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PCF ARITHMETIC WITHOUT AND WITH CHOICE
SH938

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Abstract. We deal with relatives of GCH which are provable. In particular we deal with rank version of the revised GCH. Our motivation was to find such results when only weak versions of the axiom of choice are assumed but some of the results gives us additional information even in ZFC. We also start to deal with pcf for pseudo-cofinality (in ZFC with little choice).

Annotated Content

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§ 0. Introduction

In [?] and [?], [?] we prove in ZFC = ZF + AC relatives of G.C.H. Here mainly we are interested in relatives assuming only weak forms of choice, but some results add information even working in ZFC, in particular a generalization of [?] for ranks. Always we can assume ZF + DC.

Our original motivation was

Conjecture 0.1. Assume ZF + DC and μ a limit cardinal such that $\mathrm{AC}_{<\mu}$ and μ is strong limit. For every ordinal γ, for some $\kappa < \mu$, for any $\alpha < \mu$ and κ-complete filter D on α we have $\mathrm{rk}_D(\gamma) = \gamma$.

Here we get an approximation to it, i.e. for μ a limit of measurables restricting ourselves to ultrafilters; this is conclusion 4.4 deduced by applying Theorem 3.10 to Claim 4.3. Can we do it with $\mu = \beth_\omega$?

Also we would like to weaken $\mathrm{AC}_{<\mu}$; this is interesting per se and as then we will be able to combine [?] + [?] - see below. We intend to try in [?]; starting with $\bar{J} = \langle J_n : n < \omega\rangle$ such that $\mathrm{IND}(\bar{J})$ or something similar.

It may be illuminating to compare the present result with (see [?, V]).

Claim 0.2. *If $\kappa \geq \theta > \aleph_0, \lambda \geq 2^{2^\kappa}$ then the following conditions are equivalent:*

- $(*)_1$ *for every θ-complete filter D on κ, we have $\mathrm{rk}_D(\lambda^+) = \lambda^+$*
- $(*)_2$ $\alpha < \lambda^+ \Rightarrow \mathrm{rk}_D(\alpha) < \lambda^+$ *for every θ-complete filter D on κ*
- $(*)_3$ *there is no $\mathscr{F} \subseteq {}^\kappa\lambda$ of cardinality $\geq \lambda^+$ and θ-complete filter D in κ such that $f_1 \neq f_2 \in \mathscr{F} \Rightarrow f_1 \neq_D f_2$.*

Also we can in 0.2 replace λ^+ by a cardinal of cofinality $> 2^{2^\kappa}$. So the result in [?] implies a weak version of the conjecture above, say on $|\mathrm{rk}_D(\alpha)|$, but the present one gives more precise information. On the other hand, the present conjecture is not proved for $\mu = \beth_\omega$, also it seems less accommodating to the possible results with $\aleph_\omega$ instead of $\beth_\omega$ in [?] below $2^{2^{\aleph_\omega}}$.

Question 0.3. In [?] can we prove that the rank is small?

Discussion 0.4. In 0.5 below we present examples showing some limitations.

Below part (1) of the example shows that Claim 2.3 cannot be improved too much and part (2) shows that Conclusion 4.4 cannot be improved too much. In fact, in conjecture 0.1 if we demand only "μ is a limit cardinal" then it consistently fails. This implies that we cannot improve too much other results in §3,§4.

We may wonder how to compare the result in [?] and Conjecture 0.1 even in ZFC.

Example 0.5. 1) If $D_\ell = \mathrm{dual}([\kappa_\ell]^{<\kappa_\ell})$ for $\ell = 1,2$ (so if κ_ℓ is regular then $D_\ell = \mathrm{dual}(J^{\mathrm{bd}}_{\kappa_\ell})$) and $\kappa_2 < \kappa_1$ then D_2 does not 2-commute with D_1, i.e. $\boxplus^2_{D_1,D_2}$ from Definition 2.1 fail.
2) Consistently with ZFC, for every $n, \mathrm{rk}_{J^{\mathrm{bd}}_{\aleph_n}}(\aleph_\omega) > \aleph_\omega$.

Proof. 1) Let $A = \kappa_1$ and let $f_2 \in {}^{\kappa_2}\mathrm{Ord}$ be constantly 1 hence by Definition 1.11 and Claim 1.12(3) the ideal $J_2 = J[f_2, D_2]$ is $[\kappa_2]^{<\kappa_2}$. Choose a function $h : \kappa_1 \to \kappa_2$ and $(\forall\beta < \kappa_2)(\exists^{\kappa_1}\alpha < \kappa_1)(h(\alpha) = \beta)$ and let $\langle B_\alpha : \alpha \in A\rangle$ be such that we have $B_\alpha := \kappa_2 \backslash h(\alpha)$.

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We shall show that $A, \langle B_\alpha : \alpha \in A\rangle, J_2 = J[f_2, D_2]$ exemplifies that D_2 does not commute with D_1. So assume $A_* \in D_1, B_* \in J_2^+$ and we shall show that $A_* \times B_* \nsubseteq \cup\{\{s\} \times B_s : s \in \kappa_2\}$ this suffices. As $A_* \in D_1$, clearly for some $\alpha_* < \kappa_1$ we have $A_* \supseteq \kappa_1 \backslash \alpha_*$ and $B_* \subseteq \kappa_2, |B_*| = \kappa_2$ and choose $t \in B_*$ and then choose $s \in A_*$ such that $h(s) = t+1$, such s exists by the choice of h so $(s,t) \in A_* \times B_*$ but $(s,t) \notin \{s\} \times B_s$ as promised, so we are done.
2) Assume that the sequence $\langle 2^{\aleph_n} : n < \omega\rangle$ is increasing with supremum $> \aleph_\omega$ and in $\mathrm{cf}(^{(\aleph_n)}(\aleph_n), <_{J^{\mathrm{bd}}_{\aleph_n}})$ there is an increasing sequence of length $\aleph_{\omega+1}$ for each $n \in [1,\omega)$ hence it follows that $\mathrm{rk}_{J^{\mathrm{bd}}_{\aleph_n}}(\aleph_\omega) > \mathrm{rk}_{J^{\mathrm{bd}}_{\aleph_n}}(\aleph_n) \geq \aleph_n$ for $n \in [1,\omega)$. $\square_{0.5}$

We may hope to prove interesting things in ZF + DC by division to cases: if [?] apply fine, if not then we have a strict $\mathbf{p}$. But we need $\mathrm{AC}_{<\mu}$ to prove even clause (f) in 3.1, see [?]. We may consider that even in ZFC, probably [?] indicate that we can use weaker assumptions.

Let us say something on our program on set theory with little choice of which this work is a part. We always "know" that the axiom of choice is true. In addition we had thought that there is no interesting general combinatorial set theory without AC (though equivalence of version of choice, inner model theory and some other exist). Concerning the second point, since [?] our opinion changed and have thought that there is an interesting such set theory, with "bounded choice" related to pcf. More specifically [?] seems to prove that such theory is not empty. Then [?] suggest to look at axioms of choice "orthogonal" to "$\mathbf{V} = \mathbf{L}[\mathbb{R}]$", e.g. demand then $^{\omega\geq}\alpha$ can be well ordered (and weaker relatives). The results say that the universe is somewhat similar to universes gotten by Easton like forcing, blowing up 2^λ for every regular λ without well ordering the new $\mathscr{P}(\lambda)$. Continuing this Larson-Shelah [?] generalize classical theorem on splitting a stationary subset of a regular λ consisting of ordinals of cofinality κ.

In [?] we shall continue this work. In particular, we continue §5 to get a parallel of the pcf theorem and more. Recall that in [?] in ZFC we get connections between the existence of independent sets and a strong form of [?]. We prove related theorems on rank. In [?] we continue [?].

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§ 1. Preliminaries

Context 1.1. 1) We work in ZF in all this paper.
2) We try to say when we use DC but assuming it always makes no great harm.
3) We shall certainly mention the use of any additional form of choice, mainly AC_A which means $\Pi\{X_a : a \in A\}$ for any sequence $\langle X_a : a \in A\rangle$ of non-empty sets.
4) In 1.2 - 1.12 we quote definitions and claims to be used, see [?]. The rest of §1 is used only in §5.

Definition 1.2. 1) A filter D on Y is $(\leq B)$-complete when: if $\langle A_t : t \in B\rangle \in {}^B D$ then $A := \cap\{A_t : t \in B\} \in D$. We can instead say "$|B|^+$-complete" even if $|B|^+$ is not well defined.
1A) A filter D on Y is pseudo $(\leq B)$-complete when: if $\langle A_t : t \in B\rangle \in {}^B D$ then $\cap\{A_t : t \in B\}$ is not empty (so adopt the conventions of part (1)).
2) For an ideal J on a set Y let $\mathrm{dual}(J) = \{Y \backslash X : X \in J\}$, the dual ideal and $\mathrm{Dom}(J) = Y$, abusing notation we assume J determines Y.

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3) For a filter D on a set Y let $\text{dual}(D) = \{Y \backslash X : X \in D\}$, $\text{Dom}(D) = Y$. We may use properties defined for filter D for the dual ideal (and vice versa).
4) For a filter D on Y let $D^+ = \{A \subseteq Y : Y \backslash A \notin D\}$ and for an ideal J on Y let $J^+ = (\text{dual}(J))^+$.

Remark 1.3. It may be interesting to try to assume that relevant filters are just pseudo $(\le B)$-complete instead of $(\le B)$-complete. Now 1.15 clarify the connection to some extent, but presently we do not pursue this direction.

Definition 1.4. $\mathbf{C}$ is the class of sets A such that AC_A, the axiom of choice for A non-empty sets, holds. Note that for singular μ, AC_μ is equivalent to $\text{AC}_{<\mu}$, i.e. $(\forall \alpha < \mu)\text{AC}_\alpha$.

Definition 1.5. 1) $\text{hrtg}(A) = \text{Min}\{\alpha$: there is no function from A onto $\alpha\}$.
2) $\Upsilon(A) = \text{wlor}(A) := \text{Min}\{\alpha$: there is no one-to-one function from α into $A\}$ so $\Upsilon(A) \le \text{hrtg}(A)$.

Definition 1.6. 1) For D a filter on Y and $f, g \in {}^Y\text{Ord}$ let $f <_D g$ or $f < g$ mod D means that $\{s \in Y : f(s) < g(s)\} \in D$; similarly for $\le, =, \ne$.
2) For D a filter on Y and $f \in {}^Y\text{Ord}$ and $\alpha \in \text{Ord} \cup \{\infty\}$ we define when $\text{rk}_D(f) = \alpha$ by induction on α:

- $\circledast$ For $\alpha < \infty, \text{rk}_D(f) = \alpha$ iff $\beta < \alpha \Rightarrow \text{rk}_D(f) \ne \beta$ and for every $g \in {}^Y\text{Ord}$ satisfying $g <_D f$ there is $\beta < \alpha$ such that $\text{rk}_D(g) = \beta$.

3) We can replace D by the dual ideal.

Observation 1.7. *1) Let D be a pseudo $\aleph_1$-complete filter on Y. If $f, g \in {}^Y\text{Ord}$ and $f \le_D g$ then $\text{rk}_D(f) \le \text{rk}_D(g)$ and so if $f =_D g$ then $\text{rk}_D(f) = \text{rk}_D(g)$.*
2) If D_ℓ is a pseudo $\aleph_1$-complete filter on Y for $\ell = 1, 2$ <u>then</u> $D_1 \subseteq D_2 \wedge f \in {}^Y\text{Ord} \Rightarrow \text{rk}_{D_1}(f) \le \text{rk}_{D_2}(f)$.
3) If D is a filter on Y which is not pseudo-$\aleph_1$-complete, <u>then</u> for some $f \in {}^Y\omega$ we have $\text{rk}_D(f) = \infty$.

Proof. Easy. $\square$

Claim 1.8. *Assume D is a filter on Y such that D is $\aleph_1$-complete or just pseudo $\aleph_1$-complete (see Definition 1.2(1A)).*
1) [DC] For $f \in {}^Y\text{Ord}$, in 1.6, $\text{rk}_D(f)$ is always an ordinal, i.e. $< \infty$.
2) [DC] If $\alpha \le \text{rk}_D(f)$ <u>then</u> for some $g \in \prod\limits_{t \in Y} (f(t) + 1)$ we have $\alpha = \text{rk}_D(g)$. If $\alpha < \text{rk}_D(f)$ we can add $g <_D f$ and we can demand $(\forall y \in Y)(g(y) < f(g) \vee g(y) = 0 = f(y))$.
2A) If $\text{rk}_D(f) < \infty$ then part (2) holds for f (without assuming DC).
3) If $f, g \in {}^Y\text{Ord}$ and $f <_D g$ and $\text{rk}_D(f) < \infty$ <u>then</u> $\text{rk}_D(f) < \text{rk}_D(g)$.
4) For $f \in {}^Y\text{Ord}$ we have $\text{rk}_D(f) > 0$ iff $\{t \in Y : f(t) > 0\} \in D$.
5) If $f, g \in {}^Y\text{Ord}$ and $f = g + 1$ <u>then</u> $\text{rk}_D(f) = \text{rk}_D(g) + 1$.

Proof. Straight, e.g.
2A) We prove this by induction on $\beta = \text{rk}_D(f)$. If $\beta \le \alpha$ there is nothing to prove. If $\beta = \alpha + 1$ by the definition, there is $g <_D f$ such that $\text{rk}_D(g) \ge \alpha$, now by part (3) we have $\text{rk}_D(g) < \text{rk}_D(f)$ which means $\text{rk}_D(g) < \alpha + 1$, so together $\text{rk}_D(g) = \alpha$ and let $g' \in {}^Y\text{Ord}$ be defined by $g'(s)$ is $g(s)$ if $g(s) < f(s)$ and is 0 if $g(s) \ge f(s)$ so $g' <_D f$ and $g' \le_D g \le_D g'$ hence $\text{rk}_D(g') = \text{rk}_D(g) = \alpha$ is as required.

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Lastly, if $\beta > \alpha+1$ by the definition there is $f' <_D f$ such that $\mathrm{rk}_D(f') \ge \alpha+1$ and by 1.7(1) without loss of generality $t \in Y \Rightarrow f'(t) \le f(t)$ and by part (3) $\mathrm{rk}_D(f') < \mathrm{rk}_D(f)$ so we can apply the induction hypothesis to f'. $\square_{1.8}$

Claim 1.9. *1)* $[\mathrm{AC}_{\aleph_0}]$ *If* D *is an* $\aleph_1$*-complete filter on* Y *and* $f \in {}^Y\mathrm{Ord}$ *and* $Y = \cup\{Y_n : n < \omega\}$ *<u>then</u>* $\mathrm{rk}_D(f) = \min\{\mathrm{rk}_{D+Y_n}(f) : n < \omega$ *and* $Y_n \in D^+\}$.
2) $[\mathrm{AC}_{\mathscr{W}}]$ *If* D *is a* $|\mathscr{W}|^+$*-complete filter on* $Y, \mathscr{W}$ *infinite and* $f \in {}^Y\mathrm{Ord}$ *and* $\cup\{Y_t : t \in \mathscr{W}\} \in D$ *<u>then</u>* $\mathrm{rk}_D(f) = \min\{\mathrm{rk}_{D+Y_t}(f) : t \in \mathscr{W}$ *and* $Y_t \in D^+\}$.

Remark 1.10. 1) With choice, see [?].
2) In the proof of 1.9(2) it is natural to list $\mathscr{W}_*(\subseteq \mathscr{W})$ as $\bar{u} = \langle u_\alpha : \alpha < \alpha_*\rangle$ and define $g \in {}^Y\mathrm{Ord}$ by $g(s) = g_\alpha(s)$ when $s \in u_\alpha \backslash \cup \{u_\beta : \beta < \alpha\}$ and $g(s) = 0$ if $s \in Y \backslash \cup \{u_\alpha : \alpha < \alpha_*\}$. However, we assume only $\mathrm{AC}_{\mathscr{W}}$ which does not imply the existence of such $\bar{u}$.

Proof. 1) By part (2).
2) Note that necessarily $\{t : Y_t \in D^+\}$ is non-empty, (otherwise $t \in \mathscr{W} \Rightarrow Y \backslash Y_t \in D$ hence $Y \backslash (\cup\{Y_t : t \in \mathscr{W})\}) = \cap\{Y \backslash Y_t : t \in \mathscr{W}\}$ which belongs to D as the intersection of a family of $\mathscr{W}$ sets from D, but this contradicts the assumption "D is $(\le \mathscr{W})$-complete". The inequality $\le$ is obvious (i.e. by 1.7(2)). We prove by induction on the ordinal α that $(\forall v \in \mathscr{W})[Y_v \in D^+ \Rightarrow \mathrm{rk}_{D+Y_v}(f) \ge \alpha] \Rightarrow \mathrm{rk}_D(f) \ge \alpha$.

For $\alpha = 0$ and α is limit this is trivial.

For $\alpha = \beta+1$, we assume $(\forall v \in \mathscr{W})[Y_v \in D^+ \Rightarrow \mathrm{rk}_{D+Y_v}(f) \ge \alpha > \beta]$ so by Definition 1.6 it follows that $[v \in \mathscr{W} \wedge Y_v \in D^+ \Rightarrow (\exists g)(g \in {}^Y\mathrm{Ord} \wedge g <_{D+Y_v} f \wedge \mathrm{rk}_{D+Y_v}(g) \ge \beta]$ hence, if $v \in \mathscr{W} \wedge Y_v \in D^+$ then $\{t \in Y : f(t) = 0\} = \emptyset \mod (D+Y_v)$, i.e. $\{v : f(v) = 0\} \cap Y_v = \emptyset \mod D$. As this holds for every $v \in \mathscr{W}$ and D is $|\mathscr{W}|^+$-complete clearly we have $\{t \in Y : f(t) = 0\} = \emptyset \mod D$. We can by 1.7(1) replace f by $f' \in {}^Y\mathrm{Ord}$ when $\{v \in Y : f(v) = f'(v)\} \in D$ so without loss of generality $t \in Y \Rightarrow f(t) > 0$.

But by an assumption $\mathscr{W} \in \mathbf{C}$, hence by 1.8(2A) there is a sequence $\langle g_v : v \in \mathscr{W}_*\rangle$ where $\mathscr{W}_* := \{v \in \mathscr{W} : Y_v \in D^+\}$ such that $g_v \in {}^Y\mathrm{Ord}, g_v <_{D+Y_v} f, \mathrm{rk}_{D+Y_v}(g_v) \ge \beta$ and $t \in Y \Rightarrow g_v(t) < f(t)$ so $g_v < f$.

As D is $|\mathscr{W}|^+$-complete necessarily $Y_* := \cup\{Y_v : v \in \mathscr{W} \backslash \mathscr{W}_*\} = \emptyset \mod D$, but $\cup\{Y_v : v \in \mathscr{W}\} \in D$ hence $Y_* = \cup\{Y_v : v \in \mathscr{W}_*\}$ belongs to D. Define $g \in {}^Y\mathrm{Ord}$ by $g(s) = \min\{g_u(s) : u \in \mathscr{W}_*$ satisfies $s \in Y_u\}$ if $s \in Y_*$ and 0 if $s \in Y \backslash Y_*$.

Hence $(\cup\{Y_v : v \in \mathscr{W}_*\}) \in D$ and $g \in {}^Y\mathrm{Ord}$ and $g <_D f$ (and even $g < f$) so by the induction hypothesis

- $\odot$ it suffices to prove $v \in \mathscr{W}_* \Rightarrow \mathrm{rk}_{D+Y_v}(g) \ge \beta$.

Fix $v \in \mathscr{W}_*$, and for each $u \in \mathscr{W}_*$ let $Y_{v,u} := \{t \in Y_u \cap Y_v : g(t) = g_u(t)\}$ so by the choice of $g(t)$ we have

- $\boxplus_1$ if $v \in \mathscr{W}_*, t \in Y_v$ <u>then</u> for some $u \in \mathscr{W}_*$ we have $t \in Y_{v,u} \subseteq Y_u$ and $g(t) = g_u(t)$.

Hence

- $\boxplus_2$ $\langle Y_{v,u} : u \in \mathscr{W}_*\rangle$ exists and $\cup\{Y_{v,u} : u \in \mathscr{W}_*\} = Y_v \in (D+Y_v)$.

Now

- $\boxplus_3$ if $u \in \mathscr{W}_* \wedge Y_{v,u} \in (D+Y_v)^+$ then $\mathrm{rk}_{D+Y_{v,u}}(g) \ge \beta$.

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[Why? By the choice of $Y_{v,u}$ we have $g = g_u$ $\mathrm{mod}(D + Y_{v,u})$ hence $\mathrm{rk}_{D+Y_{v,u}}(g) = \mathrm{rk}_{D+Y_{v,u}}(g_u)$, also $Y_{v,u} \subseteq Y_u$ hence $D + Y_{v,u} \supseteq D + Y_u$ which by 1.7(2) implies $\mathrm{rk}_{D+Y_{v,u}}(g_u) \geq \mathrm{rk}_{D+Y_u}(g_u)$ which is $\geq \beta$. Together we are done.]

By $\boxplus_2$+$\boxplus_3$ and the induction hypothesis it follows that $v \in \mathscr{W}_* \Rightarrow \mathrm{rk}_{D+Y_v}(g) \geq \beta$ so by $\odot$ we are done. $\square_{1.9}$

Definition 1.11. For Y, D, f in 1.6 let $J[f, D] =: \{Z \subseteq Y : Y \backslash Z \in D$ or $(Y \backslash Z) \in D^+ \wedge \mathrm{rk}_{D+Z}(f) > \mathrm{rk}_D(f)\}$.

Claim 1.12. [DC+AC$_{\mathscr{Y}}$] Assume D is an $\aleph_1$-complete $|\mathscr{Y}|^+$-complete filter on Y, recalling Definition 1.2.
1) If $f \in {}^Y\mathrm{Ord}$ <u>then</u> $J[f, D]$ is an $\aleph_1$-complete and $|\mathscr{Y}|^+$-complete ideal on Y.
2) If $f_1, f_2 \in {}^Y\mathrm{Ord}$ and $J = J[f_1, D] = J[f_2, D]$ <u>then</u> $\mathrm{rk}_D(f_1) < \mathrm{rk}_D(f_2) \Rightarrow f_1 < f_2$ mod J and $\mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2) \Rightarrow f_1 = f_2$ mod J.
3) If $f \in {}^Y\mathrm{Ord}$ is e.g. constantly 1 <u>then</u> $J[f, D] = \mathrm{dual}(D)$.
4) If $f \in {}^Y\mathrm{Ord}$ and $A \in (J[f, D])^+$ <u>then</u> ($A \in D^+$ and) $\mathrm{rk}_{D+A}(f) = \mathrm{rk}_D(f)$.

Proof. 1) By 1.9.
2) As J is an ideal on Y (by part (1)) this should be clear by the definitions; that is, let $A_0 := \{t \in Y : f_1(t) < f_2(t)\}, A_1 := \{t \in Y : f_1(t) = f_2(t)\}$ and $A_2 := \{t \in Y : f_1(t) > f_2(t)\}$. Now $\langle A_0, A_1, A_0\rangle$ is a partition of Y.

First, assume $A_0 \in J^+$, then by the definition of $J[f_1, D]$ we have $\neg(\mathrm{rk}_D(f_1) < \mathrm{rk}_{D+A_0}(f_1))$; i.e. $\mathrm{rk}_{D+A_0}(f_1) \leq \mathrm{rk}_D(f_1)$ and so by 1.7(2) we have $\mathrm{rk}_D(f_1) = \mathrm{rk}_{D+A_0}(f_1)$; the same is true for f_2. Now as $A_0 \in J^+$, by the choice of A_0, $f_1 <_{D+A_0} f_2$ hence $\mathrm{rk}_D(f_1) = \mathrm{rk}_{D+A_0}(f_1) < \mathrm{rk}_{D+A_0}(f_2) = \mathrm{rk}_D(f_2)$.
[Why? By the previous sentence, by 1.8(3), by the previous sentence respectively.]

Second, similarly if $A_2 \in J^+$ then $f_2 < f_1$ mod $(D+A_2)$ and $\mathrm{rk}_D(f_1) > \mathrm{rk}_D(f_2)$.

Lastly, if $A_1 \in J^+$ then by 1.7(1) $f_1 = f_2$ mod $(D + A_1)$ hence $\mathrm{rk}_{D+A_1}(f_1) = \mathrm{rk}_{D+A_1}(f_2)$ and $\mathrm{rk}_D(f_1) = \mathrm{rk}_{D+A_1}(f_2) = \mathrm{rk}_{D+A_1}(f_2) = \mathrm{rk}_D(f_2)$.

By the last three paragraphs at most one of A_0, A_1, A_2 belongs to J^+ and as $A_0 \cup A_1 \cup A_2 = Y$ at least one of A_0, A_1, A_2 belongs to J^+, so easily we are done.
3) Obvious.
4) Proved inside the proof of part (2). $\square_{1.12}$

Definition 1.13. 1) Let $\mathrm{FIL}^{\mathrm{cc}}_S(Y)$ or $\mathrm{FIL}^{\mathrm{pcc}}_S(Y)$ be the set of D such that:
D is a filter on the set Y which is $|S|^+$-complete and is $\aleph_1$-complete or is psuedo $|S|^+$-complete and psuedo $\aleph_1$-complete.
2) Let $\mathrm{FIL}_{\mathrm{cc}}(Y)$ or $\mathrm{FIL}_{\mathrm{pcc}}(Y)$ be $\mathrm{FIL}^{\mathrm{cc}}_\emptyset$ or $\mathrm{FIL}^{\mathrm{pcc}}_\emptyset$.
3) Omitting Y means for some Y and then we let $Y = \mathrm{Dom}(D)$.

Without enough choice, the minimal ($\leq S$)-complete filter extending a filter D is gotten in stages.

Definition 1.14. 1) For a filter D on Y and set S we define $\mathrm{comp}_{S,\gamma}(D)$ by induction on $\gamma \in \mathrm{Ord} \cup \{\infty\}$.

<u>$\gamma = 0$</u>: $\mathrm{comp}_{S,\gamma}(D) = D$

<u>$\gamma = \text{limit}$</u>: $\mathrm{comp}_{S,\gamma}(D) = \cup\{\mathrm{comp}_{S,\beta}(D) : \beta < \gamma\}$

<u>$\gamma = \beta + 1$</u>: $\mathrm{comp}_{S,\gamma}(D) = \{A \subseteq Y : A$ belongs to $\mathrm{comp}_{S,\beta}(D)$ or include the intersection of some S-sequence of members of $\mathrm{comp}_{S,\beta}(D)$, i.e. $\cap\{A_s : s \in S\}$, where $\langle A_s : s \in S\rangle$ is a sequence of members of $\mathrm{comp}_{S,\beta}(D)\}$.

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2) Similarly for a family $\mathscr{S}$ of sets replacing S by "some member of $\mathscr{S}$", e.g. we define $\text{com}_{\in \mathscr{S},\gamma}(D)$ by induction on γ using $(\in \mathscr{S})$-sequences, i.e. S-sequence for some $S \in \mathscr{S}$.
3) If $\gamma = \infty$ we may omit it. We say that D is a pseudo $(\le S,\gamma)$-complete when $\emptyset \notin \text{comp}_{S,\gamma}(D)$.

Observation 1.15. *1) If D is a filter on Y and S is a set, then:*

- (a) $\langle \text{comp}_{S,\gamma}(D) : \gamma \in \text{Ord} \cup \{\infty\}\rangle$ *is an $\subseteq$-increasing sequence of filters of Y (starting with D)*
- (b) *if* $\text{comp}_{S,\gamma+1}(D) = \text{comp}_{S,\gamma}(D)$ *then for every $\beta \ge \gamma$ we have* $\text{comp}_{S,\beta}(D) = \text{comp}_{S,\gamma}(D)$
- (c) *there is an ordinal $\gamma = \gamma_S(D) < \text{hrtg}(\mathscr{P}(Y))$ such that* $\text{comp}_{S,\gamma}(D) = \text{comp}_{S,\gamma+1}(D)$ *and* $\langle \text{comp}_{S,\beta}(D) : \beta \le \gamma\rangle$ *is strictly $\subset$-increasing.*

2) Assume AC_S. Then for any filter D on Y we have $\gamma_S(D) \le \theta$ when $\theta := \min\{\lambda : \lambda$ a cardinal such that $\text{cf}(\lambda) \ge \text{hrtg}(S)\}$.
3) Assume $DC + AC_S + |S \times S| = |S|$. Then for any filter D on Y we have $\gamma_S(D) \le 1$ and $comp_{S,1}(D)$ is an $(\le S)$-complete filter or is $\mathscr{P}(Y)$; the latter holds iff D is not pseudo $(\le S)$-complete.
4) Similarly to part (2) for "$\in \mathscr{S}$" but AC_S is replaced by $S \in \mathscr{S} \Rightarrow \text{AC}_S$ and $\theta = \min\{\kappa : \kappa$ regular and $S \in \mathscr{S} \Rightarrow \kappa \ge \text{hrtg}(\mathscr{S})\}$.

Remark 1.16. Note that in part (2) of 1.15, θ is regular and $\theta \le \text{hrtg}({}^{\omega>}S)$ but the inverse is not true, if $\text{hrtg}(S) = \aleph_0$ but holds if $\text{hrtg}(S) > \aleph_0$.

Proof. We prove the versions with $\mathscr{S}$, i.e. for (4). Let $D_\gamma = \text{comp}_{\in \mathscr{S},\gamma}(D)$ for $\gamma \in \text{Ord}$.
1) Clause (a) is by the definition; clause (b) is proved by induction on $\beta \ge \gamma$, for $\beta = \gamma$ this is trivial, for $\beta = \gamma+1$ use the assumption and for $\beta > \gamma+1$ use the definition and the induction hypothesis. As for clause (c) let $\gamma_* = \min\{\gamma \in \text{Ord} \cup \{\infty\}$; if $\gamma < \infty$ then $D_\gamma = D_{\gamma+1}\}$, so $\langle D_\gamma : \gamma \le \gamma_*\rangle$ is $\subset$-increasing continuous by clause (a), and by clause (b), $\langle D_\gamma : \gamma \ge \gamma_*\rangle$ is constant. Now define $h : \mathscr{P}(Y) \to \gamma_*$ by: $A \in D_{\gamma+1} \backslash D_\gamma \Rightarrow h(A) = \gamma$ and $h(A) = 0$ when there is no such γ. So h is onto γ_* hence $\gamma_* < \text{hrtg}(\mathscr{P}(A))$ so γ_* is as required on $\gamma_S(D)$.
2) We prove also the relevant statement in part (4), so $S \in \mathscr{S} \Rightarrow \text{AC}_S \wedge \text{cf}(\theta) \ge \text{hrtg}(S)$. Let γ be an ordinal.

Let

$$\mathscr{T}^1_n = \{\Lambda : \Lambda \text{ is a set of sequences of length } \le n, \text{ closed under initial segments such that for every non-maximal } \eta \in \Lambda \text{ for some } S \in \mathscr{S} \text{ we have } \eta\hat{\ }\langle s\rangle \in \Lambda \Leftrightarrow s \in S\}.$$

$$\mathscr{T}^2_{\gamma,n} = \{\mathbf{x} : \begin{array}{ll} (a) & \mathbf{x} \text{ has the form } \langle Y_\eta, \gamma_\eta : \eta \in \Lambda\rangle \\ (b) & \Lambda \in \mathscr{T}^1_n \text{ and } Y_\eta \subseteq Y \text{ for } \eta \in \Lambda \\ (c) & Y_\eta = \cap\{Y_{\eta\hat{\ }<s>} : s \text{ satisfies } \eta\hat{\ }\langle s\rangle \in \Lambda\} \text{ if } \eta \in \Lambda \\ & \text{but } \eta \text{ is not } \triangleleft\text{-maximal in } \Lambda \\ (d) & \eta \triangleleft \nu \in \Lambda \Rightarrow \gamma_\nu < \gamma_\eta < 1+\gamma \\ (e) & Y_\eta \in D \text{ if } \eta \in \Lambda \text{ is } \triangleleft\text{-maximal in } \Lambda \\ & \text{but } \ell g(\eta) < n\} \end{array}$$

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$$\mathscr{T}^2_n = \cup\{\mathscr{T}^2_{\gamma,n} : \gamma \text{ is an ordinal}\}.$$

Let $\mathbf{n}(\mathbf{x}) = n$ for the minimal possible n such that $\mathbf{x} \in \mathscr{T}^2_n$ and let $\mathbf{x} = \langle Y^{\mathbf{x}}_\eta, \gamma^{\mathbf{x}}_\eta : \eta \in \Lambda_{\mathbf{x}}\rangle$.

Let $\mathscr{T}^3_\gamma = \cup\{\mathscr{T}^2_{\gamma,n} : n < \omega\}$ and let $<_*$ be the natural order on $\mathscr{T}^3_\gamma$: $\mathbf{x} <_* \mathbf{y}$ iff $n(\mathbf{x}) < n(\mathbf{y}), \Lambda_{\mathbf{x}} = \Lambda_{\mathbf{y}} \cap {}^{n(\mathbf{x})\geq}(\cup\{S : S \in \mathscr{S}\})$ and $(Y^{\mathbf{x}}_\eta, \gamma^{\mathbf{x}}_\eta) = (Y^{\mathbf{y}}_\eta, \gamma^{\mathbf{y}}_\eta)$ for $\eta \in \Lambda_{\mathbf{x}}$.

Now (recalling $D_\gamma = \text{comp}_{\mathscr{S},\gamma}(D)$)

- ⊛ $A \in D_\gamma$ iff there is an ω-branch $\langle \mathbf{x}_n : n < \omega\rangle$ of $(\mathscr{T}^3_\gamma, <_*)$ such that $Y^{\mathbf{x}_n}_{<>} = A$.

[Why? We prove it by induction on the ordinal γ. For $\gamma = 0$ and γ limit this is obvious so assume we have it for γ and we shall prove it for $\gamma + 1$.

First assume $A \in D_{\gamma+1}$ and we shall find such ω-branch; if $A \in D_\gamma$ this is obvious, otherwise there are $S \in \mathscr{S}$ and a sequence $\langle A_s : s \in S\rangle$ of members of D_γ such that $A = \cap\{A_s : s \in S\}$. So $X_s := \{\bar{\mathbf{x}} : \bar{\mathbf{x}} \text{ witness } A_s \in D_\gamma\}$ is well defined and non-empty by the induction hypothesis, clearly the sequence $\langle X_s : s \in S\rangle$ exists, hence we can use AC_S to choose $\langle \bar{\mathbf{x}}_s : s \in S\rangle$ satisfying $\bar{\mathbf{x}}_s \in X_s$.

Now define $\bar{\mathbf{x}} = \langle \mathbf{x}_n : n < \omega\rangle$ as follows: $\Lambda_{\mathbf{x}_n} = \{\langle\rangle\} \cup \{\langle s\rangle\hat{\ }\eta : \eta \in \Lambda_{\mathbf{x}_{s,n-1}}$ and $s \in S\}, \gamma^{\mathbf{x}_n}_{<>} = \cup\{\gamma^{\mathbf{x}_{s,n}}_{<>} + 1 : s \in S\}$ and $Y^{\mathbf{x}}_{<>} = A$ and $Y^{\mathbf{x}_n}_{<s>\hat{\ }\eta} = Y^{\mathbf{x}_{s,n-1}}_\eta$. Now check.

Second, assume that there is such ω-branch $\langle \mathbf{x}_n : n < \omega\rangle$ of $(\mathscr{T}^3_\gamma, <_*)$ such that $Y^{\mathbf{x}_n}_{<>} = A$. Let $S = \{\eta(0) : \eta \in \Lambda_{\mathbf{x}_1}\}$ so necessarily $S \in \mathscr{S}$. For each $n < \omega$ and $s \in S$ we define $\mathbf{y}_{n,s}$ as follows: $\Lambda^{\mathbf{y}_{n,s}}_s = \{\nu : \langle s\rangle\hat{\ }\nu \in \Lambda_{\mathbf{x}_{n+1}}\}$ and for $\nu \in \Lambda^n_s$ let $\gamma^{\mathbf{y}_{n,s}}_\nu = \nu^{\mathbf{x}_{n+1}}_{<s>\hat{\ }\nu}$ and $Y^{\mathbf{y}_{n,s}}_\nu = Y^{\mathbf{x}_{n+1}}_{<s>\hat{\ }\nu}$. Now clearly $\langle \mathbf{y}_{n,s} : n < \omega\rangle$ is an ω-branch of $(\mathscr{T}^3_\gamma, \leq_*)$ so by the induction hypothesis $A^{\mathbf{x}_1}_{<s>} \in D$, $\text{comp}_{S,\gamma}(D)$ and $Y^{\mathbf{x}_0}_{<>} = A = \cap\{Y^{\mathbf{x}}_{<s>} :<>\in \Lambda_{\mathbf{x}_1}\} \in \text{comp}_{S,\gamma+1}(D)$. So we are done.]

Now toward a contradiction assume that $\gamma_S(D) > \theta$, so there is $A \in D_{\theta+1} \backslash D_\theta$ hence here is an ω-branch $\langle \mathbf{x}_n : n < \omega\rangle$ of $\mathscr{T}^3_\gamma$ witnessing that $A \in D_{\theta+1}$, let $\Lambda = \cup\{\Lambda_{\mathbf{x}_n} : n < \omega\}$ and $\gamma_\eta = \gamma^{\mathbf{x}_n}_\eta$ for every $n < \omega$ large enough. So Λ is well founded (recalling $\eta \triangleleft \nu \in \Lambda \Rightarrow \gamma_\eta > \gamma_\nu$) and we can choose $\langle \gamma'_\eta : \eta \in \Lambda\rangle$ such that $\gamma'_\eta = \sup\{\gamma_\nu + 1 : \eta \triangleleft \nu \in \Lambda \text{ and } \ell g(\nu) = \ell g(\eta) + 1\}$. If $\gamma_{<>} < \theta$ we are done otherwise let $\eta \in \Lambda$ be $\triangleleft$-maximal such that $\gamma'_\eta \geq \theta$ hence $\eta \triangleleft \nu \Rightarrow \gamma'_\nu < \theta$, so necessarily $\gamma'_\eta = \theta = \cup\{\gamma'_\nu + 1 : \eta \triangleleft \nu \in \Lambda, \ell g(\nu) = \ell g(\eta) + 1\}$. Let $S \in \mathscr{S}$ be such that $\eta\hat{\ }\langle s\rangle \in \Lambda \Leftrightarrow s \in S$, so $\{\gamma'_{\eta\hat{\ }<s>} : s \in S\}$ is an unbounded subset of θ so $\text{cf}(\theta) \leq \text{hrtg}(S) < \theta$. This takes care of the first possibility for θ so the second case is easier.

3) It suffices to show that we can replace $\mathbf{x} \in \mathscr{T}^2_2$ by $\mathbf{x} \in \mathscr{T}^2_1$. $\square_{1.15}$

Definition 1.17. 1) For a filter D on a set Y and a set S let $\gamma_S(D)$ be as in clause (c) of the Observation 1.15(1).
1A) Similarly with "$\in \mathscr{S}$" instead S.
2) D is pseudo (S, γ)-complete if $\emptyset \notin \text{comp}_{S,\gamma}(D)$.
2A) Similarly with "$\in \mathscr{S}$" instead S.

Observation 1.18. *If h is a function from S_1 onto S_2 then* $\text{hrtg}(S_1) \geq \text{hrtg}(S_2)$ *and every [pseudo] $(\leq S_1)$-complete filter is a [pseudo] $(\leq S_2)$-complete filter.*

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§ 2. Commuting ranks

The aim of this section is to sort out when two rank rk_{D_1}, rk_{D_2} do so called commute.

Definition 2.1. Assume that D_ℓ is an $\aleph_1$-complete filter on Y_ℓ for $\ell = 1, 2$. For $\iota \in \{1, 2, 3, 4, 5\}$ we say D_2 does ι-commute with D_1 <u>when</u>: $\boxplus_\iota = \boxplus^\iota_{D_1,D_2}$ holds <u>where</u>:

- $\boxplus_1$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ <u>then</u> we can find A_*, B_* such that: $A_* \in D_1, B_* \in D_2$ and $A_* \times B_* \subseteq \cup\{\{s\} \times B_s : s \in A\}$ so $A_* \subseteq A$
- $\boxplus_2$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $J_2 = J[f_2, D_2]$ for[1] some $f_2 \in {}^{Y_2}\text{Ord}$ <u>then</u> we can find A_*, B_* such that $A_* \in D_1, B_* \in J_2^+$ and $A_* \times B_* \subseteq \cup\{\{s\} \times B_s : s \in A\}$ so $A_* \subseteq A$
- $\boxplus_3$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $J_1 = J[f_1, D_1]$ for some $f_1 \in {}^{Y_1}\text{Ord}$ <u>then</u> we can find A_*, B_* such that $A_* \in J_1^+, A_* \subseteq A, B_* \in D_2$ and $s \in A_* \Rightarrow B_* \subseteq B_s$
- $\boxplus_4$ if $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $\bar{J}^1 = \langle J^1_t : t \in Y_2\rangle$ satisfies $J^1_t \in \{J[f, D_1] : f \in {}^{Y_1}\text{Ord}\}$ and $J_2 \in \{J[f, D_2] : f \in {}^{Y_2}\text{Ord}\}$ <u>then</u> we can find A_*, B_* such that $B_* \in J_2^+$ and $t \in B_* \Rightarrow A_* \in (J^1_t)^+$ and $(s, t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$ hence $A_* \subseteq A, A_* \in D_1^+$
- $\boxplus_5$ like $\boxplus_4$ but we omit the sequence $\bar{J}^1$ and the demand on A_* is $A_* \in D_1^+$.

Remark 2.2. 1) These are seemingly not commutative relations.
2) We shall first give a consequence and then give sufficient conditions.
3) We intend to generalize to systems (see 3.1 and 3.8).
4) Can we below use "$D_\ell \in \text{FIL}_{\text{pcc}}(Y_1)$, see Definition 1.13? Yes, but only when we do not use $D + A, A \in D^+$.

Claim 2.3. *1) We have* $\text{rk}_{D_1}(f) \leq \text{rk}_{D_2}(g)$ <u>*when*</u>*:*

$\oplus$ (a) $D_\ell \in \text{FIL}_{\text{cc}}(Y_\ell)$ *for* $\ell = 1, 2$
(b) $\bar{g} = \langle g_t : t \in Y_2\rangle$
(c) $g_t \in {}^{Y_1}\text{Ord}$
(d) $g \in {}^{Y_2}\text{Ord}$ *is defined by* $g(t) = \text{rk}_{D_1}(g_t)$
(e) $\bar{f} = \langle f_s : s \in Y_1\rangle$
(f) $f_s \in {}^{Y_2}\text{Ord}$ *is defined by* $f_s(t) = g_t(s)$
(g) $f \in {}^{Y_1}\text{Ord}$ *is defined by* $f(s) = \text{rk}_{D_2}(f_s)$

$\boxplus$ (a) D_2 *does 2-commute with* D_1
(b) AC_{Y_1} *holds.*

2) We have $\text{rk}_{D_1}(f) = \text{rk}_{D_2}(g)$ *when* $\oplus$ *from above and*

$\boxplus^+$ (a) D_ℓ *does 2-commute with* $D_{3-\ell}$ *for* $\ell = 1, 2$
(b) AC_{Y_1} *and* AC_{Y_2}.

---

[1]Note that if $\text{rk}_{D_2}(f_2) = \infty$ then $J[f_2, D_2] = \text{dual}(D_2)$ which usually is an easier case. Also, if D is a non-pseudo $\aleph_1$-complete filter on Y and $f \in {}^Y\text{Ord}, n < \omega \Rightarrow \{s \in Y : f(s) \geq n\} \in D$ <u>then</u> $\text{rk}_D(f) = \infty$.

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Remark 2.4. 1) In order not to use DC in the proof we should consider ∞ as a member of Ord in clauses (d),(g) of $\boxplus$.
2) If we use $\boxplus^3_{D_1,D_2}$ the problem is with $(*)_{12}$, i.e. we only have $g'_t <_{D_1+A_*} g_t$ but the desired conclusion $(\mathrm{rk}_{D_1+A_*}(g') < \mathrm{rk}_{D_1}(g_t))$ is problematic as we have to deal with $J[g_t, D_1)$ for many $t's$, this motivate $\boxplus^4_{D_1,D_2}$.

Proof. 1) We prove by induction on the ordinal ζ that:

$\boxdot_\zeta$ if $\oplus + \boxplus$ above hold for $D_1, D_2, f, g, \bar{f}, \bar{g}$ and $\mathrm{rk}_{D_1}(f) \geq \zeta$ then $\mathrm{rk}_{D_2}(g) \geq \zeta$.

The case $\zeta = 0$ is trivial and the case ζ a limit ordinal follows by the induction hypothesis. So assume that $\zeta = \xi + 1$.

Let

$(*)_1$ $A := \{s \in Y_1 : f(s) > 0\}$.

As we are assuming $\mathrm{rk}_{D_1}(f) > \xi \geq 0$ by 1.8(4) necessarily

$(*)_2$ $A \in D_1$.

For each $s \in A$, $f(s) > 0$ so applying clause (g) of $\oplus$ we get

$(*)_3$ $\mathrm{rk}_{D_2}(f_s) > 0$ when $s \in A$

hence

$(*)_4$ $B_s := \{t \in Y_2 : f_s(t) > 0\}$ belongs to D_2 when $s \in A$.

So $\langle B_s : s \in A\rangle \in {}^A(D_2)$. Recall (see $\boxplus(a)$ of the assumption) that D_2 does 2-commute with D_1, apply it to $A, \langle B_s : s \in A\rangle, J_2 := J[g, D_2]$; so we can find A_*, B_* such that

$(*)_5$ (a) $A_* \in D_1$ (and $A_* \subseteq A$)
(b) $B_* \in J_2^+$ recalling $J_2 = J[g, D_2]$ so $B_* \in D_2^+$ and (by Definition 1.11) $\mathrm{rk}_{D_2+B_*}(g) = \mathrm{rk}_{D_2}(g)$
(c) $(s,t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$.

Now by the present assumption of $\boxdot_\zeta$ we have

$(*)_6$ $\mathrm{rk}_{D_1}(f) \geq \zeta = \xi + 1$.

Hence by the definition of rk and 1.8(2) we can find f' such that:

$(*)_7$ (a) $f' \in {}^{Y_1}\mathrm{Ord}$ and $\mathrm{rk}_{D_1}(f') \geq \xi$
(b) $f' <_{D_1} f$
(c) if $s \in Y_1$ then $(f'(s) < f(s)) \vee (f'(s) = 0 = f(s))$ hence
- by $(*)_1$ without loss of generality $s \in A \Rightarrow f'(s) < f(s)$.

For each $s \in A$, clearly $f'(s) < f(s) = \mathrm{rk}_{D_2}(f_s) \leq \mathrm{rk}_{D_2+B_*}(f_s)$, by 1.8(3), clause (g) of $\oplus$ and 1.7(2) (as $D_2 \subseteq D_2 + B_*$) respectively, hence by 1.8(2) for each $s \in Y_1$ there is a function f'_s such that

$(*)_8$ (a) $f'_s \in {}^{(Y_2)}\mathrm{Ord}$,
(b) $f'_s < f_s$ mod D_2 if $s \in A$ and $t \in Y_2 \Rightarrow f'_s(t) < f_s(t) \vee f'_s(t) = 0 = f_s(t)$
(c) $\mathrm{rk}_{D_2+B_*}(f'_s) = f'(s)$; may require this only for $s \in A$.

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As $Y_1 \in \mathbf{C}$ by $\boxplus(b)$ of the assumption, clearly

$(*)_8^+$ there is such a sequence $\bar{f}' = \langle f'_s : s \in Y_1 \rangle$.

As $s \in A_* \wedge t \in B_* \Rightarrow f_s(t) > 0$, see $(*)_4 + (*)_5$, clearly

$(*)_9$ if $s \in A_*$ and $t \in B_*$ then $f'_s(t) < f_s(t)$.

We now define $\bar{g}' = \langle g'_t : t \in Y_2 \rangle$ by

$(*)_{10}$ $g'_t(s) = f'_s(t)$ for $s \in Y_1, t \in Y_2$ so $g'_t \in {}^{Y_1}\mathrm{Ord}$.

So

$(*)_{11}$ $s \in A_* \wedge t \in B_* \Rightarrow g'_t(s) = f'_s(t) < f_s(t) = g_t(s)$

hence (recalling $A_* \in D_1$ by $(*)_5(a)$ and 1.8(3))

$(*)_{12}$ if $t \in B_*$ then $g'_t <_{D_1} g_t$ hence $\mathrm{rk}_{D_1}(g'_t) < \mathrm{rk}_{D_1}(g_t)$.

Define $g' \in {}^{(Y_2)}\mathrm{Ord}$ by $g'(t) := \mathrm{rk}_{D_1}(g'_t)$ hence (recalling $\mathrm{rk}_{D_1}(g_t) = g(t)$)

$(*)_{13}$ $g' < g \bmod D_2 + B_*$.

Note that here $D_1 + A_* = D_1$, (though not so when we shall prove 2.9).

Now we apply the induction hypothesis to $g', f', \bar{f}' := \langle f'_s : s \in Y_1 \rangle, \bar{g}' := \langle g'_t : t \in Y_2 \rangle, D_1 + A_*, D_2 + B_*$ and ξ and get

$(*)_{14}$ $\xi \le \mathrm{rk}_{D_2+B_*}(g')$.

[Why is this legitimate? First, by 2.10(2) below obviously clause (a) of $\boxplus$ holds, second also clause (b) of $\boxplus$ holds, third, we have to check that clauses (a)-(g) of $\oplus$ hold in this instance.

Clause (a): First "$D_1 + A_* \in \mathrm{FIL}_{cc}(Y_1)$" as we assume $D_1 \in \mathrm{FIL}_{cc}(Y_1)$ and $A_* \in D_1$, see $(*)_5(a)$, actually $A_* \in D_1^+$ suffice (used in proving 2.9).

Second, "$D_2 + B_* \in \mathrm{FIL}_{cc}(Y_2)$" as $D_2 \in \mathrm{FIL}_{cc}(Y_2)$ and $B_* \in D_2^+$ by $(*)_5(b)$.

Clause (b): "$\bar{g}' = \langle g'_t : t \in Y_2 \rangle$" by our choice.

Clause (c): "$g'_t \in {}^{Y_1}\mathrm{Ord}$" by $(*)_{10}$.

Clause (d): "$g' \in {}^{Y_2}\mathrm{Ord}$ is defined by $g'(t) = \mathrm{rk}_{D_1}(g'_t)$" by its choice after $(*)_{12}$.

Clause (e): "$\bar{f}' = \langle f'_s : s \in Y \rangle$" by our choice in $(*)_8^+$.

Clause (f): "$f'_s \in {}^{Y_2}\mathrm{Ord}$ is defined by $f'_s(t) = g'_t(s)$ holds by $(*)_{10}$.

Clause (g): "$f' \in {}^{Y_1}\mathrm{Ord}$ is defined by $f'(s) = \mathrm{rk}_{D_2+B_*}(f'_s)$" holds by $(*)_7(a) + (*)_8(c)$.

Now $\boxdot_\xi$, the induction hypothesis, assumes "$\mathrm{rk}_{D_1+A_*}(f') \ge \xi$" which holds by $(*)_7(a) + (*)_5(a)$, actually $A_* \in D_1^+$ suffice here and its conclusion is $\xi \le \mathrm{rk}_{D_2+B_*}(g')$ as promised in $(*)_{14}$.]

Next

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$(*)_{15}$ $\xi <$ $\mathrm{rk}_{D_2}(g)$.

[Why?

- $\bullet_1$ $\xi \le$ $\mathrm{rk}_{D_2+B_*}(g')$ by $(*)_{14}$
- $\bullet_2$ $\mathrm{rk}_{D_2+B_*}(g') <$ $\mathrm{rk}_{D_2+B_*}(g)$ by $(*)_{13}$ and 1.8(3)
- $\bullet_3$ $\mathrm{rk}_{D_2+B_*}(g) =$ $\mathrm{rk}_{D_2}(g)$ by $(*)_5(b)$.

Together $(*)_{15}$ holds.]

So

$(*)_{16}$ $\zeta = \xi + 1 \le$ $\mathrm{rk}_{D_2}(g)$

as promised. Together we are done.

2) This follows by applying part (1) twice: once for (f, g, D_1, D_2) getting $\mathrm{rk}_{D_1}(f) \le \mathrm{rk}_{D_2}(g)$ and once for (g, f, D_2, D_1) getting $\mathrm{rk}_{D_2}(g) \le \mathrm{rk}_{D_1}(f)$. Together we are done. $\square_{2.3}$

Claim 2.5. *Assume $D_\ell \in \mathrm{FIL}_{cc}(Y_\ell)$ for $\ell = 1, 2$.*

If D_2 does ι_1-commute with D_1 then D_2 does ι_2-commute with D_1 when $(\iota_1, \iota_2) = (1,2), (1,3), (2,4), (1,4), (1,5), (4,5)$.

Proof. Obvious for (4,5) use 1.12(3). $\square_{2.5}$

Claim 2.6. *Assume $D_\ell \in \mathrm{FIL}_{cc}(Y_\ell)$ for $\ell = 1, 2$. If at least one of the following cases occurs, then D_2 does 1-commute (hence 2-commute) with D_1.*

Case 1: D_2 is $|Y_1|^+$-complete.

Case 2: D_1 is an ultrafilter which is $|Y_2|^+$-complete

Case 3: D_1, D_2 are ultrafilters and if $\bar{A} = \langle A_t : t \in Y_2 \rangle \in {}^{Y_2}(D_1)$ then for some $A_* \in D_1$ we have $\{t : A_t \supseteq A_*\} \in D_2$.

Proof. So let $A \in D_1$ and $\langle B_s : s \in A \rangle \in {}^{A}(D_2)$ be given.

Case 1: Let $A_* = A$ and $B_* = \cap\{B_s : s \in A\}$, so $A_* \in D_1$ by an assumption and $B_* \in D_2$ as we assume $\{B_s : s \in A\} \subseteq D_2$ and D_2 is $|Y_1|^+$-complete (and necessarily $|A| \le |Y_1|$).

Case 2: For each $t \in Y_2$ let $A'_t := \{s \in Y_1 : s \in A$ satisfies $t \in B_s\}$ and let A''_t be the unique member of $\{A'_t, Y_1 \backslash A'_t\} \cap D_1$, recalling D_1 is an ultrafilter on Y_1. Clearly the functions $t \mapsto A'_t$ and $t \mapsto A''_t$ are well defined hence the sequences $\langle A'_t : t \in Y_2 \rangle, \langle A''_t : t \in Y_2 \rangle$ exist and $\{A''_t : t \in Y_2\} \subseteq D_1$.

As D_1 is $|Y_2|^+$-complete necessarily $A_* := \cap\{A''_t : t \in Y_2\} \cap A$ belongs to D_1, and clearly $A_* \subseteq A$. Let $B_* = \{t \in Y_2 : A''_t = A'_t\}$.

So now choose any $s_* \in A_*$ (possible as $A_* \in D_1$ implies $A_* \ne \emptyset$) so $B_{s_*} \in D_2$ and $t \in B_{s_*} \Rightarrow s_* \in A'_t \Rightarrow s_* \in A'_t \cap A_* \Rightarrow A'_t \cap A_* \ne \emptyset \Rightarrow A''_t = A'_t \Rightarrow t \in B_*$ so $B_{s_*} \subseteq B_*$ but $B_{s_*} \in D_2$ hence $B_* \in D_2$. So A_*, B_* are as required.

Case 3:

Like Case 2. $\square_{2.6}$

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Claim 2.7. *Assume* $\mathrm{AC}_{\mathscr{P}(Y_2)}$.
1) Assume $D_1 \in \mathrm{FIL}_{cc}(Y_1)$ *and* $D_2 \in \mathrm{FIL}_{cc}(Y_2)$.
Then D_2 *does 3-commute with* D_1 <u>*when*</u> D_1 *is* $(\le \mathscr{P}(Y_2))$*-complete.*
2) In part (1) if $E \subseteq D_2$ *is* $(D_2, \subseteq)$*-cofinal, it suffices to assume* D_1 *is* $(\le E)$*-complete.*

Remark 2.8. 1) For part (1) in the definition of $(\le \mathscr{P}(Y_2))$-complete we can use just partitions, but not so in part (2).
2) Maybe see more in [**?**].

Proof. 1) So let $A \in D_1$ and $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$ and $J_1 = J[f_1, D_1]$ for some $f_1 \in {}^Y\mathrm{Ord}$ be given. So $s \mapsto B_s$ is a function from $A \in D_1$ to $D_2 \subseteq \mathscr{P}(Y_2)$ hence as $\mathrm{AC}_{\mathscr{P}(Y_2)}$ is assumed recalling that by 1.12(1) the ideal J_1 on Y_1 is $(\le \mathscr{P}(Y_2))$-complete, there is $B_* \in D_2$ such that $A_* := \{s \in A : B_s = B_*\} \in J_1^+$. Clearly A_*, B_* are as required.
2) For $B \in E$ let $A^*_B = \{s \in A : B \subseteq B_s\}$, so clearly $\langle A^*_B : B \in E\rangle$ is a sequence of subsets of $A \in D_1$ with union A, so again by 1.12(1) for some $B_* \in E$ the set $A_* := \{s \in A : B_* \subseteq B_s\}$ belongs to J_1^+, so we are done. $\square_{2.7}$

Claim 2.9. $\mathrm{rk}_{D_1}(f) \le \mathrm{rk}_{D_2}(g)$ <u>*when*</u>*:*

$\oplus$ *as in 2.3*

but we replace clause $(\boxplus)$ *there by*

$\boxplus'$ (a) D_2 *does 4-commute with* D_1
(b) AC_{Y_1} *holds.*

Proof. We repeat the proof of 2.3 but:

<u>First change</u>: we replace $(*)_5$ and the paragraph before it by the following:
So $\bar{B} = \langle B_s : s \in A\rangle \in {}^A(D_2)$.
Recall that D_2 does 4-commute with D_1, apply this to $A, \langle B_s : s \in A\rangle, \bar{J}^1 = \langle J^1_t : t \in Y_2\rangle$ where $J^1_t := J[g_t, D_1], J_2 := J[g, D_2]$ and we get A_*, B_* such that:

$(*)'_5$ (a) $A_* \in D_1^+$ and $A_* \subseteq A$
(b) $B_* \in J_2^+$ hence $B_* \in D_2^+$ and $\mathrm{rk}_{D_2+B_*}(g) = \mathrm{rk}_{D_2}(g)$
(c) $(s,t) \in A_* \times B_* \Rightarrow s \in A \wedge t \in B_s$
(d) if $t \in B_*$ then $A_* \in (J^1_t)^+$ hence

$$t \in B_* \Rightarrow \mathrm{rk}_{D_1+A_*}(g_t) = \mathrm{rk}_{D_1}(g_t) = g(t).$$

<u>Second change</u>: we replace $(*)_{12}$ and the line before, the line after it and $(*)_{13}$ by:
Define $g' \in {}^{Y_2}\mathrm{Ord}$ by $g'(t) = \mathrm{rk}_{D_1+A_*}(g'_t)$.
Now

$(*)'_{12}$ if $t \in B_*$ then
(a) $g'_t <_{D_1+A_*} g_t$, by $(*)_{11}$
(b) $\mathrm{rk}_{D_1+A_*}(g'_t) < \mathrm{rk}_{D_1+A_*}(g_t)$ by (a) and 1.8(3),
(c) $\mathrm{rk}_{D_1+A_*}(g_t) = \mathrm{rk}_{D_1}(g_t)$ recalling $(*)'_5(d)$ and $J^1_t = J[g_t, D_1]$
(d) $\mathrm{rk}_{D_1}(g_t) = g(t)$ by clause (d) of $\oplus$
(e) $\mathrm{rk}_{D_1+A_*}(g_t) = g(t)$ by (c), (d) above hence

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(*f*) $g'(t) < g(t)$ by the choice of g', clause (b) and clause (e).

Hence by $(*)'_{12}(f)$ we have

$(*)'_{13}$ $g' < g \mod D_2 + B_*$.

Concerning the rest, we quote $(*)_5(b)$ twice but $(*)'_5(b) = (*)_5(b)$, and quote $(*)_5(a)$ twice but noted there that $(*)'_5(a)$ suffice and g' is defined before $(*)'_{12}$ rather than apply $(*)_{12}$. $\square_{2.9}$

Claim 2.10. *1) If D is a filter on Y and $A \in D^+$ and $f \in {}^Y\text{Ord}$ <u>then</u> for some $g \in {}^Y\text{Ord}$ we have $J[g,D] = J[f, D+A]$ and $g = f \mod D+A$.*
2) Assume D_ℓ is an $\aleph_1$-complete filter on Y_ℓ and $C_\ell \in D^+_\ell$ and $E_\ell = D_\ell + C_\ell$, for $\ell = 1,2$ and $\iota \in \{1,\dots,4\}$. If $\boxplus^\iota_{D_1,D_2}$ then $\boxplus^\iota_{E_1,E_2}$.

Proof. 1) Let $\alpha = \text{rk}_D(f)$ and define $g \in {}^Y\text{Ord}$ by: $g(s)$ is $f(s)$ if $s \in A$ and is $\alpha+1$ if $r \in Y \backslash A$.
2) The proof splits to cases.

<u>Case 1</u>: $\iota = 1$

Let $A \in E_1, \bar B = \langle B_s : s \in A\rangle \in {}^A(E_2)$ and we should find A_*, B_* as promised in $\boxplus^\iota_{E_1,E_2}$. Let $A' = A \cup (Y_1 \backslash C_\ell), \bar B' = \langle B'_s : s \in A'\rangle$ where B'_s is $B_s \cup (Y_2 \backslash C_2)$ if $s \in A$ and Y_2 if $s \in A' \backslash A = (Y_1 \backslash C_1) \backslash A$. Now we apply $\boxplus^\iota_{D_1,D_2}$ to $A', \langle B'_s : s \in A'\rangle$ and get $(A'_*, B'_*) \in (D_1 \times D_2)$. Let $A_* = A'_* \cap C_1, B_* = B'_* \cap C_2$ and we shall show that A_*, B_* are as required, clearly $A_* \in E_1, B_* \in E_2$. Now if $(s,t) \in A_* \times B_*$, then $(s,t) \in A'_* \times B'_*$ hence $s \in A' \wedge t \in B'_s$ (by the choice of A'_*, B'_*).

Recalling $(s,t) \in A_* \times B_*$, first $s \in A' \cap A_* \backslash A'$ and $(A'_* \cap C_1) = A' \backslash (Y_1 \backslash C_1) \subseteq A$ and second, $t \in B'_s \cap B_* = B'_s \cap (B'_* \cap C_2) \subseteq B'_s \backslash (Y_2 \backslash C_2) \subseteq B_s$, as required.

<u>Case 2</u>: $\iota = 2$

So let $A \in E_1, \bar B = \langle B_s : s \in A\rangle \in {}^A(E_2), f_2 \in {}^{Y_2}\text{Ord}$ and $J_2 = J[f_2, E_2]$ and we should find A_*, B_*.

Let $\beta_* = \text{rk}_{E_2}(f_2)$ and define $f'_2 \in {}^{Y_2}\text{Ord}$ by $f'_2(t)$ is $f_2(t)$ if $t \in C_2$ and is $\beta_* + 1$ if $t \in Y \backslash C_2$.

By part (1)

(1) $f'_2 = f_2 \mod E_2$ hence $J_2(f'_2, E_2) = J_2[f_2, E_2]$
(2) $J_2[f'_2, D_2] = J_2[f'_2, E_2]$.

The rest is like Case 1 only easier.

<u>Case 3</u>: $\iota = 3$

Let $\alpha_* = \text{rk}_{E_1}(f_1)$ and $f'_1 \in {}^{Y_1}\text{Ord}$ be defined by $f'_1(s)$ is $f_1(s)$ if $s \in C_1$ and $\alpha_* + 1$ if $s \in Y_1 \backslash C_1$. We continue as in Case 2.

<u>Case 4</u>: $\iota = 4$

Easy by part (1). $\square_{2.10}$

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§ 3. Rank systems and A Relative of GCH

To phrase our theorem we need to define the framework.

Definition 3.1. Main Definition: We say that $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu) = (\mathbb{D}_{\mathbf{p}}, \mathrm{rk}_{\mathbf{p}}, \Sigma_{\mathbf{p}}, \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{p}})$ is a weak (rank) 1-system when:

(a) μ is singular

(b) each $\mathbf{d} \in \mathbb{D}$ is (or just we can compute from it) a pair $(Y, D) = (Y_{\mathbf{d}}, D_{\mathbf{d}}) = (Y[\mathbf{d}], D_{\mathbf{d}}) = (Y_{\mathbf{p},\mathbf{d}}, D_{\mathbf{p},\mathbf{d}})$ such that:
 (α) $\mathrm{hrtg}(Y_{\mathbf{d}}) < \mu$, on $\mathrm{hrtg}(-)$ see Definition 1.5
 (β) $D_{\mathbf{d}}$ is a filter on $Y_{\mathbf{d}}$

(c) for each $\mathbf{d} \in \mathbb{D}$, a definition of a function $\mathrm{rk}_{\mathbf{d}}(-)$ with domain ${}^{Y[\mathbf{d}]}\mathrm{Ord}$ and range $\subseteq \mathrm{Ord}$, that is $\mathrm{rk}_{\mathbf{p},\mathbf{d}}(-)$ or $\mathrm{rk}^{\mathbf{p}}_{\mathbf{d}}(-)$

(d) (α) Σ is a function with domain $\mathbb{D}$ such that $\Sigma(\mathbf{d}) \subseteq \mathbb{D}$
 (β) if $\mathbf{d} \in \mathbb{D}$ and $\mathbf{e} \in \Sigma(\mathbf{d})$ then $Y_{\mathbf{e}} = Y_{\mathbf{d}}$ [natural to add $D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$,

 this is not demanded but see 3.8(2)]

(e) (α) $\mathbf{j}$ is a function from $\mathbb{D}$ onto $\mathrm{cf}(\mu)$
 (β) let $\mathbb{D}_{\geq i} = \{\mathbf{d} \in \mathbb{D} : \mathbf{j}(\mathbf{d}) \geq i\}$ and $\mathbb{D}_i = \mathbb{D}_{\geq i} \backslash \mathbb{D}_{i+1}$
 (γ) $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow \mathbf{j}(\mathbf{e}) \geq \mathbf{j}(\mathbf{d})$

(f) for every $\sigma < \mu$ for some $i < \mathrm{cf}(\mu)$, if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where:
 ($*$) we say that $\mathbf{d}$ is $(\mathbf{p}, \leq X)$-complete (or $(\leq X)$-complete for $\mathbf{p}$) when: if $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\zeta = \mathrm{rk}_{\mathbf{d}}(f)$ and $\langle A_j : j \in X\rangle$ a partition[2] of $Y_{\mathbf{d}}$, then for some $\mathbf{e} \in \Sigma(\mathbf{d})$ and $j \in X$ we have $A_j \in D_{\mathbf{e}}$ and $\zeta = \mathrm{rk}_{\mathbf{e}}(f)$; so this is not the same as "$D_{\mathbf{d}}$ is $(\leq X)$-complete"; we may say $(\mathbf{p}, |X|^+)$-complete, or $(\mathbf{p}, < |X|^+)$-complete

(g) no hole[3]: if $\mathrm{rk}_{\mathbf{d}}(f) > \zeta$ then for some pair $(\mathbf{e}, g)$ we have: $\mathbf{e} \in \Sigma(\mathbf{d})$ and $g <_{D[\mathbf{e}]} f$ and $\mathrm{rk}_{\mathbf{e}}(g) = \zeta$

(h) if $f = g + 1 \mod D_{\mathbf{d}}$ then $\mathrm{rk}_{\mathbf{d}}(f) = \mathrm{rk}_{\mathbf{d}}(g) + 1$

(i) if $f \leq g \mod D_{\mathbf{d}}$ then $\mathrm{rk}_{\mathbf{d}}(f) \leq \mathrm{rk}_{\mathbf{d}}(g)$.

Definition 3.2. 1) We say $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ is a weak (rank) 2-system, (if we write system we mean 2-system) when in 3.1 we replace clauses (d),(f),(g) by:

(d)′ (α) Σ is a function with domain $\mathbb{D}$
 (β) for $\mathbf{d} \in \mathbb{D}$ we have $\Sigma(\mathbf{d}) \subseteq \{(\mathbf{e}, h) : \mathbf{e} \in \mathbb{D}_{\geq \mathbf{j}(\mathbf{d})}$ and $h : Y_{\mathbf{e}} \to Y_{\mathbf{d}}\}$; writing $\mathbf{e} \in \Sigma(\mathbf{d})$ means then $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ for some function h

(f)′ for every $\sigma < \mu$ for some $i < \mathrm{cf}(\mu)$, if $\mathbf{d} \in \mathbb{D}_{\geq i}$, then $\mathbf{d}$ is $(\mathbf{p}, \leq \sigma)$-complete where:

[2] as long as X (above: σ) is a well ordered set it does not matter whether we use a partition or just a covering, i.e. $\cup\{A_j : j \in X\} = Y_{\mathbf{d}}$

[3] we may use another function Σ here, as in natural examples here we use $\Sigma(\mathbf{d}) = \{\mathbf{d}\}$ and not so in clause (f)

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(∗) we say that $\mathbf{d}$ is $(\mathbf{p}, \le X)$-complete (for $\mathbf{p}$) when: if $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\zeta = \mathrm{rk}_{\mathbf{d}}(f)$ and $\langle A_j : j \in X\rangle$ a partition[4] of $Y_{\mathbf{d}}$, then for some $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $j \in X$ we have $h^{-1}(A_j) \in D_{\mathbf{e}}$ and $\zeta = \mathrm{rk}_{\mathbf{e}}(f \circ h)$; we define "$(\mathbf{p}, |X|^+)$-complete" similarly

$(g)'$ no hole: if $\mathrm{rk}_{\mathbf{d}}(f) > \zeta$ then for some $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $g \in {}^{Y[\mathbf{e}]}\mathrm{Ord}$ we have $g < f \circ h \mod D_{\mathbf{e}}$ and $\mathrm{rk}_{\mathbf{e}}(g) = \zeta$.

Definition/Claim 3.3. Let $\mathbf{p}$ be a weak rank 1-system; we can define $\mathbf{q}$ and prove it is a weak rank 2-system by $\mathbb{D}_{\mathbf{q}} = \mathbb{D}_{\mathbf{p}}$, $\mathrm{rk}_{\mathbf{q}} = \mathrm{rk}_{\mathbf{p}}, \Sigma_{\mathbf{q}}(\mathbf{d}) = \{(\mathbf{e}, \mathrm{id}_{Y[\mathbf{d}]}) : \mathbf{e} \in \Sigma_{\mathbf{p}}(\mathbf{d})\}, \mathbf{j}_{\mathbf{q}} = \mathbf{j}_{\mathbf{p}}, \mu_{\mathbf{q}} = \mu_{\mathbf{p}}$.

Convention 3.4. 1) We use $\mathbf{p}$ only for systems as in Definition 3.1 or 3.2.
2) We may not distinguish $\mathbf{p}$ and $\mathbf{q}$ in 3.3 so deal only with 2-systems.

Remark 3.5. The following is an alternative to Definition 3.2. As in 3.1 we can demand $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow Y_{\mathbf{e}} = Y_{\mathbf{d}}$ but for every $\mathbf{d}$ we have a family $\mathscr{E}_{\mathbf{d}}$, i.e. the function $\mathbf{d} \mapsto \mathscr{E}_{\mathbf{d}}$ is part of $\mathbf{p}$ and make the following additions and changes:

(α) $\mathscr{E}_{\mathbf{d}}$ is a family of equivalence relations on $Y_{\mathbf{d}}$
(β) we replace ${}^{Y[\mathbf{d}]}\mathrm{Ord}$ by $\{f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$: $\mathrm{eq}(f) := \{(s,t) : s,t \in Y_{\mathbf{d}}$ and $f(s) = f(t)\} \in \mathscr{E}_{\mathbf{d}}\}$
(γ) if E_1, E_2 are equivalence relations on $Y_{\mathbf{d}}$ such that E_2 refines E_1 then , $E_2 \in \mathscr{E}_{\mathbf{d}} \Rightarrow E_1 \in \mathscr{E}_{\mathbf{d}}$
(δ) if $\mathbf{e} \in \Sigma(\mathbf{d})$ then $Y_{\mathbf{e}} = Y_{\mathbf{d}}$ and $\mathscr{E}_{\mathbf{d}} \subseteq \mathscr{E}_{\mathbf{e}}$.

Definition 3.6. For $\iota = 1, 2$ we say that $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ is a strict ι-system when it satisfies clauses (a)-(i) from 3.1 or from 3.2 and

(j) for every $\mathbf{d} \in \mathbb{D}$ and ζ, ξ, f, j_0 satisfying ⊞ below, there[5] is $j < \mathrm{cf}(\mu)$ such that: there are no $\mathbf{e}, g$ satisfying ⊕ below, where:
⊕ •$_1$ $\mathbf{e} \in \mathbb{D}_{\ge j}$
•$_2$ $g \in {}^{Y[\mathbf{e}]}\zeta$
•$_3$ $\{g(t) : t \in Y_{\mathbf{e}}\} \subseteq [\xi, \zeta_*)$ for some $\zeta_* < \zeta$
•$_4$ $j \ge j_0$
•$_5$ $\mathrm{rk}_{\mathbf{e}}(g) \ge \zeta$,
⊞ •$_1$ $f \in {}^{Y[\mathbf{d}]}\xi$
•$_2$ $\mathrm{rk}_{\mathbf{d}}(f) = \zeta$
•$_3$ $\xi < \zeta$
•$_4$ $\mathrm{cf}(\mu) = \mathrm{cf}(\zeta)$
•$_5$ $j_0 < \mathrm{cf}(\mu)$
•$_6$ $s \in Y_{\mathbf{d}} \wedge \mathbf{e}' \in \mathbb{D}_{\ge j_0} \Rightarrow \mathrm{rk}_{\mathbf{e}'}(f(s)) = f(s)$.

Observation 3.7. *1) If* $\mathbf{p}$ *is a strict ι-system then* $\mathbf{p}$ *is a weak ι-system.*
2) In Definition 3.6, from (j)⊞•$_6$ recalling (j)⊕•$_1$+•$_4$ we can deduce: $\mathrm{rk}_{\mathbf{e}}(f(s)) = f(s)$ *for* $s \in Y_{\mathbf{d}}$.
3) In ⊕•$_5$ of (j) of 3.6 without loss of generality $\mathrm{rk}_{\mathbf{e}}(y) > \zeta + 7$ *as we can use* $g + 7$.

[4]as long as X (above: σ) is a well ordered set it does not matter whether we use a partition or just a covering, i.e. $\cup\{A_j : j \in X\} = Y_{\mathbf{d}}$

[5]can we make j depend on f (and a partition of) $Y_{\mathbf{d}}$? Anyhow later we use $\mathbf{d}' \in \Sigma(\mathbf{d})$, if $\{\mathbf{d}\} \ne \Sigma(\mathbf{d})$? Also so $\iota = 1, 2$ may make a difference.

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Definition 3.8. 1) We say that a weak ι-system $\mathbf{p}$ is weakly normal <u>when</u>:

- $\bullet_1$ in $(d)(\beta)$ of Definition 3.1 we add $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow D_{\mathbf{d}} \subseteq D_{\mathbf{e}}$
- $\bullet_2$ in $(d)'(\beta)$ of Definition 3.2 we add: if $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ then $(\forall A \in D_{\mathbf{d}})(h^{-1}(A) \in D_{\mathbf{e}})$.

2) We say $\mathbf{p}$ is normal <u>when</u> it is weakly normal and

- $\bullet_3$ in Definition 3.1, if $A \in D^+_{\mathbf{d}}, \mathbf{d} \in \mathbb{D}, f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\zeta = \mathrm{rk}_{\mathbf{d}}(f)$ <u>then</u> for some $\mathbf{e} \in \Sigma(\mathbf{d})$ we have $A \in D_{\mathbf{e}}$ and $\mathrm{rk}_{\mathbf{e}}(f) = \zeta$
- $\bullet_4$ in Definition 3.2, if $\mathbf{d} \in \mathbb{D}, f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$, $\mathrm{rk}_{\mathbf{d}}(f) = \zeta$ and $A \in D^+_{\mathbf{d}}$, then for some $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ we have $\{s \in Y_{\mathbf{e}} : h(s) \in A\} \in D_{\mathbf{e}}$ and $\mathrm{rk}_{\mathbf{e}}(f \circ h) = \zeta$.

3) We say $\mathbf{p}$ is semi-normal <u>when</u> it is weakly normal and we have $\bullet'_3, \bullet'_4$ holds where they are as above just ending with $\geq \zeta$.

Claim 3.9. *Assume $\mathbf{p}$ is a weak ι-system and $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$.*
0) If $\mathbf{p}, \mathbf{q}$ are as in 3.3, <u>then</u> $\mathbf{p}$ is [weakly] normal iff $\mathbf{q}$ is.
1) If $f, g \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $f <_{D_{\mathbf{d}}} g$ <u>then</u> $\mathrm{rk}_{\mathbf{d}}(f) < \mathrm{rk}_{\mathbf{d}}(g)$.
2) If $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ and $\mathrm{rk}_{\mathbf{d}}(f) > 0$ <u>then</u> $\{s \in Y_{\mathbf{d}} : f(s) > 0\} \in D^+_{\mathbf{d}}$.
2A) If in addition $\mathbf{p}$ is semi-normal then $\{s \in Y_{\mathbf{d}} : f(s) = 0\} = \emptyset \bmod D_{\mathbf{d}}$.
3) $\mathrm{rk}_{\mathbf{d}}(f)$ depends just on $f/D_{\mathbf{d}}$ (and $\mathbf{d}$ and, of course, $\mathbf{p}$).

Proof. 0) Easy; note that by this part, below without loss of generality $\iota_{\mathbf{p}} = 2$.
1) Let $f_1 \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ be defined by $f_1(s) = f(s) + 1$. So clearly $f_1 \leq_{D_{\mathbf{d}}} g$ hence by clause (i) of 3.1 (equivalently 3.2) we have $\mathrm{rk}_{\mathbf{d}}(f_1) \leq \mathrm{rk}_{\mathbf{d}}(g)$. Also $f_1 = f + 1 \bmod D_{\mathbf{d}}$ hence by clause (h) of 3.1 (equivalently 3.2) we have $\mathrm{rk}_{\mathbf{d}}(f_1) = \mathrm{rk}_{\mathbf{d}}(f) + 1$. The last two sentences together give the desired conclusion.
2) Toward contradiction assume the conclusion fails. Let $f' \in {}^{Y_{\mathbf{d}}}\mathrm{Ord}$ be constantly zero, so $f = f' \bmod D_{\mathbf{d}}$ hence by part (3) we have $\mathrm{rk}_{\mathbf{d}}(f') = \mathrm{rk}_{\mathbf{d}}(f) > 0$. By clause $(g)'$ of Definition 3.2, the "no hole" applied to $(f', \mathbf{d})$, there is a triple $(\mathbf{e}, h, g)$ as there, so $B := \{s : s \in Y_{\mathbf{e}}$ and $g(s) < f(h(s))\} \in D_{\mathbf{e}}$, i.e. $\{s \in Y_{\mathbf{e}} : g(s) < 0\} \in D_{\mathbf{e}}$, contradiction.
2A) Let $A = \{s \in Y_{\mathbf{a}} : f(s) = 0\}$, so toward contradiction assume $A \in D^+_{\mathbf{d}}$. As $\mathbf{p}$ is semi-normal we can find a pair $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ as in $\bullet'_4$ of 3.8(3) so $A_1 = \{t \in Y_{\mathbf{e}} : h_1(t) \in A\} \in D_{\mathbf{e}}$ and $\mathrm{rk}_{\mathbf{e}_1}(f \circ h) \geq \mathrm{rk}_{\mathbf{d}}(f) > 0$, but clearly $\{t \in Y_{\mathbf{e}} : (f \circ h)(t) = 0\} \in D_{\mathbf{e}}$, contradiction by part (2).
3) Use clause (i) of Definition 3.1 twice. $\square_{3.9}$

Theorem 3.10. [ZF] *Assume that $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ is a strict rank 1-system (see Main Definition 3.1) or just a strict 2-system. <u>Then</u> for every ordinal ζ there is $i < \mathrm{cf}(\mu)$ such that: if $\mathbf{d} \in \mathbb{D}_{\geq i}$ <u>then</u> $\mathrm{rk}_{\mathbf{d}}(\zeta) = \zeta$, i.e. $\mathrm{rk}_{\mathbf{d}}(\langle \zeta : s \in Y_{\mathbf{d}} \rangle) = \zeta$.*

Proof. We shall use the notation:

- $\odot_0$ If there is an i as required in the theorem for the ordinal ζ <u>then</u> let $\mathbf{i}(\zeta)$ be the minimal such i (otherwise, $\mathbf{i}(\zeta)$ is not well defined).

Without loss of generality,

- $\odot_1$ every $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ is $(\mathbf{p}, \leq (\mathrm{cf}(\mu)))$-complete, i.e. clause (f) of 3.1 for $\sigma_* := \mathrm{cf}(\mu)^+$ holds for every $\mathbf{d} \in \mathbb{D}$.

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[Why? Let i_* be the $i <$ cf(μ) which exists by clause (f) of Definition 3.1, 3.2 for σ_* as μ is singular (see clause (a) of Definition 3.1). Now we just replace $\mathbb{D}$ by $\mathbb{D}_{\ge i_*}$ (and $\mathbf{j}$ by $\mathbf{j}{\restriction}\mathbb{D}_{\ge i_*}$, etc).]

Clearly we have

$\odot_2$ $\text{rk}_{\mathbf{d}}(\zeta) \ge \zeta$ for ζ an ordinal and $\mathbf{d} \in \mathbb{D}$.

[Why? We can prove this by induction on ζ for all $\mathbf{d} \in \mathbb{D}$, by clauses (h) + (i) of Definition 3.1.]

As a warmup we shall note that:

$\odot_3$ if $\mathbf{d} \in \mathbb{D}$ and $\zeta < \sigma_*$ or just $\mathbf{d} \in \mathbb{D}$ and is hereditarily $(\mathbf{p}, \le \zeta)$-complete which means that every $\mathbf{e}$ in the Σ-closure of $\{\mathbf{d}\}$ is $(\mathbf{p}, \le \zeta)$-complete <u>then</u>:
- (α) $\text{rk}_{\mathbf{d}}(\zeta) = \zeta$
- (β) $f \in {}^{Y[\mathbf{d}]}\zeta \Rightarrow \text{rk}_{\mathbf{d}}(f) < \zeta$.

[Why? Note that as $\zeta < \sigma_*$ clearly $\mathbf{d}$ is $(\mathbf{p}, \le \zeta)$-complete by $\odot_1$ and clause (f) of 3.1, so we can assume that $\mathbf{d}$ is hereditarily $(\mathbf{p}, \le \zeta)$-complete. We prove the statement inside $\odot_3$ by induction on the ordinal ζ (for all hereditarily $(\mathbf{p}, \le \zeta)$-complete $\mathbf{d} \in \mathbb{D}$). Note that for $\varepsilon < \zeta$, "$\mathbf{d}$ is $(\mathbf{p}, \le \zeta)$-complete" implies "$\mathbf{d}$ is $(\mathbf{p}, \le \varepsilon)$-complete", we shall use this freely.

Arriving to ζ, to prove clause (β) let $f \in {}^{Y[\mathbf{d}]}\zeta$ and for $\varepsilon < \zeta$ we define $A_\varepsilon := \{t \in Y_{\mathbf{d}} : f(t) = \varepsilon\}$, so $\langle A_\varepsilon : \varepsilon < \zeta\rangle$ is a well defined partition of $Y_{\mathbf{d}}$ so the sequence exists, hence as "$\mathbf{d}$ is hereditarily $(\mathbf{p}, \le \zeta)$-complete" recalling $(*)$ from clause $(f)'$ of 3.2 for some triple $(\mathbf{e}, h, \varepsilon)$ we have $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $\varepsilon < \zeta$ and $h^{-1}(A_\varepsilon) \in D_{\mathbf{e}}$ and $\text{rk}_{\mathbf{e}}(f \circ h) = \text{rk}_{\mathbf{d}}(f)$.

Now $f \circ h = \langle \varepsilon : t \in Y_{\mathbf{e}}\rangle \mod D_{\mathbf{e}}$ hence by Claim 3.9(3) we have $\text{rk}_{\mathbf{e}}(f \circ h) = \text{rk}_{\mathbf{e}}(\varepsilon)$. But the assumptions on $\mathbf{d}$ holds for $\mathbf{e}$ hence by the induction hypothesis on ζ we know that $\text{rk}_{\mathbf{e}}(\varepsilon) = \varepsilon$ and $\varepsilon < \zeta$ so together $\text{rk}_{\mathbf{d}}(f \circ h) < \zeta$, so clause (β) of $\odot_3$ holds.

To prove clause (α) first consider $\zeta = 0$; if $\text{rk}_{\mathbf{d}}(\zeta) > 0$ by clause (g) of Definition 3.1, 3.2 there are $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ and $g \in {}^{Y[\mathbf{e}]}\text{Ord}$ such that $g < \langle \zeta : t \in Y_{\mathbf{e}}\rangle \mod D_{\mathbf{e}}$, so for some $t \in Y_{\mathbf{e}}$ we have $g(t) < \zeta$ but $\zeta = 0$, contradiction; this is close to 3.9(2).

Second, consider $\zeta > 0$, so by $\odot_2$ we have $\text{rk}_{\mathbf{d}}(\zeta) \ge \zeta$ and assume toward contradiction that $\text{rk}_{\mathbf{d}}(\zeta) > \zeta$, so by clause (g) of Definition 3.1, 3.2 there is a triple $(\mathbf{e}, h, g)$ as there. Now apply clause (β) of $\odot_3$ for ζ (which we have already proved) recalling $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ so also $\mathbf{e}$ is $(\mathbf{p}, \le \zeta)$-complete. We get $\text{rk}_{\mathbf{e}}(g) < \zeta$, a contradiction. So $\odot_3$ indeed holds.]

Now as in the desired equality we have already proved one inequality in $\odot_2$, we need to prove only the other inequality. We do it by induction on ζ.

<u>Case 1</u>: $\zeta < \mu$.

By clause (f) of Definition 3.1, 3.2 for some $i <$ cf(μ) we have $\mathbf{d} \in \mathbb{D} \wedge \mathbf{j}(\mathbf{d}) \ge i \Rightarrow \mathbf{d}$ is $(\mathbf{p}, \le \zeta)$-complete, hence by $\odot_3(\alpha)$ we have $\text{rk}_{\mathbf{d}}(\zeta) = \zeta$, as required.

<u>Case 2</u>: $\zeta = \xi + 1$.

By clause (h) of Definition 3.1 we have $\mathbf{d} \in \mathbb{D} \Rightarrow \text{rk}_{\mathbf{d}}(\zeta) = \text{rk}_{\mathbf{d}}(\xi) + 1$. Hence $\mathbf{d} \in \mathbb{D}_{\ge \mathbf{i}(\xi)} \Rightarrow \text{rk}_{\mathbf{d}}(\zeta) = \text{rk}_{\mathbf{d}}(\xi) + 1 = \xi + 1 = \zeta$, so $\mathbf{i}(\xi)$ exemplifies that $\mathbf{i}(\zeta)$ exists and is $\le \mathbf{i}(\xi)$ so we are done.

<u>Case 3</u>: ζ is a limit ordinal $\ge \mu$ of cofinality $\neq$ cf(μ).

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So for each $\xi < \zeta$ by the induction hypothesis $\mathbf{i}(\xi) < \text{cf}(\mu)$ is well defined. For $i < \text{cf}(\mu)$ let $u_i := \{\xi < \zeta : \mathbf{i}(\xi) \le i\}$, so is well defined; moreover the sequence $\langle u_i : i < \text{cf}(\mu)\rangle$ exists and is $\subseteq$-increasing. If $i < \text{cf}(\mu) \Rightarrow \sup(u_i) < \zeta$ then $\langle \sup(u_i) : i < \text{cf}(\mu)\rangle$ is a $\le$-increasing sequence of ordinals $< \zeta$ with limit ζ. So as $\text{cf}(\zeta) \ne \text{cf}(\mu)$ necessarily for some $i_* < \text{cf}(\mu)$ the set $S := \{\xi : \xi < \zeta \text{ and } \mathbf{i}(\xi) < i_*\}$ is an unbounded subset of ζ. We shall prove that $\mathbf{i}(\zeta)$ is well defined and $\le i_*$.

<u>Subcase 3A</u>: $\text{cf}(\zeta) \ge \mu$.

Let $\mathbf{d} \in \mathbb{D}_{\ge i_*}$ and $g \in {}^{Y[\mathbf{d}]}\zeta$ be given. Clearly $\text{Rang}(g)$ is a subset of ζ of cardinality $< \text{hrtg}(Y_{\mathbf{d}})$ which by clause $(b)(\alpha)$ of Definition 3.2 is $< \mu \le \text{cf}(\zeta)$ hence we can fix $\xi \in S$ such that $\text{Rang}(g) \subseteq \xi$, hence by clause (i) of Definition 3.1, $\text{rk}_{\mathbf{d}}(g) \le \text{rk}_{\mathbf{d}}(\xi)$ but $\mathbf{i}(\xi) = i_*$ and $\mathbf{d} \in \mathbb{D}_{\ge i_*}$ hence $\text{rk}_{\mathbf{d}}(\xi) = \xi < \zeta$ so together $\text{rk}_{\mathbf{d}}(g) < \zeta$. As this holds for every $\mathbf{d} \in \mathbb{D}_{\ge i_*}$ by the no-hole clause $(g)'$ and clause (e)(γ) of Definition 3.2 it follows that $\mathbf{d} \in \mathbb{D}_{\ge i_*} \Rightarrow \text{rk}_{\mathbf{d}}(\zeta) \le \zeta$ as required.

<u>Subcase 3B</u>: $\text{cf}(\zeta) < \mu$ (but still $\text{cf}(\zeta) \ne \text{cf}(\mu)$).

Let $\langle \zeta_\varepsilon : \varepsilon < \text{cf}(\zeta)\rangle$ be an increasing sequence of ordinals from S with limit ζ. Now let $j_* < \text{cf}(\mu)$ be such that $\mathbf{d} \in \mathbb{D}_{\ge j_*} \Rightarrow \mathbf{d}$ is $(\mathbf{p}, \text{cf}(\zeta)^+)$-complete, see clause (f) of Definition 3.1.

Now assume $\mathbf{d} \in \mathbb{D}_{\ge \max\{i_*, j_*\}}$ and $g \in {}^{Y[\mathbf{d}]}\zeta$. For $\varepsilon < \text{cf}(\zeta)$ let $A_\varepsilon = \{t \in Y_{\mathbf{d}} : g(t) < \zeta_\varepsilon \text{ but } j < \varepsilon \Rightarrow g(t) \ge \zeta_j\}$ so $\langle A_\varepsilon : \varepsilon < \text{cf}(\zeta)\rangle$ is well defined and is a partition of $Y_{\mathbf{d}}$. Hence by clause (f) of Definition 3.2 for some $\varepsilon < \text{cf}(\zeta)$ and $(\mathbf{e}, h) \in \Sigma(\mathbf{d})$ we have $h^{-1}(A_\varepsilon) \in D_{\mathbf{e}}$ and $\text{rk}_{\mathbf{d}}(g) = \text{rk}_{\mathbf{e}}(g \circ h)$; but $\mathbf{j}(\mathbf{e}) \ge \mathbf{j}(\mathbf{d}) \ge i_*, j_*$ and by the choice of A_ε and clause (i) of Definition 3.1 the latter is $\le \text{rk}_{\mathbf{e}}(\zeta_\varepsilon)$ hence as $\mathbf{i}(\zeta_\varepsilon) \le i_*$ the latter is $= \zeta_\varepsilon < \zeta$. As this holds for every $\mathbf{d} \in \mathbb{D}_{\ge \max\{i_*, j_*\}}$ and $g \in {}^{Y[\mathbf{d}]}\zeta$, by the no-hole clause $(g)'$ of Definition 3.2 necessarily $\text{rk}_{\mathbf{d}}(\zeta) \le \zeta$. So $\max\{i_*, j_*\} < \text{cf}(\mu)$ is as required, so we are done.

<u>Case 4</u>: $\zeta \ge \mu$ is a limit ordinal such that $\text{cf}(\zeta) = \text{cf}(\mu)$.

Let $\langle \zeta_i : i < \text{cf}(\zeta)\rangle$ be increasing with limit ζ. Assume toward contradiction that for every $i < \text{cf}(\mu)$ there is $\mathbf{d}_i \in \mathbb{D}_{\ge i}$ such that $\text{rk}_{\mathbf{d}_i}(\zeta) > \zeta$ <u>but</u> we do not assume that such a sequence $\langle \mathbf{d}_i : i < \text{cf}(\mu)\rangle$ exists. Choose such $\mathbf{d}_0$; as $\text{rk}_{\mathbf{d}_0}(\zeta) > \zeta$, clearly there are $f_0 \in {}^{Y[\mathbf{d}'_0]}\zeta$ and a member $\mathbf{d}'_0$ of $\Sigma(\mathbf{d}_0)$, though not necessarily $Y_{\mathbf{d}'_0} = Y_{\mathbf{d}}$, such that

$\odot_4$ $\text{rk}_{\mathbf{d}'_0}(f_0) = \zeta$

[Why? By using clause $(g)'$ of Definition 3.2.]

Note

$\odot_5$ $\mathbf{i}(f_0(t))$ is well defined for every $t \in Y_{\mathbf{d}'_0}$.

[Why holds? Because $f_0(t) < \zeta$ and the induction hypothesis.]

For $j_1 < \text{cf}(\zeta), j_0 < \text{cf}(\mu)$ let $A_{j_1,j_0} = \{t \in Y_{\mathbf{d}'_0} : f_0(t) < \zeta_{j_1} \text{ and } (\forall j < j_1)(f_0(t) \ge \zeta_j) \text{ and } \mathbf{i}(f_0(t)) = j_0\}$. By clause $(f)'$ of 3.2 applied to the pair $(\mathbf{d}'_0, f_0)$ and the partition $\langle A_{j_1,j_0} : j_1 < \text{cf}(\zeta), j_0 < \text{cf}(\mu)\rangle$, for some $(\mathbf{d}_*, h_*) \in \Sigma(\mathbf{d}'_0)$ we have $\text{rk}_{\mathbf{d}_*}(f_0 \circ h_*) = \zeta$ and for some j_0, j_1 we have $h_*^{-1}(A_{j_1,j_0}) \in D_{\mathbf{d}_*}$. By 3.9(3) for some $f = f_0 \circ h_*$ mod $D_{\mathbf{d}_*}$ and letting $\mathbf{d} := \mathbf{d}_*$ we have

$\odot_6$ (a) $\mathbf{d} \in \mathbb{D}$

(b) $f \in {}^{Y[\mathbf{d}]}\text{Ord}$

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(c) $\quad \mathrm{rk}_{\mathbf{d}}(f) = \zeta$

(d) $\quad t \in Y_{\mathbf{d}} \Rightarrow \mathbf{i}(f(t)) = j_0 \wedge f(t) < \zeta_{j_1} \wedge (\forall j < j_1)(f(t) \geq \zeta_j)$.

Next

$\odot_7$ letting $\xi := \zeta_{j_1}$, clause $\boxplus$ from 3.6 for $(\mathbf{d}, \zeta, \xi, f, j_0)$.

[Why? We check the six demands

- $\bullet_1$ "$f \in {}^{Y[\mathbf{d}]}\xi$" which holds by $\odot_6(b) + (d)$
- $\bullet_2$ "$\mathrm{rk}_{\mathbf{d}}(f) = \zeta$" which holds by $\odot_6(c)$
- $\bullet_3$ "$\xi < \zeta$" which holds as $(\forall i < \mathrm{cf}(\mu))(\zeta_i < \zeta)$
- $\bullet_4$ "$\mathrm{cf}(\zeta) = \mathrm{cf}(\mu)$" which holds by the case assumption
- $\bullet_5$ $j_0 < \mathrm{cf}(\mu)$ obvious
- $\bullet_6$ $s \in Y_{\mathbf{d}} \wedge \mathbf{e}' \in \mathbb{D}_{\geq j_0} \Rightarrow \mathrm{rk}_{\mathbf{e}'}(f(s)) = f(s)$ holds by $\odot_6(d)$.

So $\odot_7$ indeed holds.]

Now by $\odot_7$, clause (j) of Definition 3.6(1) applied with $\mathbf{d}, \zeta, \xi = \zeta_{j_1}, f, j_0$ here standing for $\mathbf{d}, \zeta, \xi, f, j_0$ there, we can find j as there. Let $i_2 = \max\{j, j_1, j_0, \mathbf{i}(\zeta_{j_1})\}$ so $i_2 < \mathrm{cf}(\mu)$ and choose $\mathbf{e}_0 \in \mathbb{D}_{\geq i_2}$ such that $\mathrm{rk}_{\mathbf{e}_0}(\zeta) > \zeta$ as in the beginning of the case. As $\mathrm{rk}_{\mathbf{e}_0}(\zeta) > \zeta$ by clause $(g)'$ of 3.2 there are $\mathbf{e}_1 \in \Sigma(\mathbf{e}_0)$ and $g \in {}^{Y[\mathbf{e}_1]}\zeta$ such that $\mathrm{rk}_{\mathbf{e}_1}(g) \geq \zeta$ so $g < \langle \zeta : t \in Y_{\mathbf{e}_1} \rangle$. Now without loss of generality

$\odot_8$ (a) $\quad \mathrm{rk}_{\mathbf{e}_1}(g) = \zeta + 1$

(b) $\quad \zeta_* = \sup\{g(t) + 1 : t \in Y_{\mathbf{e}_1}\} < \zeta$

(c) $\quad \mathbf{j}(\mathbf{e}_1) \geq i_2$.

[Why? Because we can use $g' \in {}^{Y[\mathbf{e}_1]}\zeta$ defined by $g'(t) = g(t) + 2$ for $t \in Y_{\mathbf{e}_1}$, by clause (h) of 3.1, $\mathrm{rk}_{\mathbf{e}_1}(g') = \mathrm{rk}_{\mathbf{e}_1}(g) + 2 > \zeta$. By clause $(e)(\gamma)$ we have $\mathbf{j}(\mathbf{e}_1) \geq \mathbf{j}(\mathbf{e}_0) \geq i_2$. Now we find $(\mathbf{d}_2'', h'') \in \Sigma(\mathbf{e}_1)$ and g_2 as in the proof of $\odot_6$ and rename.]

Also without loss of generality

$\odot_9$ $t \in Y_{\mathbf{e}_1} \Rightarrow g(t) \geq \zeta_{j_1}$.

[Why? Let $A_0 = \{t \in Y_{\mathbf{e}_1} : g(t) < \zeta_{j_1}\}, A_1 = \{t \in Y_{\mathbf{e}_1} : g(t) \geq \zeta_{j_1}\}$ so by clause $(f)'$ of 3.2 for some pair $(\mathbf{e}_2, h) \in \Sigma(\mathbf{e}_1)$ we have $\mathrm{rk}_{\mathbf{e}_2}(g \circ h) = \mathrm{rk}_{\mathbf{e}_1}(g) = \zeta + 1$ and $(h^{-1}(A_0) \in D_{\mathbf{e}_2}) \vee (h^{-1}(A_1) \in D_{\mathbf{e}_2})$. So if $h^{-1}(A_0) \in D_{\mathbf{e}_2}$ then by clause (i) of 3.1, $\mathrm{rk}_{\mathbf{e}_1}(g \circ h) \leq \mathrm{rk}_{\mathbf{e}_2}(\xi)$ but $\mathbf{i}(\xi)$ is well defined $\leq i_2 \leq \mathbf{j}(\mathbf{e}_1) \leq \mathbf{j}(\mathbf{e}_2)$ so $\mathrm{rk}_{\mathbf{e}_2}(\xi) = \xi$ together $\mathrm{rk}_{\mathbf{e}_2}(g \circ h) \leq \xi$ contradicting the previous sentence. Hence $h^{-1}(A_0) \notin D_{\mathbf{e}_2}$ so $h^{-1}(A_1) \in D_{\mathbf{e}_2}$. Let $g' \in {}^{Y[\mathbf{e}_2]}\mathrm{Ord}$ be defined by $g'(t)$ is $(g \circ h)(t)$ if $t \in h^{-1}(A_1)$ and is $\zeta_{j_1} + 1$ if $t \in h^{-1}(A_0)$. By Claim 3.9(3) we have $\mathrm{rk}_{\mathbf{e}_2}(g') = \mathrm{rk}_{\mathbf{e}_2}(g \circ h)$ so $(\mathbf{e}_2, g')$ satisfies all requirements on the pair $(\mathbf{e}_1, g)$ and $t \in Y_{\mathbf{e}_2} \Rightarrow g'(t) \geq \zeta_{j_1} > 0$, so we have justified the non-loss of generality.]

Recall $\xi := \zeta_{j_1}$ and let $\mathbf{e} = \mathbf{e}_1$. By the choice of j after $\odot_6$, i.e. as in clause (j) of 3.6, recalling $\mathbf{e} \in \mathbb{D}_{\geq j}$ we shall get a contradiction to the choice of $(\mathbf{d}, \xi, \zeta, f, j_0, \mathbf{e}, g, j)$. To justify it we have to recall by $\odot_7$ that the quintuple $(\mathbf{d}, \zeta, \xi, f, j_0)$ satisfies $\boxplus$ of 3.6(j) and then we prove that the triple $(\mathbf{e}, g, j)$ satisfies $\oplus$ of 3.6(j).

Now $\oplus$ of 3.6 says:

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$\bullet_1$ "$\mathbf{e} \in \mathbb{D}_{\geq j}$" as
as $j \geq i_2, \mathbf{e}_0 \in \mathbb{D}_{\geq i_2}$ and $\mathbf{e} = \mathbf{e}_1 \in \Sigma(\mathbf{e}_0)$

$\bullet_2$ "$g \in {}^{Y[\mathbf{e}]}\zeta$"
which holds as $g \in {}^{Y[\mathbf{e}]}\zeta$

$\bullet_3$ "$g(t) \in [\xi, \zeta_*)$"
holds as $g(t) < \zeta$ by $\bullet_2 + \odot_8(b)$ and $g(t) \geq \zeta_{j_1} = \xi$ by $\odot_9$

$\bullet_4$ "$j \geq j_0$"
holds as $j \geq i_2 \geq j_0$

$\bullet_5$ "$\mathrm{rk}_{\mathbf{e}}(g) > \zeta$"
holds by $\odot_8(a)$.

So we really get a contradiction. $\square_{3.10}$

Definition 3.11. 1) We say that the pair $(\mathbf{d}, \mathbf{e})$ commute (or 6-commute) for $\mathbf{p}$ <u>when</u> $\mathbf{d}, \mathbf{e} \in \mathbb{D}_{\mathbf{p}}$ and $\mathrm{rk}_{\mathbf{d}}(f) \geq \mathrm{rk}_{\mathbf{e}}(g)$ whenever $(f, g, \bar{f}, \bar{g})$ is a $(\mathbf{p}, \mathbf{d}, \mathbf{e})$-rectangle, see below; fixing f, g we may say $(\mathbf{d}, \mathbf{e})$ commute for f, g.
2) We say that $(\mathbf{d}, \mathbf{e}, f, g, \bar{f}, \bar{g})$ is $\mathbf{p}$-rectangle or $(f, g, \bar{f}, \bar{g})$ is a $(\mathbf{p}, \mathbf{d}, \mathbf{e})$-rectangle <u>when</u>:

$\circledast$ (a) $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$
(b) $\mathbf{e} \in \mathbb{D}_{\mathbf{p}}$
(c) $\bar{g} = \langle g_t : t \in Y_{\mathbf{e}} \rangle$ and $g_t \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ for $t \in Y_{\mathbf{e}}$
(d) $g \in {}^{Y[\mathbf{e}]}\mathrm{Ord}$ is defined by $g(t) = \mathrm{rk}_{\mathbf{d}}(g_t)$
(e) $f_s \in {}^{Y[\mathbf{e}]}\mathrm{Ord}$ is defined by $f_s(t) = g_t(s)$
(f) $\bar{f} = \langle f_s : s \in Y[\mathbf{d}] \rangle$
(g) $f \in {}^{Y[\mathbf{d}]}\mathrm{Ord}$ is defined by $f(s) = \mathrm{rk}_{\mathbf{e}}(f_s)$.

Claim 3.12. *[Assume* ZF + $\mathrm{AC}_{<\mu}$*] If* $\mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{i}, \mu)$ *be a weak rank 1-system* <u>*then*</u> $\mathbf{p}$ *is a strict rank 1-system* <u>*when*</u> *there is a function* Σ_1 *such that (and we may say* Σ_1 *witness it):*

$(*)_0$ Σ_1 *a function with domain* $\mathbb{D}$
$(*)_1$ $\Sigma_1(\mathbf{d}) \subseteq \Sigma(\mathbf{d})$ *is non-empty for* $\mathbf{d} \in \mathbb{D}$
$(*)_2$ *for every* $\mathbf{d}, \zeta, \xi, f, j_0$ *satisfying* $\boxplus$ *of 3.6, for some* $j < \mathrm{cf}(\mu)$ *for every* $\mathbf{e} \in \mathbb{D}_{\geq j}$ *we have*
  (a) $\mathbf{e}$ *is* $(\mathbf{p}, \leq \Sigma_1(\mathbf{d}))$*-complete*
  (b) *if* $\mathbf{d}_* \in \Sigma_1(\mathbf{d}), \mathbf{e}_* \in \Sigma_1(\mathbf{e})$ *then* $(\mathbf{d}_*, \mathbf{e}_*)$ *commute (for* $\mathbf{p}$*) see 3.11*
$(*)_3$ *we strengthen clause (g) of Definition 3.1 to*
  $(g)^+$ *add:* $\mathrm{rk}_{\mathbf{e}}(f) = \mathrm{rk}_{\mathbf{d}}(f)$ *and* $\mathbf{e} \in \Sigma_1(\mathbf{d})$
$(*)_4$ $\mathrm{AC}_{Y[\mathbf{d}]}$ *and* $\mathrm{AC}_{\Sigma_1(\mathbf{d})}$ *whenever* $\mathbf{d} \in \mathbb{D}$.

Remark 3.13. 1) In $(*)_2$, can we make j depend on f and a partition of $Y_{\mathbf{d}}$? Will be somewhat better.
2) We can similarly prove this for a weak rank 2-system. It is natural though not necessary to add $(\mathbf{e}, h) \in \Sigma_1(\mathbf{d}) \Rightarrow Y_{\mathbf{e}} = Y_{\mathbf{d}} \wedge h = \mathrm{id}_{Y_{\mathbf{d}}}$.
3) If we add $\Sigma_1(\mathbf{d}) = \{\mathbf{d}\}$ then clause $(g)^+$ means just that in (g) we add $\mathbf{e} = \mathbf{d}$ and $(*)_2(a)$ holds trivially and the second clause in $(*)_4$, i.e. $\mathrm{AC}_{\Sigma_1(\mathbf{d})}$ holds trivially.

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Proof. Let $\mathbf{d},\zeta,\xi,f,j_0$ satisfying $\boxplus$ of 3.6(j) be given and we should find $j < \text{cf}(\mu)$ such that for no pair $(\mathbf{e},g)$ clause $\oplus$ there holds. By 3.9(2) and clause (f) of Definition 3.1, without loss of generality $s \in Y_{\mathbf{d}} \Rightarrow f(s) > 0$.

Let $j < \text{cf}(\mu)$ be as in $(*)_2$ in the claim and without loss of generality $j > j_0$ and we shall prove that j is as required in clause (j) of Definition 3.6, this is enough. So assume $\mathbf{e} \in \mathbb{D}_{\geq j}, g \in {}^{Y[\mathbf{e}]}[\xi,\zeta_*), \xi < \zeta_* < \zeta$ and toward contradiction, $(j,\zeta,\xi,\mathbf{e},g)$ satisfy $\oplus$ there. For each $t \in Y_{\mathbf{e}}$ clearly $g(t) < \zeta = \text{rk}_{\mathbf{d}}(f)$ hence by clause $(g)^+$ of $(*)_3$, see (g) of Definition 3.1, "no hole", there are $g_t \in {}^{Y[\mathbf{d}]}\xi$ and $\mathbf{d}_t \in \Sigma_1(\mathbf{d})$ such that $g_t <_{D_{\mathbf{d}_t}} f$ and $\text{rk}_{\mathbf{d}_t}(g_t) = g(t)$, without loss of generality $g_t < \max(f, 1_{Y[\mathbf{d}]}) = f$ and by the $(*)_3$, "we add" also $\text{rk}_{\mathbf{d}_t}(f) = \text{rk}_{\mathbf{d}}(f)$.

As $\text{AC}_{Y_{\mathbf{e}}}$ by $(*)_4$, we can choose such sequence $\langle (g_t,\mathbf{d}_t) : t \in Y_{\mathbf{e}}\rangle$. Now $\mathbf{e}$ is $(\mathbf{p}, \leq \Sigma_1(\mathbf{d}))$-complete and $(\mathbf{d},\mathbf{e})$ commute for $\mathbf{p}$, by clauses (a),(b) respectively of $(*)_2$ (i.e. by the choice of j and as $\mathbf{e} \in \mathbb{D}_{\geq j}$), hence we can find $\mathbf{e}_* \in \Sigma_1(\mathbf{e})$ and $\mathbf{d}_* \in \Sigma_1(\mathbf{d})$ such that $\text{rk}_{\mathbf{e}_*}(g) = \text{rk}_{\mathbf{e}}(g) = \zeta$ and $\{t \in Y_{\mathbf{e}} : \mathbf{d}_t = \mathbf{d}_*\}$ belongs to $D_{\mathbf{e}_*}$. For $s \in Y_{\mathbf{d}} = Y_{\mathbf{d}_*}$ let $f_s \in {}^{Y[\mathbf{e}_*]}\text{Ord}$ be defined by $f_s(t) = g_t(s)$ so $f_s(t) = g_t(s) < \xi$ and let $f' \in {}^{Y[\mathbf{d}_*]}\text{Ord}$ be defined by $f'(s) = \text{rk}_{\mathbf{e}_*}(f_s)$ and let $\bar{f} = \langle f_s : s \in Y_{\mathbf{d}_*}\rangle$.

Fixing $s \in Y_{\mathbf{d}_*}$ we have $t \in Y_{\mathbf{e}_*} \Rightarrow f_s(t) = g_t(s) < \text{Max}\{f(s),1\} = f(s)$, i.e. $f_s < \langle f(s) : t \in Y_{\mathbf{e}_*}\rangle$ hence $\text{rk}_{\mathbf{e}_*}(f_s) < \text{rk}_{\mathbf{e}_*}(f(s))$. Now by $\boxplus\bullet_6$ from 3.6, as $j_0 \leq j \leq \mathbf{j}(\mathbf{e}_*)$ we have $s \in Y_{\mathbf{e}_*} \Rightarrow \text{rk}_{\mathbf{e}_*}(f(s)) = f(s)$ so $s \in Y_{\mathbf{e}_*} \Rightarrow \text{rk}_{\mathbf{e}_*}(f_s) < f(s)$, i.e. $f' < f$.

Clearly $(f',g,\bar{f},\bar{g})$ is a $(\mathbf{p},\mathbf{d}_*,\mathbf{e}_*)$-rectangle hence by clause (b) of $(*)_2$ of the assumptions, i.e. the choice of $(\mathbf{e},g)$ and Definition 3.11(2) we know that $\text{rk}_{\mathbf{d}_*}(f') \geq \text{rk}_{\mathbf{e}_*}(g)$.

But recall that $\text{rk}_{\mathbf{e}_*}(g) = \text{rk}_{\mathbf{e}}(g)$ by the choice of $\mathbf{e}_*$. We get a contradiction by

$$(*)\ \ \zeta = \text{rk}_{\mathbf{d}}(f) = \text{rk}_{\mathbf{d}_*}(f) > \text{rk}_{\mathbf{d}_*}(f') \geq \text{rk}_{\mathbf{e}_*}(g) = \text{rk}_{\mathbf{e}}(g) \geq \zeta.$$

[Why those inequalities? By $\bullet_2$ of $\boxplus$ from 3.6 we are assuming; as $\mathbf{d}_* \in \{\mathbf{d}_t : t \in Y_{\mathbf{e}}\}$ and the choice of the $\mathbf{d}_t$'s; as $f' <_{D_{\mathbf{d}_*}} f$ and 3.9(3); by an inequality above; by the choice of $\mathbf{e}_*$; by $\bullet_5$ of $\oplus$ of 3.6.] $\square_{3.12}$

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§ 4. Finding Systems

§(4A) Building weak rank systems and measurable

Claim 4.1. *[ZF + DC + AC$_{<\mu}$]*
If $\circledast_1$ holds and $\mathbf{p}_{\bar\kappa,\theta^*} = \mathbf{p}_{\bar\kappa} = \mathbf{p} = (\mathbb{D}, \mathrm{rk}, \Sigma, \mathbf{j}, \mu)$ is defined in $\circledast_2$ then $\mathbf{p}$ is a weak rank 1-system, even semi normal (and $(g)^+$ of 3.12 holds) where:

$\circledast_1$ (a) $\bar\kappa = \langle \kappa_i : i < \partial\rangle$ *is an increasing sequence of regular cardinals $> \partial = \mathrm{cf}(\partial)$ with limit μ such that if $i < \partial$ is a limit ordinal then $\kappa_i = (\Sigma\{\kappa_j : j < i\})^+$*

(b) θ^* *is a cardinal $\geq \mu$ or is ∞*

$\circledast_2$ (a) $\mathbb{D}_i = \{J : J$ *is a κ_i-complete ideal on some $\kappa = \kappa_J < \mu$ satisfying $\mathrm{cf}(J, \subseteq) < \theta^*$ (and if $\theta^* = \infty$ we stipulate this as the empty demand) such that $\beta < \kappa \Rightarrow \{\beta\} \in J\}$ and let $\mathbb{D} = \mathbb{D}_0$*

(b) *if $\mathbf{d} = J \in \mathbb{D}_i$, J an ideal on $\kappa_J := \cup\{A : A \in J\}$ then we let $Y_{\mathbf{d}} = \kappa_J$ and $D_{\mathbf{d}}$ be the filter dual to the ideal J*

(c) $\mathbf{j}(J) = \min\{i : J$ *is not κ^+_{i+1}-complete$\}$*

(d) $\Sigma(J) = \{J + B : B \supseteq A$ *and $\kappa_J \backslash B$ is not in $J\}$*

(e) $\mathrm{rk}_J(f)$ *is as in Definition 1.6.*

Proof. So we have to check all the clauses in Definition 3.1.

Clause (a): As $\mu = \Sigma\{\kappa_i : i < \partial\}$, the sequence $\langle \kappa_i : i < \partial\rangle$ is increasing and $\kappa_0 > \partial$ (all by $\circledast_1$) clearly μ is a singular cardinal (and $\partial = \mathrm{cf}(\mu)$).

Clause (b): Let $\mathbf{d} \in \mathbb{D}$, so $\mathbf{d} = J$.

Subclause (α): So $Y_{\mathbf{d}} = \kappa_J < \mu$ hence $\mathrm{hrtg}(Y_{\mathbf{d}}) = \mathrm{hrtg}(\kappa_J) = \kappa^+_J < \mu$ recalling μ is a limit cardinal and the definition of $\mathbb{D} = \mathbb{D}_0$ in clause (a) of $\circledast_2$.

Subclause (β): Also obvious.

Clause (c): For $f \in {}^{(\kappa_{\mathbf{d}})}\mathrm{Ord}$, $\mathrm{rk}_{\mathbf{d}}(f)$ as defined in $\circledast_2(e)$, is an ordinal recalling Claim 1.8(1).

Clause (d)(α):
Trivial.

Clause (d)(β):
Trivially $\mathbf{e} \in \Sigma(\mathbf{d}) \Rightarrow Y_{\mathbf{e}} = Y_{\mathbf{d}} \wedge D_{\mathbf{e}} \supseteq D_{\mathbf{d}}$; so "$\mathbf{p}$ is weakly normal", see Definition 3.8, moreover "$\mathbf{p}$ is semi-normal" as $\mathrm{rk}_D(f) \leq \mathrm{rk}_{D+A}(f)$ for $A \in D^+$.

Clause (e):
Obvious from the definitions.

Clause (f):
Let $\sigma < \mu$ be given and choose $i < \partial$ such that $\sigma < \kappa_i$. Let $\mathbf{d} \in \mathbb{D}$ be such that $j = \mathbf{j}(\mathbf{d}) \geq i$ hence $D = D_{\mathbf{d}}$ is a filter on some κ_J, so assume $\cup\{A_\varepsilon : \varepsilon < \varepsilon^*\} = \kappa_J$ and $\varepsilon^* < \kappa_i$. Now D is κ_i-complete and (see 1.9(2) as AC_{κ_J} is assumed) we have $\mathrm{rk}_{D_{\mathbf{d}}}(f) = \min\{\mathrm{rk}_{D+A_\varepsilon}(f) : \varepsilon < \varepsilon^*$ and $A_\varepsilon \in D^+_{\mathbf{d}}\}$ which is what is needed as $A_\varepsilon \in D^+_{\mathbf{d}} \Rightarrow \mathbf{d} + (\kappa_j \backslash A_\varepsilon) \in \Sigma(\mathbf{d})$.

Clause (g): By 1.8(2).

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Moreover, the stronger version with $\mathbf{e} = \mathbf{d}$ holds so in particular $(g)^+$ of 3.12 holds.

Clause (h):

Easy. On the one hand, as $g < f$, the definition of $\mathrm{rk}_{\mathbf{d}}(f)$, we have $\mathrm{rk}_{\mathbf{d}}(f) \geq \mathrm{rk}_{\mathbf{d}}(g) + 1$. On the other hand, if $g' < f$ mod $D_{\mathbf{d}}$ then $g' \leq g$ mod D hence by clause (i) below we have $\mathrm{rk}_{\mathbf{d}}(g') \leq \mathrm{rk}_{\mathbf{d}}(g) < \mathrm{rk}_{\mathbf{d}}(g) + 1$, as this holds for every $g' < f$ mod $D_{\mathbf{d}}$ we have $\mathrm{rk}_{\mathbf{d}}(f) \leq \mathrm{rk}_{\mathbf{d}}(g) + 1$. Together we are done.

Clause (i):

Obvious. $\square_{4.1}$

Discussion 4.2. 1) Assume μ is a singular cardinal, $\mu = \sum\limits_{i<\kappa} \mu_i, \kappa = \mathrm{cf}(\kappa) < \mu_i < \mu$ and μ_i is increasing with i. Assume that for each i there is a pair $(D, Y), D$ is a μ_i-complete ultra-filter on $Y, \mathrm{hrtg}(Y) < \mu$. This seems to be a good case, but either we have "D is a $(\leq \mathrm{hrtg}(Y))$-complete" so ${}^Y\mathrm{Ord}/D$ is "dull" or $\mathrm{hrtg}(Y) > \kappa = \mathrm{completeness}(D)$ and so there is a κ-complete non-principal ultrafilter on κ and on $\kappa < \mu$ so $\mu = \sup(\text{measurables} \cap \mu)$.
2) For μ as in 4.3, if $\kappa < \mu$ then $\mathrm{hrtg}(\mathscr{P}(\kappa)) \leq \min(\mu \cap \text{the measurables} \setminus \kappa^+)$ hence $\mathrm{AC}_{\mathrm{hrtg}(\mathscr{P}(\kappa))}$ hence $|\mathscr{P}(\kappa)| < \mu$.

Claim 4.3. [ZF + DC + $\mathrm{AC}_{<\mu}$]

Assume μ is singular and $\mu = \sup(\mu \cap$ the class of measurable cardinals), (equivalently for every $\kappa < \mu$ there is a κ-complete non-principal ultrafilter on some $\kappa' < \mu$). Let $\bar\kappa = \langle \kappa_i : i < \mathrm{cf}(\mu)\rangle$ be increasing with limit $\mu, \kappa_i > \mathrm{cf}(\mu)$ such that for i limit $\kappa_i = (\Sigma\{\kappa_j : j < i\})^+$ and κ_i is measurable for i non-limit.

Then $\mathbf{p} = \mathbf{p}^{\mathrm{uf}}_{\bar\kappa}$ is a strict rank 1-system where $\mathbf{p}$ is defined by

$\circledast$ (a) $\mathbb{D}_{\geq i} = \{J$: *dual* (J) *is a non-principal ultra-filter which is κ_i-complete on some* $\kappa = \kappa_J < \mu\}$ *so naturally* $Y_J = \kappa_J$ *and* $D_J = \mathrm{dual}(J)$

(b) $\mathbf{j}(J) = \min\{i : J$ *is not* κ^+_{i+1}*-complete*$\}$, *well defined*

(c) $\Sigma(J) = \{J\}$

(d) $\mathrm{rk}_J(f) = \mathrm{rk}_{\mathrm{dual}(J)}(f)$ *as in 1.6.*

Proof. We can check clauses (a)-(i) of 3.1 as in the proof of 4.1.

We still have to prove the "strict", i.e. we should prove clause (j) from Definition 3.6. We prove this using Claim 3.12, we choose $\Sigma_1(\mathbf{d}) := \{\mathbf{d}\} \subseteq \Sigma(\mathbf{d})$ for $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}$ so it suffices to prove $(*)_0 - (*)_4$ of 3.12.

So in Claim 3.12, we have $(*)_0, (*)_1$ hold by the choice of Σ_1, and concerning $(*)_3$ in 4.1 easily we have $(g)^+$ as said in the proof there. Next $(*)_4$ holds as for each $\kappa < \mu$ we have AC_κ as $\kappa < \mu$ by an assumption and for $\mathbf{d} \in \mathbb{D}$ we have $\mathrm{AC}_{\Sigma_1(\mathbf{d})}$, as $\Sigma_1(\mathbf{d})$ is a singleton.

Note that

$\boxplus_1$ (a) if $\kappa < \chi < \mathrm{hrtg}(\mathscr{P}(\kappa))$ then χ is not measurable hence

(b) if $\kappa < \mu$ then $\mathrm{hrtg}(\mathscr{P}(\kappa)) < \mu$.

Now we are left with proving $(*)_2$, so let $\mathbf{d} \in \mathbb{D}_{\mathbf{p}}, \zeta, \xi, f \in {}^{Y[\mathbf{d}]}\zeta$ be given as in $\boxplus$ from 3.6(j), and we should find j as there.

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Let $j < \partial =$ cf(μ) be such that hrtg$(\mathscr{P}(\kappa_{\mathbf{d}})) < \kappa_j$, and let $\mathbf{e} \in \mathbb{D}_{\geq j}$. Now clause (a) is trivial as $|\Sigma_1(\mathbf{d})| = 1$, and clause (b) says that " the pair $(\mathbf{d}, \mathbf{e})$ commute for $\mathbf{p}$", see Definition 3.11 recalling $\Sigma_1(\mathbf{d}) = \{\mathbf{d}\}$. So let $(f, g, \bar{f}, \bar{g})$ be a $(\mathbf{p}, \mathbf{d}, \mathbf{e})$-rectangle, see Definition 3.11(2), and we should prove that $\text{rk}_{\mathbf{e}}(g) \leq \text{rk}_{\mathbf{d}}(f)$; let $Y_1 = Y_{\mathbf{e}}, D_1 = D_{\mathbf{p},\mathbf{e}}, Y_2 = Y_{\mathbf{d}}, D_2 = D_{\mathbf{p},\mathbf{d}}$.

To prove this we apply 2.3 or 2.9, but the $f, \bar{f}$ are interchanged with $g, \bar{g}$; we check $\oplus(a) - (g)$ from 2.3. They hold by $\circledast(a) - (f)$ of Definition 3.11.

Concerning $\boxplus(a), (b)$ from 2.3, "AC_{Y_1}" holds as $\text{AC}_{<\mu}$ holds and the definition of $\mathbf{p}$. Lastly, we should prove $\boxplus(a)$ there which says "$D_{\mathbf{d}}$ does 2-commute with $D_{\mathbf{e}}$" which holds by Case 2 of Claim 2.6. $\square_{4.3}$

Conclusion 4.4. *[$AC_{<\mu}, \mu$ a singular cardinal] Assume $\mu = \sup\{\lambda < \mu : \lambda$ is a measurable cardinal$\}$. Then for every ordinal ζ for some $\kappa < \lambda$ we have $\text{rk}_D(\zeta) = \zeta$ for every κ-complete ultrafilter D on some cardinal $< \mu$.*

Proof. It suffices to prove this for the case μ has cofinality $\aleph_0$. Now we can apply Claim 4.3 and Theorem 3.10. $\square_{4.4}$

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§ 5. Pseudo true cofinality

Pseudo PCF

We try to develop pcf theory with little choice. We deal only with $\aleph_1$-complete filters, and replace cofinality and other basic notions by pseudo ones, see below. This is quite reasonable as with choice there is no difference.

This section main result are 5.9, existence of filters with pseudo-true-cofinality; 5.19, giving a parallel of $J_{<\lambda}[\alpha]$.

In the main case we may (in addition to ZF) assume DC + $\mathrm{AC}_{\mathscr{P}(\mathscr{P}(Y))}$; this will be continued in [**?**].

Hypothesis 5.1. ZF

Definition 5.2. 1) We say that a partial order P is $(<\kappa)$-directed <u>when</u> every subset A of P of power $<\kappa$ has a common upper bound.
1A) Similarly P is $(\leq S)$-directed.
2) We say that a partial order P is pseudo $(<\kappa)$-directed <u>when</u> it is $(<\kappa)$-directed and moreover every subset $\cup\{P_\alpha : \alpha < \delta\}$ has a common upper bound <u>when</u>:

- (a) if $\delta < \kappa$ is a limit ordinal
- (b) $\bar{P} = \langle P_\alpha : \alpha < \delta\rangle$ is a sequence of non-empty subsets of P
- (c) if $\alpha_1 < \alpha_2, p_1 \in P_{\alpha_1}$ and $p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$.

2A) For a set S we say that the partial order P is pseudo $(\leq S)$-directed <u>when</u> $\cup\{P_s : s \in S\}$ has a common upper bound whenever

- (a) $\langle P_s : s \in S\rangle$ is a sequence
- (b) $P_s \subseteq P$
- (c) if $s \in S$ then P_s has a common upper bound.

Definition 5.3. We say that a partial (or quasi) order P has pseudo true cofinality δ <u>when</u>: δ is a limit ordinal and there is a sequence $\langle P_\alpha : \alpha < \delta\rangle$ such that

- (a) $P_\alpha \subseteq P$ and $\delta = \sup\{\alpha < \delta : P_\alpha$ non-empty$\}$
- (b) if $\alpha_1 < \alpha_2 < \delta, p_1 \in P_{\alpha_1}, p_2 \in P_{\alpha_2}$ then $p_1 <_P p_2$
- (c) if $p \in P$ then for some $\alpha < \delta$ and $q \in P_\alpha$ we have $p \leq_P q$.

Remark 5.4. 0) See 5.2(2) and 5.8(1).
1) We could replace δ by a partial order Q.
2) The most interesting case is in Definition 5.6.
3) We may in Definition 5.3 demand δ is a regular cardinal.
4) Usually in clause (a) of Definition 5.3 without loss of generality $\bigwedge\limits_\alpha P_\alpha \neq \emptyset$, as without loss of generality $\delta = \mathrm{cf}(\delta)$ using $P'_\alpha = P_{f(\alpha)}$ where $f(\alpha)$ = the α-th member of C where C is an unbound subset of $\{\beta < \delta : P_\beta \neq \emptyset\}$ of order type $\mathrm{cf}(\delta)$. Why do we allow $P_\alpha = \emptyset$? as it is more natural in 5.17(1), but can usually ignore it.

Example 5.5. Suppose we have a limit ordinal δ and a sequence $\langle A_\alpha : \alpha < \delta\rangle$ of sets with $\prod\limits_{\alpha<\delta} A_\alpha = \emptyset$; moreover $u \subseteq \delta = \sup(u) \Rightarrow \prod\limits_{\alpha\in u} A_\alpha = \emptyset$. Define a partial order P by:

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(a) its set of elements is $\{(\alpha, a) : a \in A_\alpha \text{ and } \alpha < \delta\}$

(b) the order is $(\alpha_1, a_1) <_P (\alpha_2, a_2)$ iff $\alpha_1 < \alpha_2$ (and $a_\ell \in A_{\alpha_\ell}$ for $\ell = 1, 2$).

It seems very reasonable to say that P has true cofinality but there is no increasing cofinal sequence.

Definition 5.6. 1) For a set Y and sequence $\bar{\alpha} = \langle \alpha_t : t \in Y \rangle$ of ordinals and cardinal κ we define

$$\text{ps-tcf-fil}_\kappa(\bar{\alpha}) = \{D : \ D \text{ a } \kappa\text{-complete filter on } Y \text{ such that } (\Pi\bar{\alpha}/D) \text{ has a pseudo true cofinality}\};$$

see below.

2) We say that $\Pi\bar{\alpha}/D$ or $(\Pi\bar{\alpha}, D)$ or $(\Pi\bar{\alpha}, <_D)$ has pseudo true cofinality γ <u>when</u> D is a filter on $Y = \text{Dom}(\bar{\alpha})$ and γ is a limit ordinal and the partial order $(\Pi\bar{\alpha}, <_D)$ essentially does[6], i.e., there is a sequence $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \gamma \rangle$ satisfying:

$\circledast_{\bar{\mathscr{F}}}$ (a) $\mathscr{F}_\beta \subseteq \{f \in {}^Y\text{Ord} : f <_D \bar{\alpha}\}$

(b) $\mathscr{F}_\beta \neq 0$

(c) if $\beta_1 < \beta_2, f_1 \in \mathscr{F}_{\beta_1}$ and $f_2 \in \mathscr{F}_{\beta_2}$ then $f_1 < f_2 \mod D$

(d) if $f \in {}^Y\text{Ord}$ and $f < \bar{\alpha} \mod D$ then for some $\beta < \gamma$ we have $g \in \mathscr{F}_\beta \Rightarrow f < g \mod D$ (by clause (c) this is equivalent to: for some $\beta < \gamma$ and some $g \in \mathscr{F}_\beta$ we have $f \leq g \mod D$).

3) $\text{ps-pcf}_\kappa(\bar{\alpha}) = \text{ps-pcf}_{\kappa\text{-comp}}(\bar{\alpha}) := \{\gamma$: there is a κ-complete filter D on Y such that $\Pi\bar{\alpha}/D$ has pseudo true cofinality γ and γ is minimal for $D\}$.

4) $\text{pcf-fil}_{\kappa,\gamma}(\bar{\alpha}) = \{D : D$ a κ-complete filter on Y such that $\Pi\bar{\alpha}/D$ has true pseudo-cofinality $\gamma\}$.

5) In part (2) if γ is minimal we call it $\text{ps-tcf}(\Pi\bar{\alpha}, D)$ or simply $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$; note that it is a well defined (regular cardinal).

Claim 5.7. *1) If $\lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, <u>then</u> $(\Pi\bar{\alpha}, <_D)$ is pseudo $(< \lambda)$-directed.*
1A) If $\text{hrtg}(S) < \lambda = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ <u>then</u> $(\Pi\bar{\alpha}, <_D)$ is pseudo $(\leq S)$-directed.
2) Similarly for any quasi order.
3) If $\text{cf}(\alpha_t) \geq \lambda = \text{cf}(\lambda)$ for $t \in Y$ <u>then</u> $(\Pi\bar{\alpha}, <_D)$ is λ-directed.
4) Assume AC_α for $\alpha < \lambda$. If $\text{cf}(\alpha_s) \geq \lambda$ for $s \in Y$ <u>then</u> $(\Pi\bar{\alpha}, <_D)$ is pseudo λ-directed.

Proof. 1), 1A), 2) As in 5.8(1) below.

3) So assume $\mathscr{F} \subseteq \Pi\bar{\alpha}$ satisfies $|\mathscr{F}| < \lambda$, so there is a sequence $\langle f_\alpha : \alpha < \mu \rangle$ listing $\mathscr{F}$ for some $\mu < \lambda$. Let $f \in \Pi\bar{\alpha}$ be defined by $f(s) = \sup\{f_\alpha(s) : \alpha < \mu\}$, now $f(s) < \alpha(s)$ as $\text{cf}(\alpha_s) \geq \lambda > \mu$.

4) So assume $\bar{P} = \langle P_\alpha : \alpha < \delta \rangle, \delta$ a limit ordinal $< \lambda$ and $P_\alpha \subseteq \Pi\bar{\alpha}$ non-empty and $\alpha < \beta < \delta \wedge f \in P_\alpha \wedge g \in P_\beta \Rightarrow f <_D g$. As AC_δ holds we can find a sequence $\bar{f} = \langle f_\alpha : \alpha \in \delta \rangle \in \prod\limits_{\alpha < \beta} P_\alpha$ and apply part (3). $\square_{5.7}$

[6] so necessarily $\{s \in Y : \alpha_s > 0\}$ belongs to D but is not necessarily empty; if it is $\neq Y$ then $\Pi\bar{\alpha} = \emptyset$, so pedantically this is wrong, $(\Pi\bar{\alpha}, <_D)$ does not have any pseudo true cofinality hence we say "essentially" but usually we shall ignore this <u>or</u> assume $\bigwedge\limits_t \alpha_t \neq 0$ when not said otherwise.

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Claim 5.8. *Let $\bar{\alpha} = \langle \alpha_s : s \in Y \rangle$ and D is a filter on Y.*
0) If $\Pi\bar{\alpha}/D$ has pseudo true cofinality <u>then</u> $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is a regular cardinal; similarly for any partial order.
1) If $\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_1 and true cofinality γ_2 <u>then</u> $\text{cf}(\gamma_1) = \text{cf}(\gamma_2) = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$, similarly for any partial order.
2) $\text{ps-pcf}_\kappa(\bar{\alpha})$ is a set of regular cardinals so if $\Pi\bar{\alpha}/D$ has pseudo true cofinality <u>then</u> $\text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ is γ where $\gamma = \text{cf}(\gamma)$ and $\Pi\bar{\alpha}/D$ has pseudo cofinality γ.
3) Always $\text{ps-pcf}_\kappa(\bar{\alpha})$ has cardinality $< \text{hrtg}(\{D : D$ a κ-complete filter on $Y\})$.
4) If $\bar{\beta} = \langle \beta_s : s \in Y \rangle \in {}^Y\text{Ord}$ and $\{s : \beta_s = \alpha_s\} \in D$ <u>then</u> $\text{ps-tcf}(\Pi\bar{\alpha}/D) = \text{ps-tcf}(\Pi\bar{\beta}/D)$ so one is well defined iff the other is.

Proof. 0) By the definitions.
1) Let $\langle \mathscr{F}^\ell_\beta : \beta < \gamma_\ell \rangle$ exemplify "$\Pi\bar{\alpha}/D$ has pseudo true cofinality γ_ℓ" for $\ell = 1, 2$. Now

- $(*)$ if $\ell \in \{1, 2\}$ and $\beta_\ell < \gamma_\ell$ then for some $\beta_{3-\ell} < \gamma_{3-\ell}$ we have $g_1 \in \mathscr{F}^\ell_{\beta_\ell} \wedge g_2 \in \mathscr{F}^{3-\ell}_{\beta_{3-\ell}} \Rightarrow g_1 <_D g_2$.

[Why? Choose $g^\ell \in \mathscr{F}^\ell_{\beta_\ell+1}$, choose $\beta_{3-\ell} < \gamma_{3-\ell}$ and $g_{3-\ell} \in \mathscr{F}^{3-\ell}_{\beta_{3-\ell}}$ such that $g^\ell < g^{3-\ell} \mod D$. Clearly $f \in \mathscr{F}^\ell_{\beta_\ell} \Rightarrow f <_D g^\ell <_D g^{3-\ell}$ so $g^{3-\ell}$ is as required.]

Hence

- $(*)$ $h_1 : \gamma_1 \to \gamma_2$ is well defined when
  $h_1(\beta_1) = \text{Min}\{\beta_2 < \gamma_2 : (\forall g_1 \in \mathscr{F}^1_{\beta_1})(\forall g_2 \in \mathscr{F}^2_{\beta_2})(g_1 < g_2 \mod D)\}$.

Clearly h is non-decreasing and it is not eventually constant (as $\cup\{\mathscr{F}^1_\beta : \beta < \gamma_1\}$ is cofinal in $\Pi\bar{\alpha}/D$) and has range unbounded in γ_2 (similarly).

The rest should be clear.
2) Follows.
3),4) Easy. $\square_{5.8}$

Concerning [?]

Claim 5.9. <u>*The Existence of true cofinality filter*</u> *$[\kappa > \aleph_0 + \text{DC} + \text{AC}_{<\kappa}]$ If*

- *(a) D is a κ-complete filter on Y*
- *(b) $\bar{\alpha} \in {}^Y\text{Ord}$*
- *(c) $\delta := \text{rk}_D(\bar{\alpha})$ satisfies $\text{cf}(\delta) \geq \text{hrtg}(\text{Fil}^1_\kappa(Y))$, see below.*

<u>Then</u> for some D' we have

- *(α) D' is a κ-complete filter on Y*
- *(β) $D' \supseteq D$*
- *(γ) $\Pi\bar{\alpha}/D'$ has pseudo true cofinality, in fact, $\text{ps-tcf}(\Pi\bar{\alpha}, <_{D'}) = \text{cf}(\text{rk}_D(\bar{\alpha}))$.*

Recall from [?]

Definition 5.10. 1) $\text{Fil}^1_\kappa(Y) = \{D : D$ a κ-complete filter on $Y\}$ and if $D \in \text{Fil}^1_\kappa(Y)$ then $\text{Fil}^1_\kappa(D) = \{D' \in \text{Fil}^1_\kappa(Y) : D \subseteq D'\}$.
2) $\text{Fil}^4_\kappa(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are κ-complete filters on $Y\}$.
3) $J[f, D]$ where D is a filter on Y and $f \in {}^Y\text{Ord}$ is $\{A \subseteq Y : A = \emptyset \mod D$ or $\text{rk}_{D+A}(f) > \text{rk}_D(f)\}$.

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Remark 5.11. 1) On the Definition of pseudo $(< \kappa, 1+\gamma)$-complete D see 1.14; we may consider changing the definition of $\mathrm{Fil}^1_\kappa(Y)$ to D is an $\aleph_1$-complete and pseudo$(< \kappa, 1+\gamma))$-complete filter on Y.

Proof. Proof of the Claim 5.9

Recall $\{y \in Y : \alpha_y = 0\} = \emptyset \mod D$ by 1.8(4) as $\mathrm{rk}_D(\langle \alpha_y : y \in Y\rangle) = \delta > 0$ but $f_1, f_2 \in {}^Y\mathrm{Ord} \wedge (f_1 = f_2 \mod D) \Rightarrow \mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2)$ hence without loss of generality $y \in Y \Rightarrow \alpha_y > 0$.

Let $\mathbb{D} = \{D' : D'$ is a filter on Y extending D which is κ-complete$\}$. So $\mathrm{hrtg}(\mathbb{D}) \le \mathrm{hrtg}(\mathrm{Fil}^1_{\aleph_1}(Y)) \le \mathrm{cf}(\delta)$. For any $\gamma < \mathrm{rk}_D(\bar\alpha)$ and $D' \in \mathbb{D}$ let

$(*)_2$ (a) $\mathscr{F}_{\gamma,D'} = \{f \in \Pi\bar\alpha : \mathrm{rk}_D(f) = \gamma$ and D' is $\mathrm{dual}(J[f,D])\}$

(b) $\mathscr{F}_{D'} = \cup\{\mathscr{F}_{\gamma,D'} : \gamma < \mathrm{rk}_D(\bar\alpha)\}$

(c) $\Xi_{\bar\alpha,D'} = \{\gamma < \mathrm{rk}_D(\bar\alpha) : \mathscr{F}_{\gamma,D'} \ne \emptyset\}$

(d) $\mathscr{F}_\gamma = \cup\{\mathscr{F}_{\gamma,D''} : D'' \in \mathbb{D}\}$.

Now

$(*)_3$ if $\gamma < \mathrm{rk}_D(\bar\alpha)$ then $\mathscr{F}_\gamma \ne \emptyset$.

[Why? By 1.8(2) there is $g \in {}^Y\mathrm{Ord}$ such that $g < \bar\alpha \mod D$ and $\mathrm{rk}_D(g) = \gamma$ and without loss of generality $g \in \Pi\bar\alpha$. Now let $D' = \mathrm{dual}(J[g,D])$, so $(D,D') \in \mathrm{Fil}^4_\kappa(Y)$ by 1.12(1) (using $\mathrm{AC}_{<\kappa}$) the fitler D' is κ-complete so $D' \in \mathbb{D}$ and clearly $g \in \mathscr{F}_{\gamma,D'}$, see 1.8(2), but $\mathscr{F}_{\gamma,D'} \subseteq \mathscr{F}_\gamma$ so $\mathscr{F}_\gamma \ne 0$; here we use $\mathrm{AC}_{<\kappa}$.]

$(*)_4$ $\{\sup(\Xi_{\bar\alpha,D'}) : D' \in \mathbb{D}$ and $\Xi_{\bar\alpha,D'}$ is bounded in $\mathrm{rk}_D(\bar\alpha)\}$ is a subset of $\mathrm{rk}_{D'}(\bar\alpha)$ which has cardinality $< \mathrm{hrtg}(\mathbb{D}) \le \mathrm{hrtg}(\mathrm{Fil}^1_\kappa(Y)) \le \mathrm{cf}(\delta)$.

[Why? The function $D' \mapsto \sup(\Xi_{\bar\alpha,D'})$ witness this.]

$(*)_5$ the set in $(*)_4$ is bounded below $\mathrm{rk}_D(\bar\alpha)$ so let $\gamma(*) < \mathrm{rk}_D(\bar\alpha)$ be its supremum.

[Why? By $(*)_4$.]

$(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar\alpha,D'}$ is unbounded in $(\Pi\bar\alpha, <_{D'})$.

[Why? Choose $\gamma < \delta = \mathrm{rk}_D(\bar\alpha)$ such that $\gamma > \gamma(*)$. By $(*)_3$ there is $f \in \mathscr{F}_{\gamma(*)}$ and by $(*)_2(d)$ for some $D' \in \mathbb{D}$ we have $f \in \mathscr{F}_{\gamma(*),D'}$ so by the choice of $\gamma(*)$ the set $\Xi_{\bar\alpha,D'}$ cannot be bounded in $\mathrm{rk}_D(\bar\alpha)$.]

$(*)_7$ if $\gamma_1 < \gamma_2$ are from $\Xi_{\bar\alpha,D'}$ and $f_1 \in \mathscr{F}_{\gamma_1,D'}, f_2 \in \mathscr{F}_{\gamma_2,D'}$ <u>then</u> $f_1 <_{D'} f_2$.

[Why? By 1.8.]

Together we are done: by $(*)_6$ there is $D' \in \mathbb{D}$ such that $\Xi_{\bar\alpha,D'}$ is unbounded in $\mathrm{rk}_D(\bar\alpha)$. Hence $\bar{\mathscr{F}} = \langle \mathscr{F}_{\gamma,D'} : \gamma \in \Xi_{\bar\alpha,D'}\rangle$ witness that $(\Pi\bar\alpha, <_{D'})$ has pseudo true cofinality by $(*)_7$, noting that $\mathscr{F}_{\gamma,D'} \ne \emptyset$ by the definition of $\Xi_{\bar\alpha,D'}$ and $\mathscr{F}_{\gamma,D'} \subseteq \Pi\bar\alpha$ by its definition. So $\text{ps-tcf}(\Pi\bar\alpha, <_D) = \mathrm{cf}(\mathrm{otp}(\Xi_{\bar\alpha,D'})) = \mathrm{cf}(\mathrm{rk}_D(\bar\alpha))$, so we are done. $\square_{5.9}$

So we have

Definition/Claim 5.12. 1) We say that $\delta = \text{ps-tcf}_{\bar D}(\bar\alpha)$, where δ is a limit ordinal <u>when</u>, for some set Y:

(a) $\bar\alpha \in {}^Y\mathrm{Ord}$

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(*b*) $\bar{D} = (D_1, D_2)$

(*c*) $D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y

(*d*) $\text{rk}_{D_1}(\bar{\alpha}) = \delta = \sup(\Xi_{\bar{D},\bar{\alpha}})$ where $\Xi_{\bar{D},\bar{\alpha}} = \{\gamma <\ \text{rk}_{D_1}(\bar{\alpha})$: for some $f < \bar{\alpha}$ mod D_1, we have $\text{rk}_{D_1}(f) = \gamma$ and $D_2 = \text{dual}(J[f, D_1]\}$.

2) [DC] If D_1 is $\aleph_1$-complete filter on $Y, \bar{\alpha} = \langle \alpha_t : t \in Y\rangle$ and $\text{cf}(\text{rk}_{D_1}(f)) \geq \text{hrtg}(\text{Fil}^1_{\aleph_1}(Y))$ then for some $\aleph_1$-complete filter D_2 on Y extending D_1 we have $\text{ps-tcf}_{(D_1,D_2)}(\bar{\alpha})$ is well defined.

3) [DC] Moreover in part (2) there is a definition giving for any $(Y, D_1, D_2, \bar{\alpha})$ as there, a sequence $\langle \mathscr{F}_\gamma : \gamma < \delta\rangle$ exemplifying the value of $\text{ps-tcf}_{\bar{D}}(\bar{\alpha})$.

Proof. 2), 3) Let $\delta := \text{rk}_{D_1}(f)$, so $\text{cf}(\delta) \geq \text{hrtg}(\text{Fil}^1_{\aleph_1}(Y))$ hence by Claim 5.9 above and its proof the conclusion holds: the inequality is needed for "$\delta = \sup(\Xi_{\bar{D},\alpha})$". $\square_{5.12}$

Observation 5.13. *Assume D is an $\aleph_1$-complete filter on Y and $f, f_n \in {}^Y\text{Ord}$ for $n < \omega$ and $f(t) = \sup\{f_n(t) : n < \omega\}$. Then $\text{rk}_D(f) \geq \sup\{\text{rk}_D(f_n) : n < \omega\}$.*

Remark 5.14. Similarly for other amounts of completeness, see 5.18.

Proof. As $f \geq f_n$ clearly $\text{rk}_D(f) \geq \text{rk}_D(f_n)$ for each n; hence $\text{rk}_D(f) \geq \sup\{\text{rk}_D(f_n) : n < \omega\}$. $\square_{5.13}$

Remark 5.15. Also in 1.9(2) we can use AC_Y only, i.e. omit the assumption DC, a marginal point here.

Claim 5.16. *[$\text{AC}_{<\theta}$] The ordinal δ has cofinality $\geq \theta$ when:*

$\circledast$ (*a*) $\delta = \text{rk}_D(\bar{\alpha})$

(*b*) $\bar{\alpha} = \langle \alpha_y : y \in Y\rangle \in {}^Y\text{Ord}$

(*c*) *D is an $\aleph_1$-complete filter on Y*

(*d*) $y \in Y \Rightarrow \text{cf}(\alpha_y) \geq \theta$.

Proof. Note that $y \in Y \Rightarrow \alpha_y > 0$. Toward contradiction assume $\text{cf}(\delta) < \theta$ so δ has a cofinal subset C of cardinality $< \theta$. For each $\beta < \delta$ for some $f \in {}^Y\text{Ord}$ we have $\text{rk}_D(f) = \beta$ and $f <_D \bar{\alpha}$ and without loss of generality $f \in \prod\limits_{y\in Y} \alpha_y$. By $\text{AC}_{<\theta}$ there is a sequence $\langle f_\beta : \beta \in C\rangle$ such that $f_\beta \in \prod\limits_{y\in Y} \alpha_y, f_\beta <_D \bar{\alpha}$ and $\text{rk}_D(f_\beta) = \beta$. Define $g \in \prod\limits_{y\in Y} \alpha_y$ by $g(y) = \cup\{f_\beta(y) : \beta \in C$ and $f_\beta(y) < \alpha_t\}$. By clause (d) we have $[y \in Y \Rightarrow g(y) < \alpha_y]$, so $g <_D \bar{\alpha}$, hence $\text{rk}_D(\bar{g}) <\ \text{rk}_D(\alpha)$ but by the choice of g we have $\beta \in C \Rightarrow f_\beta \leq_D g$ hence $\beta \in C \Rightarrow \beta =\ \text{rk}_D(f_\beta) \leq\ \text{rk}_D(g)$ hence $\delta = \sup(C) \leq\ \text{rk}_D(g)$, contradiction. $\square_{5.16}$

Observation 5.17. *1) Assume $(\bar{\alpha}, D)$ satisfies*

(*a*) *D a filter on Y and $\bar{\alpha} = \langle \alpha_t : t \in Y\rangle$ and each α_t is an ordinal and let α'_t be $\alpha_t + 1$ if $\alpha_t = 0 \vee \text{cf}(\alpha_t) = \aleph_0$ and be α_t if otherwise*

(*b*) *$\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \partial\rangle$ exemplify $\partial = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ so we demand just $\partial = \sup\{\beta < \partial : \mathscr{F}_\beta \neq \emptyset\}$*

(*c*) *$\mathscr{F}'_\beta = \{f \in \prod\limits_{t\in Y} \alpha'_t$: for some $g \in \mathscr{F}_\beta$ we have $f = g$ mod $D\}$.*

Then: $\langle \mathscr{F}'_\beta : \beta < \partial\rangle$ exemplify $\partial = \text{ps-tcf}(\Pi\bar{\alpha}, <_D)$ that is

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(α) $\bigcup\limits_{\beta<\gamma} \mathscr{F}'_\beta$ *is cofinal in* $(\Pi\bar\alpha, <_D)$

(β) *for every* $\beta_1 < \beta_2 < \partial$ *and* $f_1 \in \mathscr{F}'_{\beta_1}$ *and* $f_2 \in \mathscr{F}'_{\beta_2}$ *we have* $f_1 < f_2$ mod D.

2) Similarly, if $D, \bar{\mathscr{F}}$ *satisfies clauses (a),(b) above and* D *is* $\aleph_1$*-complete and* $\partial = \text{cf}(\partial) > \aleph_0$ *<u>then</u> we can "correct"* $\bar{\mathscr{F}}$ *to make it* $\aleph_0$*-continuous that is* $\langle \mathscr{F}''_\beta : \beta < \partial\rangle$ *defined in* $(c)_1 + (c)_2$ *below satisfies* $(\alpha) + (\beta)$ *above and* (γ) *below and so is* $\aleph_0$*-continuous, (see below) where*

$(c)_1$ *if* $\beta < \partial$ *and* $\text{cf}(\beta) \neq \aleph_0$ *then* $\mathscr{F}''_\beta = \mathscr{F}'_\beta$

$(c)_2$ *if* $\beta < \partial$ *and* $\text{cf}(\beta) = \aleph_0$ *then* $\mathscr{F}''_\beta = \{\sup\langle f_n : n < \omega\rangle$*: for some increasing sequence* $\langle \beta_n : n < \omega\rangle$ *with limit* β *we have* $n < \omega \Rightarrow f_n \in \mathscr{F}'_{\beta_n}\}$*, see below*

(γ) *if* $\beta < \partial$ *and* $\text{cf}(\beta) = \aleph_0$ *and* $f_1, f_2 \in \mathscr{F}''_\beta$ *then* $f_1 = f_2$ *mod* D.

3) This applies to any increasing sequence $\langle \mathscr{F}_\beta : \beta < \delta\rangle, \mathscr{F}_\beta \subseteq {}^Y\text{Ord}, \delta$ *a limit ordinal.*

Proof. Straightforward. $\square_{5.17}$

Definition 5.18. 0) If $f_n \in {}^Y\text{Ord}$ for $n < \omega$, <u>then</u> $\sup\langle f_n : n < \omega\rangle$ is defined as the function f with domain Y such that $f(t) = \cup\{f_n(t) : n < \omega\}$.
1) We say $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \lambda\rangle$ exemplifying $\lambda =$ ps-tcf$(\Pi\bar\alpha, <_D)$ is weakly $\aleph_0$-continuous <u>when</u>:

if $\beta < \partial$, $\text{cf}(\beta) = \aleph_0$ and $f \in \mathscr{F}_\beta$ then for some sequence $\langle(\beta_n, f_n) : n < \omega\rangle$ we have $\beta = \cup\{\beta_n : n < \omega\}, \beta_n < \beta_{n+1} < \beta, f_n \in \mathscr{F}_{\beta_n}$ and $f = \sup\langle f_n : n < \omega\rangle$; so if D is $\aleph_1$-complete then $\{f/D : f \in \mathscr{F}_\beta\}$ is a singleton.
2) We say it is $\aleph_0$-continuous if we can replace the last "then" by "iff".

Theorem 5.19. *<u>The Canonical Filter Theorem</u> Assume* DC *and* $\text{AC}_{\mathscr{P}(Y)}$.

Assume $\bar\alpha = \langle \alpha_t : t \in Y\rangle \in {}^Y\text{Ord}$ *and* $t \in Y \Rightarrow \text{cf}(\alpha_t) \geq \text{hrtg}(\mathscr{P}(Y))$ *and* $\partial \in \text{ps-pcf}_{\aleph_1\text{-comp}}(\bar\alpha)$ *hence is a regular cardinal. <u>Then</u> there is* $D = D^{\bar\alpha}_\partial$*, an* $\aleph_1$*-complete filter on* Y *such that* $\partial = \text{ps-tcf}(\Pi\bar\alpha/D)$ *and* $D \subseteq D'$ *for any other such* $D' \in \text{Fil}^1_{\aleph_1}(D)$.

Remark 5.20. 1) By 5.9 there are some such ∂.
2) We work to use just $\text{AC}_{\mathscr{P}(Y)}$ and not more.
3) If $\kappa > \aleph_0$ we can replace "$\aleph_1$-complete" by "κ-complete".

Proof. Let

$\boxplus_1$ (a) $\mathbb{D} = \{D : D$ is an $\aleph_1$-complete filters on Y such that $(\Pi\bar\alpha/D)$ has pseudo true cofinality $\partial\}$,

(b) $D_* = \cap\{D : D \in \mathbb{D}\}$.

Now obviously

(c) D_* is an $\aleph_1$-complete filter on Y.

For $A \subseteq Y$ let $\mathbb{D}_A = \{D \in \mathbb{D} : A \notin D\}$ and let $\mathscr{P}_* = \{A \subseteq Y : \mathbb{D}_A \neq \emptyset\}$. As $\text{AC}_{\mathscr{P}(Y)}$ we can find $\langle D_A : A \in \mathscr{P}_*\rangle$ such that $D_A \in \mathbb{D}_A$ for $A \in \mathscr{P}_*$. Let $\mathbb{D}_* = \{D_A : A \in \mathscr{P}_*\}$, clearly

$\boxplus_2$ $D_* = \cap\{D : D \in \mathbb{D}_*\}$ and $\mathbb{D}_* \subseteq \mathbb{D}$ is non-empty.

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As $\mathrm{AC}_{\mathscr{P}_*}$ holds clearly

$(*)_0$ we can choose $\langle \bar{\mathscr{F}}^A : A \in \mathscr{P}_* \rangle$ such that $\bar{\mathscr{F}}_A$ exemplifies $D_A \in \mathbb{D}$ as in 5.17(1),(2), so in particular is $\aleph_0$-continuous.

For each $\beta < \partial$ let $\mathscr{F}^*_\beta = \cap\{\mathscr{F}^A_\beta : A \in \mathscr{P}_*\}$, now

$(*)_1$ $\mathscr{F}^*_\beta \subseteq \Pi\bar{\alpha}$.

[Why? As by 5.17(1)(c) we have $\mathscr{F}^A_\beta \subseteq \Pi\bar{\alpha}$ for each $A \in \mathscr{P}_*$.]

$(*)_2$ if $\beta_1 < \beta_2 < \partial, f_1 \in \mathscr{F}^*_{\beta_1}$ and $f_2 \in \mathscr{F}^*_{\beta_2}$ then $f_1 < f_2 \mod D_*$.

[Why? Note that $A \in \mathscr{P}_* \Rightarrow f_1 <_{D_A} f_2$ by the choice of $\langle \mathscr{F}^*_\beta : \beta < \partial \rangle$, hence the set $\{t \in Y : f_1(t) < f_2(t)\}$ belongs to D_A for every $A \in \mathscr{P}_*$ hence by $\boxplus_2$ it belongs to D_* which means that $f_1 <_{D_*} f_2$ as required.]

$(*)_3$ if $f \in \Pi\bar{\alpha}$ then for some $\beta_f < \partial$ we have $f' \in \cup\{\mathscr{F}^*_\beta : \beta \in [\beta_f, \partial)\} \Rightarrow f < f'$ mod D_*.

[Why? For each $A \in \mathscr{P}_*$ there are β, g such that $\beta < \partial, g \in \mathscr{F}^A_\beta$ and $f < g$ mod D hence $\beta' \in [\beta+1, \partial) \wedge f' \in \mathscr{F}^A_{\beta'} \Rightarrow f < g < f'$ mod D_A. Let β_A be the minimal such ordinal $\beta < \delta$. As $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y)) \geq \mathrm{hrtg}(\mathscr{P}_*)$, clearly $\beta_* = \sup\{\beta_A + 1 : A \in \mathscr{P}_*\}$ is $< \delta$. So $A \in \mathscr{P}_* \wedge g \in \cup\{\mathscr{F}^*_\beta : \beta \in [\beta_*, \delta)) \Rightarrow f <_{D_A} g$. By $\boxplus_2$ the ordinal α_* is as required on β_f.]

Moreover

$(*)_4$ there is a function $f \mapsto \beta_f$ in $(*)_3$.

[Why? As we can (and will) choose β_f as the minimal β such that ...]

$(*)_5$ for every $\beta_* < \partial$ there is $\beta \in (\beta_*, \partial)$ such that $\mathscr{F}^*_\beta \neq \emptyset$.

[Why? We choose by induction on n, a sequence $\bar{\beta}_n = \langle \beta_{n,A} : A \in \mathscr{P}_* \rangle$ and a sequence $\bar{f}_n = \langle f_{n,A} : A \in \mathscr{P}_* \rangle$ and a function f_n such that

(α) $\beta_n < \partial$ and $m < n \Rightarrow \beta_m < \beta_n$
(β) $\beta_0 = \beta_*$ and for $n > 0$ we let $\beta_n = \sup\{\beta_{m,A} : m < n, A \in \mathscr{P}_*\}$
(γ) $\beta_{n,A} \in (\beta_n, \partial)$ is minimal such that there is $f_{n,A} \in \mathscr{F}^A_{\beta_{n,A}}$ satisfying $n = m+1 \Rightarrow f_m < f_{\beta_n,A}$ mod D_A
(δ) $\langle f_{n,A} : A \in \mathscr{P}_* \rangle$ is a sequence such that each $f_{n,A}$ are as in clause (γ)
(ε) $f_n \in \Pi\bar{\alpha}$ is defined by $f_n(t) = \sup\{f_{m,A}(t) + 1 : A \in \mathscr{P}_*$ and $m < n\}$.

[Why can we carry the induction? Arriving to n first, f_n is well defined $\in \Pi\bar{\alpha}$ by clause (ε) as $\mathrm{cf}(\alpha_t) \geq \mathrm{hrtg}(\mathscr{P}_*)$ for $t \in Y$. Second by clause (γ) and the choice of $\langle \langle \bar{\mathscr{F}}^A_\beta : \beta < \partial \rangle : A \in \mathscr{P}_* \rangle$ in $(*)_0$ the sequence $\langle \beta_{n,A} : A \in \mathscr{P}_* \rangle$ is well defined. Third by clause (δ) we can choose $\langle f_{m,A} : A \in \mathscr{P}_* \rangle$ because we have $\mathrm{AC}_{\mathscr{P}_*}$. Fourth, β_n is well defined by clause (β) as $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}_*)$.

Lastly, the inductive construction is possibly by DC.]

Let $\beta^* = \cup\{\beta_n : n < \omega\}$ and $f = \sup\langle f_n : n < \omega \rangle$. Easily $f \in \cap\{\mathscr{F}^A_{\beta^*} : A \in \mathscr{P}_*\}$ as each $\langle \mathscr{F}^A_\beta : \beta < \partial \rangle$ is $\aleph_0$-continuous.]

$(*)_6$ if $f \in \Pi\bar{\alpha}$ then for some $\beta < \gamma$ and $f' \in \mathscr{F}^*_\beta$ we have $f < f'$ mod D^*.

[Why? By $(*)_3 + (*)_5$.]

So we are done. $\square_{5.19}$

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