mathfrak{B}}_{c,k+1}]$ the ISO player wins"

$(*)_7$ let $k(*) = F^{\mathfrak{B}}_2(d_0)$

$(*)_8$ we fix a winning strategy $\mathbf{st}_k$ for ISO in $\mathfrak{B}$'s sense in the game $\beth_{\mathrm{qf}(\tau),\theta_*,d}[N^{\mathfrak{B}}_{d_0,k}, N^{\mathfrak{B}}_{d_0,k+1}]$.

Now for each $k < k(*)$ by induction on $n < \omega$ we choose $\mathbf{s}^{n,k} \in \mathfrak{B}$ such that:

$(*)_9$ (a) $\mathfrak{B} \models$ "$\bar{\mathbf{s}}^{n,k} = \langle \mathbf{s}_{0,k}, \ldots, \mathbf{s}_{n,k} \rangle$ is an initial segment of a play of the game $\beth_{\mathrm{qf}(\tau),\theta_*,d_0}[M^{\mathfrak{B}}_{d_0,k}, M^{\mathfrak{B}}_{d_0,k+1}]$"

(b) $\mathfrak{B} \models$ "in this initial segment $\mathbf{s}^{n,k}$ the ISO player uses his winning strategy $\mathbf{st}_k$"

(c)   the AIS player chooses:

(α)   if n is even then $\iota_n = 1$ and $A'_n = \{e \in \mathfrak{B} : e \in M^{\mathfrak{B}}_{d_0,k}$ that is $\mathfrak{B} \models$ "$P'_{i(*)}(e, F_1(d_0, k))$" and $\mathfrak{B} \models$ "eRa_n"

(β)   if n is odd then $\iota_n = 2$ and $A'_n = \{e \in \mathfrak{B} : e \in M^{\mathfrak{B}}_{d_0,k+1}$ and $\mathfrak{B} \models$ "eRa_n"$\}$

(γ)   the "ordinal" $\beta_{\mathbf{s}_{n,k}} = d_{n+1}$.

This can be done and $g_k = \{(a_1, a_2):$ for some n we have $\mathfrak{B} \models$ "$g^{\mathbf{s}_{n,k}}_n(a_1) = a_2$"$\}$, it is an isomorphism from $M^{\mathfrak{B}}_{d_0,k}$ onto $M^{\mathfrak{B}}_{d_0,k+1}$. As this holds for every $k < k(*)$ we get that $M^{\mathfrak{B}}_{d_0,0}, M^{\mathfrak{B}}_{d_0,k(*)}$ are isomorphic so by $(*)_6(b)$ we get a contradiction. $\square_{3.4}$

Definition 3.6. Let $\Theta_{\theta,R}$ be the sentence (for R a binary predicate) such that $M = (|M|, R^M) \models \Theta_{\theta,R}$ iff $\mathscr{P} = \{\{a : aR^M b\} : b \in M\}$ is a $(\theta, \aleph_0)$-cover of $\mathrm{Dom}(P^M)$ which means: it is a family of subsets of $\mathrm{Dom}(R) = \{a : aR^M b$ for some $b \in M\}$ each of cardinality $\le \theta$ such that any $u \in [\mathrm{Dom}(R^M)]^{\le\theta}$ is included in the union of countably many such sets.

Claim 3.7. *1)* $\Theta_{\theta,R}$ *can be expressed by a sentence in* $\mathbb{L}^1_{\le\theta^+}(\{R\})$.
2) In fact, if M_1, M_2 *are* $\mathscr{E}^1_{\mathrm{qf}(\{R\}),\theta^+,\omega+\omega+1}$*-equivalent* $\{R\}$*-models then* $M_1 \models \Theta_{\theta,R} \Leftrightarrow M_2 \models \Theta_{\theta,R}$.
3) If $M_1 \mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha} M_2$ *and* $\alpha \ge \omega+\omega+1$ *and* $\mathrm{cf}(\theta) \ge \aleph_0$ *then* $\|M_1\| = \|M_1\|$ *or* $\|M_1\|, \|M_2\| > \theta$.
4) If $M_1 \mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha} M_2$ *and* $\|M_1\| \le \theta$ *and* $\alpha \ge \omega+\omega+1$ *then* $M_1 \cong M_2$.

Proof. 1) By (2).

2) Toward contradiction, assume this fails; and without loss of generality $M_1 \mathscr{E}^0_{\mathrm{qf}(\{R\}),\theta^+,\omega+\omega+1} M_2$. By symmetry, without loss of generality assume $M_2 \models \Theta_{\theta,R}$ but $M_1 \models \neg\Theta_{\theta,R}$. We simulate a play of the game $\partial_{\mathrm{qf}(\{R\}),\theta^+,\omega+\omega+1}(M_1, M_2)$ in which the ISO player uses a (fixed) winning strategy $\mathbf{st}$.

<u>Case 1</u>: There is $B_* \in [M_1]^{\leq\theta}$ not included in a countable union of sets from $\{\{a : aR^{M_1}b\} : b \in M_1\}$. Here $\partial_{\mathrm{qf}(\{R\}),\theta,\omega+\omega+1}[M_1, M_2]$ suffice; we now simulate a play in which the ISO player uses $\mathbf{st}$.

In the first move we let the AIS player choose $\beta_{\mathbf{s}_1} = \omega + \omega$ and choose $\iota_{\mathbf{s}_1} = 1, A^1_{\mathbf{s}_1} = B_* \in [M_1]^{\leq\theta}$, see above. Let the ISO player (using $\mathbf{st}$) complete the choice of the state $\mathbf{s}_1$ (which is a winning state for itself, of course).

Let $n(*) > 7$ be minimal such that:

(∗) $B := \{a \in A^1_{\mathbf{s}_1} : h^1_{\mathbf{s}_1}(a) < n(*) - 7\}$ is not included in a countable union of sets from $\{\{a : aR^{M_1}b\} : b \in B\}$.

By the case assumption, $n(*)$ is well defined. In the following $n(*)-2$ moves, the AIS player takes care that $\beta_{\mathbf{s}_2} = \omega+n(*), \ldots, \beta_{\mathbf{s}_{n(*)}} = \omega+1$ (and $\iota_{\mathbf{s}_2} = \ldots = \iota_{\mathbf{s}_{n(*)}} = 1$).

So $g_{\mathbf{s}_{n(*)}}$ is a function whose domain includes $\{a \in A^1_{\mathbf{s}_1} : h^1_{\mathbf{s}_1}(a) < n(*)\}$ hence includes B. Now as $M_2 \models \Theta_{\theta,R}$ we can find $c_n \in M_2$ for $n < \omega$ such that $(\forall b \in B) \bigvee_n g_{\mathbf{s}_{n(*)}}(b)R^{M_2}c_n$ and AIS player takes care that $\beta_{\mathbf{s}_{n(*)+1}} = \omega$ and $\{c_n : n < \omega\} \subseteq A^2_{\mathbf{s}_{n(*)+1}}$. We shall prove now that for each n there is $d_n \in M^1$ such that:

$(∗)_{d_n}$ if $b \in B$ and $g_{\mathbf{s}_{\eta(*)}}(b)R^{M_2}c_n$ <u>then</u> $bR^{M_1}d_n$.

Let us use a play $\mathbf{s}_{n(*)+2}$ with the AIS choosing $\beta_{n(*)+1} = h^2_{\mathbf{s}_{n(*)_1}}(c_n) + 7$. Then continue the play by $s_{n(*)+3} \ldots \mathbf{s}_{n(*)+1}, \mathbf{s}_{n(*)+\beta_{n(*)+1}}$ with $B_{\mathbf{s}_{n(*)+\ell+1}} = B_{\mathbf{s}_{n(*)+\ell}-1}$.

The rest should be clear.

<u>Case 2</u>: There is $b_* \in M_1$ such that $B_* = \{a \in M_1 : aR^{M_1}b_*\}$ has cardinality $> \theta$.

Here $\partial_{\mathrm{qf}(\{R\}),\theta^+,\omega+1}$ suffice, and let $B \subseteq B_*$ be of cardinality θ^+. We simulate a play in which the AIS player takes care that $\beta_{\mathbf{s}_1} = \omega, \iota_{\mathbf{s}_1} = 1, A^1_{\mathbf{s}_1} = \{b_*\} \cup B$.

We let n_* be minimal such that $h^1_{\mathbf{s}_1}(b_*) < n_* - 7$ and $|\{b \in B : h^1_{\mathbf{s}_1}(b) < n_* - 7\}| \geq \theta^+$.

The rest should be clear.

3),4) Should be clear. $\square_{3.7}$

Recall

Definition 3.8. 1) A logic $\mathscr{L}$ is Δ-closed <u>when</u> : for vocabularies $\tau_1 \subseteq \tau_2$ with $\tau_2 \backslash \tau_1$ finite, and sentences $\psi, \vartheta \in \mathscr{L}(\tau_2)$ <u>if</u> $K_0 = \{M\upharpoonright\tau_1 : M \models \psi\}, K_1 = \{M\upharpoonright\tau_1 : M \models \vartheta\}$ are complementary classes of τ_1-models <u>then</u> some $\varphi \in \mathscr{L}(\tau_1)$ define K_1.

2) A logic $\mathscr{L}$ is strongly Δ-closed <u>when</u> : for relational vocabularies $\tau_1 \subseteq \tau_2$ with $\tau_2 \backslash \tau_1$ finite, unary predicate $P \in \tau_2 \backslash \tau_1$ and sentences $\psi, \vartheta \in \mathscr{L}(\tau_2)$, <u>if</u> $K_0 := \{(M\upharpoonright\tau_1)\upharpoonright P^M : M \models \psi$ and $P^M \neq \emptyset\}$ and $K_1 := \{(M\upharpoonright\tau_1)\upharpoonright P^M : M \models \vartheta$ and $P^M \neq \emptyset\}$ are complementary classes of τ_1-models <u>then</u> some $\varphi \in \mathscr{L}(\tau_1)$ define K_1.

3) A logic $\mathscr{L}$ has dullness-elimination <u>when</u> : if τ_1, τ_2 are relational vocabularies, i.e. with predicates only, $\tau_2 = \tau_1 \cup \{P\}, P$ a unary predicate not from $\tau_1, \psi_2 \in \mathscr{L}(\tau_2)$ and for every $M \models \psi_2$ we have $Q \in \tau_1 \Rightarrow Q^M = Q^M\upharpoonright P^M$ and $P^M \neq \emptyset$ <u>then</u>

for some $\psi_1 \in \mathscr{L}(\tau_1)$ we have $\{M : M$ a τ_1-model of $\psi_1\} = \{(M{\restriction}\tau_1){\restriction}P^M : M$ a τ_2-model of $\psi_2\}$.

4) We say the logic $\mathscr{L}$ is Δ-closed in $\mathbf{C}$ for $\mathbf{C}$ a class of cardinals <u>when</u>: if in part (1) we assume just that $K_0 \cap K_{\mathbf{C}}^\tau, K_0 \cap K_{\mathbf{C}}^\tau$ are complimentary in $K_{\mathbf{C}}^\tau = \{M : M$ a τ-model of cardinality from $\mathbf{C}\}$ <u>then</u> for some $\psi \in \mathscr{L}(\tau_1)$ for every τ_1-model M of cardinality $\in \mathbf{C}$ we have $M \models \psi \Leftrightarrow M \in K_0$.

Claim 3.9. *1) The logic $\mathbb{L}^1_{\leq \theta}$ has dullness-elimination.*
2) If τ is empty, M_1, M_2 are τ-models of cardinality $> \theta$ <u>then</u> M_1, M_2 are $\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$-equivalent for $\alpha < \theta^+$.
3) Also $\|M_1\|, \|M_2\| \geq \theta, \mathrm{cf}(\theta) = \aleph_0$ is O.K. in part (2).

Proof. 1) Easy (or use the sum theorem (see 3.16) and part (2)).
2),3) Easy. $\square_{3.9}$

Theorem 3.10. <u>Second Characterization Theorem</u>
   Let $\kappa = \beth_\kappa$, then $\mathbb{L}^1_\kappa$ is the minimal logic $\mathscr{L}$ (up to equivalence) such that:

  $\oplus$ (a) $\mathscr{L}$ is a nice logic,
   (b) $\mathbb{L}^{-1}_\kappa \leq \mathscr{L}$, i.e. $\mathbb{L}_{\theta^+,\aleph_0} \leq \mathscr{L}$ for $\theta < \kappa$,
   (c) $\Theta_{\theta,R}$ is equivalent to some sentence in $\mathscr{L}$ for $\theta < \kappa$,
   (d) $\mathscr{L}$ is Δ-closed,
   (e) $\mathscr{L}$ has dullness-elimination.

Remark 3.11. Putting together 3.4, 3.10 we get full characterization.

Proof. First $\mathbb{L}^1_\kappa$ satisfies clause (a) by 2.15(2), clause (b) by 2.15, clause (c) by 3.7(1) and clause (d) by Theorem 3.13(1) and Observation 3.12(1) below, and lastly $\mathbb{L}^1_\kappa$ has dullness-elimination by Claim 3.9(1) so together $\mathbb{L}^1_\kappa$ satisfies $\oplus$ of 3.10. Second, we shall assume $\mathscr{L}$ satisfies $\oplus$ and $\psi \in \mathbb{L}^1_\kappa(\tau)$ and we shall find $\psi' \in \mathscr{L}(\tau)$ equivalent to it (it will be φ_1 below). By "$\mathscr{L}$ satisfies weak substitution" without loss of generality τ has predicates only.

   We deal with and define naturally $\psi^{[P]} \in \mathbb{L}^1_\kappa(\tau), P$ a new unary predicate and let $\tau' = \tau \cup \{P\}$ so $\psi^{[P]}$ says $(M{\restriction}\tau){\restriction}P^M$ satisfies ψ.

   Let $\theta, \alpha < \kappa$ and let $\psi \in \mathbb{L}^1_\kappa(\tau)$ be defined using $(\mathrm{qf}(\tau), \theta, \alpha)$, so $|\tau| < \kappa$. We let $\partial = \beth_{\alpha+1}(\theta + |\tau|)$ and $\mu = 2^\partial$. Let $\tau^+ = \tau \cup \{R_1, R\}$, $\mathrm{arity}(R_1) = 3$, $\mathrm{arity}(R) = 2$ and $R, R_1 \notin \tau$. For $i = 0, 1$ let K_ℓ be the class of τ^+-models M such that:

  $\boxplus_\ell$ (a) $(|M|, R^M) \models \Theta_{\mu,R}$ and $P^M \neq \emptyset$
   (b) $\{B_b : b \in M\}$ is $\subseteq$-directed where $B_b = \{a \in M : aR^M b\}$
   (c) for each $b \in M$ the set $\{(a_1, a_2) : (a_1, a_2, b) \in R_1^M\}$ is a well ordering
          of B_b of order type $\leq \mu$
   (d) if $b_1, b_2 \in M$ and $B_{b_1} \subseteq B_{b_2} \subseteq P^M$ then
          $(M{\restriction}\tau){\restriction}B_{b_1} \prec_{\mathbb{L}_{\partial^+,\partial^+}} (M{\restriction}\tau){\restriction}B_{b_2}$ in particular B_{b_1} is
          closed under F^M for any function symbol
          $F \in \tau$
   (e) if $b \in M$, then for some c we have $B_c = B_b \cap P^M$
   (f) if $b \in M$ and $B_b \subseteq P$ <u>then</u> $(M{\restriction}\tau){\restriction}B_b \models \psi^{[P]}$ iff $\ell = 1$
   (g) if $Q \in \tau$ then $Q^M = Q^M {\restriction} P^M$.

Now

$(*)_1$ clauses (b),(c),(d),(e),(g) of $\boxplus_\ell$ can be described by some $\varphi \in \mathbb{L}_{(2^\mu)^+,\aleph_0}(\tau_1)$.

[Why? Note clause (c), so obvious. E.g. for clause (d), let $X = \{(M_1,<_1,M_2,<_2) : M_1, M_2$ are τ-models of cardinality $\leq \mu$ and universe $\leq M$ such that $<_\ell$ is a well ordering of M_ℓ for $\ell = 1,2$ and $M_1 \prec_{\mathbb{L}_{\partial^+,\partial^+}} M_2\}$. Clearly X has cardinality $\subseteq 2^{\mu+|\tau|} = 2^\mu$ so there is a list $\langle (M_{\alpha,1},<_{\alpha,1},M_{\alpha,2},<_{\alpha,2}) : \alpha < 2^\mu \rangle$ of X. The sentence says that: if $b_1, b_2 \in M, B_{b_1} \subseteq B_{b_2}$ then for some $\alpha < 2^\mu$ and isomorphism π from $M_{\alpha,2}$ onto $(M{\upharpoonright}\tau){\upharpoonright}B_2, \pi$ maps $M_{\alpha,1}$ to $(M{\upharpoonright}\tau){\upharpoonright}B_{b_1}$ and it maps $<_{\alpha,2}$ to $\{(a_1,a_2) : (a_1,a_2) \in R_1^M\}$.]

$(*)_2$ clause (f) can be expressed by a sentence from $\mathscr{L}(\tau^+)$.

[Why? Let $\langle N_\alpha : \alpha < 2^\mu \rangle$ list the τ-models with universe $\subseteq \mu$. For each $b \in M$ such that $B_b \subseteq P^M$ let g_b be the unique one-to-one function from B_b onto some ordinal $\gamma_b \leq \mu$ such that

- if $a_1, a_2 \in B_b$ then $(a_1,a_2,b) \in R_1^M \Leftrightarrow g_b(a) < g_b(a_2)$.

Hence for such b there is a unique α_b such that g_b is an isomorphism from $(M{\upharpoonright}\tau){\upharpoonright}B_b$ onto N_α. For each α there is a formula $\psi_\alpha(x) \in \mathbb{L}_{\mu^+,\aleph_0}(\tau_M)$ such that for every $b \in M$ we have $M \models \psi_\alpha[b]$ iff b is as above and $\alpha_b = \alpha$.

So for every $u \subseteq 2^\mu, \psi'_u(x) = \bigvee_{\alpha \in u} \psi_\alpha(x)$ belongs to $\mathbb{L}_{(2^\mu)^+,\aleph_0}(\tau_M)$ hence is equivalent to some $\psi''_u(x) \in \mathscr{L}(\tau_M)$. Now $(*)_2$ should be clear.]

$(*)_3$ K_ℓ is definable by some $\varphi_\ell \in \mathscr{L}(\tau^+)$ for $\ell = 1,2$.

[Why? By clause (b) of the assumption, some $\varphi' \in \mathscr{L}(\tau^+)$ is equivalent to φ where φ is from $(*)_1$, i.e. says that $\boxplus(b) - (e), (g)$ hold. By clause (c) of the assumption some $\varphi'' \in \mathscr{L}(\tau^+)$ is equivalent to $\Theta_{\theta,R}$. But by clause (a) of the assumption $\mathscr{L}$ is a nice logic, hence (see clauses $(b)_2, (b)_3, (c)$ of Definition 1.12) there is a sentence $\varphi'_\ell \in \mathscr{L}(\tau^+)$ equivalent to φ_ℓ from $(*)_2$. By clause $(b)_2$ of Definition 1.12, there is $\varphi_\ell \in \mathscr{L}(\tau^+)$ equivalent to $\varphi' \wedge \varphi'' \wedge \varphi'_\ell$ so we are done.]

$(*)_4$ for $\ell = 1,2$ the classes $\{(M_1{\upharpoonright}\tau){\upharpoonright}P^{M_1} : M_1 \in K_1\}, \{(M_2{\upharpoonright}\tau)P_i^M, M_2 \in K_2\}$ are equal to $\{M : M$ a τ-model of $\psi\}, \{M : M$ a τ-model of $\neg\psi\}$, respectively.

By $\mathscr{L}$ being Δ-closed and having dullness-elimination (see clauses (d),(e) of the theorem's assumption) we are done. $\square_{3.10}$

We can note and recall

Observation 3.12. *1) If the logic $\mathscr{L}$ satisfies interpolation <u>then</u> it is Δ-closed.*
2) $\mathscr{L}$ is Δ-closed and has dullness-elimination <u>iff</u> $\mathscr{L}$ is strongly Δ-closed, recalling Definition 3.8(2).
3) If $\mathscr{L}$ satisfies interpolation with finitely many sorts (so the interpolant mentions only the common sorts) <u>then</u> $\mathscr{L}$ is strongly Δ-closed.
4) The logic $\mathscr{L}$ satisfies interpolation and dullness-elimination <u>iff</u> $\mathscr{L}$ satisfies interpolation for finitely many sort models.
5) In 3.10 we can replace clauses (d),(e) by

$(d)'$ *for every $\mu < \kappa, \mathscr{L}$ is Δ-closed in $\{\lambda : \lambda = \lambda^\mu\}$, see Definition 3.8.*

Proof. Should be clear. $\square_{3.12}$

Theorem 3.13. *1) For $\kappa = \beth_\kappa$ the logic $\mathbb{L}^1_\kappa$ satisfies interpolation.*
2) Also if $\psi \in \mathbb{L}^1_{\le\theta}(\tau_1)$ where $\theta < \kappa$, the vocabularies $\tau_0 \subseteq \tau_1$ have cardinality $\le \theta$
and $\partial = \beth_{\theta^+}, \mu = 2^\partial$ then we can find a sequence $\langle \psi_\alpha : \alpha < (2^\partial)^+ \rangle$ of members of
$\mathbb{L}^1_{\le 2^\partial}(\tau_0)$ such that:

    (a) $\vdash \psi \to \psi_\alpha$ for $\alpha < (2^\partial)^+$

    (b) if $\tau_0 = \tau_1 \cap \tau_2$ and $\vartheta \in \mathbb{L}^1_{\le\theta}(\tau_2)$ a sentence such that $\vdash \psi \to \vartheta$ then $\vdash \psi_\alpha \to \vartheta$
        for some $\alpha < (2^\partial)^+$.

Proof. 1) This means that we should prove the existence of $\psi \in \mathbb{L}^1_\kappa(\tau_0)$ such that
$\vdash \psi_1 \to \psi$ and $\vdash \psi \to \psi_2$ when we assume:

    $\boxplus_1$ *(a)* $\tau_1 \cap \tau_2 = \tau_0$

            (b) $\psi_\ell \in \mathbb{L}^1_\kappa(\tau_\ell)$ for $\ell = 1, 2$

            (c) $\vdash \psi_1 \to \psi_2$.

Toward contradiction we assume this fails. Without loss of generality $\tau_\ell (\ell < 3)$ are
vocabularies of cardinality $< \kappa$.

Let θ_* be such that $\theta_* < \kappa, \theta_* \ge |\tau_1| + |\tau_2|$ and $\psi_\ell \in \mathbb{L}^1_{\le\theta_*}$ for $\ell = 1, 2$. For
each $\theta \in [\theta_*, \kappa)$ as $\mathbb{L}^1_{\le\theta}(\tau_0)$ is closed under conjunction (and conjunctions) of $\le \theta$
formulas and negations, clearly

    $\boxplus_2$ for every $\alpha < \kappa$ we can choose M^1_α, M^2_α such that

        (a) M^1_α is a τ_1-model

        (b) M^2_α is a τ_2-model

        (c) $M^1_\alpha {\restriction} \tau_0, M^2_\theta {\restriction} \tau_0$ are $\mathscr{E}^1_{\mathrm{qf}(\tau_0),|\alpha|,\alpha}$-equivalent

        (d) $M^1_\alpha \models \psi_1$

        (e) $M^2_\alpha \models \neg\psi_2$

and continue naturally as in the proof of 3.4.
2) Similarly. $\square_{3.13}$

Theorem 3.14. *If $\kappa = \beth_\kappa$ then $\mathbb{L}^1_\kappa$ satisfies the addition and product theorems,*
that is (all the models are τ-models for a fixed vocabulary τ; for the sum case we
assume τ has no function symbols (in particular no individual constants; or only
unary functions) and the relevant models have disjoint universes):

    (a)   *sum:* $\mathrm{Th}_{\mathbb{L}^1_\kappa}(M_1 + M_2) = \mathrm{Th}_{\mathbb{L}^1_\kappa}(M_1) + \mathrm{Th}_{\mathbb{L}^1_\kappa}(M_2)$ *that is,*

    (a)'   *if $\mathrm{Th}_{\mathbb{L}^1_\kappa}(M_\ell) = \mathrm{Th}_{\mathbb{L}^1_\kappa}(N_\ell)$ for $\ell = 1, 2$ then $\mathrm{Th}_{\mathbb{L}^1_\kappa}(M_1 + M_2) = \mathrm{Th}_{\mathbb{L}^1_\kappa}(N_1 + N_2)$*

    (b)   *product - $\mathrm{Th}_{\mathbb{L}^1_\kappa}(M_1 \times M_2) = \mathrm{Th}_{\mathbb{L}^1_\kappa}(M_1) \times \mathrm{Th}_{\mathbb{L}^1_\kappa}(M_2)$ that is,*

    (b)'   *if $\mathrm{Th}_{\mathbb{L}^1_\kappa}(M_\ell) = \mathrm{Th}_{\mathbb{L}^1_\kappa}(N_\ell)$ for $\ell = 1, 2$ then $\mathrm{Th}_{\mathbb{L}^1_\kappa}(M_1 \times M_2) = \mathrm{Th}_{\mathbb{L}^1_\kappa}(N_1 \times N_2)$*

    (c)   *moreover, we can replace $\mathrm{Th}_{\mathbb{L}^1_\kappa}(N)$ by $N/\mathscr{E}^1_{\mathrm{qf}(\tau),\theta,\alpha}$ when $\theta < \kappa, \alpha < \theta^+$.*

Proof. It suffices to prove clause (c). We prove it for products. For the sums this is easier.

Clause (c) for product:

Clearly it suffices to prove

⊞ assume $M_\iota = M_1^\iota \times M_2^\iota$ for $\iota = 1, 2$ are τ-models and $M_\ell^1 \mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^1 M_\ell^2$ for $\ell = 1, 2$ <u>then</u> $M_1 \mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^1 M_2$.

So let $\langle M_{\ell,k} : k \leq k(\ell,\iota) \rangle$ be such that $M_{1,0} = M_\ell^1, M_{2,k} = M_\ell^2$ and $M_{\ell,k}\mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^0 M_{\ell,k+1}$ for $k < k(\ell)$. Let $k(*) = \max\{k(1), k(2)\}$ and let $M_{\ell,k} = M_{\ell,k(\ell)}$ if $k(\ell) < k \leq k(*)$. As $\mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^0$ is reflexive, still we have $M_{\ell,k}\mathscr{E}_{\mathrm{qf}(\tau),\alpha}^0 M_{\ell,k+1}$ for $\ell = 1, 2$ and $k < k(*)$.

Let $M_k = M_{1,k} \times M_{2,k}$ for $k \leq k(*)$ we have to prove $M_0 \mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^0 M_{k(*)}$; by the definition of $\mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^1$ it suffices to prove for each $k < k(*)$ that

⊕$_k$ $M_k \mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^0 M_{k+1}$.

So we have to find a winning strategy for the ISO player for the game $\partial_{\mathrm{qf}(\tau),\theta,\alpha}[M_k, M_{k+1}]$.

The ISO player restricts itself, in the n-th move to (θ, n)-states $\mathbf{s}_n$ for a play of $\partial_{\mathrm{qf}(\tau),\theta,\alpha}[M_k, M_{k+1}]$ satisfying, for some pair $(\mathbf{s}_{n,1}, \mathbf{s}_{n,2})$, that for $\ell = 1, 2$

(a) $\mathbf{s}_{n,\ell}$ is a winning n-state for the game $\partial_{\mathrm{qf}(\tau),\theta,\alpha}[M_{\ell,k}, M_{\ell,k+1}]$

(b) $A_{\mathbf{s}_n}^\ell = A_{\mathbf{s}_{n,1}}^\ell \times A_{\mathbf{s}_{n,2}}^\ell$

(c) $h_{\mathbf{s}_n}((b, c)) = \max\{h_{\mathbf{s}_{n,1}}^1(b), h_{\mathbf{s}_{n,\ell}}^2(c)\}$.

The rest should be clear. $\square_{3.14}$

*   *   *

Remark 3.15. 1) Theorem 3.14(a) applies to the monadic version, too.

2) Why $\kappa = \beth_\kappa$? As for any θ and $\alpha < \theta^+$ there is a sentence $\psi \in \mathbb{L}_{\theta^+}$ and model M_ψ of ψ of cardinality $\beth_\alpha$ such that any other model N of ψ can be embedded into M_ψ.

3) For infinite addition, i.e. $\sum_{s \in I} M_s^\ell$, we have a problem: passing to $\mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^0$ we do not know how to uniformize $k(s)$, of course: if $k(s)$ is constant or bounded there, then a parallel claim holds, still see 3.16 below.

4) Even overcoming this obstacle, for infinite product we have a problem during a play of $\partial_{\mathrm{qf}(\tau),\theta,\alpha}[M, N]$. The point is about translating the h's from here to there and back. There may well be a difference between product (or reduced product) and generalized product. Anyhow unlike part (2) this cannot be remedied using $\mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}^0$ we know that, see 4.10.

Theorem 3.16. *1) Assume, for simplicity that τ is a relational vocabulary and $\langle M_s^\ell : s \in I \rangle$ is a sequence of τ-models with pairwise disjoint universes, for $\ell = 1, 2$ and $M_1 = \Sigma\{M_s^1 : s \in S\}, M_2 = \Sigma\{M_s^2 : s \in S\}$. If $\kappa = \beth_\kappa$ and M_s^1, M_s^2 are $\mathbb{L}_\kappa^1$-equivalent for $s \in S$ <u>then</u> M_1, M_2 are $\mathbb{L}_\kappa^1$-equivalent.*

2) Moreover for every $\theta, \alpha < \kappa$ there are $\partial, \beta < \kappa$ such that for any set S and sequence $\langle M_s^\ell : s \in S \rangle$ for $\ell \in \{1, 2\}$ of pairwise disjoint τ-models we have: if M_s^1, M_s^2 are $\mathscr{E}_{\mathrm{qf}(\tau),\partial,\beta}$-equivalent for $s \in S$ then the models $M_1 = \Sigma\{M_s^1 : s \in S\}, M_2 = \Sigma\{M_s^2 : s \in S\}$ are $\mathscr{E}_{\mathrm{qf}(\tau),\theta,\alpha}$-equivalent.

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3) In part (2) if $M_s^b \mathscr{E}^0_{\mathrm{qf}(\tau),\theta,\alpha} M_j^2$ for every $s \in S$ <u>then</u> $M_1 \mathscr{E}^0_{\mathrm{qf}(\tau),\theta,\alpha} M_2$.

Proof. 1),2) By the proof of 3.4.
3) As in the proof of 3.14. $\hfill\square_{3.16}$

Remark 3.17. This proof indicates that there are better versions of the game for which we can choose $(\partial, \beta) = (\theta, \alpha)$. However, it is not clear whether a more complicated definition is worth the gain.

§ 4. $\mathbb{L}^1_\kappa$ IS STRONG

Our first aim is:

Question 4.1. How strong is $\mathbb{L}^1_\kappa$? It is more like $\mathbb{L}^{-1}_\kappa$ or $\mathbb{L}^0_\kappa$? E.g. upward LST? Downward LST to which cardinals?

So far we have given indication to its being similar to $\mathbb{L}^{-1}_\kappa$, however we shall show below that for the upward LST, $\mathbb{L}^1_\kappa$ is closer to $\mathbb{L}^0_\kappa$ (see 4.8) and fail the theorem on the theory of an infinite product (which both $\mathbb{L}^0_\kappa, \mathbb{L}^{-1}_\kappa$ satisfies). But restricting ourselves to cardinals $\lambda = \lambda^{\aleph_0} \geq \kappa$, the situation for the downward LST is similar to the one for $\mathbb{L}^0_\kappa$, see below.

Definition 4.2. For a relational vocabulary τ and cardinal θ of cofinality $> \aleph_0$ we say **K** is a (τ, θ, ω)-class of structures <u>when</u>:

    (a) **K** is a class of τ-models each of cardinality θ

    (b) **K** is closed under submodels of cardinality θ

    (c) **K** is closed under isomorphism

    (d) **K** is closed under increasing union of ω-chains.

2) For $\tau, \theta, \mathbf{K}$ as above let $\psi_{\mathbf{K}} \in \mathbb{L}^0_{(2^\theta)^+, \theta^+}(\tau)$ be a sentence such that for τ-models M we have: $M \models \psi_{\mathbf{K}}$ <u>iff</u> for any $A \subseteq M$ of cardinality θ there is $h : A \to \omega$ such that for every large enough n, $M {\restriction} h^{-1}\{n\} \in \mathbf{K}$.

Example 4.3. Let $\mathbf{K}^{\mathrm{wo}}_\theta = \{(A, <_*) :<_*$ is a linear order of A and for some sequence $\langle \mathscr{U}_n : n < \omega \rangle$ we have $(\mathscr{U}_n, <_* {\restriction} \mathscr{U}_n) \cong (\theta, <)$ and $\mathscr{U}_n \subseteq \mathscr{U}_{n+1}$ and $A = \cup\{\mathscr{U}_n : n < \omega\}$.

Claim 4.4. *For $\tau, \mathbf{K}$ as in 4.2.*
1) The sentence $\psi_{\mathbf{K}}$ belongs to $\mathbb{L}_{(2^\theta)^+, \theta^+}(\tau)$ indeed.
2) If $\theta + |\mathbf{K}/\cong| = \mu$ <u>then</u> moreover $\psi_{\mathbf{K}} \in \mathbb{L}^0_{\mu^+, \theta^+}$.
3) Moreover $\psi_{\mathbf{K}} \in \mathbb{L}^1_{\leq \theta}$.

Proof. 1),2) Obvious.
3) Easy. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square_{4.4}$

Claim 4.5. *1) If $\lambda \geq 2^\theta$ <u>then</u> $\Theta_{\theta, R}$ has a model of cardinality λ iff $\mathrm{cov}(\lambda, \theta^+, \theta^+, \aleph_0) = \lambda$, see Remark 4.7 below (or [?, II]).*
2) If $\lambda = \lambda^{\aleph_0} \geq 2^{\aleph_0}$ <u>then</u> $\Theta_{\theta, R}$ has a model of cardinality λ iff $\lambda = \lambda^\theta$.

Proof. 1) Read the definitions.
2) Check. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square_{4.5}$

Complementary is

Claim 4.6. *Assume ($\kappa = \beth_\kappa$ and) $\lambda = \mathrm{cov}(\lambda, \kappa, \kappa, \aleph_1)$ and τ is a vocabulary of cardinality $< \lambda$. For every τ-model M of cardinality $> \lambda$ <u>there</u> is $N \subseteq M$ of cardinality λ such that $N \equiv_{\mathbb{L}^1_\kappa} M$.*
2) Assume $\lambda = \mathrm{cov}(\lambda, \beth_\gamma, \beth_\gamma, \aleph_1)$ for every $\gamma < \theta^+$. If M is a τ-model, $|\tau| \leq \beth_{\theta^+}$ and $\|M\| > \lambda$ <u>then</u> there is $N \subseteq M$ of cardinality λ such that $N \equiv_{\mathbb{L}^1_{\leq \theta}} M$.

Proof. Similar to 2.15(4)(c), but in ω stages deal simultaneously with λ submodels in each stage. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square_{4.6}$

Remark 4.7. Recall $\mathrm{cov}(\lambda, \mu, \theta, \sigma) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{<\mu}$ and for every $u \in [\lambda]^{<\theta}$ there are $\alpha < \sigma$ and $u_i \in \mathscr{P}$ for $i < \alpha$ such that $u \subseteq \cup\{u_i : i < \alpha\}\}$ for $\lambda \geq \mu \geq \theta \geq \sigma$.

Concerning the upward LST theorem, the logic $\mathbb{L}^1_\kappa$ fail it badly.

Claim 4.8. *There is a sentence ψ from $\mathbb{L}^1_{\leq \aleph_1}$ such that: ψ has a model of cardinality λ iff $\lambda \nrightarrow (\aleph_1)^{<\omega}_2$.*

Proof. Easy, recalling 4.4(3). $\square_{4.8}$

Discussion 4.9. We can phrase other relatives. Also it points to the restrictions when we are looking for such logics with upward LST.

Theorem 4.10. *1) For every θ there are models $\langle M_n, N_n : n < \omega \rangle$ with a fixed countable vocabulary such that M_n, N_n are $\mathbb{L}^1_{\leq\theta}$-equivalent for $n < \omega$ but $\prod_{n<\omega} M_n, \prod_{n<\omega} N_n$ are not $\mathbb{L}^1_{\leq 2^{\aleph_0}}$-equivalent.*
2) Moreover in $\beth_{\mathrm{qf}(\tau), 2^{\aleph_0}, \omega+1}[\prod_{n<\omega} M_n, \prod_{n<\omega} N_n]$ the AIS player has a winning strategy.

Proof. Let $\tau = \{c, <\} \cup \{F_n : n < \omega\}$ where F_n is a unary function, c is an individual constant and $<$ is a binary relation.

For every α let M^n_α be the τ-model with universe $1 + \alpha$, $c^{M^n_\alpha} = 0$, $<^{M^n_\alpha} = \{(i,j) : i = 0 = j$ or $i < j < 1 + \alpha\}$ and $F_k^{M^n_\alpha}$ is the identity when $k = n$ and is constantly 0 otherwise.

Fix $\theta \geq 2^{\aleph_0}$, by 3.1 for some $\alpha = \alpha(\theta)$ large enough for each n there is a model N_n which is $\mathbb{L}^1_{\leq\theta}$-equivalent to M^n_α, but $<^{N_n} \restriction (|N_n| \backslash \{c^{N_n}\})$ is not well ordered; let $M_n = M^n_\alpha$ and $\langle d_{n,k} : k < \omega \rangle$ be a $<^{N_n}$-decreasing sequence of members of N_n which are $\neq c^{N_n}$.

Let $M = \prod_{n<\omega} M_n$ and $N = \prod_{n<\omega} N_n$. For $\eta \in {}^\omega\omega$ let $d_\eta = \langle d_{n,\eta(n)} : n < \omega \rangle \in \prod_{n<\omega} N_n = N$.

Now

⊛ if $\Lambda \subseteq {}^\omega\omega$ is non-meagre or just not bounded in $({}^\omega\omega, <_{J^{\mathrm{bd}}_\omega})$ then there is no partial isomorphism f from N into M with domain $\{d_\eta : \eta \in \Lambda\}$, i.e. f should preserve all quantifier-free formulas.

This clearly suffices as if $\Lambda = \cup\{\Lambda_n : n < \omega\}$ then at least for one n, the set Λ is non-meagre (or just unbounded in $({}^\omega\omega, <_{J^{\mathrm{bd}}_\omega})$).

Why is ⊛ true? Toward contradiction assume that f is such a partial isomorphism. By the assumption for some $n(*) < \omega$ the set $\{\eta(n(*)) : \eta \in \Lambda\}$ is infinite, so choose $\bar{\eta} = \langle \eta_k : k < \omega \rangle$ such that

$(*)_0$ $\langle \eta_k(n(*)) : k < \omega \rangle$ is strictly decreasing in $N_{n(*)}$.

Now:

$(*)_1$ if $m(1) < m(2) < \omega$ then for every $n < \omega$ we have $N_n \models$ "$F_{n(*)}(d_{n,\eta_{m(2)}(n(*))}) < F_{n(*)}(d_{n,\eta_{m(1)}(n(*))})$".

[Why? If $n = n(*)$ then $F_{n(*)}^{N_n}$ is the identity and this means $N_n \models$ "$d_{n,m(2)} < d_{n,m(1)}$" which holds by the choice of $\langle d_{n,k} : k < \omega \rangle$ recalling $(*)_0$. If $n < \omega \wedge n \neq n(*)$ then $F_{n(*)}^{N_n}$ is constantly 0 so this means $N_n \models$ "$0 < 0$" which holds by the choice of N_n (and $<^{M_\alpha^n}$), so we are done.]

$(*)_2$ $N \models$ "$F_{n(*)}(d_{\eta_{m(2)}}) < F_{n(*)}(d_{\eta_{m(1)}})$" for $m(1) < m(2) < \omega$.

[Why? By $(*)_1$ and the definition of product.]
    Also

$(*)_3$ if $m < \omega$ then $N_{n(*)} \models$ "$c \neq F_{n(*)}(d_{n(*),\eta_m(n(*))})$" hence $N \models$ "$c \neq F_{n(*)}(d_{\eta_n})$"

$(*)_4$ let $\nu_k = f(d_{\eta_k})$ so $\nu_k = \langle a_{k,n} : n < \omega \rangle \in \prod_{n<\omega} M_n$.

But f is a partial isomorphism so by $(*)_2 + (*)_3 + (*)_4$ we have

$(*)_5$ $M \models$ "$F_{n(*)}(\nu_{m(2)}) < F_{n(*)}(\nu_{m(1)}) \neq c$" for $m(1) < m(2) < \omega$.

Hence for every $n, M_n \models$ "$F_{n(*)}(\nu_{m(2)}(n)) < F_{n(*)}(\nu_{m(1)}(n))$" for $m(1) < m(2) < \omega$. Also by the choice of $F_{n(*)}^{M_n}$ we have $n \neq n(*) \Rightarrow M_n \models$ "$c = F_{n(*)}(\nu_{m(1)}(n))$" but $M \models$ "$c \neq F_{n(*)}(\nu_{m(1)})$" hence $M_{n(*)} \models$ "$c \neq F_{n(*)}(\nu_{m(1)}(n(*)))$", i.e. $\nu_{m(1)}(n(*)) \neq 0$. Together $\langle \nu_m(n(*)) : m < \omega \rangle$ is $<^{M_n}$-decreasing in $M_{n(*)} \backslash \{0\}$ contradiction. $\square_{4.10}$

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