MODULES AND INFINITARY LOGICS
SH977

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Dedicated to Rüdiger Göbel for this 70th birthday

ABSTRACT. We deal with Abelian groups and R-modules. We consider theories in infinitary logic of the form $\mathbb{L}_{\lambda,\theta}$ of such structures M and prove they have elimination of quantifiers up to positive existential formulas, so ones defining subgroups of some power of M. Hence in the appropriate sense those theories are stable and understood to some extent.

Date: October 28, 2013.

2010 *Mathematics Subject Classification.* Primary 03C60, 13C99; Secondary: 03C10, 03C45, 03C48, 03C75, 20K99, 13C05.

Key words and phrases. model theory, modules, stability, infinitary logics, elimination of quantifiers.

The author thanks Alice Leonhardt for the beautiful typing. First version done April 2010. The author would like to thank the Israel Science Foundation and German-Israeli Science Foundation for partial support. Paper Number 977.

§ 0. Introduction

Much is known on classes of R-modules and first order logic. Szmielew [?] proved the decidability of the theory of Abelian groups. Szmielew [?] prove an elimination of quantifiers in the theory of Abelian groups up to Boolean combinations of pe (= positive existential) formulas.

Eklof [?] proved the existence of universal homogeneous R-models in λ if $\lambda = \lambda^{<\gamma}$ where γ depending on R only. Fisher improved this to saturated models of elementary classes (see his review of [?]), this implies stability by a general criterion ([?, §0],[?, Ch.III]).

Baur [?] proved that for the class of R-modules any first order formula is equivalent to a Boolean combination of positive existential formulas and also proved stability (of $\mathrm{Th}(M)$) for M and R-module.

We like to know for a given ring R how complicated the class of R-modules which are models of a sentence ψ in an infinitary logic.

Question 0.1. Given a ring R, for the class Mod_R of left R-modules:
1) Does it have for the logic $\mathbb{L}_{\lambda,\mu}$ a kind of elimination of quantifiers (say up to some depth).
2) Is it stable? (say no formula $\varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\infty,\infty}(\tau_R)$ linearly ordering arbitrarily long sequence of tuples in some models of ψ)?
3) Can we define something like non-forking?

Question 0.2. Do we have a parallel of the main gap, i.e. proving that either every $M \in \mathrm{Mod}_\psi$ can be characterized by some suitable cardinal invariants <u>or</u> there are many complicated $M \in \mathrm{Mod}_\psi$?

Here we first show that for any R-module, in $\mathbb{L}_{\lambda,\theta}(\tau_R)$ or better $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ we have a version of eliminating quantifiers up to positive existential formulas <u>however</u> we add parameters. Second, by this we can prove some versions and consequences of stability. More specifically

- after expanding by enough individual constants, every formula in $\mathbb{L}_{\infty,\theta,\gamma}(\tau_R)$ is equivalent to a Boolean combination of positive existential such formulas
- the number of added individual constants is reasonable: $\leq \beth_\gamma(|\tau|^{<\theta})$
- stability, i.e. no long sequences of linearly ordered $(<\theta)$-tuples
- $(\Lambda^{\mathrm{ep}}_{\varepsilon,\alpha}, 2)$-indiscernible implies $\Lambda^{\mathrm{ep}}_{\varepsilon,\alpha}$-indiscernible
- convergence follows, see Definition 3.4

§ 1. Preliminaries

Notation 1.1. 1) Let θ^- be σ if $\theta = \sigma^+$ and θ if θ is a limit cardinal.

Definition 1.2. 1) A vocabulary τ consists of function symbols (e.g. individual constants) and predicates (= relation symbols), in addition the vocabulary assign generally to each of them its arity = number of places $\text{arity}_\tau(-)$; here it can be an infinite ordinal; an individual constant is a 0-place function.
2) For a vocabulary τ we say M is a τ-structure when M, writing $\tau_M = \tau(M) = \tau$, consisting of:

- (a) $|M|$, the universe of M, a non-empty set of the so-called elements of M, so we may write $a \in M, \bar{a} \in {}^\varepsilon M$ and $A \subseteq M$, etc., instead $a \in |M|, \bar{a} \in {}^\varepsilon(|M|)$ and $A \subseteq |M|$, etc.
- (b) F^M a function from ${}^\varepsilon M$ to M, possibly partial where ε is the ordinal $\text{arity}_\tau(F)$, for F a function symbol from τ
- (c) $P^M \subseteq {}^\varepsilon M$ where ε is the ordinal $\text{arity}_\tau(P)$ for P a predicate from τ.

Definition 1.3. 1) We say τ is a θ-additive (or a θ-Abelian) vocabulary when τ has the two-place function symbols $x + y, x - y$, the individual constant 0 and the other predicates and function symbols has arity $< \theta$.
2) M is a θ-additive structure (or model) when:

- (a) τ_M, the vocabulary of M is a θ-additive vocabulary
- (b) $G_M := (|M|, +^M, -^M, 0^M)$ is an Abelian group
- (c) if $P \in \tau_M$ is an ε-place predicate then P^M is a sub-group of $(G_M)^\varepsilon$
- (e) if $F \in \tau_M \backslash \{+, -, 0\}$ is an ε-place function symbol then F^M is a partial ε-place function from M to M and $\text{graph}(F^M) = \{\bar{a}\hat{\ }\langle F^M(\bar{a})\rangle : \bar{a} \in \text{Dom}(F^M)\}$ is a subgroup of $(G_M)^{\varepsilon+1}$.

Remark 1.4. Fisher [?] defines and deals with "Abelian structure" in other directions.

Definition 1.5. 1) We consider an R-module M as a $\tau(R)$-structure, where $\tau_R = \tau(R)$ be the vocabulary of R-modules, i.e. have binary functions $x + y, x - y$, individual constant 0 and unary function symbol F_a, interpreted as multiplication by a from the left for every $a \in R$.
2) If $\bar{x}, \bar{y}$ has length ε then we let $\bar{x} + \bar{y} = \langle x_\zeta + y_\zeta : \zeta < \varepsilon\rangle, \bar{x} - \bar{y} = \langle x_\zeta - y_\zeta : \zeta < \varepsilon\rangle$ and similarly $a\bar{x}$ for $a \in R$, and when we replace $\bar{x}$ and/or $\bar{y}$ by a member of ${}^\varepsilon M$.

Observation 1.6. *1) For a ring R, an R-module is an $\aleph_0$-additive structure in the vocabulary τ_R.*
2) For a τ-additive model M, for every τ-term $\sigma(\bar{x})$ we have

- (a) $M \models$ "$\sigma(\bar{a} \pm \bar{b}) = \sigma(\bar{a}) \pm \sigma(\bar{b})$" *meaning (when F is partial), if two of the terms are well defined then so is the third and the equality hold*
- (b) $M \models P(\bar{a} \pm \bar{b})$ *when* $M \models P(\bar{a}) \wedge P(\bar{b})$.

§ 2. Eliminating quantifiers

Context 2.1. 1) R is a fixed ring $\tau = \tau_R$, see 1.5(1) <u>or</u> just τ is an θ-additive vocabulary, see 1.3(1), 1.6(1).
2) $\mathbf{K}$ is the class of R-modules <u>or</u> of τ-additive models.
3) M, N will denote R-modules <u>or</u> are τ-additive models.
4) $\theta = \operatorname{cf}(\theta)$.

Definition 2.2. For $\varepsilon < \theta$ and ordinal α (and τ as in 2.1). We shall define $\Lambda^{\rm pe}_{\alpha,\varepsilon} = \Lambda^{{\rm pe},\theta}_{\alpha,\varepsilon} = \Lambda^{{\rm pe},\theta}_{\alpha,\varepsilon}(\tau)$, a set of formulas $\varphi(\bar x)$ in $\mathbb{L}_{\infty,\theta}(\tau)$ in fact in $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$ with $\ell g(\bar x) = \varepsilon < \theta$, so $\bar x = \langle x_\xi : \xi < \varepsilon\rangle$ if not said otherwise, by induction on the ordinal α.

For $\zeta < \theta$ we write $\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta}$ for the set of $\varphi = \varphi(\bar x, \bar y), \ell g(\bar x) = \varepsilon, \ell g(\bar y) = \zeta$ (so $\bar y = \langle y_\xi : \xi < \zeta\rangle$ if not said otherwise) with $\varphi \in \Lambda^{\rm pe}_{\alpha,\varepsilon+\zeta}$ and $\Lambda^{\rm pe}_\alpha = \cup\{\Lambda^{\rm pe}_{\alpha,\varepsilon} : \varepsilon < \theta\}, \Lambda^{\rm pe}_{\alpha,\varepsilon,<\theta} = \cup\{\Lambda^{\rm pe}_{\alpha,\varepsilon,\zeta} : \zeta < \theta\}$. If $\tau = \tau_R$ we may write $\Lambda^{\rm pe}_{\alpha,\varepsilon}(R)$.

The definition is as follows:

<u>Case 1</u>: $\alpha = 0$

<u>For R-modules</u>:

It is the set of $\varphi = \varphi(\bar x)$ of the form: $\sum\limits_{\ell<n} a_\ell x_{\zeta(\ell)} = 0$ with $\zeta(\ell) < \ell g(\bar x)$ or better, $\sum\limits_{\zeta<\varepsilon} a_\zeta x_\zeta = 0$ where $a_\zeta \in R$ is 0_R for all but finitely many ζ's.

<u>For general τ, so here the τ-additive case:</u>

It is the set of $\varphi(\bar x)$ has the form $P(\bar\sigma(\bar x)), \bar\sigma$ a sequence of length $\text{arity}_\tau(P)$ of terms (in the variables $\bar x$), P may be equality or any predicate from τ of arity the length of $\bar\sigma$.

<u>Case 2</u>: α a limit ordinal

It is $\cup\{\Lambda^{\rm pe}_{\beta,\varepsilon}(R) : \beta < \alpha\}$.

<u>Case 3</u>: $\alpha = \beta + 1$

For some $\zeta < \theta$ and $\Phi \subseteq \Lambda^{\rm pe}_{\beta,\varepsilon+\zeta}$ we have $\psi(\bar x) = \exists \bar y(\bigwedge\{\varphi(\bar x\hat{\ }\bar y) : \varphi(\bar x, \bar y) \in \Phi\})$.

Claim 2.3. *1) In 2.2, $\Lambda^{\rm pe}_{\alpha,\varepsilon}$ is $\subseteq$- increasing with α and is of cardinality $\le \beth_\varepsilon(|\tau| + \aleph_0)$ if $\theta = \aleph_0$ and $\beth_\varepsilon(|\tau|^{<\theta})$ in general.*
2) For $M \in \mathbf{K}$ and $\varphi(\bar x) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}(\tau)$, the set $\varphi(\bar M) = \{\bar b \in {}^\varepsilon M : M \models \varphi[\bar b]\}$ is a sub-Abelian group of ${}^\varepsilon M$ and the set $\{\bar b \in {}^\varepsilon M : M \models \varphi[\bar b - \bar a]\}$ is affine (= closed under $\bar x - \bar y + \bar z$) for any $\bar a \in {}^\varepsilon M$.

Proof. Easy. $\square$

Theorem 2.4. *For every α for every $M \in \mathbf{K}$ there is a subset $\mathbf{I} = \mathbf{I}_\alpha$ of ${}^{\theta>}M$ of cardinality $\le \kappa_\alpha = \beth_\alpha(|\tau|^{<\theta})$ such that: in M every formula $\psi(\bar x)$ from $\mathbb{L}_{\infty,\theta,\alpha}(\tau)$, so $\ell g(\bar x) < \theta$, is equivalent in M to a Boolean combination of formulas of the form $\varphi(\bar x - \bar a)$ with $\varphi(\bar x) \in \Lambda^{\rm pe}_{\alpha,\ell g(\bar x)}(\tau)$ and $\bar a \in \mathbf{I} \cap {}^{\ell g(\bar x)}M$.*

Before we shall prove

Conclusion 2.5. *For every $M \in \mathbf{K}$, limit ordinal $\alpha, \varepsilon < \theta$ and $\bar a \in {}^\varepsilon M$, for some $i(*), j(*) \le \kappa_\alpha$ and $\varphi_i(\bar x_\varepsilon), \psi_j(\bar x_\varepsilon) \in \Lambda^{\rm pe}_{\alpha,\varepsilon}$ for $i < i(*), j < j(*)$ we have*

$\{\bar a' \in {}^\varepsilon M : \mathrm{tp}_{\mathbb{L}^{\mathrm{pe}}_{\infty,\theta,\alpha}}(\bar a', \emptyset, M) = \mathrm{tp}_{\mathbb{L}^{\mathrm{pe}}_{\infty,\theta,\alpha}}(\bar a, \emptyset, M)\}$ *is equal to* $\{\bar a' \in {}^\varepsilon M : M \models \bigwedge\limits_{i<i(*)} \varphi_i(\bar a' - \bar a) \wedge \bigwedge\{\neg\psi_j(\bar a' - \bar a'') : j < j(*)$ *and* $\bar a'' \in I_\gamma \cap {}^\varepsilon M\}\}$.

Definition 2.6. 1) We say $\bar b_1, \bar b_2 \in {}^\varepsilon M$ are α-equivalent over $\mathbf{I} \subseteq {}^{\theta>}M$ <u>when</u> $\varphi(\bar x_\varepsilon) \in \Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}(R), \bar a \in \mathbf{I} \Rightarrow M \models$ "$\varphi[\bar b_1 - a] \equiv \varphi[\bar b_2 - \bar a]$".
2) Replacing $\mathbf{I}$ by A means $\mathbf{I} = \cup\{{}^\varepsilon A : \varepsilon < \theta\}$.

We shall use freely

Observation 2.7. *The sequence* $\bar b_1, \bar b_2 \in {}^\varepsilon M$ *are* α*-equivalent over* $\mathbf{I} \subseteq {}^\varepsilon M$ <u>*iff*</u> *for any* $\varphi(\bar x) \in \Lambda^{\mathrm{pe}}_{\alpha,\varepsilon}$ *we have* $(a) \vee (b)$ *where:*

- (a) *for some* $\bar a \in \mathbf{I} \cap {}^\varepsilon M$ *we have* $M \models \varphi[\bar b_1 - \bar a] \wedge \varphi[\bar b_2 - \bar a]$
- (b) *for every* $\bar a \in \mathbf{I} \cap {}^\varepsilon M$ *we have* $M \models \neg\varphi[\bar b_1 - \bar a] \wedge \neg\varphi[\bar b_2 - \bar a]$

Proof. Straight. $\square_{2.7}$

Proof. <u>Proof of 2.4</u>
By induction on α we choose $\mathbf{I}_\alpha$ and prove the statement. For $\alpha = 0$ choose $\mathbf{I}_\alpha = \{0^M\}$ and for α a limit ordinal this is obvious, use $\cup\{\mathbf{I}_\beta : \beta < \alpha\}$ so assume $\alpha = \beta + 1$ and we shall choose $\mathbf{I}_\alpha$.
Choose $\mathbf{I}_\alpha$ such that

$\boxplus_\alpha$ (a) $\mathbf{I}_\alpha$ is a subset of ${}^{\theta>}M$
- (b) $|\mathbf{I}_\alpha| \le 2^{\kappa_\beta}$ where $\kappa_\beta = \beth_\beta(|\tau|^{<\theta})$
- (c) $\mathbf{I}_\beta \subseteq \mathbf{I}_\alpha$
- (d) If $\varepsilon < \theta$ and $\varphi_i(\bar x) \in \Lambda^{\mathrm{ep}}_{\beta,\varepsilon}$ and $\bar a_i \in \mathbf{I}_\beta \cap {}^\varepsilon M$ for $i < i(*) \le \kappa_\beta$ and there is $\bar d \in {}^\varepsilon M$ such that $M \models \varphi_i[\bar d - \bar a_i]$ for $i < i(*)$ <u>then</u> there is such $\bar d \in \mathbf{I}_\alpha$
- (e) Assume $\varepsilon < \theta, \bar x = \bar x_\varepsilon, \psi(\bar x)$ is a conjunction of formulas from $\Lambda^{\mathrm{ep}}_{\beta,\varepsilon}$ and $\varphi_i(\bar x) \in \Lambda^{\mathrm{ep}}_{\beta,\varepsilon}$ for $i < \kappa_\beta$ and apply 4.1 with $\lambda_\alpha = (2^{\kappa_\beta})^+, \kappa_\beta, \psi({}^\varepsilon M), \psi({}^\varepsilon M) \cap \varphi_i({}^\varepsilon M)$ for $i < \kappa_\beta$ here standing for $\lambda, S, G, G_s (s \in S)$ there; (i.e. the subgroups of $({}^\varepsilon|M|, +^M)$ with universes as above) getting the ideal I on κ_β and further assume $\kappa_\beta \notin I$
  - (α) there are $\bar d_\iota \in \mathbf{I}_\alpha \cap \varphi_i({}^\varepsilon M)$ for $\iota < \iota(*) \le 2^{\kappa_\beta}$ such that for every $\bar a \in \psi({}^\varepsilon M)$ there is $\iota < \iota(*)$ satisfying $\{i < \kappa : \bar a - \bar d_\iota \notin \varphi_i({}^\varepsilon M)\} \in I$
  - (β) for any $u \in I$ there are u_* such that $u \subseteq u_* \in I$ and $\bar d_\iota \in \cap\{\varphi_i({}^\varepsilon M) : i \in \kappa_\beta \backslash u_*\} \cap \psi({}^\varepsilon M) \cap \mathbf{I}_\alpha$ for $\iota < (2^{\kappa_\beta})^+$ such that: $i \in u_* \wedge \iota(1) \le \iota(2) < (2^{\kappa_\beta})^+ \Rightarrow \bar d_{\iota(1)} - \bar d_{\iota(2)} \notin \varphi_i({}^\varepsilon M)$
- (f) if $\varepsilon < \theta$ and $\bar d_1, \bar d_2 \in \mathbf{I}_\alpha \cap {}^\varepsilon M$ <u>then</u> $\bar d_1 + \bar d_2 \in \mathbf{I}_\alpha, \bar d_1 - \bar d_2 \in \mathbf{I}_\alpha$ and $\xi < \theta \Rightarrow \bar 0_\xi{}^\frown \bar d_1 \in \mathbf{I}_\alpha$.

This is possible for $(e)(\alpha)$ by clause (c) of 4.1 and for $(e)(\beta)$ by clause (d) of 4.1.
To prove the induction statement for α clearly it suffices to prove:

$\boxdot$ assume $\varepsilon, \xi < \theta$; if $\bar{b}_1, \bar{b}_2 \in {}^{\varepsilon}M$ are α-equivalent over $\mathbf{I}_\alpha$ and $\bar{c}_1 \in {}^{\xi}M$ <u>then</u> for some $\bar{c}_2 \in {}^{\xi}M$ the sequences $\bar{b}_1{}^\frown\bar{c}_1, \bar{b}_2{}^\frown\bar{c}_2 \in {}^{(\varepsilon+\xi)}M$ are β-equivalent over $\mathbf{I}_\beta$.

Why $\boxdot$ holds? Let $\bar{x}$ be of length ε and $\bar{y}$ of length ξ. Let $\Phi_1 = \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm ep}_{\beta,\varepsilon+\xi}$: for some $\bar{a} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi}M$ we have $M \models \varphi[\bar{b}_1{}^\frown\bar{c}_1 - \bar{a}]\}$ and for $\varphi(\bar{x}, \bar{y}) \in \Phi_1$ choose $\bar{a}_{\varphi(\bar{x},\bar{y})} \in \mathbf{I}_\beta \cap {}^{\varepsilon+\xi}M$ such that $M \models \varphi[\bar{b}_1{}^\frown\bar{c}_1 - \bar{a}_{\varphi(\bar{x},\bar{y})}]$. Let $\Phi_2 = \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm ep}_{\beta,\varepsilon+\xi} : \varphi(\bar{x}, \bar{y}) \notin \Phi_1\}$.

So by $\boxplus_\alpha(d)$ there is a $\bar{b}^*{}^\frown\bar{c}^* \in \mathbf{I}_\alpha$ be such that $\ell g(\bar{b}^*) = \ell g(\bar{b}_1), \ell(\bar{c}^*) = \ell g(\bar{c}_1)$ and $\varphi(\bar{x}, \bar{y}) \in \Phi_1 \Rightarrow M \models \varphi[\bar{b}^*{}^\frown\bar{c}^* - \bar{a}_{\varphi(\bar{x},\bar{y})}]$. For transparency note that if $\Phi_2 = \emptyset$ then as the formula $\wedge\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_1\} \in \Lambda^{\rm pe}_{\alpha,\varepsilon+\xi}$ clearly by the assumption of $\boxdot$ there is $\bar{c}_2 \in {}^{\xi}M$ such that $\varphi(\bar{x}, \bar{y}) \in \Phi_1 \Rightarrow M \models \varphi(\bar{b}_2{}^\frown\bar{c}_2 - \bar{a}_{\varphi(\bar{x},\bar{y})})$, so $\bar{c}_2$ is as required, hence we are done so without loss of generality $\Phi_2 \neq \emptyset$. Clearly $|\Phi_2| \le \kappa_\beta$ and let $\Phi'_\ell = \{\varphi(\bar{0}_\varepsilon, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_\ell\}$ for $\ell = 1, 2$.

Let $\{\neg\varphi_i(\bar{x}{}^\frown\bar{y} - \bar{a}_i) : i < \kappa_\beta\}$ list possibly with repetitions the set of formulas $\neg\varphi(\bar{x}{}^\frown\bar{y} - \bar{a})$ satisfied by $\bar{c}_1{}^\frown\bar{b}_1$ with $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm ep}_{\beta,\varepsilon,\zeta}, \bar{a} \in \mathbf{I}_\beta$ and let $\varphi'_i(\bar{y}) = \varphi_i(0_\varepsilon, \bar{y})$. Let $\psi'(\bar{y}) = \wedge\{\varphi(\bar{y}) : \varphi(\bar{y}) \in \Phi'_1\}$.

Let the ideal I on κ_β be defined as in 4.1 with $G = \psi'({}^{\varepsilon}M)$ where $\psi'(\bar{x}_\xi) = \bigwedge\{\varphi(\bar{y}) : \varphi(\bar{y}) \in \Phi'_1\}$ and $G_i = G \cap \varphi'_i({}^{\xi}M)$ for $i \in S := \kappa_\beta, \lambda = (2^{\kappa_\beta})^+$.

<u>Case 1</u>: $\kappa_\beta \in I$.

So clearly $M \models \varphi[\bar{b}_1 - b^*, \bar{c}_1 - \bar{c}^*]$ for every $\varphi(\bar{x}, \bar{y}) \in \Phi_1$.

Let $\psi_*(\bar{x}, \bar{y}) = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \Phi_1\}$, so clearly it $\in \Lambda^{\rm ep}_{\alpha,\varepsilon,\zeta}$ and $M \models \psi_*[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^*]$ hence $M \models (\exists\bar{y})\psi_*[\bar{b}_1 - b^*, \bar{y}]$. But $(\exists\bar{y})\psi(\bar{x}, \bar{y}) \in \Lambda^{\rm ep}_{\alpha,\varepsilon}$ so by the assumption on $\bar{b}_1, \bar{b}_2$ we have $M \models (\exists\bar{y})\psi_*[\bar{b}_2 - \bar{b}^*, \bar{y}]$ hence for some $\bar{c}'_2$ we have $M \models \psi_*[\bar{b}_2 - \bar{b}^*, \bar{c}'_2]$ and let $\bar{c}''_2 = \bar{c}'_2 + \bar{c}^*$, so $M \models \psi_*[\bar{b}_2 - \bar{b}^*, \bar{c}''_2 - \bar{c}^*]$. As we are in case 1, by $\boxplus_\alpha(e)(\beta)$ there is a sequence $\langle\bar{e}_\iota : \iota < \lambda\rangle$ of members of G, i.e. of $\{\bar{a} \in {}^{\xi}M : M \models \psi_*(\bar{0}_\varepsilon, \bar{a})\}$ such that $i < \kappa_\beta \wedge (\iota(1) < \iota(2) < \lambda) \Rightarrow \bar{e}_{\iota(2)} - \bar{e}_{\iota(1)} \notin G_i$.

So for every $\iota < \lambda$, the sequence $(\bar{b}_2 - \bar{b}^*){}^\frown(\bar{c}''_2 - \bar{c}^* + \bar{e}_\iota)$ belongs to $\psi_*({}^{\varepsilon+\xi}M)$ and for each $i < \kappa_\beta$ the set $\{\iota < \lambda : (\bar{b}_2 - \bar{b}^*){}^\frown(\bar{c}''_2 - \bar{c}^* + \bar{e}_\iota)$ belongs to $(\bar{a}_i - \bar{b}^*{}^\frown\bar{c}^*) + G_i\}$ has at most one member. As $\kappa_\beta < \lambda$ for some $\iota < \lambda, (\bar{b}_2{}^\frown\bar{b}){}^\frown(\bar{c}''_2 - c^* + \bar{e}_\iota) \notin \cup\{\bar{a}_i - \bar{b}^*{}^\frown\bar{c}^* + G_i : i < \kappa_\beta\}$.

So $\bar{c}_2 := \bar{c}''_2 + \bar{e}_\iota$ is as required.

<u>Case 2</u>: $\kappa_\beta \notin I$

So there is a sequence $\langle\bar{d}_\iota : \iota < \iota(*)\rangle$ of members of $\mathbf{I}_\alpha$ as in $\boxplus_\alpha(e)(\alpha)$ for $\xi, G, G_i(i < \kappa_\beta)$ as above, i.e. with $\bar{\psi}'(\bar{y}), \langle\varphi'_i(\bar{y}) : i < \kappa_\beta\rangle$ here stands for $\psi(\bar{x}), \langle\varphi_i(\bar{x}) : i < \kappa_\beta\rangle$ there; so $\iota(*) < (2^{\kappa_\alpha})^+$ and $\iota < \iota(*) \Rightarrow \bar{d}_\iota \in \mathbf{I}_\alpha \cap {}^{\varepsilon}M$. As clearly $\bar{c}_1 - \bar{c}^* \in G$ necessarily for some $\iota < \iota(*)$ the set $u := \{i < \kappa_\beta : (\bar{c}_1 - \bar{c}^* - \bar{d}_\iota) \notin G_i\}$ belongs to I and, of course, $\bar{b}^*{}^\frown(\bar{c}^* + \bar{d}_\iota) \in \mathbf{I}_\alpha \cap {}^{\varepsilon+\xi}M$ and we have:

$(*)_1$ $M \models \varphi[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^* - d_\iota]$ for $\varphi \in \Phi_1$

$(*)_2$ if $i \in \kappa_\beta \backslash u$ then $M \models \varphi_i[\bar{b}_1 - \bar{b}^*, \bar{c}_1 - \bar{c}^* - \bar{d}_\iota]$.

As in Case 1 there is $\bar{c}''_2 \in {}^{\xi}M$ such that

$(*)_3$ $M \models \varphi[\bar{b}_2 - \bar{b}^*, \bar{c}''_2 - \bar{c}^* - \bar{d}_\iota]$ for $\varphi \in \Phi_1$

$(*)_4$ if $i \in \kappa_\beta \backslash u$ then $M \models \varphi_i[\bar{b}_2 - \bar{b}^*, \bar{c}''_2 - \bar{c}^* - \bar{d}_\iota]$.

As $u \in I$ by $\boxplus_\alpha(e)(\beta)$ that is, by 4.1 there are $\bar{\mathbf{e}}, u_*$ such that $\bar{\mathbf{e}}$ is a sequence of the form $\langle \bar{e}_j : j < \kappa^+_\beta \rangle$ and $u \subseteq u_* \in I$ such that:

- $(*)_5$ $\bar{e}_j \in G_i$ for $i \in \kappa_\beta \backslash u_*$
- $(*)_6$ $e_{j_2} - \bar{e}_{j_1} \notin G_i$ for $j_1 < j_2 < \kappa^+_\beta, i \in u_*$.

So

- $(*)_7$ $(\bar{b}_2 - \bar{b}^*)\hat{}(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ belongs to $\cap\{\varphi({}^{\varepsilon+\xi}M) : \varphi \in \Phi_1\}$
- $(*)_8$ if $i \in \kappa_\beta \backslash u_*$ then also $i \in \kappa_\beta \backslash u$ so by $(*)_4 + (*)_5$ the sequence $(\bar{b}_2 - \bar{b}^*)\hat{}(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ satisfies $\varphi_i(\bar{x}\hat{}\bar{y} - \bar{a}_i)$ in M hence $\bar{b}_2\hat{}(\bar{c}''_2 - \bar{e}_j)$ satisfies the formula $\neg\varphi_i(\bar{x}\hat{}\bar{y} - \bar{a}_i)$ in M.

Lastly, by $(*)_6$

- $(*)_9$ for each $i \in u_*$, there is $j_i < \kappa^+_\beta$ such that for every $j \in \kappa^+_\beta \backslash \{j_i\}$ the sequence $(\bar{b}_2\hat{}\bar{b}^*)\hat{}(\bar{c}''_2 - c^* - e_j)$ satisfies $\neg\varphi_i(\bar{x}\hat{}\bar{y} - \bar{a}_i)$, so for some $j, (\bar{c}''_2 - \bar{c}^* - \bar{e}_j)$ this holds for every $i \in u_*$.

Putting together $(*)_7 + (*)_8 + (*)_9$ clearly $(\bar{c}''_2 - \bar{c}^* - \bar{d}_\iota - \bar{e}_j)$ is as required in $\boxdot$ so we are done. $\square_{2.4}$

Definition 2.8. Let $\theta = \operatorname{cf}(\theta), \gamma$ an ordinal, $\bar{\lambda} = \langle \lambda_\beta : \beta < \gamma \rangle$.
1) For an R-module M we say $\bar{\mathbf{I}}$ is a (θ, γ)-witness for M <u>when</u> $\bar{\mathbf{I}} = \langle \mathbf{I}_\beta : \beta \leq \gamma \rangle$ and for each $\alpha \leq \gamma, \mathbf{I}_\alpha$ satisfies the conclusion of 2.4.
2) We say $\bar{\mathbf{I}}$ is a $(\bar{\lambda}, \theta, \gamma)$-witness when if in addition $\bar{\lambda} = \langle \lambda_\beta : \beta \leq \gamma \rangle$ and $\beta \leq \gamma \Rightarrow \lambda_\beta > |\mathbf{I}_\beta|$.

§ 3. Stability

Context 3.1. 1)

(a) R a fixed ring, $\tau = \tau_R$ <u>or</u>

(b) τ is a θ-additive vocabulary; $\mathbf{K}$ the class of τ-additive models.

2) $M \in \mathbf{K}$ a fixed R-module.
3) $\theta = \mathrm{cf}(\theta)$ and an ordinal, $\gamma(*)$ limit for simplicity.
4) $\bar{\lambda} = \langle \lambda_\alpha : \alpha \le \gamma(*)\rangle, \lambda_\alpha > \kappa_\alpha := \beth_\alpha(|R| + \theta^-)$.
5) $\bar{\mathbf{I}}^*$ is a $(\bar{\lambda}, \theta, \gamma(*))$-witness, see 2.8.
6) $A_* = \cup\{\bar{a} : \bar{a} \in \mathbf{I}_{\gamma(*)}\}$.
7) $\Lambda_\varepsilon = \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon}$ for $\varepsilon < \theta$ and $\Lambda = \cup\{\Lambda_\varepsilon : \varepsilon < \theta\}$.
8) $M_* = M_{A_*} := (M, a)_{a \in A_*}$.

Definition 3.2. Assume $\varepsilon < \theta, \Lambda \subseteq \Lambda^{\mathrm{pe}}_{\theta,\gamma(*)}$ and $A_* \subseteq A \subseteq M \in \mathbf{K}$ and $\bar{a} \in {}^\varepsilon M$.
1) $\mathbf{S}^\varepsilon_\Lambda(A, M) = \{\mathrm{tp}_\Lambda(\bar{a}, A, M) : \bar{a} \in {}^\varepsilon M\}$, see below.
2) For $\bar{a} \in {}^\varepsilon M$ let $\mathrm{tp}_\Lambda(\bar{a}, A, M) = \{\varphi(\bar{x}{}^\frown\bar{b} - \bar{c}) : \bar{b} \in {}^\xi A$ and $\bar{c} \in {}^{\varepsilon+\xi}M$ and $M \models \varphi[\bar{a}_1{}^\frown\bar{b} - \bar{c}]$ and $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma,\varepsilon+\xi} \cap \Lambda\}$.

The Stability Theorem 3.3. *Assume $\Lambda \subseteq \Lambda^{\mathrm{pe}}_{\gamma(*)}$ and $A \subseteq M \in \mathbf{K}$.*
1) The set $\mathbf{S}^\varepsilon_\Lambda(A, M)$ has cardinality $\le ((|A|)^{<\theta})^{|\Lambda|}$.
2) For any $\kappa \ge 4$, yes! four, there are no $\bar{a}_\alpha \in {}^\varepsilon M, \bar{b}_\alpha \in {}^\xi M$ for $\alpha < \kappa$ and[1] *$\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon,\xi}$ such that for $\alpha < \beta < \kappa$ we have $M \models$ "$\varphi[\bar{a}_\alpha, \bar{b}_\beta] \wedge \neg\varphi[\bar{a}_\beta, \bar{b}_\alpha]$".*
3) If the formula $\varphi(\bar{x}, \bar{y})$ from $\mathbb{L}_{\infty,\theta,\gamma(*)}$ or just is a Boolean combination of such formulas and $\kappa \ge \beth_{\gamma(*)+2}(|\tau|^{<\theta})^+$ <u>then</u> there are no $M \in \mathbf{K}, \bar{a}_\alpha \in {}^\varepsilon M, \bar{b}_\alpha \in {}^\zeta M$ for $\alpha < \kappa$ such that $M \models \varphi[\bar{a}_\alpha, \bar{b}_\beta] \wedge \neg\varphi[\bar{a}_\beta, \bar{b}_\alpha]$ whenever $\alpha < \beta < \kappa$. Actually $\kappa \ge \beth_{\gamma(*)+1}(|\tau|^{<\theta})^+$ suffice.
4) If $p \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ and $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{ep}}_{\gamma(*),\varepsilon,\xi}$ and $p \cap \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^\xi A\} \ne \emptyset$ <u>then</u> for some $\bar{a}_\varphi \in {}^\varepsilon A$ and $\bar{b} \in {}^\xi A$ we have $\varphi(\bar{x} - \bar{a}_\varphi, \bar{b}) \vdash p{\restriction}\{\pm\varphi\}$ and $\varphi(\bar{x} - \bar{a}_\varphi, \bar{b}) \in p$.

Proof. 1) Consider the statement

$(*)$ if $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon,\xi} \cap \Lambda$ and $p_\ell(\bar{x}) = \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{a}_\ell, A, M) \in \mathbf{S}^\varepsilon_{\{\varphi(\bar{x},\bar{y})\}}(A, M)$ for $\ell = 1, 2$ and $\bar{b} \in {}^\xi A, \bar{c} \in {}^{\varepsilon+\xi}A$ and $\varphi(\bar{x}{}^\frown\bar{b} - \bar{c}) \in p_1(\bar{x}) \cap p_2(\bar{x})$ <u>then</u> $p_1(\bar{x}) = p_2(\bar{x})$.

Why $(*)$ is true? Assume $\varphi(\bar{x}{}^\frown\bar{b}' - \bar{c}') \in p_1(\bar{x})$, so $\bar{a}_1{}^\frown\bar{b}' - \bar{c}' \in \varphi(\bar{M})$. But we are assuming $\varphi(\bar{x}{}^\frown\bar{b} - \bar{c}) \in p_\ell(\bar{x}) = \mathrm{tp}_{\{\varphi(\bar{x},\bar{y})\}}(\bar{a}_\ell, A, M)$ hence $\bar{a}_\ell{}^\frown\bar{b} - \bar{c} \in \varphi(M)$ for $\ell = 1, 2$. Together $\bar{a}_2{}^\frown\bar{b}' - \bar{c}' = (\bar{a}_2{}^\frown\bar{b} - \bar{c}) - (\bar{a}_1{}^\frown\bar{b} - \bar{c}) + (\bar{a}_1{}^\frown\bar{b}' - \bar{c}')$ belongs to $\varphi(M)$, hence $\varphi(x{}^\frown\bar{b}' - c') \in p_2(x)$. So $\varphi(\bar{x}{}^\frown\bar{b}' - \bar{c}') \in p_1 \Rightarrow \varphi(\bar{x}{}^\frown b' - \bar{c}') \in p_2$ and by symmetry we have $\Leftrightarrow$ hence $p_1(\bar{x}) = p_2(\bar{x})$, i.e. we have proved $(*)$.

Why $(*)$ is sufficient? For every $\xi < \theta, \varphi(\bar{x}, \bar{y}) \in \Lambda^{\mathrm{pe}}_{\gamma(*),\varepsilon,\xi} \cap \Lambda$ and $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ choose $(\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}, \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})})$ such that

$\oplus_1$
- $\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^\varepsilon A$ and $\bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} \in {}^{\varepsilon+\xi}A$
- if possible $\varphi(\bar{x}{}^\frown\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) \in p(\bar{x})$.

[1]This holds also for $\neg\varphi(\bar{x}, \bar{y})$ but for κ finite we can invert the order.

For $p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ let $\Phi_{p(\bar{x})} = \{\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm pe}_{\gamma,\varepsilon,\xi}$: in $\oplus_1$ we have "possible"$\}$ and let $q_{p(\bar{x})} = \{\varphi(\bar{x}\hat{\ }\bar{b}_{p(\bar{x}),\varphi(\bar{x},\bar{y})} - \bar{c}_{p(\bar{x}),\varphi(\bar{x},\bar{y})}) : \varphi(\bar{x}, \bar{y}) \in \Phi_{p(\bar{x})}\}$.

Now

$\oplus_2$ if $p_1(\bar{x}), p_2(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)$ and $\Phi_{p_1(\bar{x})} = \Phi_{p_2(\bar{x})}$ and $q_{p_1(\bar{x})} = q_{p_2(\bar{x})}$ then $p_1(\bar{x}) = p_2(\bar{x})$.

[Why? Just think.]

$\oplus_3$ the set $\{(\Phi_{p(\bar{x})}, q_{p(\bar{x})}) : p(\bar{x}) \in \mathbf{S}^\varepsilon_\Lambda(A, M)\}$ has cardinality $\leq 2^{|\Lambda|} + (|A|^{<\theta})^{|\Lambda|}$.

[Why? Straightforward.]

Clearly we are done.

2) Note that $\varphi(\bar{x}, \bar{y}) \in \Lambda^{\rm pe}_{\gamma,\varepsilon,\xi}$ implies that

$\boxplus$ if $M \models \varphi[\bar{a}, \bar{b}] \wedge \varphi[\bar{a}, \bar{b}'] \wedge \varphi[\bar{a}', \bar{b}]$ then $M \models \varphi[\bar{a}', \bar{b}']$.

[Why? As $\varphi({}^{\varepsilon+\zeta}M)$ is a subgroup of ${}^{\varepsilon+\zeta}M$ and $\bar{a}\hat{\ }\bar{b}, \bar{a}'\hat{\ }\bar{b}, \bar{a}\hat{\ }\bar{b}'$ belongs to it then so does $\bar{a}'\hat{\ }\bar{b}' = \bar{a}'\hat{\ }\bar{b} + (\bar{a}\hat{\ }\bar{b}') - (\bar{a}\hat{\ }\bar{b})$ but the latter is equal to $\bar{a}'\hat{\ }\bar{b}'$.]

So we can choose $\bar{a} = \bar{a}_0, \bar{a}' = \bar{a}_3, \bar{b} = \bar{b}_1, \bar{b}' = \bar{b}_2$ and get a contradiction.

3) Toward contradiction let $\langle \bar{a}_\alpha : \alpha < \kappa\rangle, \bar{a}_\alpha \in {}^\varepsilon M$ form a counterexample. By Erdös-Rado theorem $\beth_{\gamma(*)+2}(|\tau|^{<\theta})^+ \rightarrow (4)^2_{\beth_{\gamma(*)+1}(|\tau|^{<\theta})}$. Now for $\alpha < \beta < \kappa$ let $p_{\alpha,\beta} = {\rm tp}_{\Lambda^{\rm pe}_{\gamma(*),\varepsilon,\varepsilon}}(\bar{a}_\alpha\hat{\ }\bar{a}_\beta; \emptyset, M)$ so $\{p_{\alpha,\beta} : \alpha < \beta\}$ has cardinality $\leq \beth_{\gamma(*)+1}(|\tau|^{<\theta})$ hence by the arrow above for some $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ and $p, \ell < m < u \Rightarrow p_{\alpha_\ell,\alpha_n} = p$; we get contradiction by part (2). If κ is just $\geq \beth_{\gamma(*)+1}(|\tau|^{<\theta})^+$, use $\boxplus$ from the proof of part (2) and repeat a proof of the Erdös-Rado theorem.

4) Should be clear. $\square_{3.2}$

Recall ([?])

Definition 3.4. For $\Phi \subseteq \Lambda$ we say $\mathbf{I} \subseteq {}^\varepsilon M$ is (μ, Φ)-convergent <u>when</u> $|\mathbf{I}| \geq \mu$ and for every $\xi < \theta$ and $\varphi(\bar{x}) \in \Phi_{\varepsilon+\xi}$ and $\bar{b} \in {}^\xi M, \bar{c} \in {}^{\xi+\varepsilon} M$ for all but $< \mu$ of the $\bar{a} \in I$ the truth value of $\bar{a}\hat{\ }\bar{b} - \bar{c} \in \varphi(M)$ is constant.

Claim 3.5. *1) A sufficient condition for $\mathbf{I} = \{\bar{a}_i : i < \lambda\} \subseteq {}^\varepsilon M$ to be (μ, Φ)-convergent is: for some $\varepsilon, \mathbf{I} \subseteq {}^\varepsilon M$ and $i < j < \lambda \wedge \varphi(\bar{x}) \in \Phi \cap \Lambda_\varepsilon \Rightarrow \bar{a}_j - \bar{a}_i \in \varphi(M)$.*
2) If $\varepsilon < \theta, \lambda = {\rm cf}(\lambda) > \mu \geq \mu_{\gamma(*)}$ and $(\forall i < \lambda)(|i|^{\mu_{\gamma(*)}} < \lambda)$ and $\bar{a}_i \in {}^\varepsilon M$ for $i < \lambda$ with no repetition <u>then</u> for some stationary $S \subseteq \lambda, \{\bar{a}_i : i \in S\}$ is (μ^+, Φ)-convergent.

Remark 3.6. 1) Note that being $(\mu, \mathbf{I})$-convergent is very close to being $(< \omega)$-indiscernible, and sometimes is the reasonable generalization of indiscernible.
2) So 3.5(1) says that 2-indiscernible almost implies $(< \omega)$-indiscernible.
2) Also 3.5(2) says there are $(< \omega)$-indiscernibles.

Proof. Should be clear. $\square_{3.5}$

§ 4. How much does the subgroup exhaust a group

Claim 4.1. *Assume the groups G_s (for $s \in S$) are subgroups of the group G and $\lambda > |S|^+$. There is an ideal I on S (possibly $I = \mathscr{P}(S)$) such that:*

- (a) *for every $u \in I$ there is a sequence $\bar{g} = \langle g_\alpha : \alpha < \lambda \rangle$ of members of G such that $s \in u \wedge \alpha < \beta < \lambda \Rightarrow g_\alpha G_s \neq g_\beta G_s$*
- (b) *for $u \in \mathscr{P}(S) \backslash I$, clause (a) fails*
- (c) *if $S \notin I$, $\mathrm{cf}(\lambda) > 2^{|S|}$ and $\alpha < \lambda \Rightarrow |\alpha|^{|S|} < \lambda$, e.g. $(\exists \mu)(\lambda = (\mu^{|S|})^+)$ <u>then</u> there is $A \subseteq G$ of cardinality $< \lambda$ such that for every $g \in G$ for some $a \in A$ we have $\{s \in S : gG_s \neq aG_s\} \in I$*
- (d) *under the assumptions of clause (c) and in addition λ is regular <u>then</u> for every $u \in I$ for some $\bar{g}$ and v we have*
  - $u \subseteq v \in I$
  - $\bar{g} = \langle g_\alpha : \alpha < \lambda \rangle$
  - $g_\alpha G_s = g_0 G_s$ *moreover* $g_\alpha \in G_s$ *for* $s \in S \backslash v$
  - *if* $s \in v, \alpha < \beta < \lambda$ *then* $g_\alpha G_s \neq g_\beta G_s$.
- (e) *$I \subseteq \mathscr{P}(S), I$ is closed under subsets*
- (f) *I is an ideal provided that G is Abelian <u>or</u> just each G_s is a normal subgroup.*

Definition 4.2. For G and $\bar{G} = \langle G_s : s \in S \rangle$ as in 4.1 and $\lambda \geq \aleph_0$ let $I = I_{G,\bar{G},\lambda}$ be as defined in clauses (a),(b) of 4.1, it is an ideal (but may be $\mathscr{P}(S)$).

Proof. Let I be the set of $u \subseteq S$ such that clause (a) holds.
Now

$(*)$ (α) $I \subseteq \mathscr{P}(\kappa)$
(β) I is $\subseteq$-downward closed, i.e. is closed under subsets.

[Why? Obvious.]

Now for 4.1, we have chosen I such that clauses (a),(b),(e) hold.

Toward proving clause (c) of 4.1 for each $u \in I^+ := \mathscr{P}(S) \backslash I$, let $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha(u) \rangle$ be a maximal sequence of members of G such that $\alpha < \beta < \alpha(u) \wedge s \in u \Rightarrow g_{u,\alpha} G_s \neq g_{u,\beta} G_s$. By the definition of I as $u \notin I$, necessarily $\alpha(u) < \lambda$, and as we are assuming $\mathrm{cf}(\lambda) > 2^{|S|}$, clearly $\alpha(*) = \sup\{\alpha(u) : u \in I^+\} < \lambda$. So $B := \{g_{u,\alpha} : u \in I^+$ and $\alpha < \alpha(u)\}$ is a subset of G of cardinality $< \lambda$. For every $u \in I$ and $h : S \backslash u \to B$ choose $g_h \in G$ such that, if possible, $(\forall s \in S \backslash u)(g_h G_s = h(s) G_s)$, so $A = \{g_h : h$ is a function from $S \backslash u$ into B and $u \in I\}$ is a subset of G of cardinality $\leq |B|^{|S|} < \lambda$, recalling we are assuming now $(\forall \alpha < \lambda)(|\alpha|^{|S|} < \lambda)$.

We shall show that A is as required (in clause (c)), then we are done. Let $g_* \in G$. Let $u = \{s \in S$: for no $w \in I^+$ and $\alpha < \alpha(w)$ do we have $gG_s = g_{u,\alpha} G_s\}$. Now if $u \in I^+$ then $\bar{g}_u = \langle g_{u,\alpha} : \alpha < \alpha(u) \rangle$ is well defined and g_* satisfies the demand on $g_{u,\alpha(u)}$ contradicting the maximality of $\bar{g}_u$. So $u \in I$ and we can find $h : (S \backslash u) \to B$ such that $s \in S \backslash u \Rightarrow g_* G_s = h(s) G_s$. So g_h is well defined and $\in A$ and is as required, so we are done.

For clause (d) let $u \in I$ be given and let $\langle g_\alpha : \alpha < \lambda \rangle$ witness that $u \in I$. For each $\alpha < \beta$ let $u_\alpha = \{s \in S$: there is $\beta < \alpha$ such that $g_\alpha G_s = g_\beta G_s\}$, clearly $u_\alpha \cap u = \emptyset$ and let $h_\alpha : u_\alpha \to \alpha$ be such that $s \in u_\alpha \Rightarrow g_\alpha G_s = g_{h_\alpha(s)}$.

As λ is regular, recalling $(\forall \alpha < \lambda)(|\alpha|^{|S|} < \lambda)$ by the present assumption on λ, for some $h : u_* \to \lambda$, the set $\mathscr{W}$ is a stationary subset of λ where $\mathscr{W} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = |S|^+$ and $h_\alpha = h, u_\alpha = u_*\}$. Clearly $\alpha, \beta \in \mathscr{W} \wedge s \in u_* \Rightarrow g_\alpha G_s = g_{h(s)} G_s = g_\beta G_s$ and $\alpha \neq \beta \in \mathscr{W} \wedge s \in S \backslash u_* \Rightarrow g_\alpha G_s \neq g_\beta G_s$. Letting $\langle \alpha_i : i < \lambda \rangle$ list $\mathscr{W}$ and $g_i = g_{\alpha_i}$ for $\alpha < \lambda$, clearly $v = u_*, \langle g'_i : i < \lambda \rangle$ are as promised in clause (d). Well for the "moreover" and, i.e. use $\langle g_0^{-1} g_{1+i} : i < \lambda \rangle$.

We are left with clause (f).

I is an ideal <u>when</u> the assumption of clause (f) holds. Let $u_1, u_2 \in I$ be disjoint and we shall prove that $u := u_1 \cup u_2 \in I$. Let $\langle g_{\ell,\alpha} : \alpha < \lambda \rangle$ witness $u_\ell \in I$ for $\ell = 1, 2$. We try to choose $g_{3,\varepsilon} \in G$ such that $\zeta < \varepsilon \wedge s \in u \Rightarrow g_{3,\varepsilon} G_s \neq g_{3,\zeta} G_s$; we can add $g_{3,\varepsilon} \in \{g_{1,i} g_{2,j} : i, j < \lambda\}$. Arriving to ε, if for some $i < \lambda \wedge j < \lambda$ we can choose $g_{3,\varepsilon} := g_{1,i} g_{2,j}$ fine.

Otherwise there are $f : \lambda \times \lambda \to \varepsilon$ and $g : \lambda \times \lambda \to u$ such that for $(i, j) \in \lambda \times \lambda$ we have $g_{1,i} g_{2,j} G_{g(i,j)} = g_{3,f(i,j)} G_{g(i,j)}$.

For each $i < \lambda, \zeta < \varepsilon$ and $s \in u \subseteq S$ let $\mathscr{U}^2_{i,\zeta,s} = \{j < \lambda : f(i,j) = \zeta, g(i,j) = s\}$. Now $j \in \mathscr{U}^2_{i,\zeta,s} \Rightarrow g_{1,i} g_{2,j} G_s = g_{3,\zeta} G_s \Rightarrow g_{2,j} G_s = g_{1,i}^{-1} g_{3,\zeta} G_s$ hence if $s \in u_2$ then $j(1) \neq j(2) \in \mathscr{U}^2_{i,\zeta,s} \Rightarrow g_{2,j(1)} G_s = (g_{1,i}^{-1} g_{3,\zeta}) G_s = g_{2,j(2)} G_s$ contradiction. Hence $\mathscr{U}^2_{i,\zeta,s}$ has cardinality ≤ 1 when $i < \lambda, \zeta < \varepsilon, s \in u_2$.

For $j < \lambda, \zeta < \varepsilon$ and $s \in u$ let

$$\mathscr{U}^1_{j,\zeta,s} = \{i < \lambda : f(i,j) = \zeta \text{ and } g(i,j) = s\}.$$

If G is Abelian, as above we have $\zeta < \varepsilon \wedge j < \lambda \wedge s \in u_1 \Rightarrow |\mathscr{U}^1_{j,\zeta,s}| \leq 1$. If not but still every G_s is a normal subgroup of G then for any $j < \lambda, \zeta < \mu, s \in u_1$ we have $i \in \mathscr{U}^1_{j,\zeta,s} \Rightarrow g_{1,i} g_{2,j} G_s = g_{3,\zeta} G_s \Rightarrow g_{1,i}(G_s g_{2,j}) = g_{1,i}(g_{2,j} G_s) = g_{3,\zeta} G_s \Rightarrow g_{1,i} G_s = g_{3,\zeta}(G_s g_{2,j}^{-1})$ hence $i(1) \neq i(2) \in \mathscr{U}^1_{j,\zeta,s} \Rightarrow g_{1,i(1)} G_s = g_{3,\zeta}(G_s g_{2,j}^{-1}) = g_{1,i(2)} G_s$, a contradiction so again $\mathscr{U}^1_{j,\zeta,s}$ has at most one member.

For $\ell \in \{1, 2\}$ and $i < \lambda$ let $\mathscr{U}^\ell_i = \cup \{\mathscr{U}^\ell_{i,\zeta,s} : \zeta < \varepsilon$ and $s \in u_\ell\}$, so as $|u_\ell| \leq |S|$ clearly $|\mathscr{U}^\ell_i| \leq |S|$. As $\lambda > |S|^+$ there are $i, j < \lambda$ such that $i \notin \mathscr{U}^1_j \wedge j \notin \mathscr{U}^2_i$; hence the member $g_{1,i} g_{2,j}$ of G satisfies the demand on $g_{3,\varepsilon}$.

So we can carry the induction on $\varepsilon < \lambda$, so we are done proving clause (f). $\square_{4.1}$

Claim 4.3. *In 4.1 there is a $W \subseteq S$ such that*

(a) *there is a sequence $\bar{s} = \langle s_i : i < i(*) \rangle$ listing W satisfying $(\bigcap_{i<j} G_{s_i}, \bigcap G_{s_i})$ is finite for $j < i(*)$ stipulating $\bigcap_{i<0} G_{s_i} = G$*

(b) *if $W' \subseteq S$ satisfies (A) then $W' \subseteq W$.*

Proof. Immediate. $\square$

§ 5. Concluding Remark

Example 5.1. An example of additive structure is a ring satisfying $xy = -yx$, i.e. if $(R, +^R)$ is $\oplus\{\mathbb{Z}x_s : s \in I\}, f$ is a function from $I \times I$ into R is such that $f(x,y) = -f(y,x)$ and $f(x,x) = 0$ and we have

$$(\sum_{\ell<\ell(*)} a_\ell x_{s_\ell})(\sum_{m<n(*)} b_m x_{t_n}) = \Sigma\{a_\ell b_m x_{f(s_\ell,t_m)} : \ell < \ell(*), m < m(*)\}.$$

Remark 5.2. 1) We may use $\tau \supseteq \{+,-,0,1\} \cup \{P_i : i < i(*)\}, P_i$ unary and instead modules use τ-models M such that $|M|$ is the disjoint union $\cup\{P_i^M : i < i(*)\}, +^M$ is a partial two-place function, $+^M = \cup\{+^M \restriction P_i^M : i < i(*)\}, (P_i^M, +^M)$ an Abelian group, all relations and functions commute with $+$ or at least every relation is affine, i.e. let $F_*(x,y,z) = x - y + z$, and demand $G(\ldots, F_*(x_i,y_i,z_i),\ldots)_{i<i(*)} = F_*(G(\bar{x}), G(\bar{y}), G(\bar{z}))$ and $\bar{a}, \bar{b}, \bar{c} \in P^M \Rightarrow F_*(\bar{a}, \bar{b}, \bar{c}) = \langle F_*(a_i,b_i,c_i) : i < \text{arity}(P)\rangle \in P^M$.
2) However, as we use infinitary logics, if M is the disjoint union of Abelian groups $G_i^M := (P_i^M, +_i^M)$ for $i < i(*)$ and we define G_M as the direct sum having predicate for those subgroups then we have bi-interpretability. Concerning having "affine structure" only, we can expand by choosing an element in each to serve as zero.
3) It is natural to extend our logic by cardinality quantifiers saying "the definable subgroup G divided by the definable subgroup H has cardinality $\geq \lambda$".

Remark 5.3. Concerning 2.4 note
1) Note that instead of an R-module M we can use $(M, c_\alpha)_{\alpha<\kappa}$, i.e. expand M by κ individual constants; the only difference is using $\beth_\alpha(|R|^{<\theta} + \kappa)$ instead $\beth_\alpha(|R|^{<\theta})$.
2) The theorem 2.4 has an arbitrary choice: the $\mathbf{I}_\alpha$, so e.g. not every formula $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\theta,\gamma}$ and $\bar{a} \in \mathbf{I}_\partial$ is $\varphi(\bar{x}, \bar{a}_\gamma)$ equivalent to a formula without parameters. Instead of using extra individual constants, in the proof (see $\boxplus_\alpha$ in the proof of 2.4) for any $\psi(\bar{x}), \psi(\bar{x}) \wedge \varphi_i(\bar{x})$ for $i < i(*) < \kappa_\beta, I, G, G_i(i < i(*))$ and the ideal I on κ_β can expand M by:

- (a) $P^M = \{\bar{a} : M \models \psi[\bar{a}]$ and $\{i < \kappa_\beta : \bar{a} \notin G_i\} \in I\}$ is a subgroup
- (b) predicates for the set $\{\bar{a} + P^M : \bar{a} \in \psi(M)\}$.

So the proof shows that we can in M eliminate quantifier to quantifier-free formulas in this expansion.
3) Also this may give too much information. Still the result gives elimination of quantifiers: not as low as in the first order case.
4) We can now define non-forking and hopefully [?] will deal with this.

Question 5.4. 1) Are there arbitrarily large Abelian groups G which are not only indecomposable, but even potentially so, i.e. absolutely, even after any forcing G is indecomposable.
2) Relatives, e.g. no potential non-trivial automorphism.

Discussion 5.5. We know that for the minimal $\lambda, \lambda \to (\omega)_{\aleph_0}^{<\omega}$, up to λ the answer is yes (and more) but if $|G| \geq \lambda$ then potentially it has non-trivial endomorphisms and even non-trivial embedding of G into itself (Eklof-Shelah [?], Göbel-Shelah [?]). We can improve this to "for some $a_1 \neq a_2$ from G", potentially there are an embeddings f_1, f_2 of G into itself such that $f_1(a_1) = a_2, f_2(a_2) = a_1$, see [?].

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