WHEN FIRST ORDER T HAS LIMIT MODELS SH868

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ABSTRACT. We to a large extent sort out when does a (first order complete theory) T have a superlimit model in a cardinal λ. Also we deal with related notions of being limit.

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Anotated Content

0. Introduction

§(0A) Background and Content

Recall that ([?, Ch.III]). If T is (first order complete and) superstable then for $\lambda \geq 2^{|T|}, T$ has a saturated model M of cardinality λ and moreover

(∗) if $\langle M_\alpha : \alpha < \delta\rangle$ is $\prec$-increasing, δ a limit ordinal $< \lambda^+$ and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $\cup\{M_\alpha : \alpha < \delta\}$ is isomorphic to M.

When investigating categoricity of an a.e.c. (abstract elementary classes) $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$, the following property turns out to be central: M is $\leq_{\mathfrak{k}}$-universal model of cardinality λ with the property (∗) above (called superlimit) - possibly with addition parameter $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$); we also consider some relatives, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [?, 3.1] or see the revised version [?, 3.3] and see [?] or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete T, we know $\{\lambda : T$ has a saturated model of T of cardinality $\lambda\}$, that is, it is $\{\lambda : \lambda^{<\lambda} \geq |D(T)|$ or T is stable in $\lambda\}$, on the definitions of $D(T)$ and other notions see §(0B) below. What if we replace saturated by superlimit (or some relative)? Let $\mathrm{EC}_\lambda(T)$ be the class of models M of T of cardinality λ.

If there is a saturated $M \in \mathrm{EC}_\lambda(T)$ we have considerable knoweldge on the existence of limit model for cardinal λ, this was as mentioned in [?, 3.6] by [?], see 0.9(1),(2). E.g. for superstable T in $\lambda \geq 2^{|T|}$ there is a superlimit model (the saturated one). It seems a natural question on [?, 3.6] whether it exhausts the possibilities of $(\lambda, *)$-superlimit and (λ, κ)-superlimit models for elementary classes.

Clearly the cases of the existence of such models of a (first order complete) theory T where there are no saturated (or special) models are rare, because even the weakest version of Definition [?, 3.1] = [?, 3.3] or here Definitino 0.7 for λ implies that T has a universal model of cardinality λ, which is rare (see Kojman Shelah [?] which includes earlier history and recently Djamonza [?]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for $\lambda < \lambda^{<\lambda}$), i.e., those for $\lambda = \aleph_1$.

E.g. a sufficient condition for some versions is the existence of $T' \supseteq T$ of cardinality λ such that $\mathrm{PC}(T', T)$ is categorical in λ, see 0.4(3). By [?] we have consistency results for such T_1 so naturally we first deal with the consistency results from [?]. In §1 we deal with the case of the countable superstable T_0 from [?] which is not $\aleph_0$-stable. By [?] consistently $\aleph_1 < 2^{\aleph_0}$ and for some $T_0' \supseteq T_0$ of cardinality $\aleph_1$, $\mathrm{PC}(T_0', T_0)$ is categorical in $\aleph_1$. We use this to get the consistency of "T_0 has a superlimit model of cardinality $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$".

In §2 we prove that for some stable not superstable countable T_1 we have a parallel but weaker result. We relook at the old consistency results of "some $\mathrm{PC}(T_1', T_1), |T_1'| = \aleph_1 > |T_1|$, is categorical in $\aleph_1$" from [?]. From this we deduce that in this universe, T_1 has a strongly $(\aleph_1, \aleph_0)$-limit model.

It is a reasonable thought that we can similarly have a consistency result on the theory of linear order, but this is still not clear.

In §3 we show that if T has a superlimit model in $\lambda \geq |T| + \aleph_1$ then T is stable and T is superstable except possibly under some severe restrictions on the cardinal λ (i.e., $\lambda < \beth_\omega$ and $\lambda < \lambda^{\aleph_0}$). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by 1.1 + 3.1 we have:

Conclusion 0.1. *Assume $\lambda \geq |T| + \beth_\omega$. Then T has a superlimit model of cardinality λ iff T is superstable and $\lambda \geq |D(T)|$.*

In subsequent work we shall show that for some unstable T (e.g. the theory of linear orders), if $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa)$, then T has a medium (λ, κ)-limit model, whereas if T has the independence property even weak (λ, κ)-limit models do not exist; see [?] and more in [?], [?], [?], [?].

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§(0B) Basic Definitions

Notation 0.2. 1) Let T denote a complete first order theory which has infinite models but T_1, T', etc. are not necessarily complete.
2) Let M, N denote models, $|M|$ the universe of M and $\|M\|$ its cardinality and $M \prec N$ means M is an elementary submodel of N.
3) Let $\tau_T = \tau(T), \tau_M = \tau(M)$ be the vocabulary of T, M respectively.
4) Let $M \models$ "$\varphi[\bar{a}]^{\underline{if}(\mathrm{stat})}$" means that the model M satisfies $\varphi[\bar{a}]$ iff the statement stat is true (or is 1 rather than 0)).

Definition 0.3. 1) For $\bar{a} \in {}^{\omega>}|M|$ and $B \subseteq M$ let $\mathrm{tp}(\bar{a}, B, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{b} \in {}^{\ell g(\bar{y})}B$ and $M \models \varphi[\bar{a}, \bar{b}]\}$.
2) Let $D(T) = \{\mathrm{tp}(\bar{a}, \emptyset, M) : M$ a model of T and $\bar{a}$ a finite sequence from $M\}$.
3) If $A \subseteq M$ then $\mathbf{S}^m(A, M) = \{\mathrm{tp}(\bar{a}, A, N) : M \prec N$ and $\bar{a} \in {}^m N\}$, if $m = 1$ we may omit it.
4) A model M is λ-saturated when: if $A \subseteq M, |A| < \lambda$ and $p \in \mathbf{S}(A, M)$ then p is realized by some $a \in M$, i.e. $p \subseteq \mathrm{tp}(a, A, M)$; if $\lambda = \|M\|$ we may omit it.
5) A model M is special when letting $\lambda = \|M\|$, there is an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ of cardinals with limit λ and a $\prec$-increasing sequence $\langle M_i : i < \mathrm{cf}(\lambda)\rangle$ of models with union M such that M_{i+1} is λ_i-saturated of cardinality λ_{i+1} for $i < \mathrm{cf}(\lambda)$.

Definition 0.4. 1) For any T let $\mathrm{EC}(T) = \{M : M$ is a τ_T-model of $T\}$.
2) $\mathrm{EC}_\lambda(T) = \{M \in \mathrm{EC}(T) : M$ is of cardinality $\lambda\}$.
3) For $T \subseteq T'$ let

$$\mathrm{PC}(T', T) = \{M \restriction \tau_T : M \text{ is model of } T'\}$$

$$\mathrm{PC}_\lambda(T', T) = \{M \in \mathrm{PC}(T', T) : M \text{ is of cardinality } \lambda\}.$$

4) We say M is λ-universal for T_1 when it is a model of T_1 and every $N \in \mathrm{EC}_\lambda(T)$ can be elementarily embedded into M; if $T_1 = \mathrm{Th}(M)$ we may omit it.
5) We say $M \in \mathrm{EC}(T)$ is universal when it is λ-universal for $\lambda = \|M\|$.

We are here mainly interested in

Definition 0.5. Given T and $M \in \mathrm{EC}_\lambda(T)$ we say that M is a superlimit or λ-superlimit model when: M is universal and if $\delta < \lambda^+$ is a limit ordinal, $\langle M_\alpha : \alpha \le \delta \rangle$ is $\prec$-increasing continuous, and M_α is isomorphic to M for every $\alpha < \delta$ then M_δ is isomorphic to M.

Remark 0.6. Concerning the following definition we shall use strongly limit in 2.14(1), medium limit in 2.14(2).

Definition 0.7. Let λ be a cardinal $\ge |T|$. For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice and $\mathbf{F}$ is a class function; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$.
1) For non-empty $\Theta \subseteq \{\mu : \aleph_0 \le \mu < \lambda$ and μ is regular$\}$ and $M \in \mathrm{EC}_\lambda(T)$ we say that M is a (λ, Θ)-superlimit when: M is universal and
if $\langle M_i : i \le \mu \rangle$ is $\prec$-increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$
then $\cup\{M_i : i < \mu\} \cong M$.
2) If Θ is a singleton, say $\Theta = \{\theta\}$, we may say that M is (λ, θ)-superlimit.
3) Let $S \subseteq \lambda^+$ be stationary. A model $M \in \mathrm{EC}_\lambda(T)$ is called S-strongly limit or (λ, S)-strongly limit when for some function: $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ we have:

- (a) for $N \in \mathrm{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
- (b) if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is a $\prec$-increasing continuous sequence [1] in $\mathrm{EC}_\lambda(T)$ and $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$, then $M \cong \cup\{M_i : i < \delta\}$.

4) Let $S \subseteq \lambda^+$ be stationary. $M \in \mathrm{EC}_\lambda(T)$ is called S-limit or (λ, S)-limit if for some function $\mathbf{F} : \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ we have:

- (a) for every $N \in \mathrm{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$
- (b) if $\langle M_i : i < \lambda^+ \rangle$ is a $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ such that $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ for $i < \lambda^+$ then for some closed unbounded [2] subset C of λ^+,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

5) We define[3] "S-weakly limit", "S-medium limit" like "S-limit", "S-strongly limit" respectively by demanding that the domain of $\mathbf{F}$ is the family of $\prec$-increasing continuous sequence of members of $\mathrm{EC}_\lambda(T)$ of length $< \lambda^+$ and replacing "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \le i+1 \rangle) \prec M_{i+2}$".
6) If $S = \lambda^+$ then we may omit S (in parts (3), (4), (5)).
7) For non-empty $\Theta \subseteq \{\mu : \mu \le \lambda$ and μ is regular$\}$, M is (λ, Θ)-strongly limit[4] if M is $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strongly limit. Similarly for the other notions. If we do not write λ we mean $\lambda = \|M\|$.
8) We say that $M \in K_\lambda$ is invariantly strong limit when in part (3), $\mathbf{F}$ is just a subset of $\{(M, N)/\cong : M \prec N$ are from $\mathrm{EC}_\lambda(T)\}$ and in clause (b) of part (3) we

[1] no loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$, i.e., only $\mathbf{F}(M')$ for $M' \cong M$ is required

[2] alternatively, we can use as a parameter a filter on λ^+ extending the co-bounded filter

[3] Note that M is (λ, S)-strongly limit iff M is $(\{\lambda, \mathrm{cf}(\delta) : \delta \in S\})$-strongly limit.

[4] in [?] we consider: we replace "limit" by "limit$^-$" if "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$", "$M_{i+1} \prec \mathbf{F}(\langle M_j : j \le i+1 \rangle) \prec M_{i+2}$" are replaced by "$\mathbf{F}(M_i) \prec M_{i+1}$", "$M_i \prec \mathbf{F}(\langle M_j : j \le i \rangle) \prec M_{i+1}$" respectively. But $(\mathrm{EC}(T), \prec)$ has amalgamation.

replace "$\mathbf{F}(M_{i+1}) \prec M_{i+2}$" by "$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M,N)/\cong) \in \mathbf{F})$". But abusing notation we still write $N = \mathbf{F}(M)$ instead $((M,N)/\cong) \in \mathbf{F}$. Similarly with the other notions, so we use the isomorphism type of $\bar{M}\char`^\langle N\rangle$ for "weakly limit" and "medium limit".
9) In the definitions above we may say "$\mathbf{F}$ witness M is ..."

Observation 0.8. *1) Assume $\mathbf{F}_1, \mathbf{F}_2$ are as above and $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$ (or $\mathbf{F}_1(\bar{N}) \prec \mathbf{F}_2(\bar{N})$) whenever defined. If $\mathbf{F}_1$ is a witness then so is $\mathbf{F}_2$.*
2) All versions of limit models implies being a universal model in $\mathrm{EC}_\lambda(T)$.

3) The Obvious implications diagram: For non-empty $\Theta \subseteq \{\theta : \theta \text{ is regular } \leq \lambda\}$ and stationary $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$:

$$\begin{array}{ccccc}
 & & \text{superlimit } = (\lambda, \{\mu : \mu \leq \lambda \text{ regular}\})\text{-superlimit} & & \\
 & & \downarrow & & \\
 & & (\lambda, \Theta)\text{-superlimit} & & \\
 & & \downarrow & & \\
 & & S_1\text{-strongly limit} & & \\
 & \downarrow & & \downarrow & \\
S_1\text{-medium limit,} & & & & S_1\text{-limit} \\
 & \downarrow & & \downarrow & \\
 & & S_1\text{-weakly limit.} & &
\end{array}$$

Lemma 0.9. *Let T be a first order complete theory.*
1) If λ is regular, M a saturated model of T of cardinality λ, <u>then</u> M is (λ,λ)-superlimit.
*2) If T is stable, and M is a saturated model of T of cardinality $\lambda \geq \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \leq \mu \leq \lambda$ and μ is regular$\})$, <u>then</u> M is (λ,Θ)-superlimit (on $\kappa(T)$-see [**?**, III,§3]).*
3) If T is stable in λ and $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ <u>then</u> T has an invariantly strongly (λ,κ)-limit model.

Remark 0.10. Concerning 0.9(2), note that by [**?**] if λ is singular or just $\lambda < \lambda^{<\lambda}$ and T has a saturated model of cardinality λ <u>then</u> T is stable (even stable in λ) and $\mathrm{cf}(\lambda) \geq \kappa(T)$).

Proof. 1) Let M_i be a λ-saturated model of T of cardinality λ for $i < \lambda$ and $\langle M_i : i < \lambda\rangle$ is $\prec$-increasing and $M_\lambda = \bigcup\limits_{i<\lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $< \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$ hence every $p \in \mathbf{S}(A, M_\lambda)$ is realized in M_i hence in M_λ; so clearly M_λ is λ-saturated. Remembering the uniqueness of a λ-saturated model of T of cardinality λ we finish.
2) Use [**?**, III,3.11]: if M_i is a λ-saturated model of $T, \langle M_i : i < \delta\rangle$ increasing $\mathrm{cf}(\delta) \geq \kappa(T)$ <u>then</u> $\bigcup\limits_{i<\delta} M_i$ is λ-saturated.
3) Let $\mathbf{K}_{\lambda,\kappa} = \{\bar{M} : \bar{M} = \langle M_i : i \leq \kappa\rangle$ is $\prec$-increasing continuous, $M_i \in \mathrm{EC}_\lambda(T)$ and $(M_{i+2}, c)_{c \in M_{i+1}}$ is saturated for every $i < \kappa\}$. Clearly $\bar{M}, \bar{N} \in \mathbf{K}_{\lambda,\kappa} \Rightarrow M_\kappa \cong N_\kappa$. Also for every $M \in \mathrm{EC}_\lambda(T)$ there is N such that $M \prec N$ and $(N,c)_{c\in M}$

is saturated, as also $\mathrm{Th}((M,c)_{c\in M})$ is stable in λ; so there is an invariant $\mathbf{F}: \mathrm{EC}_\lambda(T) \to \mathrm{EC}_\lambda(T)$ such that $M \prec \mathbf{F}(M)$ and $(\mathbf{F}(M),c)_{c\in M}$ is saturated; such $\mathbf{F}$ witness the desired conclusion. $\square_{0.9}$

Definition 0.11. 0) For regular $\kappa < \lambda$ let $S^\lambda_\theta = \{\delta < \lambda : \mathrm{cf}(\delta) = \lambda\}$.
1) For a regular uncountable cardinal λ let $\check{I}[\lambda] = \{S \subseteq \lambda$: some pair $(E,\bar{a})$ witnesses $S \in \check{I}[\lambda]$, see below$\}$.
2) We say that $(E,\bar{u})$ is a witness for $S \in \check{I}[\lambda]$ <u>iff</u>:

- (a) E is a club of the regular cardinal λ
- (b) $\bar{u} = \langle u_\alpha : \alpha < \lambda\rangle, u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$
- (c) for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $\mathrm{cf}(\delta)$ (and δ is a limit ordinal).

By [**?**, §1]

Claim 0.12. *If $\kappa^+ < \lambda$ and κ,λ are regular then some stationary $S \subseteq \{\delta < \lambda : cf(\delta) = \kappa\}$ belongs to $\check{I}[\lambda]$.*

By [**?**]

Claim 0.13. *If $\lambda = \mu^+, \theta = \mathrm{cf}(\theta) \le \mathrm{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\theta} \le \mu$ <u>then</u> $S^\lambda_\theta \in \check{I}[\lambda]$.*

1. On superstable not $\aleph_0$-stable T

We first note that superstable T tend to have superlimit models.

Claim 1.1. *Assume T is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then T has a superlimit model of cardinality λ iff T has a saturated model of cardinality λ iff T has a universal model of cardinality λ iff $\lambda \geq |D(T)|$.*

Proof. By [?, III,§5] we know that T is stable in λ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of T of cardinality λ hence no saturated model and no superlimit model, etc., recalling 0.8(2). If $\lambda \geq |D(T)|$, then T is stable in λ hence has a saturated model of cardinality λ by [?, III] (hence universal) and the class of λ-saturated models of T is closed under increasing elementary chains by [?, III] so we are done. $\square_{1.1}$

The following are the prototypical theories which we shall consider.

Definition 1.2. 1) $T_0 = \mathrm{Th}({}^\omega 2, E^0_n)_{n<\omega}$ when $\eta E^0_n \nu \Leftrightarrow \eta \restriction n = \nu \restriction n$.
2) $T_1 = \mathrm{Th}({}^\omega(\omega_1), E^1_n)_{n<\omega}$ where $\eta E^1_n \nu \Leftrightarrow \eta \restriction n = \nu \restriction n$.
3) $T_2 = \mathrm{Th}(\mathbb{R}, <)$.

Recall

Observation 1.3. *0) T_ℓ is a countable complete first order theory for $\ell = 0, 1, 2$.*
1) T_0 is superstable not $\aleph_0$-stable.
2) T_1 is strictly stable, that is, stable not superstable.
3) T_2 is unstable.
4) T_ℓ has elimination of quantifiers for $\ell = 0, 1, 2$.

Claim 1.4. *It is consistent with* ZFC *that $\aleph_1 < 2^{\aleph_0}$ and some $M \in \mathrm{EC}_{\aleph_1}(T_0)$ is a superlimit model.*

Proof. By [?], for notational simplicity we start with $\mathbf{V} = \mathbf{L}$.

So T_0 is defined in 1.2(1) and it is the T from Theorem [?, 1.1] and let S be the set of $\eta \in ({}^\omega 2)^{\mathbf{L}}$. We define T' (called T_1 there) as the following theory:

- $\circledast_1$ (*i*) T_0, or just for each n the sentence saying E_n is an equivalence relation with 2^n equivalence classes, each E_n equivalence class divided to two by E_{n+1}, E_{n+1} refine E_n, E_0 is trivial
  - (*ii*) the sentences saying that
    - (α) for every x, the function $z \mapsto F(x,z)$ is one-to-one and
    - (β) $xE_n(F(x,z))$ for each $n < \omega$
  - (*iii*) $E_n(c_\eta, c_\nu)^{\mathrm{if}(\eta \restriction n = \nu \restriction n)}$ for $\eta, \nu \in S$.

In [?] it is proved that in some forcing[5] extension $\mathbf{L}^{\mathbb{P}}$ of $\mathbf{L}$, $\mathbb{P}$ an $\aleph_2$-c.c. proper forcing of cardinality $\aleph_2$, in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$, the class $\mathrm{PC}(T', T_0) = \{M \restriction \tau_{T_0} : M$ is a τ-model of $T'\}$ is categorical in $\aleph_1$.

However, letting M^* be any model from $\mathrm{PC}(T', T_0)$ of cardinality $\aleph_1$, it is easy to see that (in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$):

- $\circledast_2$ the following conditions on M are equivalent
  - (a) M is isomorphic to M^*

[5] We can replace $\mathbf{L}$ by any $\mathbf{V}_0$ which satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$.

(b) $M \in \mathrm{PC}(T', T_0)$

(c) (α) M is a model of T_0 of cardinality $\aleph_1$

(β) M^* can be elementarily embedded into M

(γ) for every $a \in M$ the set $\cap\{a/E_n^M : n < \omega\}$ has cardinality $\aleph_1$.

But

$\circledast_3$ every model M_1 of T of cardinality $\leq \aleph_1$ has a proper elementary extension to a model satisfying (c), i.e., $(\alpha), (\beta), (\gamma)$ of $\circledast_2$ above

$\circledast_4$ if $\langle M_\alpha : \alpha < \delta \rangle$ is an increasing chain of models satisfying (c) of $\circledast_2$ and $\delta < \omega_2$ then also $\cup\{M_\alpha : \alpha < \delta\}$ does.

Together we are done. $\square_{1.4}$

Naturally we ask

Question 1.5. What occurs to T_0 for $\lambda > \aleph_1$ but $\lambda < 2^{\aleph_0}$?

Question 1.6. Does the theory T_2 of linear order consistently have an $(\aleph_1, \aleph_0)$-superlimit? (or only strongly limit?) but see §3.

Question 1.7. What is the answer for T when T is countable superstable not $\aleph_0$-stable and $D(T)$ countable for $\aleph_1 < 2^{\aleph_0}$ for $\aleph_2 < 2^{\aleph_0}$?

So by the above for some such T, in some universe, for $\aleph_1$ the answer is yes, there is a superlimit.

2. A strictly stable consistent example

We now look at models of T_1 (redefined below) in cardinality $\aleph_1$; recall

Definition 2.1. $T_1 = \text{Th}({}^\omega(\omega_1), E_n)_{n<\omega}$ where $E_n = \{(\eta,\nu) : \eta,\nu \in {}^\omega(\omega_1)$ and $\eta \restriction n = \nu \restriction n\}$.

Remark 2.2.

- (a) Note that T_1 has elimination of quantifiers.
- (b) If $\lambda = \Sigma\{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, <u>then</u> T_1 has a $(\lambda,\aleph_0)$-superlimit model in λ (see 2.15).

Definition/Claim 2.3. 1) Any model of T_1 of cardinality λ is isomorphic to $M_{A,h} := (\{(\eta,\varepsilon) : \eta \in A, \varepsilon < h(\eta)\}, E_n)_{n<\omega}$ for some $A \subseteq {}^\omega\lambda$ and $h : {}^\omega\lambda \to (\text{Car} \cap \lambda^+)\backslash\{0\}$ where $(\eta_1,\varepsilon_1)E_n(\eta_2,\varepsilon_2) \Leftrightarrow \eta_1 \restriction n = \eta_2 \restriction n$, pedantically we should write $E_n^{M_{A,h}} = E_n {\restriction} |M_{A,n}|$.
2) We write M_A for $M_{A,h}$ when A is as above and $h : A \to \{|A|\}$, so constantly $|A|$ when A is infinite.
3) For $A \subseteq {}^\omega\lambda$ and h as above the model $M_{A,h}$ is a model of T_1 <u>iff</u> A is non-empty and $(\forall \eta \in A)(\forall n < \omega)(\exists^{\aleph_0} \nu \in A)(\nu{\restriction}n = \eta{\restriction}n \wedge \nu(n) \neq \eta(n))$.
4) Above $M_{A,h}$ has cardinality λ iff $\Sigma\{h(\eta) : \eta \in A\} = \lambda$.

Definition 2.4. 1) We say that A is a (T_1,λ)-witness <u>when</u>

- (a) $A \subseteq {}^\omega\lambda$ has cardinality λ
- (b) if $B_1, B_2 \subseteq {}^\omega\lambda$ are (T_1,A)-big (see below) of cardinality λ <u>then</u> $(B_1 \cup {}^{\omega>}\lambda, \triangleleft)$ is isomorphic to $(B_2 \cup {}^{\omega>}\lambda, \triangleleft)$.

2) A set $B \subseteq {}^\omega\lambda$ is called (T_1,A)-big <u>when</u> it is $(\lambda,\lambda) - (T_1,A)$-big; see below.
3) B is $(\mu,\lambda) - (T_1,A)$-big means: $B \subseteq {}^\omega\lambda, |B| = |A| = \mu$ and for every $\eta \in {}^{\omega>}\lambda$ there is an isomorphism f from $({}^{\omega\geq}\lambda, \triangleleft)$ onto $(\{\eta\hat{}\nu : \nu \in {}^{\omega\geq}\lambda\}, \triangleleft)$ mapping A into $\{\nu : \eta\hat{}\nu \in B\}$.
4) $A \subseteq {}^\omega(\omega_1)$ is $\aleph_1$-suitable <u>when</u>:

- (a) $|A| = \aleph_1$
- (b) for a club of $\delta < \omega_1, A \cap {}^\omega\delta$ is everywhere not meagre in the space ${}^\omega\delta$, i.e., for every $\eta \in {}^{\omega>}\delta$ the set $\{\nu \in A \cap {}^\omega\delta : \eta \triangleleft \nu\}$ is a non-meagre subset of ${}^\omega\delta$ (that is what really is used in [**?**]).

Claim 2.5. *It is consistent with ZFC that $2^{\aleph_0} > \aleph_1 +$ there is a $(T_1,\aleph_1)$-witness; moreover every $\aleph_1$-suitable set is a $(T_1,\aleph_1)$-witness.*

Proof. By [**?**, §2]. $\square_{2.5}$

Remark 2.6. The witness does not give rise to an $(\aleph_1,\aleph_0)$-limit model, as for the union of any "fast enough" $\prec$-increasing ω-chain of members of $\text{EC}_{\aleph_1}(T_1)$, the relevant sets are meagre.

Definition 2.7. Let A be a (T_1,λ)-witness. We define $K^1_{T_1,A}$ as the family of $M = (|M|, <^M, P^M_\alpha)_{\alpha\leq\omega}$ such that:

- (α) $(|M|, <^M)$ is a tree with $(\omega+1)$ levels
- (β) P^M_α is the α-th level; let $P^M_{<\omega} = \cup\{P^M_n : n < \omega\}$

(γ) M is isomorphic to M^1_B for some $B \subseteq {}^\omega\lambda$ of cardinality λ where M^1_B is defined by $|M^1_\beta| = ({}^{\omega>}\lambda) \cup B, P_n^{M^1_B} = {}^n\lambda, P_\omega^{M^1_B} = B$ and $<^{M^1_B} = \triangleleft \restriction |M^1_B|$, i.e., being an initial segment

(δ) moreover B is such that some f satisfies:

$\circledast$ (a) $f : {}^{\omega>}\lambda \to \omega$ and $f(<>) = 0$ for simplicity

(b) $\eta \trianglelefteq \nu \in {}^{\omega>}\lambda \Rightarrow f(\eta) \le f(\nu)$

(c) if $\eta \in B$ then $\langle f(\eta \restriction n) : n < \omega\rangle$ is eventually constant

(d) if $\eta \in {}^{\omega>}\lambda$ then $\{\nu \in {}^\omega\lambda : \eta^\frown\nu \in B$ and $m < \omega \Rightarrow f(\eta^\frown(\nu \restriction m)) = f(\eta)\}$ is (T_1, A)-big

(e) for $\eta \in {}^{\omega>}\lambda$ and $n \in [f(\eta), \omega)$ for λ ordinals $\alpha < \lambda$, we have $f(\eta^\frown\langle\alpha\rangle) = n$.

Claim 2.8. *[The Global Axiom of Choice] If A is a $(T_1, \aleph_1)$-witness then*

(a) $K^1_{T_1,A} \ne \emptyset$

(b) *any two members of $K^1_{T_1,A}$ are isomorphic*

(c) *there is a function $\mathbf{F}$ from $K^1_{T_1,A}$ to itself (up to isomorphism, i.e., $(M, \mathbf{F}(M))$ is defined only up to isomorphism) satisfying $M \subseteq \mathbf{F}(M)$ such that $K^1_{T_1,A}$ is closed under increasing unions of sequence $\langle M_n : n < \omega\rangle$ such that $\mathbf{F}(M_n) \subseteq M_{n+1}$.*

Proof. Clause (a): Trivial.

Clause (b): By the definition of "A is a $(T_1, \aleph_1)$-witness" and of $K^1_{T_1,A}$.

Clause (c):

We choose $\mathbf{F}$ such that

$\circledast$ if $M \in K^1_{A,T_1}$ then $M \subseteq \mathbf{F}(M) \in K^1_{A,T_1}$ and for every $k < \omega$ and $a \in P_k^M$, the set $\{b \in P_{k+1}^{\mathbf{F}(M)} : a <_{\mathbf{F}(M)} b$ and $b \notin M\}$ has cardinality $\aleph_1$.

Assume $M = \cup\{M_n : n < \omega\}$ where $\langle M_n : n < \omega\rangle$ is $\subseteq$-increasing}, $M_n \in K^1_{A,T_1}, \mathbf{F}(M_n) \subseteq M_{n+1}$. Clearly M is as required in the beginning of Definition 2.7, that is, satisfies clauses $(\alpha), (\beta), (\gamma)$ there. To prove clause (δ), we define $f : P^M_{<\omega} \to \omega$ by $f(a) = \text{Min}\{n : a \in M_n\}$. Pendantically, $\mathbf{F}$ is defined only up to isomorphism.
So we are done. $\square_{2.8}$

Claim 2.9. *[The Global Axiom of Choice]*
If A is a (T_1, λ)-witness then

(a) $K^1_{T_1,A} \ne \emptyset$

(b) *any two members of $K^1_{T_1,A}$ are isomorphic*

(c) *if $M_n \in K^1_{T_1,A}$ and $n < \omega \Rightarrow M_n \subseteq M_{n+1}$ then $M := \cup\{M_n : n < \omega\} \in K^1_{T_1,A}$.*

Remark 2.10. If we omit clause (b), we can weaken the demand on the set A.

Proof. Assume $M = \cup\{M_n : n < \omega\}, M_n \subseteq M_{n+1}, M_n \in K^1_{T_1,A}$ and f_n witnesses $M_n \in K^1_{T_1,A}$. Clearly M satisfies clauses $(\alpha),(\beta),(\gamma)$ from Definition 2.7, we just have to find a witness f as in clause (δ) there.

For each $a \in M$ let $n(a) = \text{Min}\{n : a \in M_n\}$, clearly if $M \models$ "$a < b < c$" then $n(a) \le n(b)$ and $n(a) = n(c) \Rightarrow n(a) = n(b)$. Let $g_n : M \to M$ be defined by: $g_n(a) = b$ iff $b \le^M a, b \in M_n$ and b is $\le^M$-maximal under those restrictions; clearly it is well defined. Now we define $f'_n : M_n \to \omega$ by induction on $n < \omega$ such that $m < n \Rightarrow f'_m \subseteq f'_n$, as follows.

If $n = 0$ let $f'_n = f_n$.

If $n = m+1$ and $a \in M_n$ we let $f'_n(a)$ be $f'_m(a)$ if $a \in M_m$ and be $(f_n(a) - f_n(g_m(a))) + f'_m(g_m(a)) + 1$ if $a \in M_n \backslash M_m$. Clearly $f := \cup\{f'_n : n < \omega\}$ is a function from M to $\omega, a \le^M b \Rightarrow f(a) \le f(b)$, and for any $a \in M$ the set $\{b \in M : a \le^M b$ and $f(b) = f(a)\}$ is equal to $\{b \in M_{n(a)} : f_{n(a)}(a) = f_{n(a)}(b)$ and $a \le^M b\}$.
So we are done. $\square_{2.9}$

Definition 2.11. Let A be a (T_1,λ)-witness. We define $K^2_{T_1,A}$ as in Definition 2.7 but f is constantly zero.

Claim 2.12. *[The Global Axiom of Choice] If A is a $(T_1,\aleph_1)$-witness* <u>*then*</u>

- (a) $K^2_{T_1,A} \ne \emptyset$
- (b) *any two members of $K^2_{T_1,A}$ are isomorphic*
- (c) *there is a function $\mathbf{F}$ from $\cup\{^{\alpha+2}(K^2_{T_0,A}) : \alpha < \omega_1\}$ to $K^2_{T_1,A}$ which satisfies:*
  - $\boxtimes$ (α) *if $\bar M = \langle M_i : i \le \alpha+1\rangle$ is an $\prec$-increasing sequence of models of T then $M_{\alpha+1} \subseteq \mathbf{F}(\bar M) \in K^2_{T_1,A}$*
  - (β) *the union of any increasing ω_1-sequence $\bar M = \langle M_\alpha : \alpha < \omega_1\rangle$ of members of $K^2_{T_1,A}$ belongs to $K^2_{T_1,A}$ when $\omega_1 = \sup\{\alpha : \mathbf{F}(\bar M \restriction (\alpha+2)) \subseteq M_{\alpha+2})$ and is a well defined embedding of M_α into $M_{\alpha+2}\}$.*

Remark 2.13. Instead of the global axiom of choice, we can restrict the models to have universe a subset of λ^+ (or just a set of ordinals).

Proof. <u>Clause (a)</u>: Easy.

<u>Clause (b)</u>: By the definition.

<u>Clause (c)</u>: Let $\langle \mathscr{U}_\varepsilon : \varepsilon < \omega_1\rangle$ be an increasing sequence of subsets of ω_1 with union ω_1 such that $\varepsilon < \omega_1 \Rightarrow |\mathscr{U}_\varepsilon \backslash \bigcup\limits_{\zeta<\varepsilon} \mathscr{U}_\zeta| = \aleph_1$. Let $M^* \in K^2_{T_1,A}$ be such that $^{\omega>}(\omega_1) \subseteq |M^*| \subseteq {}^{\omega\ge}(\omega_1)$ and $M^*_\varepsilon =: M^* \restriction {}^{\omega\ge}(\mathscr{U}_\varepsilon)$ belongs to $K^2_{T_1,A}$ for every $\varepsilon < \omega_1$.

We choose a pair $(\mathbf{F},\mathbf{f})$ of functions with domain $\{\bar M : \bar M$ an increasing sequence of members of $K^2_{T_1,A}$ of length $< \omega_1\}$ such that:

- (α) $\mathbf{F}(\bar M)$ is an extension of $\cup\{M_i : i < \ell g(\bar M)\}$ from $K^2_{T_1,A}$
- (β) $\mathbf{f}(\bar M)$ is an embedding from $M^*_{\ell g(\bar M)}$ into $\mathbf{F}(\bar M)$
- (γ) if $\bar M^\ell = \langle M_\alpha : \alpha < \alpha_\ell\rangle$ for $\ell = 1,2$ and $\alpha_1 < \alpha_2, \bar M^1 = \bar M^2 \restriction \alpha_1$ and $\mathbf{F}(\bar M^1) \subseteq M_{\alpha_1}$ then $\mathbf{f}(\bar M^1) \subseteq \mathbf{f}(\bar M^2)$
- (δ) if $a \in \mathbf{F}(\bar M)$ and $n < \omega$ then for some $b \in M^*_{\ell g(\bar M)}$ we have $\mathbf{F}(M) \models aE_n(\mathbf{f}(\bar M)(b))$.

Now check. $\square_{2.12}$

Conclusion 2.14. *Assume there is a $(T_1, \aleph_1)$-witness (see Definition 2.4) for the first-order complete theory T_1 from 2.1:*
1) T_1 has an $(\aleph_1, \aleph_0)$-strongly limit model.
2) T_1 has an $(\aleph_1, \aleph_1)$-medium limit model.
3) T_1 has a $(\aleph_1, \aleph_0)$-superlimit model.

Proof. 1) By 2.8 the reduction of problems on $(\mathrm{EC}(T_1), \prec)$ to $K^1_{T_1,A}$ (which is easy) is exactly as in [**?**].
2) By 2.12.
3) Like part (1) using claim 2.9. $\square_{2.14}$

Claim 2.15. *If $\lambda = \Sigma\{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, <u>then</u> T_1 has a $(\lambda, \aleph_0)$-superlimit model in λ.*

Proof. Let M_n be the model M_{A_n,h_n} where $A_n = {}^\omega(\lambda_n)$ and $h_n : A_n \to \lambda_n^+$ is constantly λ_n.

Clearly

- $(*)_1$ M_n is a saturated model of T_1 of cardinality λ_n
- $(*)_2$ $M_n \prec M_{n+1}$
- $(*)_3$ $M_\omega = \cup\{M_n : n < \omega\}$ is a special model of T_1 of cardinality λ.

The main point:

- $(*)_4$ M_ω is $(\lambda, \aleph_0)$-superlimit model of T_1.

[Why? Toward this assume

- (a) N_n is isomorphic to M_ω say $f_n : M_\omega \to N_n$ is such isomorphic
- (b) $N_n \prec N_{n+1}$ for $n < \omega$.

Let $N_\omega = \cup\{N_n : n < \omega\}$ and we should prove $N_\omega \cong M_\omega$, so just N_ω is a special model of T_1 of cardinality λ suffice.

Let $N'_n = N_\omega{\restriction}(\cup\{f_n(M_k) : k \le n\})$. Easily $N'_n \prec N'_{n+1} \prec N_\omega$ and $\cup\{N'_n : n < \omega\} = N_{\omega_*}$ and $\|N'_n\| = \lambda_n$. So it suffices to prove that N'_n is saturated and by direct inspection shows this. $\square_{2.15}$

3. On non-existence of limit models

Naturally we assume that non-existence of superlimit models for unstable T is easier to prove. For other versions we need to look more. We first show that for $\lambda \geq |T|+\aleph_1$, if T is unstable then it does not have a superlimit model of cardinality λ and if T is unsuperstable, we show this for "most" cardinals λ. On "Φ proper for $K_{\rm or}$ or $K^\omega_{\rm tr}$", see [?, VII] or [?] or hopefully some day in [?, III]. We assume some knowledge on stability.

Claim 3.1. *1) If T is unstable, $\lambda \geq |T|+\aleph_1$, <u>then</u> T has no superlimit model of cardinality λ.*
2) If T is stable not superstable and $\lambda \geq |T|+\beth_\omega$ or $\lambda = \lambda^{\aleph_0} \geq |T|$ <u>then</u> T has no superlimit model of cardinality λ.

Remark 3.2. 1) We assume some knowledge on EM models for linear orders I and members of $K^\omega_{\rm tr}$ as index models, see, e.g. [?, VII].
2) We use the following definition in the proof, as well as a result from [?] or [?].

Definition 3.3. For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal μ such that for some, equivalently for every set A of cardinality λ there is $\mathscr{P}_A \subseteq [A]^{\leq\kappa} = \{B \subseteq A : |B| \leq \kappa\}$ of cardinality λ such that any $B \in [\lambda]^{\leq\kappa}$ is the union of $< \kappa$ members of $\mathscr{P}_A$.

Proof. 1) Towards a contradiction assume M^* is a superlimit model of T of cardinality λ. As T is unstable we can find $m, \varphi(\bar{x},\bar{y})$ such that

- $(*)$ $\varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\tau(T)}$ linearly orders some infinite $\mathbf{I} \subseteq {}^m M, M \models T$ so $\ell g(\bar{x}) = \ell g(\bar{y}) = m$.

We can find a Φ which is proper for linear orders (see [?, VII]) and $F_\ell(\ell < m)$ such that $F_\ell \in \tau_\Phi \backslash \tau_T$ is a unary function symbol for $\ell < m, \tau_T \subseteq \tau(\Phi)$ and for every linear order I, ${\rm EM}(I,\Phi)$ has Skolem functions and its τ_T-reduct ${\rm EM}_{\tau(T)}(I,\Phi)$ is a model of T of cardinality $|T|+|I|$ and $\tau(\Phi)$ is of cardinality $|T|+\aleph_0$ and $\langle a_s : s \in I\rangle$ is the Skeleton of ${\rm EM}(I,\Phi)$, that is, it is an indiscernible sequence in ${\rm EM}(I,\Phi)$ and ${\rm EM}(I,\Phi)$ is the Skolem hull of $\{a_s : s \in I\}$, and letting $\bar{a}_s = \langle F_\ell(a_s) : \ell < m\rangle$ in ${\rm EM}(I,\Phi)$ we have ${\rm EM}_{\tau(T)}(I,\Phi) \models \varphi[\bar{a}_s,\bar{a}_t]^{{\rm if}(s<t)}$ for $s,t \in I$.

Next we can find Φ_n (for $n < \omega$) such that:

- $\boxplus$ (a) Φ_n is proper for linear order and $\Phi_0 = \Phi$
  - (b) ${\rm EM}_{\tau(\Phi)}(I,\Phi_n) \prec {\rm EM}_{\tau(\Phi)}(I,\Phi_{n+1})$ for every linear order I and $n < \omega$; moreover
  - $(b)^+$ $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and ${\rm EM}(I,\Phi_n) \prec {\rm EM}_{\tau(\Phi_n)}(I,\Phi_{n+1})$ for every $n < \omega$ and linear order I
  - (c) if $|I| \leq n$ then ${\rm EM}_{\tau(\Phi)}(I,\Phi_n) = {\rm EM}_{\tau(\Phi)}(I,\Phi_{n+1})$ and ${\rm EM}_{\tau(T)}(I,\Phi_n) \cong M^*$
  - (d) $|\tau(\Phi_n)| = \lambda$.

This is easy. Let Φ_ω be the limit of $\langle \Phi_n : n < \omega\rangle$, i.e. $\tau(\Phi_\omega) = \cup\{\tau(\Phi_n) : n < \omega\}$ and if $k < \omega$ then ${\rm EM}_{\tau(\Phi_k)}(I,\Phi_\omega) = \cup\{{\rm EM}_{\tau(\Phi_k)}(I,\Phi_n) : n \in [k,\omega)\}$. So as M^* is a superlimit model, for any linear order I of cardinality $\lambda, {\rm EM}_{\tau(T)}(I,\Phi_\omega)$ is the direct limit of $\langle {\rm EM}_{\tau(T)}(J,\Phi_\omega) : J \subseteq I \text{ finite}\rangle$, each isomorphic to M^*, so as we have assumed that M^* is a superlimit model it follows that ${\rm EM}_{\tau(T)}(I,\Phi_\omega)$ is isomorphic to M^*. But by [?, III] or [?] which may eventually be [?, III] there are

2^λ many pairwise non-isomorphic models of this form varying I on the linear orders of cardinality λ, contradiction.
2) First assume $\lambda = \lambda^{\aleph_0}$. Let $\tau \subseteq \tau_T$ be countable such that $T' = T \cap \mathbb{L}(\tau)$ is not superstable. Clearly if M^* is $(\lambda, \aleph_0)$-limit model then $M^* \restriction \tau'$ is not $\aleph_1$-saturated. [Why? As in [**?**, Ch.VI,§6], but we shall give full details. There are $N_* \models T, p = \{\varphi_n(\lambda, \bar{a}_n) : n < \omega\}$ a type in $N_*, \bar{a}_n \triangleleft \bar{a}_{n+1}, \bar{a}_{<>}$ empty and $\varphi_{n+1}(x, \bar{a}_{n+1})$ forks over $\bar{a}_n$. Let $\mathbf{F}(M)$ be such that if $n < \omega$ and $\bar{b}_n \subseteq M$ realizes $\mathrm{tp}(\bar{a}_n, \emptyset, N_*)$ then for some $\bar{b}_{n+1}$ from $\mathbf{F}, M$ realizing $\mathrm{tp}(\bar{a}_{n+1}, \emptyset, N_*)$, the type $\mathrm{tp}(\bar{b}_{n+1}, M, \mathbf{F}(M))$ does not fork over b_n.] But if $\kappa = \mathrm{cf}(\kappa) \in [\aleph_1, \lambda]$ and M^* is a (λ, κ)-limit then $M^* \restriction \tau'$ is $\aleph_1$-saturated, contradiction.

The case $\lambda \geq |T| + \beth_\omega$ is more complicated (the assumption $\lambda \geq \beth_\omega$ is to enable us to use [**?**] or see [**?**] for a simpler proof; we can use weaker but less transparent assumptions; maybe $\lambda \geq 2^{\aleph_0}$ suffices).

As T is stable not superstable by [**?**] for some $\bar{\Delta}$:

$\circledast_1$ for any μ there are M and $\langle a_{\eta,\alpha} : \eta \in {}^\omega\mu$ and $\alpha < \mu\rangle$ such that
- (a) M is a model of T
- (b) $\mathbf{I}_\eta = \{a_{\eta,\alpha} : \alpha < \mu\} \subseteq M$ is an indiscernible set (and $\alpha < \beta < \mu \Rightarrow a_{\eta,\alpha} \neq a_{\eta,\beta}$)
- (c) $\bar{\Delta} = \langle \Delta_n : n < \omega\rangle$ and $\Delta_n \subseteq \mathbb{L}_{\tau(T)}$ infinite
- (d) for $\eta, \nu \in {}^\omega\mu$ we have $\mathrm{Av}_{\Delta_n}(M, \mathbf{I}_\eta) = \mathrm{Av}_{\Delta_n}(M, \mathbf{I}_\nu)$ iff $\eta \restriction n = \nu \restriction n$.

Hence by [**?**, VIII], or see [**?**] assuming M^* is a universal model of T of cardinality λ :

$\circledast_{2.1}$ there is Φ such that
- (a) Φ is proper for $K^\omega_{\mathrm{tr}}, \tau_T \subseteq \tau(\Phi), |\tau(\Phi)| = \lambda \geq |T| + \aleph_0$
- (b) for $I \subseteq {}^{\omega\geq}\lambda$, $\mathrm{EM}_{\tau(\Phi)}(I, \Phi)$ is a model of T and $I \subseteq J \Rightarrow \mathrm{EM}(I, \Phi) \prec \mathrm{EM}(J, \Phi)$
- (c) for some two-place function symbol F if for $I \in K^\omega_{\mathrm{tr}}$ and $\eta \in P^I_\omega, I$ a subtree of ${}^{\omega\geq}\lambda$ for transparency we let $\mathbf{I}_{I,\eta} = \{F(a_\eta, a_\nu) : \nu \in I\}$ then $\langle \mathbf{I}_{I,\eta} : \eta \in P^I_\omega\rangle$ are as in $\circledast_1(b), (d)$.

Also

$\circledast_{2.2}$ if Φ_1 satisfies (a),(b),(c) of $\circledast_{2.1}$ and M is a universal model of T then there is Φ^*_2 satisfying (a),(b),(c) of $\circledast_{2.1}$ and $\Phi_1 \leq \Phi^*_2$ see $\circledast_{2.3}(a)$ and for every finitely generated $J \in K^\omega_{\mathrm{tr}}$, see $\circledast_{2.3}(b)$ below, there is $M' \cong M$ such that $\mathrm{EM}_{\tau(T)}, (J, \Phi_1) \prec M' \prec \mathrm{EM}_{\tau(T)}(J, \Phi^*_2)$

$\circledast_{2.3}$ (a) we say $\Phi_1 \leq \Phi_2$ when $\tau(\Phi_1) \subseteq \tau(\Phi_2)$ and $J \in K^\omega_{\mathrm{tr}} \Rightarrow \mathrm{EM}(J, \Phi_1) \prec \mathrm{EM}_{\tau(\Phi_1)}(J, \Phi_2)$

(b) we say $J \subseteq I$ is finitely generated if it has the form $\{\eta_\ell : \ell < n\} \cup \{\rho$: for some n, ℓ we have $\rho \in P^I_n$ and $\rho <^I \eta_\ell\}$ for some $\eta_0, \dots, \eta_{n-1} \in P^I_\omega$

$\circledast_{2.4}$ if $M_* \in \mathrm{EC}_\lambda(T)$ is superlimit (or just weakly S-limit, $S \subseteq \lambda^+$ stationary) <u>then</u> there is Φ as in $\circledast_{2.1}$ above such that $\mathrm{EM}_{\tau(T)}(J, \Phi) \cong M_*$ for every finitely generated $J \in K^\omega_{\mathrm{tr}}$

$\circledast_{2.5}$ we fix Φ as in $\circledast_{2.4}$ for $M_* \in \mathrm{EC}_\lambda(T)$ superlimit.

Hence (mainly by clause (b) of $\circledast_{2.1}$ and $\circledast_{2.4}$ as in the proof of part (1))

$\circledast_3$ if $I \in K^\omega_{\rm tr}$ has cardinality $\le \lambda$ then ${\rm EM}_{\tau(\Phi)}(I,\Phi)$ is isomorphic to M^*.

Now by [?], we can find regular uncountable $\kappa < \beth_\omega$ such that $\lambda = \lambda^{[\kappa]}$, see Definition 3.3.

Let $S = \{\delta < \kappa : {\rm cf}(\delta) = \aleph_0\}$ and $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ be such that η_δ an increasing sequence of length ω with limit δ.

For a model M of T let ${\rm OB}_{\bar\eta}(M) = \{\bar{\mathbf{a}} : \bar{\mathbf{a}} = \langle a_{\eta_\delta,\alpha} : \delta \in W$ and $\alpha < \kappa\rangle, W \subseteq S$ and in M they are as in $\circledast_1(b),(d)\}$.
For $\bar{\mathbf{a}} \in {\rm OB}_{\bar\eta}(M)$ let $W[\bar{\mathbf{a}}]$ be W as above and let

$$\begin{aligned}\Xi(\bar{\mathbf{a}},M) = \{\eta \in {}^\omega\kappa : \ & \text{there is an indiscernible set} \\ & \mathbf{I} = \{a_\alpha : \alpha < \kappa\} \text{ in } M \text{ such that for every } n \\ & \text{for some } \delta \in W[\bar{\mathbf{a}}], \eta \restriction n = \eta_\delta \restriction n \text{ and} \\ & {\rm Av}_{\Delta_n}(M,\mathbf{I}) = {\rm Av}_{\Delta_n}(M,\{a_{\eta_\delta,\alpha} : \alpha < \kappa\})\}.\end{aligned}$$

Clearly

$\circledast_4$ (a) if $M \prec N$ then ${\rm OB}_{\bar\eta}(M) \subseteq {\rm OB}_{\bar\eta}(N)$
(b) if $M \prec N$ and $\bar{\mathbf{a}} \in {\rm OB}_{\bar\eta}(M)$ then $\Xi(\bar{\mathbf{a}},M) \subseteq \Xi(\bar{\mathbf{a}},N)$.

Now by the choice of κ it should be clear that

$\circledast_5$ if $M \models T$ is of cardinality λ then we can find an elementary extension N of M of cardinality λ such that for every $\bar{\mathbf{a}} \in {\rm OB}_{\bar\eta}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of κ, for some stationary $W' \subseteq W[\bar{\mathbf{a}}]$ the set $\Xi[\bar{\mathbf{a}},N]$ includes $\{\eta \in {}^\omega\kappa : (\forall n)(\exists \delta \in W')(\eta \restriction n = \eta_\delta \restriction n)\}$, (moreover we can even find $\varepsilon^* < \kappa$ and $W_\varepsilon \subseteq W$ for $\varepsilon < \varepsilon^*$ satisfying $W[\bar{\mathbf{a}}] = \cup\{W_\varepsilon : \varepsilon < \varepsilon^*\}$)

$\circledast_6$ we can find $M \in {\rm EC}_\lambda(T)$ isomorphic to M^* such that for every $\bar{\mathbf{a}} \in {\rm OB}_{\bar\eta}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of κ, we can find a stationary subset W' of $W[\bar{\mathbf{a}}]$ such that the set $\Xi[\bar{\mathbf{a}},M]$ includes $\{\eta \in {}^\omega\mu : (\forall n)(\exists \delta \in W')(\eta \restriction n = \eta_\delta \restriction n)\}$.

[Why? We choose (M_i,N_i) for $i < \kappa^+$ such that

(a) $M_i \in {\rm EC}_\lambda(T)$ is $\prec$-increasing continuous
(a) M_{i+1} is isomorphic to M^*
(a) $M_i \prec N_i \prec M_{i+1}$
(a) (M_i,N_i) are like (M,N) in $\circledast_5$.

Now $M = \cup\{M_i : i < \kappa^+\}$ is as required.
Now the model M is isomorphic to M^* as M^* is superlimit.]

Now the model from $\circledast_6$ is not isomorphic to $M' = {\rm EM}_{\tau(T)}({}^{\omega>}\lambda \cup \{\eta_\delta : \delta \in S\},\Phi)$ where Φ is from $\circledast_{2.1}$. But $M' \cong M^*$ by $\circledast_3$.
Together we are done. $\square_{3.1}$

The following claim says in particular that if some not unreasonable pcf conjectures holds, the conclusion holds for every $\lambda \ge 2^{\aleph_0}$.

Claim 3.4. *Assume T is stable not superstable, $\lambda \ge |T|$ and $\lambda \ge \kappa = \mathrm{cf}(\kappa) > \aleph_0$.*
1) T has no (λ, κ)-superlimit model provided that $\kappa = \mathrm{cf}(\kappa) > \aleph_0, \lambda \ge \kappa^{\aleph_0}$ and $\lambda = \mathbf{U}_D(\lambda) := \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^\kappa$ and for every $f : \kappa \to \lambda$ for some $u \in \mathscr{P}$ we have $\{\alpha < \kappa : f(\alpha) \in u\} \in D^+$, where D is a normal filter on κ to which $\{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ belongs.
2) Similarly if $\lambda \ge 2^{\aleph_0}$ and letting $J_0 = \{u \subseteq \kappa : |u| \le \aleph_0\}, J_1 = \{u \subseteq \kappa : u \cap S^\kappa_{\aleph_0}$ non-stationary$\}$ we have $\lambda = \mathbf{U}_{J_1,J_0}(\lambda) := \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{\aleph_0}$, if $u \in J_1, f : (\kappa \backslash u) \to \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathscr{P}$, $\mathrm{Rang}(f{\restriction}w) \subseteq v\}$.

Proof. Like 3.1(2). $\square$

Claim 3.5. *1) Assume T is unstable and $\lambda \ge |T| + \beth_\omega$. Then for at most one regular $\kappa \le \lambda$ does T have a weakly (λ, κ)-limit model and even a weakly (λ, S)-limit model for some stationary $S \subseteq S^\lambda_\kappa$.*
2) Assume T is unsuperstable and $\lambda \ge |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \mathrm{cf}(\kappa_2)$. Then T has no model which is a weak (λ, S)-limit where $S \subseteq \lambda$ and $S \cap S^\lambda_{\kappa_\ell}$ is stationary for $\ell = 1, 2$.

Proof. 1) Assume $\kappa_1 \ne \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{[\kappa]}$, see Definition 3.3 and $\kappa \notin \{\kappa_1, \kappa_2\}$. Let $m, \varphi(\bar{x}, \bar{y})$ be as in the proof of 3.1

- $(*)$ if $M \in \mathrm{EC}_\lambda(T)$ then there is N such that
  - (a) $N \in \mathrm{EC}_\lambda(T)$
  - (b) $M \prec N$
  - (c) if $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa\rangle \in {}^\kappa({}^m M)$ for $\alpha < \kappa$ then for some $\mathscr{U} \in [\kappa]^\chi$ for every uniform ultrafilter D on κ to which $\mathscr{U}$ belongs there is $\bar{a}_D \in {}^n N$ such that $\mathrm{tp}(\bar{a}_D, N, N) = \mathrm{Av}(\bar{\mathbf{a}}/D, M) = \{\psi(\bar{x}, \bar{c}) : \psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{c} \in {}^{\ell g(\bar{z})}M$ and $\{\{\alpha < \kappa : N \models \psi[\bar{a}_{i_\alpha}, \bar{c}]\} \in D\}$.

Similarly

- $\boxplus_1$ for every function $\mathbf{F}$ with domain $\{\bar{M} : \bar{M}$ an $\prec$-increasing sequence of models of T of length $< \lambda^+$ each with universe $\in \lambda^+\}$ such that $M_i \prec \mathbf{F}(\bar{M})$ for $i < \ell g(\bar{M})$ and $\mathbf{F}(\bar{M})$ has universe $\in \lambda^+$ there is a sequence $\langle M_\varepsilon : \varepsilon < \lambda^+\rangle$ obeying $\mathbf{F}$ such that: for every $\varepsilon < \lambda^+$ and $\bar{\mathbf{a}} \in {}^\kappa({}^m(M_\varepsilon))$ for $\alpha < \kappa$, there is $\mathscr{U} \in [\kappa]^\kappa$ such that for every ultrafilter D on κ to which $\mathscr{U}$ belongs, for every $\zeta \in (\varepsilon, \lambda^+)$ there is $\bar{\mathbf{a}}_{D,\zeta} \in {}^m(M_{\zeta+1})$ realizing $\mathrm{Av}(\bar{\mathbf{a}}/D, M_\zeta)$ in $M_{\zeta+1}$.

Hence

- $\boxplus_2$ for $\langle M_\alpha : \alpha < \lambda^+\rangle$ as in $\boxplus_1$ for every limit $\delta < \lambda^+$ of cofinality $\ne \kappa$ for every $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa\rangle \in {}^\kappa({}^m(M_\delta))$, there is $\mathscr{U} \in [\kappa]^\kappa$ such that for every ultrafilter D on κ to which $\mathscr{U}$ belongs, there is a sequence $\langle \bar{b}_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\rangle \in {}^{\mathrm{cf}(\delta)}({}^m(M_\delta))$ such that for every $\psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}(M_\delta)$ for every $\varepsilon < \mathrm{cf}(\delta)$ large enough, $M_\delta \models \psi[\bar{b}_\varepsilon, \bar{c}]$ iff $\psi(\bar{x}, \bar{c}) \in \mathrm{Av}(\bar{\mathbf{a}}/D, M_\delta)$.

The rest should be clear.
2) Combine the above and the proof of 3.1(2). $\square_{3.5}$

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