LONG ITERATIONS FOR THE CONTINUUM SH707

SAHARON SHELAH

Abstract. We deal with an iteration theorem for proper $\aleph_2$-c.c. forcing notions with a kind of countable support. We then look at some special cases ($\mathbb{Q}_D$'s preceded by random forcing).

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Annotated content

0. Introduction

This is a modest try to investigate iterations $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \alpha^* \rangle$ which increase the continuum arbitrarily. The support is countable, but defining $p \le q$, only for finitely many $\alpha \in \mathrm{Dom}(p)$ we are allowed to fail to have pure extension. More explicitly, every $p \in \mathbb{Q}_\alpha$ has a "trunk" $\mathrm{tr}(p)$, the apure part, and we demand that $\langle \mathrm{tr}(p(\alpha)) : \alpha \in \mathrm{Dom}(p) \rangle$ is an "old" element, i.e. a function from $\mathbf{V}$. In this context we have a quite explicit form of properness which guarantees $\aleph_1$ is not collapsed. Assuming CH there are reasonable conditions guaranteeing the $\aleph_2$-c.c.

We may be more liberal in the first step of the iteration. We then concentrate on more specific context. We let $\mathbb{Q}_0$ be Random_A, adding a sequence of random reals $\langle \underset{\sim}{\nu}_\gamma : \gamma \in A \rangle$, and each $\mathbb{Q}_\alpha = \mathbb{Q}_{1+\beta}$ is $\mathbb{Q}_{\underset{\sim}{\bar{D}}_\alpha, \bar{D}_\alpha} = \langle \underset{\sim}{D}^\alpha_\eta : \eta \in {}^{\omega>}\omega \rangle$, $\underset{\sim}{D}^\alpha_\eta$ a $\mathbb{P}_\alpha$-name of a non-principal ultrafilter on ω. However, for the results we have in mind, $\underset{\sim}{D}^\alpha_\eta$ should satisfy some special properties: in the direction of being a Ramsey ultrafilter. If $\mathbb{Q}_0 = \mathrm{Random}_\lambda$, we may try to demand that for every $\mathbf{r} \in \mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}})}$, for "most" $\beta < \lambda, \underset{\sim}{\nu}_\beta$ is random over $\mathbf{V}[\mathbf{r}]$. We do not know to do it, <u>but</u> if we can restrict ourselves to measure 1 sets of the form $\cup\{\lim(T^{<n>}) : n < \omega\}, T$ a subtree of ${}^{\omega>}2$ with the fastness of convergence of $\langle |2^n \cap T|/2^n : n < \omega \rangle$ to $\mathrm{Leb}(\lim(T))$ bounded by $g \in \mathbf{V}$, moreover this holds above any $\eta \in {}^{\omega>}2$. This is a "poor relative" of the "Borel conjecture + $\mathfrak{b}$ large".

The method seems to me more versatile than the method of first forcing whatever and then forcing with the random algebra.

Lastly in §7 we deal with a relative of [**?**]. We thank the referee and Andrzej Roslanowski for infinite many helpful remarks and corrections.

1. Trunk Controllers

We define in 2.1 the notion "$\mathscr{F}$ is based on $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$", note that it is used in iterations $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \alpha^*, \beta < \alpha^* \rangle$ with $\underset{\sim}{\mathbb{Q}}_\beta$ an $\mathscr{F}_\beta$-forcing notion.

The reader may use only the fully based case, and ignore 1.20 (associativity).

Definition 1.1. 1) A trunk controller $\mathscr{F}$ is a set or a class with quasi-orders $\le = \le^{\mathscr{F}}$ and $\le_{\rm pr} = \le^{\mathscr{F}}_{\rm pr}$ (pr for the pure) and $\le_{\rm apr} = \le^{\mathscr{F}}_{\rm apr}$ (apr for the apure) such that: $\le_{\rm pr} \subseteq \le$ and $\le_{\rm apr} \subseteq \le$.
2) We may denote $\le^{\mathscr{F}}$ by $\le_{\rm us} = \le^{\mathscr{F}}_{\rm us}$ (us for the usual).
3) A trunk controller $\mathscr{F}$ is $\aleph_1$-complete if $(\mathscr{F}, \le_{\rm pr})$ is $\aleph_1$-complete.
4) A trunk controller $\mathscr{F}$ is an iteration of $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$ <u>if</u>:

- (a) each $\mathscr{F}_\beta$ is a trunk controller,
- (b) if $f \in \mathscr{F}$ then [1] ${\rm Dom}(f) \subseteq \alpha$ is countable and $f \in \mathscr{F} and \beta \in {\rm Dom}(f) \Rightarrow f(\beta) \in \mathscr{F}_\beta$
- (c) if $f_1, f_2 \in \mathscr{F}$ <u>then</u> $f_1 \le^{\mathscr{F}}_{\rm pr} f_2$ iff ${\rm Dom}(f_1) \subseteq {\rm Dom}(f_2) and (\forall \beta \in {\rm Dom}(f_1))[\mathscr{F}_\beta \models f_1(\beta) \le_{\rm pr} f_2(\beta)]$
- (d) $f_1 \le^{\mathscr{F}}_{\rm us} f_2$ <u>iff</u>
  - (i) ${\rm Dom}(f_1) \subseteq {\rm Dom}(f_2)$,
  - (ii) $\beta \in {\rm Dom}(f_1) \Rightarrow \mathscr{F}_\beta \models f_1(\beta) \le_{\rm us} f_2(\beta)$ and
  - (iii) $\{\beta \in {\rm Dom}(f_1) : \mathscr{F}_\beta \models f_1(\beta) \nleq_{\rm pr} f_2(\beta)\}$ is finite
- (e) if $f, g \in \mathscr{F}, {\rm Dom}(f) \subseteq \beta < \alpha, g \restriction \beta \le_{\rm x} f$, <u>then</u> $f \cup (g \restriction [\beta, \alpha)) \in \mathscr{F}$ is $\le_{\rm x}$-lub of $\{f, g\}$ for $x \in \{{\rm us}, {\rm pr}\}$, also if $f_n \subseteq f_{n+1} \in \mathscr{F}$ for $n < \omega$ and $\bigcup\limits_n f_n \in \mathscr{F}$ then $\bigcup\limits_n f_n$ is a $\le_{\rm pr}$-lub of $\{f_n : n < \omega\}$
- (f) $f_1 \le^{\mathscr{F}}_{\rm apr} f_2$ <u>iff</u>
  - (i) ${\rm Dom}(f_1) = {\rm Dom}(f_2)$,
  - (ii) $\beta \in {\rm Dom}(f_1) \Rightarrow \mathscr{F}_\beta \models f_1(\beta) \le f_2(\beta)$, moreover
  - (iii) the set $\{\beta \in {\rm Dom}(f_1) : f_1(\beta) \ne f_2(\beta)\}$ is finite and for those β's, $\mathscr{F}_\beta \models f_1(\beta) \le_{\rm apr} f_2(\beta)$
- (g) if $f \in \mathscr{F}$ and $\beta < \alpha$ <u>then</u> $f \restriction \beta \in \mathscr{F}$.

4A) In part (4), for $\beta < \alpha$ let $\mathscr{F}^{[\beta]} = \mathscr{F}[\beta]$ be $\mathscr{F}_\beta$, (clearly normally uniquely defined). If $\mathscr{F}$ is a trunk controller, an iteration of $\bar{\mathscr{F}} = \langle \mathscr{F}_\gamma : \gamma < \alpha \rangle$ and $\beta \le \alpha$ then let $\mathscr{F} \restriction \beta = \{f \in \mathscr{F} : {\rm Dom}(f) \subseteq \beta\}$.
5) We say a trunk controller $\mathscr{F}$ is a <u>full</u> iteration of $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$, when:

- (α) $\mathscr{F}$ is an iteration of $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$,
- (β) whenever f is a function with domain a countable subset of α such that $\beta \in {\rm Dom}(f) \Rightarrow f(\beta) \in \mathscr{F}_\beta$ then $f \in \mathscr{F}$.

6) We say a trunk controller $\mathscr{F}$ is finitely based on $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$ <u>when</u>:

- (α) $\mathscr{F}$ is an iteration of $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$,
- (β) $0 \in \mathscr{F}_\beta$ minimal (for every $\beta < \alpha$),
- (γ) $f \in \mathscr{F}$ iff f is a function with domain a countable subset of α and $\{\beta \in {\rm Dom}(f) : \neg(0 \le_{\rm pr} f(\beta))\}$ is finite.

[1]note that we did not say "iff", this is a reasonable assumption, see part (5)

7) We say $\mathscr{F}$ is the trivial trunk controller if: its set of elements is $\mathscr{H}(\aleph_0)$ and $\le = \le_{\rm pr} = \le_{\rm apr}$ are the equality on $\mathscr{H}(\aleph_0)$.
8) In part (4), (5), replacing $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$ by α means "for some $\langle \mathscr{F}_\beta : \beta < \alpha \rangle$".

We now define when a trunk controller is "simple". The aim of simple is helping with proving a forcing in $\aleph_2$-c.c.

Definition 1.2. 1) We say a trunk controller $\mathscr{F}$ is simple (or satisfies the pure $S^{\aleph_2}_{\aleph_1}$-c.c.) if: for any sequence $\langle y_\beta : \beta < \omega_2 \rangle$ of members of $\mathscr{F}$ for some club E of ω_2 and pressing down function $h : E \to \omega_2$ we have: for any ordinals $\varepsilon < \zeta$ from E of cofinality $\aleph_1$ we have $h(\varepsilon) = h(\zeta) \Rightarrow y_\varepsilon, y_\zeta$ have a common $\le^{\mathscr{F}}_{\rm pr}$-upper bound.
2) We say the trunk controller $\mathscr{F}$ is almost simple (or satisfies the $\aleph_2$-c.c.) if: for any sequence $\langle y_\gamma : \gamma < \omega_2 \rangle$ of members of $\mathscr{F}$ for some $\varepsilon < \zeta < \omega_2$, there is a common $\le$-upper bound of y_ε, y_ζ.
3) We say the trunk controller $\mathscr{F}$ is a semi-simple iteration of $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \alpha^* \rangle$ (or $\bar{\mathscr{F}}$ is) when it is an iteration of $\bar{\mathscr{F}}, \mathscr{F}_0$ is almost simple and every $\mathscr{F}_{1+\beta}$ is simple. We say "simple iteration" if also $\mathscr{F}_0$ is simple.

Claim 1.3. *Suppose that $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \alpha^* \rangle$ is a sequence of trunk controllers.*
1) There is a unique trunk controller $\mathscr{F}$ which is the full iteration of $\bar{\mathscr{F}}$.
2) Assume CH. If $\bar{\mathscr{F}}$ is semi-simple, i.e. $\mathscr{F}_0$ is an almost simple trunk controller and each $\mathscr{F}_{1+\beta}$ is a simple trunk controller whenever $1+\beta < \alpha^*$, then the $\mathscr{F}$ from part (1) is almost simple.
3) In part (2) if, $\bar{\mathscr{F}}$ is simple, i.e. also $\mathscr{F}_0$ is simple, then $\mathscr{F}$ is simple.
4) If each $\mathscr{F}_\beta$ is $\aleph_1$-complete, see 1.1(3), then in part (1) also $\mathscr{F}$ is $\aleph_1$-complete.

Proof. 1),4) Are straightforward.
2), 3) For part (2) let $\gamma(*) = 0$ and for part (3) let $\gamma(*) = -1$.
Let $f_\varepsilon \in \mathscr{F}$ for $\varepsilon < \omega_2$ and for each $\gamma \in \cup\{{\rm Dom}(f_\varepsilon) : \varepsilon < \omega_2\}$ define $\bar{y}_\gamma = \langle y_{\gamma,\varepsilon} : \varepsilon < \omega_2 \rangle$ by

$$y_{\gamma,\varepsilon} = \begin{cases} f_\varepsilon(\gamma) \ \underline{\rm if} \gamma \in {\rm Dom}(f_\varepsilon) \\ f_{\min\{\zeta<\omega_2 : \gamma \in {\rm Dom}(p_\zeta)\}}(\gamma) \ \underline{\rm if} \gamma \notin {\rm Dom}(f_\varepsilon). \end{cases}$$

So $y_{\gamma,\varepsilon} \in \mathscr{F}_\gamma$ for $\varepsilon < \omega_2$, hence by the assumption if $\gamma \ne \gamma(*)$ then $\mathscr{F}_\gamma$ is simple, so there is a club E_γ of ω_2 and a regressive function h_γ on S^2_1 such that:

$(*)$ if $\varepsilon_1, \varepsilon_2 \in E_\gamma \cap S^{\aleph_2}_{\aleph_1}$ and $h_\gamma(\varepsilon_1) = h_\gamma(\varepsilon_2)$ then $y_{\gamma,\varepsilon_1}, y_{\gamma,\varepsilon_2}$ has a common $\le^{\mathscr{F}_\gamma}_{\rm pr}$-upper bound.

Let cd be a one-to-one function from the following set into ω_2:

$$\{g : g \text{ is a function into } \omega_2 \text{ from some ountable subset of } \cup\{{\rm Dom}(f_\varepsilon) : \varepsilon < \omega_2\}\}.$$

Lastly, let E be the set of ordinals ε satisfying

$\odot$ (a) $\varepsilon < \omega_2$ is a limit ordinal,
(b) $\varepsilon \in \cap\{E_\gamma : \gamma \in \cup\{{\rm Dom}(f_\zeta) : \zeta < \varepsilon\}\}$
(c) if $\zeta < \varepsilon$ and g is a function from a countable subset of $\cup\{{\rm Dom}(f_\xi) : \xi < \zeta\}$ into ζ then ${\rm cd}(g) < \varepsilon$.

Clearly E is a club of ω_2. For $\varepsilon \in S^{\aleph_2}_{\aleph_1} \cap E$ let g_ε be the function with domain $\mathrm{Dom}(f_\varepsilon) \cap (\cup\{\mathrm{Dom}(f_\zeta) : \zeta < \varepsilon\})$ satisfying $g_\varepsilon(\gamma) := h_\gamma(\varepsilon)$ for $\gamma \in \mathrm{Dom}(g_\varepsilon)$.

Lastly, we define the function h with domain $E \cap S^{\aleph_2}_{\aleph_1}$ by $h(\varepsilon) = \mathrm{cd}(g_\varepsilon)$.

Easily,

$(*)_1$ if $\varepsilon \in S^{\aleph_2}_{\aleph_1} \cap E$ then $h(\varepsilon)$ is an ordinal $< \varepsilon$

$(*)_2$ if $\varepsilon_1 < \varepsilon_2$ belong to $S^{\aleph_2}_{\aleph_1} \cap E$ and $h(\varepsilon_1) = h(\varepsilon_2)$ then
- (a) $g_{\varepsilon_1} = g_{\varepsilon_2}$,
- (b) $\mathrm{Dom}(f_{\varepsilon_1}) \cap \mathrm{Dom}(f_{\varepsilon_2}) = \mathrm{Dom}(g_{\varepsilon_1}) = \mathrm{Dom}(g_{\varepsilon_2})$,
- (c) if $\gamma \in \mathrm{Dom}(f_{\varepsilon_1}) \cap \mathrm{Dom}(f_{\varepsilon_2})$ and $\gamma \neq \gamma(*)$ <u>then</u> $f_{\varepsilon_1}(\gamma), f_{\varepsilon_2}(\gamma)$ have a common $\leq^{\mathscr{F}_\gamma}_{\mathrm{pr}}$-upper bound (they are $y_{\gamma,\varepsilon_1}, y_{\gamma,\varepsilon_2}$, of course).

$(*)_3$ If $\varepsilon_1 < \varepsilon_2$ are as in $(*)_2$ and satisfy $\circledast$ below, <u>then</u> the function f defined in $(*)_4$ below is a common $\leq^{\mathscr{F}}_{\mathrm{pr}}$-upper bound of $f_{\varepsilon_1}, f_{\varepsilon_2}$ where
- $\circledast$ if $\gamma(*) = 0$ and it belongs to $\mathrm{Dom}(f_{\varepsilon_1}) \cap \mathrm{Dom}(f_{\varepsilon_2})$ then $f_{\varepsilon_1}(0), f_{\varepsilon_2}(0)$ has a common $\leq^{\mathscr{F}_0}_{\mathrm{pr}}$-upper bound

$(*)_4$ we choose f as follows:
- (a) $\mathrm{Dom}(f) = \mathrm{Dom}(f_{\varepsilon_1}) \cup \mathrm{Dom}(f_{\varepsilon_2})$,
- (b) if $\ell \in \{1,2\}$ and $\gamma \in \mathrm{Dom}(f_{\varepsilon_\ell}) \backslash \mathrm{Dom}(f_{3-\ell})$ then $f(\gamma) = f_{\varepsilon_\ell}(\gamma)$,
- (c) if $\gamma \in \mathrm{Dom}(f_{\varepsilon_1}) \cap \mathrm{Dom}(f_{\varepsilon_2})$ <u>then</u> $f(\gamma) \in \mathscr{F}_\gamma$ is a common $\leq^{\mathscr{F}_\gamma}_{\mathrm{pr}}$-upper bound of $f_{\varepsilon_1}(\gamma), f_{\varepsilon_2}(\gamma)$.

For part (3) we are done, for part (2) clearly there are $\varepsilon_1 < \varepsilon_2$ in $S^{\aleph_2}_{\aleph_1} \cap E$ such that $y_{0,\varepsilon_1}, y_{0,\varepsilon_2}$ have a common $\leq^{\mathscr{F}_0}$-upper bound if $0 \in \cup\{\mathrm{Dom}(f_\zeta) : \zeta < \omega_1\}$. Now we choose (f) such that

$(*)_5$ $(a), (b)$ of $(*)_4$
- (c) like (c) of $(*)_4$ for $\gamma \neq 0$
- (d) $f(0)$ is a common $\leq^{\mathscr{F}_0}$-upper bound of $y_{0,\varepsilon_1}, y_{0,\varepsilon}$.

So we are done. $\square_{1.3}$

Claim 1.4. *1) If a trunk controller $\mathscr{F}$ is an iteration of $\bar{\mathscr{F}} = \langle \mathscr{F}_\gamma : \gamma < \alpha \rangle$ and $\beta \leq \alpha$ <u>then</u> $\mathscr{F} \restriction \beta$ is a trunk controller, an iteration of $\langle \mathscr{F}_\gamma : \gamma < \beta \rangle$.*
1A) Similarly for "full iteration", "being simple iteration", "being semi-simple iteration".
2) If $\mathscr{F}$ is simple, <u>then</u> $\mathscr{F}$ is almost-simple.

Convention 1.5. Let $\leq^{\mathscr{F}}_{\mathrm{us}} = \leq^{\mathscr{F}}$ and we write $\leq^{\mathscr{F}}_{\mathrm{x}}$ for x varying on $\{\mathrm{us}, \mathrm{pr}, \mathrm{apr}\}$. Similarly in Definition 1.6 below.

Below we define "$\mathbb{Q}$ is an $\mathscr{F}$-forcing", the intention is that $\mathbb{Q}$ is a possible iterand. Note we define below "very clear" and "weakly clear" as conditions on $\mathbb{Q}$ helping to prove the $\aleph_2$-c.c. Now weakly clear suffices whereas very clear is preserved in the iterations (the problem in preserving "weakly clear" is when defining a common upper bound to have its val being an "old" function not just a name).

Definition 1.6. 1) Let $\mathscr{F}$ be a trunk controller. An $\mathscr{F}$-forcing notion $\mathbb{Q}$ is a tuple $(Q, \leq, \leq_{\mathrm{pr}}, \leq_{\mathrm{apr}}, \mathrm{val})$, we may put superscript $\mathbb{Q}$ to clarify, satisfying:

- (a) Q is a non-empty set (- the set of conditions) (we may write $p \in \mathbb{Q}$ instead of $p \in Q$ and say $\mathbb{Q}$-names, etc. and $(\mathbb{Q}, \leq)$ instead of $(Q, \leq)$)

(b) $\leq, \leq_{\rm pr}, \leq_{\rm apr}$ are quasi-orders on $\mathbb{Q}$ (called the usual, the pure and the apure)

(c) $\leq_{\rm pr} \subseteq \leq$ and $\leq_{\rm apr} \subseteq \leq$

(d) val is a function from Q to (and usually but not always [2] onto) $\mathscr{F}$, the trunk controller

(e) $\mathbb{Q} \models p \leq_x q \Rightarrow \mathscr{F} \models \mathrm{val}^{\mathbb{Q}}(p) \leq_x \mathrm{val}^{\mathbb{Q}}(q)$ for $x = $ us, pr, apr.

2) An $\mathscr{F}$-forcing $\mathbb{Q}$ is very clear (as an $\mathscr{F}$-forcing) or is a very clear $\mathscr{F}$-forcing <u>if</u>:

(∗) if $p_0, p_1 \in \mathbb{Q}$ and $\mathrm{val}^{\mathbb{Q}}(p_0)$, $\mathrm{val}^{\mathbb{Q}}(p_1)$ have a common $\leq^{\mathscr{F}}_{\rm pr}$-upper bound y <u>then</u> for some $q \in \mathbb{Q}$ we have $p_0 \leq_{\rm pr} q, p_1 \leq_{\rm pr} q$ and $\mathrm{val}^{\mathbb{Q}}(q) = y$.

3) An $\mathscr{F}$-forcing $\mathbb{Q}$ is weakly clear <u>when</u>:

if $p_0, p_1 \in \mathbb{Q}$ and $\mathrm{val}^{\mathbb{Q}}(p_0), \mathrm{val}^{\mathbb{Q}}(p_1)$ are $\leq_{\rm pr}$-compatible in $\mathscr{F}$, <u>then</u> p_0, p_1 are $\leq_{\rm pr}$-compatible.

4) An $\mathscr{F}$-forcing $\mathbb{Q}$ is apurely clear when: if $p_0, p_1 \in \mathbb{Q}$ and $\mathrm{val}^{\mathbb{Q}}(p_0), \mathrm{val}^{\mathbb{Q}}(p_1)$ are $\leq$-compatible in $\mathscr{F}$ then p_0, p_1 are $\leq$-compatible in $\mathbb{Q}$.

Discussion 1.7. We can consider some variants: if $p \leq_{\rm pr} q_\ell$ for $\ell = 1, 2$, do we just ask q_1, q_2 compatible? Does it suffice to demand "$\mathrm{val}(q_1), \mathrm{val}(q_2)$ are $\leq^{\mathscr{F}}_{\rm pr}$-compatible"? The natural examples satisfy this but the general theorems do not need it.

Claim 1.8. *1) For an $\mathscr{F}$-forcing $\mathbb{Q}$: very clear implies weakly clear.*
2) Assume $\mathbb{Q}$ is an apurely clear $\mathscr{F}$-forcing and $\mathscr{F}$ is almost simple, <u>then</u> $\mathbb{Q}$ satisfies the $\aleph_2$-c.c.
3) Assume $\mathbb{Q}$ is an $\mathscr{F}$-forcing, $\mathbb{Q}$ is weakly clear and $\mathscr{F}$ is simple, <u>then</u> $\mathbb{Q}$ and even $(\mathbb{Q}, \leq_{\rm pr})$ satisfies the regressive $S^{\aleph_2}_{\aleph_1}$-c.c., see Definition 1.9 below.

Proof. Straight. □

Definition 1.9. 1) We say that a quasi order $\mathbb{P}$ satisfies the regressive S-c.c. where S is a stationary subset of some regular uncountable cardinal κ <u>when</u> for every sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of members of $\mathbb{P}$ there are a club C of κ and a regressive function f on $S \cap C$ (i.e. $\mathrm{Dom}(f) = S \cap C$ and for every $\alpha \in S \cap C$ we have $f(\alpha) < \alpha$) such that :
if $\alpha, \beta \in S \cap C$ and $h(\alpha) = h(\beta)$ then p_α, p_β are compatible in $\mathbb{P}$.
2) We say "$\mathbb{P}$ satisfies purely regressive S-c.c" if $(\mathbb{P}, \leq^{\mathbb{P}}_{\rm pr})$ satisfies the regressive S-c.c.

Definition 1.10. Let trunk controller $\mathscr{F}$ be an iteration of $\langle \mathscr{F}_\beta : \beta < \alpha^* \rangle$. We define by induction on the ordinal $\alpha \leq \alpha^*$ what is an $\mathscr{F}$-iteration $\bar{\mathbb{Q}}$ of length α and what is $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$.

(a) $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration of length α <u>when</u>:
  (α) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$
  (β) if $\beta < \alpha$ then $\bar{\mathbb{Q}} \restriction \beta$ is an $\mathscr{F}$-iteration of length β

[2]Needed in the iteration, so actually what we need is that the range of the function val from $\underset{\sim}{\mathbb{Q}}_\alpha$ is an object not $\mathbb{P}_\alpha$-name. We waive it for the first step in the iteration, but this may cause us extra demand in associativity of the iteration if phrased not carefully.

(γ) if $\beta < \alpha$ is a limit ordinal then $\mathbb{P}_\beta = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}} \restriction \beta)$

(δ) if $\alpha = \beta + 1$ then $\underset{\sim}{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name of an $\mathscr{F}_\beta$-forcing notion

(ε) $\mathrm{Rang}(\mathrm{val}^{\underset{\sim}{\mathbb{Q}}_\beta})$ is an object not just a $\mathbb{P}_\beta$-name

(b) for an $\mathscr{F}$-iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ of length α we define the $\mathscr{F}$-forcing notion $\mathbb{P}_\alpha = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$ as follows (see 1.13):

(α) the set of elements of $\mathbb{P}_\alpha$ is the set of p such that for some $f \in \mathscr{F}$ we have

(i) p is a function

(ii) $\mathrm{Dom}(p) = \mathrm{Dom}(f)$, so it is a countable subset of α,

(iii) if $\beta \in \mathrm{Dom}(p)$ then $p(\beta)$ is a $\mathbb{P}_\beta$-name of a member of $\underset{\sim}{\mathbb{Q}}_\beta$,

(iv) $\Vdash_{\mathbb{P}_\beta}$ "$\mathrm{val}^{\underset{\sim}{\mathbb{Q}}_\beta}(p(\beta)) = f(\beta)$" for $\beta \in \mathrm{Dom}(p)$,

(v) $\beta < \alpha \Rightarrow p \restriction \beta \in \mathbb{P}_\beta$.

Clearly f is unique and we call it $\mathrm{val}^{\mathbb{P}_\alpha}(p)$.

(β) $\leq^{\mathbb{P}_\alpha}_{\mathrm{pr}}$ is defined by:

$p \leq^{\mathbb{P}_\alpha}_{\mathrm{pr}} q$ <u>iff</u> ($p, q \in \mathbb{P}_\alpha$ and) $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$ and

$$\beta \in \mathrm{Dom}(p) \Rightarrow q \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}p(\beta) \leq^{\underset{\sim}{\mathbb{Q}}_\beta}_{\mathrm{pr}} q(\beta)\text{”}.$$

(γ) $\leq^{\mathbb{P}_\alpha}$ is defined by:

$p \leq^{\mathbb{P}_\alpha} q$ <u>iff</u> ($p, q \in \mathbb{P}_\alpha$ and)

(i) $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$ and

(ii) $\beta \in \mathrm{Dom}(p) \Rightarrow q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p(\beta) \leq^{\underset{\sim}{\mathbb{Q}}_\beta} q(\beta)$" and

(iii) for some finite $w \subseteq \mathrm{Dom}(p)$ for every $\beta \in \mathrm{Dom}(p) \backslash w$ we have $q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p(\beta) \leq^{\underset{\sim}{\mathbb{Q}}_\beta}_{\mathrm{pr}} q(\beta)$"

(δ) $\leq^{\mathbb{P}_\alpha}_{\mathrm{apr}}$ is defined by

$p \leq^{\mathbb{P}_\alpha}_{\mathrm{apr}} q$ <u>iff</u> ($p, q \in \mathbb{P}_\alpha$ and)

(i) $\mathrm{Dom}(p) = \mathrm{Dom}(q)$ and

(ii) $p \leq q$ and (actually follows from the rest)

(iii) $\beta \in \mathrm{Dom}(p) \Rightarrow q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p(\beta) \leq_{\mathrm{apr}} q(\beta)$ in $\underset{\sim}{\mathbb{Q}}_\beta$"

(iv) for all but finitely many $\beta \in \mathrm{Dom}(p)$ we have $q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p(\beta) \leq^{\underset{\sim}{\mathbb{Q}}_\beta}_{\mathrm{pr}} q(\beta)$".

Remark 1.11. Note the difference between Clause (b)(δ)(iv) of Definition 1.10 which deals with iterated forcing and clause (f)(iii) of Definition 1.1(4) which deals with iterated trunks.

Convention 1.12. If $\mathscr{F}$ and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ are as in 1.10 then $\mathbb{P}_\alpha = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$.

Claim 1.13. *Let $\mathscr{F}$ be a trunk controller iteration of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$. If $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration and $\beta \leq \ell g(\bar{\mathbb{Q}})$, <u>then</u> $\mathbb{P}_\beta$ is a ($\mathscr{F} \restriction \beta$)-forcing.*

Proof. This is proved by induction on β. The proof is straight. $\square_{1.13}$

Claim 1.14. *Assume $\mathscr{F}$ is a trunk controller iteration of $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \alpha^* \rangle$; moreover is a full iteration of $\bar{\mathscr{F}}$ (see Definition 1.1(5)) and $\alpha \leq \alpha^*$ and $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration of length α and $\gamma \leq \beta \leq \alpha$.*

1) If $p \in \mathbb{P}_\beta$ <u>then</u> $p \restriction \gamma \in \mathbb{P}_\gamma$ and $\mathbb{P}_\beta \models$ "$p \restriction \gamma \le_{\rm pr} p$".
2) $\mathbb{P}_\gamma \subseteq \mathbb{P}_\beta$, i.e. $p \in \mathbb{P}_\gamma \Rightarrow p \in \mathbb{P}_\beta$ and $\le^{\mathbb{P}_\gamma}_x = \le^{\mathbb{P}_\beta}_x \restriction \mathbb{P}_\gamma$ (see convention 1.5).
3) If $p \in \mathbb{P}_\beta, x \in \{{\rm us}, {\rm pr}, {\rm apr}\}, p \restriction \gamma \le^{\mathbb{P}_\gamma}_x q \in \mathbb{P}_\gamma$ and $r = q \cup (p \restriction [\gamma, \beta))$, <u>then</u> r is $\le^{\mathbb{P}_\beta}_{\rm x}$-lub of $\{p, q\}$ when $x \in \{{\rm us}, {\rm pr}\}$ and $p \le_{\rm apr} r$ and $q \le_{\rm pr} r$ when ${\rm x} = {\rm apr}$.
4) If $p \le^{\mathbb{P}_\beta}_x q$ then $(p{\restriction}\gamma) \le^{\mathbb{P}_\gamma}_x (q{\restriction}\gamma 0.$
5) $\mathbb{P}_\gamma \lessdot \mathbb{P}_\beta$ and $\mathbb{P}_{\gamma+1}/\mathbb{P}_\gamma$ is equivalent (and even isomorphic) to $\underset{\sim}{\mathbb{Q}}_\gamma$.

Proof. Straight. $\square$

Claim 1.15. *Assume $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration of length α and $\mathbb{P} = {\rm Lim}(\bar{\mathbb{Q}})$ where $\mathscr{F}$ is the iteration of $\bar{\mathscr{F}} = \langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$.*
1) The property "very clear", 1.6(4) is preserved, i.e. if each $\underset{\sim}{\mathbb{Q}}_\beta\,(\beta < \alpha)$ is very clear, then so is $\mathbb{P} = {\rm Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$.
2) If $\mathscr{F}$ is [almost] simple and $\bar{\mathbb{Q}}$ is very clear $\mathscr{F}$-iteration, see Definition 1.6(4) <u>then</u> ${\rm Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$ is [almost] simple hence satisfies the $\aleph_2$-c.c.
3) The property "weakly clear" is preserved.

Proof. Straight. $\square$

Remark 1.16. However in $\mathscr{F}$-iterations where $\mathbb{Q}_0$ is only apurely clear (the case holds by the following we use), no clarity is preserved, but $\aleph_2$-c.c. still holds.

So putting things together we get

Conclusion 1.17. The forcing notion $\mathbb{P} = {\rm Lim}(\bar{\mathbb{Q}})$ satisfies the $\aleph_2$-c.c. <u>when</u>

- (a) the trunk controller $\mathscr{F}$ is the full iteration of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$
- (b) each $\mathscr{F}_\alpha$ is simple (see Definition 1.2(1))
- (c) $\bar{\mathbb{Q}}$ is a $\mathscr{F}$-iteration (see Definition 1.6(1))
- (d) each $\underset{\sim}{\mathbb{Q}}_\alpha$ is very clear (see Definition 2.5(3)) and weakly clear for $\alpha = 0$.

Proof. By Claim 1.3(3) we know that $\mathscr{F}$ is simple. By Definition 1.10 we know that $\mathbb{P}$ is an $\mathscr{F}$-forcing. By Claim 1.15 we know that $\mathbb{P}$ is very clear. By Claim 1.8(1) we know that $\mathbb{P}$ is weakly clear.

Lastly, by claim 1.8(2) we know that $\mathbb{P}$, even satisfies pure regressive $\aleph_2$-c.c. $\square_{1.17}$

We can weaken the hypothesis of 1.17

Conclusion 1.18. The forcing notion $\mathbb{P} = {\rm Lim}(\bar{\mathbb{Q}})$ satisfies the $\aleph_2$-c.c. <u>when</u>

- (a) $\mathscr{F}$ is the full iteration of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$
- $(b)_1$ $\mathscr{F}_\alpha$ is simple if $\alpha > 0$
- $(b)_2$ $\mathscr{F}_0$ is almost simple (see Definition 1.2(2))
- (c) $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration (see Definition 1.6)
- $(c)_1$ $\mathbb{Q}_\alpha$ is very clear if $\alpha > 0$ (see Definition 2.5)
- $(c)_2$ $\mathbb{Q}_0$ is apurely clear (see Definition 2.5(5)).

Proof. In the proof we use 1.3(2) instead of 1.3(3). $\square_{1.18}$

Claim 1.19. *Assume $\mathscr{F}$ is a trunk controller iteration of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$.*
0) The empty sequence is an $\mathscr{F}$-iteration. For every $\mathscr{F}_0$-forcing $\mathbb{Q}$ there is an $\mathscr{F}$-iteration $\bar{\mathbb{Q}}$ of length 1 such that $\mathbb{Q}_0 := \mathbb{Q}$.
1) If $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration of length $\alpha, \alpha + 1 \le \alpha^*, \mathbb{P}_\alpha = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name of an $\mathscr{F}_\alpha$-forcing notion, <u>then</u> there is a $\mathscr{F}$-iteration $\bar{\mathbb{Q}}'$ of length $\alpha+1$ such that $\bar{\mathbb{Q}}' \restriction \alpha = \bar{\mathbb{Q}}$ and $\underset{\sim}{\mathbb{Q}}'_\alpha = \underset{\sim}{\mathbb{Q}}$ that is $\bar{\mathbb{Q}}\hat{\ }\langle \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}), \underset{\sim}{\mathbb{Q}} \rangle$ is an $\mathscr{F}$-iteration.
2) If $\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ and α is a limit ordinal and $\bar{\mathbb{Q}} \restriction \beta$ is an $\mathscr{F}$-iteration for every $\beta < \alpha$ <u>then</u> $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration.
3) For any function $\mathbf{F}$ and ordinal $\alpha \le \alpha^*$ there is a unique $\mathscr{F}$-iteration $\bar{\mathbb{Q}}$ such that:

- (α) $\ell g(\bar{\mathbb{Q}}) \le \alpha$
- (β) $\beta < \ell g(\bar{\mathbb{Q}}) \Rightarrow \underset{\sim}{\mathbb{Q}}_\beta = \mathbf{F}(\bar{\mathbb{Q}} \restriction \beta)$
- (γ) *if $\beta := \ell g(\bar{\mathbb{Q}}) < \alpha$ <u>then</u> $\mathbf{F}(\bar{\mathbb{Q}})$ is not a $(\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}))$-name of an $\mathscr{F}_\beta$-forcing.*

Proof. Straight. $\square_{1.20}$

Not really necessary, but natural and aesthetic, is

Claim 1.20. *Associativity holds, that is assume*

- (a) *$\mathscr{F}$ is the iteration of $\langle \mathscr{F}_\beta : \beta < \alpha^* \rangle$*
- (b) *$\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha^* \rangle$ is an $\mathscr{F}$-iteration so $\mathbb{P}_{\alpha^*} = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$*
- (b) *$\langle \alpha_\varepsilon : \varepsilon \le \varepsilon^* \rangle$ is increasing continuous, $\alpha_0 = 0, \alpha_{\varepsilon^*} = \alpha^*$*
- (c) *for $\gamma \le \beta \le \alpha^*$ we define $\mathbb{P}_\beta / \mathbb{P}_\gamma$, an $\mathscr{F}$-forcing, naturally: it is a $\mathbb{P}_\gamma$-name and for $G_\gamma \subseteq \mathbb{P}_\gamma$ generic over $\mathbf{V}$ its interpretation is:*
  - (α) *the set of elements is $\{p \in \mathbb{P}_\beta : \mathrm{Dom}(p) \subseteq [\gamma, \beta)\}$*
  - (β) val*: inherited from $\mathbb{P}_\beta$ that is $\mathrm{val}^{\mathbb{P}_\beta/\mathbb{P}_\gamma}(p) = \mathrm{val}\mathbb{P}_\beta(p \restriction [\beta, \gamma])$*
  - (γ) $\le_x$: *$p \le_x q$ iff for some $r \in G_\gamma$ we have $\mathbb{P}_\beta \models r \cup p \le_x r \cup q$*
- (d) (α) *let $\mathscr{F}' = \{f :$ for some $g \in \mathscr{F}, f$ is a function with domain $\{\varepsilon < \varepsilon^* : \mathrm{Dom}(g) \cap [\alpha_\varepsilon, \alpha_{\varepsilon+1}) \neq \emptyset\}$ and $\varepsilon \in \mathrm{Dom}(f) \Rightarrow f(\varepsilon) = g \restriction [\alpha_\varepsilon, \alpha_{\varepsilon+1})\}$, the orders are natural*
- (e) (α) $\mathscr{F}_\varepsilon = \{f \in \mathscr{F} : \mathrm{Dom}(f) \subseteq [\alpha_\varepsilon, \alpha_{\varepsilon+1})\}$
  - (β) $\le_x^{\mathscr{F}_\varepsilon} = \le_x^{\mathscr{F}} \restriction \mathscr{F}_\varepsilon$.

<u>Then</u> $\Vdash$ "$\mathbb{P}_{\alpha_{\varepsilon+1}} / \mathbb{P}_{\alpha_\varepsilon}$ is an $\mathscr{F}_\varepsilon$-forcing" and we can find an $\mathscr{F}'$-iteration $\bar{\mathbb{Q}}' = \langle \mathbb{P}'_\varepsilon, \underset{\sim}{\mathbb{Q}}'_\varepsilon : \varepsilon < \varepsilon^* \rangle$ and $\langle \underset{\sim}{F}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ such that

- (α) *F_ε is an isomorphism from $\mathbb{P}_{\alpha_\varepsilon}$ onto $\mathbb{P}'_\varepsilon$*
- (β) *when $\varepsilon < \varepsilon^*, F_\varepsilon$ maps the $\mathbb{P}_{\alpha_\varepsilon}$-name $\mathbb{P}_{\alpha_{\varepsilon+1}} / \mathbb{P}_{\alpha_\varepsilon}$ to the $\mathbb{P}'_\varepsilon$-name $\underset{\sim}{\mathbb{Q}}'_\varepsilon$.*

Proof. Straight. $\square$

2. Being $\mathscr{F}$-Pseudo c.c.c. is preserved by $\mathscr{F}$-iterations

Our aim is a sufficient condition for not collapsing $\aleph_1$ preserved by our iteration. We would like to define, in Definition 2.1, what is a $\mathscr{F}$-pseudo c.c.c. forcing. We will also define a function $\mathbf{H}$, as a witness. Note that:

- (a) the point of $\mathbf{H}$ is that it may be in the ground model (as is $\mathscr{F}$ but not $\mathbb{Q}$)
- (b) $\mathbf{H}$ really stands for three functions but as we shall use $\langle \mathbf{H}_\alpha : \alpha < \alpha^* \rangle$ corresponding to the length of the iteration we prefer not to use $\langle \mathbf{H}_\ell : \ell < 3 \rangle$.

In the main cases, $\mathbf{H}$ disappears but ∂_p is needed for proving properties of (the limit of the) iteration.

Definition 2.1. 1) Let $\mathscr{F}$ be a trunk controller, $\mathbb{Q}$ be an $\mathscr{F}$-forcing notion. We say that $\mathbb{Q}$ is $\mathscr{F}$-psc ($\mathscr{F}$-pseudo c.c.c. in full) forcing (notion) as witnessed by $\mathbf{H}$ <u>if</u>:

for every $p \in \mathbb{Q}$, in the following game $\partial_p = \partial_{p,\mathbb{Q},\mathbf{H}} = \partial_p[\mathbb{Q}, \mathbf{H}]$ between two players, Interpolator and Extender, which lasts ω_1 moves, the Interpolator has a winning strategy.

In the ζ-th move:

- $\boxtimes$ the Interpolator chooses a condition p'_ζ such that $p \leq_{\rm pr} p'_\zeta, \varepsilon < \zeta \Rightarrow \mathscr{F} \models {\rm val}^{\mathbb{Q}}(p_\varepsilon) \leq_{\rm pr} {\rm val}^{\mathbb{Q}}(p'_\zeta)$ and[3] ${\rm val}^{\mathbb{Q}}(p'_\zeta) = \mathbf{H}((\langle {\rm val}^{\mathbb{Q}}(p_\xi),\ {\rm val}^{\mathbb{Q}}(q_\xi)) : \xi < \zeta \rangle)$ and then the Extender chooses $q_\zeta \in \mathbb{Q}$ (we <u>do not</u> required the natural[4] demand $p'_\zeta \leq q_\zeta$) and lastly the Interpolator chooses a condition p_ζ such that $p'_\zeta \leq_{\rm pr} p_\zeta$ and $p'_\zeta \leq q_\zeta \Rightarrow p_\zeta \leq_{\rm apr} q_\zeta$ and $p'_\zeta \leq_{\rm pr} q_\zeta \Rightarrow p_\zeta \leq_{\rm pr} q_\zeta$ and ${\rm val}^{\mathbb{Q}}(p_\zeta) = \mathbf{H}(\langle ({\rm val}^{\mathbb{Q}}(p_\xi), {\rm val}^{\mathbb{Q}}(q_\xi)) : \xi < \zeta \rangle \hat{} \langle {\rm val}^{\mathbb{Q}}(q_\zeta) \rangle)$. [For future notation let $q_{-1} = p$].

A play is won by the Interpolator if:

- (α) for any stationary $A \subseteq \omega_1$, for some $B \subseteq A$ we have
  - $(*)$ B is a stationary subset of ω_1 and $\mathbf{H}(\langle ({\rm val}^{\mathbb{Q}}(p_\xi), {\rm val}^{\mathbb{Q}}(q_\xi)) : \xi < \omega_1 \rangle \hat{} \langle B \rangle) = 1$
- (β) if $B \subseteq \omega_1$ satisfies $(*)$ then: for every $\varepsilon < \zeta$ from B we have: q_ε, q_ζ are compatible in $\mathbb{Q}$ <u>if</u> ${\rm val}^{\mathbb{Q}}(q_\varepsilon), {\rm val}^{\mathbb{Q}}(q_\zeta)$ are compatible in $\mathscr{F}$ and ${\rm val}(p_\varepsilon) \leq {\rm val}(q_\varepsilon), {\rm val}(p_\zeta) \leq {\rm val}(q_\zeta)$.
  (in the case the Extender chooses a weird q_ζ).
- (γ) for $E = \omega_1$ or just E a club of ω_1 computed from $\langle ({\rm val}^{\mathbb{Q}}(p_\varepsilon), {\rm val}^{\mathbb{Q}}(q_\varepsilon)) : \varepsilon < \omega_1 \rangle$ we have: if $\varepsilon < \zeta$ are from E, $p_\varepsilon \leq_{\rm pr} q_\varepsilon$ and $p_\zeta \leq_{\rm pr} q_\zeta$, <u>then</u> q_ε, q_ζ have a common $\leq_{\rm pr}$-upper bound q, with ${\rm val}^{\mathbb{Q}}(q) = \mathbf{H}(\varepsilon, \zeta, \langle ({\rm val}^{\mathbb{Q}}(p_\xi), {\rm val}^{\mathbb{Q}}(q_\xi)) : \xi \leq \zeta \rangle)$.
  [Yes, we use $\leq_{\rm pr}$; true, we have demanded $p_\varepsilon \leq_{\rm apr} q_\varepsilon$ (and $p_\zeta \leq_{\rm apr} q_\zeta$) but this does not exclude $p_\varepsilon \leq_{\rm pr} q_\varepsilon$ and even $p_\varepsilon = q_\varepsilon$.
  Why not just $\varepsilon < \zeta$ from B? For the iteration theorem 2.12.
  Note that this is a requirement on $\mathscr{F}$. Note that $p_\varepsilon \leq_{\rm pr} q_\varepsilon, p_\zeta \leq_{\rm pr} q_\zeta$ is not guaranteed.]

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[3] note that $p_\varepsilon \leq_{\rm pr} p'_\zeta$ is not demanded; the following demand is needed just in order to show that if $\mathscr{F}$ is as in 2.3, then clause (β) below is not empty

[4] our not demanding "$p'_\zeta \leq q_\zeta$" is used in the proof of 2.12(1)

2) We define "$\mathbb{Q}$ is $(\mathscr{F}, \mathscr{P})$-psc as witnessed by $(\mathbf{H}, \mathscr{P})$" as above (when $\mathscr{P}$ includes, among other things, some stationary subsets of ω_1; usually, $\mathscr{P}$ is $\mathbf{V}$, or some inner model $\mathbf{V}'$) but at the end defining when a play is won by the Interpolator, we make the changes:

- $(\alpha)'$ in every limit stage the Interpolator has a legal move <u>or</u> the sequence $\langle (\mathrm{val}^{\mathbb{Q}}(p_\xi), \mathrm{val}^{\mathbb{Q}}(q_\xi)) : \xi < \zeta \rangle$ is not in $\mathscr{P}$ and he wins immediately
- $(\beta)'$ if $\langle (\mathrm{val}^{\mathbb{Q}}(p_\zeta), \mathrm{val}^{\mathbb{Q}}(q_\zeta)) : \zeta < \omega_1 \rangle \in \mathscr{P}$ <u>then</u> for every stationary set $A \in \mathscr{P}$ of ω_1, there is a stationary subset $B \in \mathscr{P}$ of ω_1 as there.

(We may use $(\mathscr{F}, \mathbf{V}'), \mathbf{V}'$ an inner model. In this case normally $\mathbf{H}$ and $\mathscr{F}$ are from $\mathbf{V}'$. This means that the Interpolator player does not "cheat" making the play end prematurely because he has to "obey" $\mathbf{H}$, whereas the Extender player is "not motivated" to cheat as then he loses the play.)
3) In the description of the game, we can replace $\langle (\mathrm{val}^{\mathbb{Q}}(p_\varepsilon),\mathrm{val}^{\mathbb{Q}}(q_\varepsilon)) : \varepsilon < \zeta \rangle$ by $\langle \mathrm{val}^{\mathbb{Q}}(q_\varepsilon) : -1 \leq \varepsilon < \zeta \rangle$.
4) If we omit $\mathbf{H}$ and, to stress it we may say <u>bare</u>, this means that: we just omit the relevant demands on the Interpolator in $\boxtimes$ and in $(*)$ of (α) of part (1), just requiring that (β) and (γ) hold.
4A) If we omit clause (γ) of (1) we say "weakly $\mathscr{F}$-psc".

Remark 2.2. 1) Clause (γ) is used in the proof of the iteration claim 2.12, so we need it there on each $\underset{\sim}{\mathbb{Q}}_\beta$.
2) To prove clause (γ) it on the limit $\mathbb{P}_\alpha$ is not really needed (as clause (γ) of 2.1(2) is needed only for the iteration claim, i.e., so we need it about the $\underset{\sim}{\mathbb{Q}}_\beta$'s, but for the limit $\mathbb{P}_\beta$ it will be needed only if we like to deal with the associativity law).

Definition 2.3. $\mathscr{F}$ satisfies the apure c.c.c. when: if $\langle y_\varepsilon : \varepsilon < \omega_1 \rangle$ is $\leq^{\mathscr{F}}_{\mathrm{pr}}$-increasing and $y_\varepsilon \leq^{\mathscr{F}}_{\mathrm{apr}} z_\varepsilon$ for every $\varepsilon < \omega_1$ <u>then</u> there are $\varepsilon < \zeta < \omega_1$ such that z_ε, z_ζ are compatible in $\mathscr{F}$.

Remark 2.4. We may combine Definition 2.1(1), 2.3, that is in 2.1(1) we omit in $\boxtimes$ the demand "$\varepsilon < \zeta \Rightarrow \mathscr{F} \models \mathrm{val}^{\mathbb{Q}}(p_\varepsilon) \leq_{\mathrm{pr}} \mathrm{val}^{\mathbb{Q}}(p_\zeta)$" but to clause (β) we add:

- $(\beta)'$ if $B \subseteq \omega_1$ satisfies $(*)$ then for some $\varepsilon < \zeta$ from $B, \mathrm{val}^{\mathbb{Q}}(p_\varepsilon), \mathrm{val}^{\mathbb{Q}}(p_\zeta)$ are compatible in $\mathscr{F}$.

Definition 2.5. We say $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-psc iteration as witnessed by $\bar{\mathbf{H}}$ <u>if</u>:

- (a) $\mathscr{F}$ is a trunk controller, a full iteration of length α'
- (b) $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration so $\ell g(\bar{\mathbb{Q}}) \leq \alpha'$
- (c) for every $\beta < \ell g(\bar{\mathbb{Q}})$ we have $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is an $(\mathscr{F}^{[\beta]}, \mathbf{V})$-psc forcing notion as witnessed by $\mathbf{H}_\beta$" and $\bar{\mathbf{H}} = \langle \mathbf{H}_\beta : \beta < \ell g(\bar{\mathbb{Q}}) \rangle$; note that $\mathbf{H}_\beta \in \mathbf{V}$ and is an object, not a $\mathbb{P}_\beta$-name.

Definition 2.6. Let $\mathscr{F}$ be a trunk controller and $\mathbb{Q}$ be an $\mathscr{F}$-forcing.
1) We say that $\mathbb{Q}$ is a strong $\mathscr{F}$-psc forcing notion as witnessed by $\mathbf{H}$ <u>when</u> for every $p \in \mathbb{Q}$ in the game $\eth'_p = \eth'_{p,\mathbb{Q},\mathbf{H}} = \eth'_p[\mathbb{Q}, \mathbf{H}]$ the Interpolator has a winning strategy, where the game is defined as in 2.1 except that in addition we demand

$\circledast$ $\varepsilon < \zeta \Rightarrow \mathbb{Q} \models p_\varepsilon \leq_{\mathrm{pr}} p'_\zeta$.

(Recall that by Definition 2.1(1) $\varepsilon < \zeta \Rightarrow \mathscr{F} \models$ "$\mathrm{val}(p_\varepsilon) \le_{\mathrm{pr}} \mathrm{val}(p'_\zeta)$").
2) We say strong* <u>when</u> we change $\circledast$ to

$\circledast'$ $\varepsilon < \zeta \Rightarrow \mathbb{Q} \models p_\varepsilon \le p'_\zeta$ and recall $p \le_{\mathrm{pr}} p'_\zeta$.

3) Saying "an iteration $\bar{\mathbb{Q}}$ is strong$^{(*)}$" in Definition 2.5 means that this holds for each $\underset{\sim}{\mathbb{Q}}_\beta$.

Definition 2.7. A forcing notion $\mathbb{Q}$ is purely proper <u>when</u>:

- (a) $\mathbb{Q} = (Q, \le, \le_{\mathrm{pr}})$ where $\le_{\mathrm{pr}} \subseteq \le$
- (b) if χ is large enough $\mathbb{Q} \in N \prec (\mathscr{H}(\chi), \in)$ and N is countable and $p \in N \cap Q$ <u>then</u> there is q such that $p \le_{\mathrm{pr}} q \in Q$ which is $(N, \mathbb{Q})$-generic.

Claim 2.8. *1) If $\mathbb{Q}$ is strong $\mathscr{F}$-psc, <u>then</u> $\mathbb{Q}$ is strong* $\mathscr{F}$-psc.*
2) Assume

- (a) *$\mathbb{Q}$ is a σ-centered forcing notion, i.e. $\mathbb{Q} = \bigcup\limits_{n<\omega} R_n$ each R_n directed and for simplicity may assume $n \neq m \Rightarrow R_m \cap R_m = \emptyset$ and each R_n is non-empty*
- (b) *$\mathscr{F}$ is such that its set of elements is ω and $\mathscr{F} \models (\forall n < \omega)(\forall y \in \mathscr{F})[n \le_{\mathrm{pr}} y \Rightarrow n = y]$ similarly for $\le_{\mathrm{apr}}$*
- (c) *$\mathbb{Q}$, i.e. $(Q, \le, \le_{\mathrm{pr}}, \le_{\mathrm{apr}}, \mathrm{val})$ is defined by:*
  - (α) *$(Q^{\mathbb{Q}}, \le^{\mathbb{Q}})$ is $\mathbb{Q}$,*
  - (β) *$\le^{\mathbb{Q}}_{\mathrm{pr}}$ is equality*
  - (γ) *$\le^{\mathbb{Q}}_{\mathrm{apr}}$ is $\le^{\mathbb{Q}}$*
  - (δ) *$\mathrm{val}^{\mathbb{Q}}(q) = \mathrm{Min}\{n : q \in R_n\}$.*

<u>Then</u> $\mathscr{F}$ is a simple trunk controller satisfying the apure c.c.c. and $\mathbb{Q}$ is a very clear $\mathscr{F}$-forcing which is $\mathscr{F}$-psc and purely proper. (See Definition 2.7).
3) Assume

- (a) *$\mathbb{Q}$ is a forcing notion, α^* is an ordinal, $\bar{h} = \langle h_q : q \in \mathbb{Q}\rangle$ is such that*
  - (α) *h_q is a finite (partial) function from α^* to ω*
  - (β) *if $h = h_{q_1} \cup h_{q_2}$ is a function then q_1, q_2 has a least common upper bound q with $h_q = h_{q_1} \cup h_{q_2}$*
  - (γ) *if $q_1 \le q_3$ then for some $q_2, q_1 \le q_2 \le q_3$ and $h_{q_1} \subseteq h_{q_2} \wedge \mathrm{Dom}(h_{q_1}) = \mathrm{Dom}(h_{q_2})$*
  - (δ) *if $q_1 \le q_2$ then $\mathrm{Dom}(h_{q_1}) \subseteq \mathrm{Dom}(h_{q_2})$*
- (b) *$\mathscr{F}$ is a trunk controller whose set of elements is $\{h_q : q \in \mathbb{Q}\}$ such that $h_{q_1} \subseteq h_{q_2} \Leftrightarrow \mathscr{F} \models h_{q_1} \le h_{q_2} \Leftrightarrow \mathscr{F} \models h_{q_1} \le_{\mathrm{pr}} h_{q_2}$*
- (c) *$(Q, \le^{\mathbb{Q}}, \le^{\mathbb{Q}}, \le^{\mathbb{Q}}_{\mathrm{pr}}, \le^{\mathbb{Q}}_{\mathrm{apr}}, \mathrm{val})$ is defined by*
  - (α) *$(Q, \le^{\mathbb{Q}})$ is $\mathbb{Q}$, the forcing*
  - (β) *$p \le^{\mathbb{Q}}_{\mathrm{pr}} q$ <u>iff</u> $p \le q \wedge h_p \subseteq h_q$*
  - (γ) *$p \le^{\mathbb{Q}}_{\mathrm{apr}} q$ <u>iff</u> $p \le q$*
  - (δ) *$\mathrm{val}^{\mathbb{Q}}(q) = h_q$.*

<u>Then</u> $\mathbb{Q}$ is $\mathscr{F}$-psc.

Proof. 1) Trivial.
2) See the proof of 5.3.
3) The Interpolator choose $p_\varepsilon = p = p'_\varepsilon$. $\square_{??}$

Claim 2.9. *Assume $\mathscr{F}$ is an apure c.c.c. trunk controller, $\mathbb{Q}$ is $\mathscr{F}$-psc.*
1) If $p \in \mathbb{Q}$ and $\underset{\sim}{\tau}_m$ is a $\mathbb{Q}$-name of an ordinal for $m < \omega$, then for some q and $\langle \alpha_n : n < \omega \rangle$ we have:

- (a) $p \le_{\rm pr} q$
- (b) $q \Vdash$ "$\underset{\sim}{\tau}_m \in \{\alpha_n : n < \omega\}$ *for* $m < \omega$".

2) If $\mathbb{Q}$ is a strong* $\mathscr{F}$-psc, then $\mathbb{Q}$ is purely proper.
3) Moreover assuming $\mathbb{Q}$ is strong $\mathscr{F}$-psc, if $\mathbb{Q} \in N \prec (\mathscr{H}(\chi), \in), N$ countable, $p \in \mathbb{Q} \cap N$ then we can find q such that $p \le_{\rm pr} q$ and for every $\mathbb{Q}$-name $\underset{\sim}{\tau}$ of a member of $\mathbf{V}, \mathscr{I}^q_{\underset{\sim}{\tau}} = \{r$: for some p' we have $p \le_{\rm pr} p' \le_{\rm pr} q, p' \in N, p' \le_{\rm apr} r$ and r forces a value to $\underset{\sim}{\tau}\}$ is predense above q.
4) Assume $\mathbb{Q}$ is $\mathscr{F}$-psc; $p \in \mathbb{Q}, \bar N = \langle N_\varepsilon : \varepsilon < \omega_1 \rangle$ is an increasing continuous sequence of countable elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$ such that $\mathscr{F}, \mathbb{Q}, p$ belong to N_0 and $\bar N \restriction (\varepsilon + 1) \in N_{\varepsilon+1}$. Then for a club of $\varepsilon < \omega_1$, there is $p' \in \mathbb{Q} \cap N_{\varepsilon+1}$ such that $p \le_{\rm pr} p'$ and p' is $(N_\varepsilon, \mathbb{Q})$-generic.
5) If $\mathbb{Q}$ is strong* $\mathscr{F}$-psc and $\bar N$ is as in part (4), then for every $\varepsilon < \omega_1$ there is $p' \in \mathbb{Q} \cap N_{\varepsilon+1}$ such that $p \le_{\rm pr} p'$ and p' is $(\bar N \restriction (\varepsilon+1), \mathbb{Q})$-generic; hence $\mathbb{Q}$ is purely ε-proper for every $\varepsilon < \omega_1$.

Remark 2.10. 1) Note that if ${\rm Ax}^+(\aleph_1\text{-complete})$ or just ${\rm Ax}^+({\rm Levy}(\aleph_1, 2^{|\mathbb{Q}|}))$ see e.g. [?, XVII,§1] (slightly more than every stationary $\mathscr{S} \subseteq [2^{|\mathbb{Q}|}]^{\aleph_0}$ reflects in some $A \subseteq 2^{|\mathbb{Q}|}$ of cardinality $\aleph_1$) then by 2.9(4) also in 2.9(1) we can get purely properness. So the difference is very small and still strange.
2) Note also that, by 2.9(4), if $\bar{\mathbb{Q}}$ is an iterated $\mathscr{F}$-forcing, $\mathscr{F}$ is the trunk controller iteration of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$ and $\ell g(\bar{\mathbb{Q}}) = \delta$ is a limit ordinal of cofinality $> \aleph_0$ then $\Vdash_{\mathbb{P}_\delta}$ "$\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_\delta]} = \cup\{\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_\alpha]} : \alpha < \delta\}$".

Proof. Let $\mathbf{H}$ be a witness for "$\mathbb{Q}$ is psc".
1) Assume not and simulate a play of the game $\eth_p = \eth_{p,\mathbb{Q},\mathbf{H}}$, where the Interpolator plays using a fixed winning strategy whereas the Extender chooses q_ζ such that:

- (α) $p'_\zeta \le q_\zeta$ (see notation in 2.1(1))
  (i.e. a legal move)
- (β) for some $m_\zeta < \omega$ the condition q_ζ forces a value to $\underset{\sim}{\tau}_{m_\zeta}$, call it j_ζ
- (γ) $j_\zeta \notin \{j_\varepsilon : \varepsilon < \zeta\}$.

If the Extender can choose q_ζ for every $\zeta < \omega_1$, in the end $\varepsilon < \zeta and m_\varepsilon = m_\zeta \Rightarrow q_\varepsilon, q_\zeta$ are incompatible (as $j_\varepsilon \ne j_\zeta$) so there exists a stationary $B \subseteq \omega_1$, such that $\{\varepsilon, \zeta\} \subseteq B \Rightarrow q_\varepsilon \perp q_\zeta$ but the Interpolator has to win the play (as he has used his winning strategy); contradiction by (β) of 2.1(1) because $\mathscr{F}$ is apurely c.c.c. therefore in the sequence $\langle {\rm val}^{\mathbb{Q}}(q_\varepsilon) : \varepsilon \in B \rangle$ there are pairs of compatible members of $\mathscr{F}$ (recalling Definition 2.3).

So necessarily for some $\zeta < \omega_1$ there is no q_ζ as required. Let p^* be p'_ζ so $p \le_{\rm pr} p^*$ by the definition of the game. By our assumption toward contradiction, for some $m < \omega$ we have $p^* \nVdash$ "$\underset{\sim}{\tau}_m \in \{j_\varepsilon : \varepsilon < \zeta\}$" hence for some q we have $p^* \le q$ and $q \Vdash$ "$\underset{\sim}{\tau}_m \notin \{j_\varepsilon : \varepsilon < \zeta\}$". Let an ordinal $j \notin \{j_\varepsilon : \varepsilon < \zeta\}$ and condition q' be such that $q \le q' \in \mathbb{Q}$ we have $q' \Vdash$ "$\underset{\sim}{\tau}_m = j$". But then the Extender could have chosen $q_\zeta = q', j_\zeta = j$ and $m_\zeta = m$ and so clauses $(\alpha), (\beta), (\gamma)$ hold, a contradiction.

Note that we could have replaced clause (γ) by

$(\gamma)^-$ q_ζ is incompatible with q_ε whenever $\varepsilon < \zeta$ and $m_\zeta = m_\varepsilon$.

2) Let $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be countable, $\mathbb{Q} \in N$ and $\mathbf{H}, \mathbf{st} \in N$ and $p \in N \cap \mathbb{Q}$. Let $\langle \underset{\sim}{\tau}_n : n < \omega \rangle$ list the $\mathbb{Q}$-names of ordinals which belong to N. We define a strategy $\mathbf{St}_e$ for the Extender in the game ∂_p; so in stage ζ he has to choose q_ζ such that $p'_\zeta \le q_\zeta$. If for some n, for no countable set X of ordinals, $p'_\zeta \Vdash$ "$\underset{\sim}{\tau}_n \in X$" let $n(\zeta)$ be the minimal such n. As in the proof of the first part there are (q, j) such that: $p'_\zeta \le q$ and $q \Vdash$ "$\underset{\sim}{\tau}_{n(\zeta)} = j$" and $j \notin \{i :$ for some $\varepsilon < \zeta$ we have $q_\varepsilon \Vdash$ "$\underset{\sim}{\tau}_{n(\zeta)} = i$"$\}$, choose a $<^*_\chi$-minimal such pair, call it (q_ζ, j_ζ), so the Extender will choose q_ζ. If there is no such n, let $n(\zeta) = \omega$ and $q_\zeta = p'_\zeta$. Now in ∂_p the Interpolator has a winning strategy $\mathbf{St}_i$, without loss of generality $\mathbf{St}_i \in N$. Let $\langle p'_\zeta, p_\zeta, q_\zeta : \zeta < \omega_1 \rangle$ be a play where the Interpolator uses the strategy $\mathbf{St}_i$ and the Extender uses the strategy $\mathbf{St}_e$, clearly it exists and the Interpolator wins. Clearly $\varepsilon < \zeta < \omega_1 \Rightarrow n(\varepsilon) \le n(\zeta)$ (read the choices above, that is, if $\varepsilon < \zeta$ then $p_\varepsilon \le p'_\zeta$ (by $\circledast'$ of Definition 2.6, i.e. by "strong*") and $p'_\varepsilon \le_{\rm pr} p_\varepsilon$ (by $\boxtimes$ of 2.1(1)) hence $p'_\varepsilon \le p'_\zeta$). Now we prove by induction on n that for some $\zeta < \omega_1, n(\zeta) > n$ and let ζ_n be the minimal such ζ, so $p'_{\zeta_n} \Vdash$ "$\underset{\sim}{\tau}_n \in X_n$" for some countable set X_n of ordinals. If we fail for n, then: if ζ, ε satisfy $\bigcup\limits_{m<n} \zeta_m < \zeta < \varepsilon < \omega_1$ then $(q_\zeta \Vdash \underset{\sim}{\tau}_n = j_\zeta) and (q_\varepsilon \Vdash \underset{\sim}{\tau}_n = j_\varepsilon)$. But necessarily $j_\zeta \ne j_\varepsilon$ hence q_ζ, q_ε are incompatible, but this contradicts the use of $\mathbf{St}_i$.

Now for each $n < \omega$ the sequence $\langle p'_\zeta, p_\zeta, q_\zeta : \zeta < \zeta_n \rangle$ can be defined from p, $\mathbf{St}_i, \langle \underset{\sim}{\tau}_\ell : \ell \le n \rangle$ and $\mathbf{H}$ (read the definition of $\mathbf{St}_e$) hence $\langle p'_\zeta, p_\zeta, q_\zeta : \zeta < \zeta_n \rangle \in N$, so $\zeta_n \in N$, and similarly $p'_{\zeta_n} \in N$. So as $p'_{\zeta_n} \Vdash$ "$\underset{\sim}{\tau}_n \in X_n$", the set $\{\xi : p'_{\zeta_n} \nVdash$ "$\underset{\sim}{\tau}_n \ne \xi$"$\}$ is countable and it belongs to N. So $p'_{\zeta_n} \Vdash$ "$\underset{\sim}{\tau}_n \in N \cap {\rm Ord}$". Now if $\zeta < \omega_1$ is $\ge \bigcup\limits_{n<\omega} \zeta_n$ then $n < \omega \Rightarrow p'_{\zeta_n} \le p_\zeta$ hence p'_ζ is $(N, \mathbb{Q})$-generic, so as $p \le_{\rm pr} p'_\zeta$ clearly p'_ζ witnesses the desired conclusion required by "purely proper".
3) As in the proof of (2) above but now $\zeta < \varepsilon \Rightarrow \mathbb{Q} \models p_\zeta \le_{\rm pr} p'_\varepsilon$ (by the "strong") hence the proof of (2) gives the desired conclusion.
4),5) Let $\delta_\varepsilon = N_\varepsilon \cap \omega_1$, for $\varepsilon < \omega_1$. We can find a sequence $\bar{\underset{\sim}{\tau}} = \langle \underset{\sim}{\tau}_\alpha : \alpha < \omega_1 \rangle$ such that we have $\bar{\underset{\sim}{\tau}} \restriction \delta_\varepsilon \in N_{\varepsilon+1}$ and $\bar{\underset{\sim}{\tau}} \restriction \delta_\varepsilon$ list the $\mathbb{Q}$-names of members of $\mathbf{V}$ from N_ε and $\alpha < \delta_\varepsilon \Rightarrow \bar{\tau}^\varepsilon \restriction \alpha \in N_\varepsilon$.
[Why? Let $\bar{\underset{\sim}{\tau}}^\varepsilon$ be the $<^*_\chi$-be the first sequence of length δ_ε enumerating the $\mathbb{Q}$-names of members of $\mathbf{V}$ from N_ε such that $\zeta < \varepsilon \Rightarrow \bar{\underset{\sim}{\tau}}^\zeta \triangleleft \bar{\underset{\sim}{\tau}}^\varepsilon$.]

We again simulate a play of the game such that

- (a) the Interpolator uses a fix winning strategy which belongs to N_0
- (b) the Extender chooses q_α so that:
  - (α) if possible for some $\gamma < \alpha$ the condition q_α forces a value x_α to $\underset{\sim}{\tau}_\gamma$ which is not forced by any q_β for $\beta < \alpha$
  - (β) modulo clause $(\alpha), \gamma$ is minimal and then (q_α, x_α) is $\le^*_\chi$-minimal.

Clearly the play up to the $(\delta_\varepsilon + 1)$-th move belongs to $N_{\varepsilon+1}$.

For part (4) let $S_1 = \{\alpha < \omega_1 : q_\alpha$ is not $(N_\alpha, \mathbb{Q})$-generic and $N_\alpha \cap \omega_1 = \alpha\}$.

Assume toward contradiction that S_1 is stationary. So for each $\alpha \in S_1$ there is $\gamma_\alpha < \alpha$ such that q_α forces a value to $\underset{\sim}{\tau}_{\gamma_\alpha}$ not forced by any q_β for $\beta < \alpha$ hence (by Fodor's lemma) there is a stationary $S_2 \subseteq S_1$ such that $\alpha \in S_2 \Rightarrow \gamma_\alpha = \gamma_*$. But we can find $\alpha_1 < \alpha_2$ in S_2 such that $q_{\alpha_1}, q_{\alpha_2}$ are compatible, easy contradiction.
For part (5), let $A = \{\varepsilon < \omega_1 : p'_\varepsilon$ is not $(N_\varepsilon, \mathbb{Q})$-generic$\}$. If A is stationary we get contradiction to "the Interpolator has won the play because he uses a winning strategy". So we are done. $\square_{2.9}$

Remark 2.11. Of course, if $(Q^{\mathbb{Q}}, \leq^{\mathbb{Q}}_{\text{pr}})$ is $\aleph_1$-complete, part (2) of 2.9 follows from part (1).
[Why? Let $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be countable such that $\mathbb{Q} \in N$ and $p \in \mathbb{Q} \cap N$. Let $\langle \underset{\sim}{\tau}_n : n < \omega \rangle$ list the $\mathbb{Q}$-names of members of $\mathbf{V}$ which belongs to N. We now choose (p_n, X_n) by induction on $n < \omega$ such that

$\circledast$ (*a*) $p_0 = p$
(*b*) $p_n \in \mathbb{Q} \cap N$
(*c*) $p_m \leq_{\text{pr}} p_n$ if $m < n$
(*d*) $X_n \in N$ is a countable subset of N so $\in \mathbf{V}$
(*e*) if $n = m + 1$ then $p_n \Vdash$ "$\underset{\sim}{\tau}_m \in X_n$".

For $n = 0$ this is trivial.
For $n = m + 1$ there is a pair (p_n, X_n) as required by clauses (c) + (e) by 2.9(1), so as $\underset{\sim}{\tau}_m, p_n, \mathbb{Q} \in N$ there is such a pair in N, so we are done. Now by the assumption there is $q \in \mathbb{Q}$ such that $n < \omega \Rightarrow p_n \leq_{\text{pr}} q$, so q is as required.]

Lemma 2.12. *Assume $\mathscr{F}$ is a full trunk controller iteration of $\bar{\mathscr{F}}$.*
1) If $\bar{\mathbb{Q}}$ is a $\mathscr{F}$-psc iteration and $\mathscr{F}$ has the apure c.c.c., then for every $\alpha \leq \ell g(\bar{\mathbb{Q}})$ the forcing notion $\mathbb{P}_\alpha$ is a $\mathscr{F}$-psc forcing notion.
2) Similarly with strong.
3) Saharon: you need also, similarly for strong$^+$ for claim 3.3 in the next section.

Proof. 1) Let $\bar{\mathbf{H}} = \langle \mathbf{H}_\alpha : \alpha < \ell g(\bar{\mathbb{Q}}) \rangle$ be a witness for "$\bar{\mathbb{Q}}$ is an $\mathscr{F}$-psc iteration". Let $\alpha \leq \ell g(\bar{\mathbb{Q}})$. We prove this by induction on α and let $p \in \mathbb{P}_\alpha$, and define $\mathbf{H}^\alpha$ naturally composing the $\langle \mathbf{H}_\gamma : \gamma < \alpha \rangle$ and we shall describe a winning strategy for the Interpolator in the game $\partial_p = \partial_{p, \mathbb{P}_\alpha, \mathbf{H}^\alpha}$. He just guarantees that:

$(*)_0$ (*a*) $\text{Dom}(p'_\zeta) = \cup\{\text{Dom}(p_\varepsilon) : \varepsilon < \zeta\}$
(*b*) $\text{Dom}(p_\zeta) = \text{Dom}(q_\zeta)$ if $p'_\zeta \leq q_\zeta$ and $\text{Dom}(p_\zeta) = \text{Dom}(p'_\zeta)$ otherwise
(*c*) if $p_\zeta(\gamma) \neq q_\zeta(\gamma)$ then $\gamma \in \bigcup\limits_{\varepsilon < \zeta} \text{Dom}(p_\varepsilon)$
(*d*) if $\gamma \in \bigcup\limits_{\zeta < \omega_1} \text{Dom}(p_\zeta)$ and $\xi(\gamma) = \xi_\gamma = \text{Min}\{\zeta : \gamma \in \text{Dom}(p_\zeta)\}$
<u>then</u> $\Vdash_{\mathbb{P}_\gamma}$ "$\langle p'_{\xi(\gamma)+1+\zeta}(\gamma), q_{\xi(\gamma)+1+\zeta}(\gamma), p_{\xi(\gamma)+1+\zeta}(\gamma) : \zeta < \omega_1 \rangle$ is a play of $\partial_{p_{\xi(\gamma)}(\gamma)}[\mathbb{Q}_\gamma, \mathbf{H}^\gamma]$ in which the Interpolator uses a fixed winning strategy $\underset{\sim}{\mathbf{St}}^\gamma_{p_{\xi_\gamma}(\gamma)}$".

[Why can he does it? The main point is to check that p'_ζ, p_ζ is well defined and belongs to $\mathbb{P}_\alpha$ (and satisfies the appropriate inequalities). The point is that even if the Extender plays "reasonably", i.e. $p'_\zeta \leq^{\mathbb{P}_\alpha}_{\text{pr}} q_\zeta$, we know that for every $\gamma \in \text{Dom}(p'_\zeta), q_\zeta \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$p'(\zeta) \leq q(\zeta)$" but this does not mean that $p_\zeta \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$

"$p'(\gamma) \le q(\gamma)$". However, the game was defined in Definition 2.1 such that, in particular for the game for $\mathop{\mathbb{Q}}\limits_{\sim}{}_\gamma$ it is not required that $\Vdash_{\mathbb{P}_\gamma}$ "$p'(\gamma) \le q(\gamma)$".

Let us elaborate. There is no problem for the Interpolator to choose p'_ζ.

If the Extender has chosen q_ζ, clearly p_ζ is a function with domain $\mathrm{Dom}(q_\zeta)$ and $p_\zeta(\beta)$ is a $\mathbb{P}_\beta$-name of a member of $\mathop{\mathbb{Q}}\limits_{\sim}{}_\beta$ with $\mathrm{val}^{\mathop{\mathbb{Q}}\limits_{\sim}{}_\beta(p_\zeta(\beta))}$ given by $\mathbf{H}_\beta$.]

Now

$(*)_1$ $p'_\zeta \le^{\mathbb{P}_\alpha}_{\mathrm{pr}} p_\zeta$.

[Why? $\mathrm{Dom}(p'_\zeta) \subseteq \mathrm{Dom}(p_\zeta)$ in both cases of clause (b) of $(*)_0$ and for every $\beta \in \mathrm{Dom}(p'_\zeta)$ we know that $p_\zeta(\beta)$ is as dictated by the strategy $\mathop{\mathbf{St}}\limits_{\sim}{}^\beta_{p_{\xi_\beta}(\beta)}$, hence purely extends $p'_\zeta(\beta)$.]

$(*)_2$ $p_\zeta \le^{\mathbb{P}_\alpha}_{\mathrm{apr}} q_\zeta$ if $\mathbb{P}_\alpha \models$ "$p'_\zeta \le q_\zeta$".

[Why? So assume $\mathbb{P}_\alpha \models$ "$p'_\zeta \le q_\zeta$". Now the conditions p_ζ, q_ζ have the same domain by clause (b) of $(*)_0$ and if $\beta \in \mathrm{Dom}(q_\zeta)\backslash\mathrm{Dom}(p'_\zeta)$, then $p_\zeta(\beta) = q_\zeta(\beta)$. If $\beta \in \mathrm{Dom}(p'_\zeta)$ then $q_\zeta \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p'_\zeta(\beta) \le q_\zeta(\beta)$ hence $p'_\zeta(\beta) \le_{\mathrm{pr}} p_\zeta(\beta) \le_{\mathrm{apr}} q_\zeta(\beta)$", by the strategy choice, so in particular $q_\zeta \restriction \beta \Vdash p_\zeta(\beta) \le_{\mathrm{apr}} q_\zeta(\beta)$.

Let $w_\zeta := \{\beta \in \mathrm{Dom}(p'_\zeta(\beta)) : q_\zeta \restriction \beta \nVdash_{\mathbb{P}_\beta}$ "$p'_\zeta(\beta) \le_{\mathrm{pr}} q_\zeta(\beta)$"$\}$, we know that w_ζ is finite by the definition of $\le^{\mathbb{P}_\alpha}$. Now for $\beta \in \mathrm{Dom}(p'_\zeta)\backslash w_\zeta$ we have $q_\zeta \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p'_\zeta(\beta) \le_{\mathrm{pr}} q_\zeta(\beta)$", this implies $q_\zeta \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p_\zeta(\beta) \le_{\mathrm{pr}} q_\zeta(\beta)$" by $\boxtimes$ of Definition 2.1(1). Together all the demands for the satisfaction of $\mathbb{P}_\alpha \models$ "$p_\zeta \le_{\mathrm{apr}} q_\zeta$" holds.]

$(*)_3$ $p_\zeta \le^{\mathbb{P}_\alpha}_{\mathrm{pr}} q_\zeta$ if $p'_\zeta \le^{\mathbb{P}_\alpha}_{\mathrm{pr}} q_\zeta$.

[Why? Assume $\mathbb{P}_\alpha \models$ "$p'_\zeta \le_{\mathrm{pr}} q_\zeta$". The proof above works with $w_\zeta = \emptyset$.]

By $(*)_0 - (*)_3$ all the demand in $\boxtimes$ of 2.1(1) holds, so the Interpolator can use this strategy, we still have to prove that it is a winning strategy.

Let $\langle p'_\zeta, q_\zeta, p_\zeta : \zeta < \omega_1\rangle$ be a play of the game $\partial_p[\mathbb{P}_\alpha]$ in which the Interpolator uses the strategy described above. To prove that the Interpolator wins the play, let $A \subseteq \omega_1$ be stationary and $A \in \mathbf{V}$, of course.

We shall use freely

$\circledast$ if $A \in \mathbf{V}$ is a stationary subset of ω_1 in $\mathbf{V}$ and $\gamma < \alpha$ then $\Vdash_{\mathbb{P}_\gamma}$ "A is a stationary subset of ω_1".

[Why? Note that 2.9(1) is not enough and 2.9(2) has an extra assumption but 2.9(4) is enough.]

For $\zeta \in \bigcup\limits_{\varepsilon<\omega_1} \mathrm{Dom}(p_\varepsilon)$ let E_ζ be as in clause (γ) of 2.1(1) for $\mathop{\mathbb{Q}}\limits_{\sim}{}_\zeta$ and let $E = \{\delta : \delta$ limit and $\zeta \in \bigcup\limits_{\varepsilon<\delta} \mathrm{Dom}(p_\varepsilon) \Rightarrow \delta \in E_\zeta\}$. Note that E is a club of ω_1, so $A \cap E$ is a stationary subset of ω_1. Define $w_\zeta = \{\gamma \in \mathrm{Dom}(p_\zeta) : q_\zeta \restriction \gamma \nVdash_{\mathbb{P}_\gamma}$ "$p_\zeta(\gamma) \le_{\mathrm{pr}} q_\zeta(\gamma)$"$\}$, so by $(*)_2$ we have "$p_\zeta \le_{\mathrm{apr}} q_\zeta$", hence by the definition of $\mathscr{F}$-iteration, w_ζ is finite, and by the strategy of the Interpolator we have $w_\zeta \subseteq \bigcup\limits_{\varepsilon<\zeta} \mathrm{Dom}(p_\varepsilon)$. So by Fodor lemma for some stationary $A_0 \subseteq A \cap E$ we have $\zeta \in A_0 \Rightarrow w_\zeta = w^*$.

Letting $w^* = \{\gamma_\ell : \ell < k\}$ be such that $\gamma_\ell < \gamma_{\ell+1}$, we choose by induction on $\ell \le k$ a stationary set $A_\ell \subseteq \omega_1$ (from $\mathbf{V}$, of course) such that $A_{\ell+1} \subseteq A_\ell$ and $\mathbf{H}_{\gamma_\ell}(\langle \mathrm{val}^{\mathop{\mathbb{Q}}\limits_{\sim}{}_{\gamma_\ell}}(q_{\xi(\gamma_\ell)+\zeta}(\gamma_\ell)) : -1 \le \zeta < \omega_1\rangle, A_{\ell+1})$ is 1. For $\ell = 0$, A_0 has already been chosen, for $\ell+1$, in $\mathbf{V}^{\mathbb{P}_{\gamma_\ell}}$, we know that $\langle (p'_{\xi(\gamma_\ell)+1+\zeta}(\gamma_\ell), p_{\xi(\gamma_\ell)+1+\zeta}(\gamma_\ell), q_{\xi(\gamma_\ell)+1+\zeta}(\gamma_\ell)) :$

$\zeta < \omega_1\rangle$ is a play of the game $\partial_{p_{\xi(\gamma_\ell)}(\gamma_\ell)}[\mathbb{Q}_{\gamma_\ell}, \mathbf{H}_{\gamma_\ell}]$ in which the Interpolator uses a winning strategy and $\langle(\mathrm{val}^{\mathbb{Q}_{\gamma_\ell}}(p_{\xi(\gamma_\ell)+1+\zeta}(\gamma_\ell)), \mathrm{val}^{\mathbb{Q}_{\gamma_\ell}}(q_{\xi(\gamma_\ell)+1+\zeta}(\gamma_\ell)) : \zeta < \omega_1\rangle$ is a sequence of pairs from $\mathscr{F}^{[\gamma_\ell]}$, and the sequence is from $\mathbf{V}$; recall we use $\bar{\mathbf{H}}$. So by clause (α) of Definition 2.1(1) there is a stationary $A_{\ell+1} \subseteq A_\ell$ as required above which means that it satisfies $(*)$ from clause (α) from 2.1(1) hence clause (β) applies. Lastly, let $B =: A_k$, we shall prove that B is as required, we concentrate on (β) of 2.1(1); the other clause, (γ), is similar.

Let $\varepsilon < \zeta$ from B be such that $\mathrm{val}^{\mathbb{P}_\alpha}(q_\varepsilon)$, $\mathrm{val}^{\mathbb{P}_\alpha}(q_\zeta)$ are compatible in $\mathscr{F}$ and we shall find r as required, i.e. is above q_ε, q_ζ. Stipulate $\gamma_k = \alpha$ and we now let $Y = \mathrm{Dom}(q_\zeta) \cup \{\alpha\}$ and we choose by induction on $\gamma \in Y$ a condition $r_\gamma \in \mathbb{P}_\gamma$ such that

$\circledast$ (i) $q_\varepsilon \restriction \gamma \le r_\gamma$ and

(ii) $q_\zeta \restriction \gamma \le r_\gamma$ and

(iii) if $\gamma = \beta + 1$ and $\beta \in \mathrm{Dom}(r_\gamma) \backslash w^*$ <u>then</u> $r_\gamma \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q_\varepsilon(\beta) \le_{\mathrm{pr}} r_\gamma(\beta)$ if $\beta \in \mathrm{Dom}(q_\varepsilon)$ and $q_\zeta(\beta) \le_{\mathrm{pr}} r_\gamma(\beta)$ if $\beta \in \mathrm{Dom}(q_\zeta)$"

(iv) if $\beta \in Y \cap \gamma$ and $[\beta, \gamma] \cap w^* = \emptyset$ then $r_\beta = r_\gamma \restriction \beta$.

<u>Case 1</u>: $\gamma = \mathrm{Min}(Y)$.

Let $r_\gamma = \emptyset$, the empty function.

<u>Case 2</u>: $\gamma \ne \mathrm{Min}(Y)$ but $\gamma \cap Y$ has no last element.

Let $r_\gamma = \cup\{r_\beta : \beta \in \gamma \cap Y$ and $\beta > \max(w^* \cap \gamma)$ if $w^* \cap \gamma \ne \emptyset\}$, now check; it is a well defined function by clause (iv). We use here "$\mathscr{F}$ is a full iteration", see Definition 1.1(5) to show $r_\gamma \in \mathbb{P}_\gamma$. Lastly note that for checking $q_\zeta \restriction \gamma \le r_\gamma$ and $q_\varepsilon \restriction \gamma \le r_\gamma$ we are using clause (iii).

<u>Case 3</u>: $Y \cap \gamma$ has last element $\beta, \beta \notin w^*$ and $\beta \notin \mathrm{Dom}(q_\varepsilon)$.

We let $\mathrm{Dom}(r_\gamma) = \mathrm{Dom}(r_\beta) \cup \{\beta\}, r_\gamma(\beta) = q_\zeta(\beta)$ and of course $r_\gamma \restriction \beta = r_\beta$.

<u>Case 4</u>: $Y \cap \gamma$ has last element $\beta, \beta \notin w^*$ but $\beta \in \mathrm{Dom}(q_\varepsilon)$.

Now, r_β necessarily forces that

$(*)_4$ $p_\zeta(\beta) \le_{\mathrm{pr}} q_\zeta(\beta)$ and $p_\varepsilon(\beta) \le_{\mathrm{pr}} q_\varepsilon(\beta)$.

So by clause (γ) of Definition 2.1(1), r_β forces that in $\mathbb{Q}_\beta$ there is r such that $q_\zeta(\beta) \le_{\mathrm{pr}} r, q_\varepsilon(\beta) \le_{\mathrm{pr}} r$, and $\mathrm{val}^{\mathbb{Q}_\beta}(r)$ is as it should be by clause (γ) of 2.1(1) (so it is an object, not just a $\mathbb{P}_\beta$-name). Lastly let $r_\gamma(\beta)$ be a $\mathbb{P}_\beta$-name of such a condition.

<u>Case 5</u>: $Y \cap \gamma$ has last element $\beta, \beta = \gamma_\ell \in w^*$.

By the choice of $A_{\ell+1}$, we know $\Vdash_{\mathbb{P}_\beta}$ "$q_\varepsilon(\beta), q_\zeta(\beta)$ are $\le^{\mathbb{Q}_\beta}$ compatible". So, for some $r'_\beta, r_\beta \le^{\mathbb{P}_\beta} r'_\beta$, for some $x \in \mathscr{F}^{[\beta]}$ the condition r'_β forces that some $\le^{\mathbb{Q}_\beta}$-common upper bound r'' of $p_\zeta(\beta), q_\zeta(\beta)$, satisfies $\mathrm{val}^{\mathbb{Q}_\beta}(r'') = x$, and let $r_\gamma = r'_\beta \cup \{(\beta, r'')\}$.

So we are done.

2) The proof for the "strong $\mathscr{F}$-psc" version is similar. In fact, the only additional point is $\mathbf{St}^{\gamma}_{p_{\xi_\gamma}}(\gamma)$ used in $(*)_0$ is a winning strategy for the Interpolator in the game

$\Game'_{p_{\xi_\gamma}(\gamma),\underset{\sim}{\mathbb{Q}}_\gamma,\mathbf{H}_\gamma}$. Then, of course, we have checked that the strategy we have defined satisfied also ⊛ of Definition 2.6(1), i.e. $\varepsilon < \zeta \Rightarrow p_\varepsilon \le_{\rm pr} p'_\zeta$. But this is easy. $\square_{2.12}$

3. Nicer pure properness and pure decidability

Is pure decidability preserved by the iteration? We give sufficient conditions.

Definition 3.1. 1) Let $\mathscr{F}$ be a trunk controller. We say that an $\mathscr{F}$-forcing $\mathbb{Q}$ is strongly$^+$ $\mathscr{F}$-psc forcing when in Definition 2.1, we:

- (a) strengthen $\boxtimes$ of 2.1(1) demanding (as in Definition 2.6) $\varepsilon < \zeta \Rightarrow p_\varepsilon \leq_{\rm pr} p'_\zeta$ (i.e. $\mathbb{Q}$ is strong $\mathscr{F}$-psc)
- (b) strengthen clause (β) of 2.1(1) adding: if $\varepsilon < \zeta$ are from the set B and $\mathrm{val}^{\mathbb{Q}}(q_\varepsilon), \mathrm{val}^{\mathbb{Q}}(q_\zeta)$ are compatible in $\mathscr{F}$ <u>then</u> (in addition to q_ε, q_ζ has a common upper bound) there[5] is r such that
  - (i) $p_\zeta \leq_{\rm apr} r$
  - (ii) $q_\varepsilon \leq r$
  - (iii) r is a lub of p_ζ, q_ε
  - (iv) if $p_\varepsilon \leq_{\rm pr} q_\varepsilon$ then $p_\zeta \leq_{\rm pr} r$.

2) For $\mathscr{F}$ an iterated trunk controller of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$ and a $\mathscr{F}$-psc iteration $\bar{\mathbb{Q}}$ we say $\bar{\mathbb{Q}}$ is strongly$^+$-psc (for $\bar{\mathbf{H}}$) when $\Vdash_{\mathbb{P}_\beta}$ "$\mathbb{Q}_\alpha$ is a strongly$^+$ $\mathscr{F}_\beta$-psc for $\mathbf{H}_\beta$" for every $\beta < \ell g(\bar{\mathbb{Q}})$.

Claim 3.2. *Assume*

- (a) *$\mathbb{Q}$ is forcing by a measure algebra or just $\mathbb{Q}$ satisfies the stronger version of the c.c.c. implicit in 2.1(1)(α): if $p_\alpha \in \mathbb{Q}$ for $\alpha < \omega_1$ and $A \subseteq \omega_1$ is stationary then for some stationary $B \subseteq A, \langle p_\alpha : \alpha \in B\rangle$ are pairwise compatible*
- (b) *the trunk controller $\mathscr{F}$ is defined by*
  - (α) *set of elements in $\mathbb{Q}$*
  - (β) *$\leq^{\mathscr{F}} = \leq^{\mathscr{F}}_{\rm apr}$ is the order of $\mathbb{Q}$*
  - (γ) *$\leq^{\mathscr{F}}_{\rm pr}$ is equality*
- (c) *$\mathbb{Q}$ is an $\mathscr{F}$-psc defined by:*
  - (α) *the orders are as in $\mathscr{F}$*
  - (β) *$\mathrm{val}^{\mathbb{Q}}(-)$ is the identity.*

<u>Then</u> *$\mathbb{Q}$ is a $\mathscr{F}-\mathrm{psc}$, even strong$^+$ and is apurely clear and $\mathscr{F}$ is a purely c.c.c.*

Proof. To show that $\mathbb{Q}$ is $\mathscr{F}-\mathrm{psc}$, we now define a strategy for the Interpolator player.

In the game $\partial_{p,\mathbb{Q}}$:

$\boxdot$ the Interpolator chooses $p_\zeta = p$ and $p'_\zeta = p$.

The demands in $\boxtimes$ of 2.1(1) are easy: as $\mathrm{val}^{\mathbb{Q}}(-)$ is constant the demands involving it are satisfied trivially. Also $\mathbb{Q} \models$ "$p'_\zeta \leq_{\rm pr} p_\zeta$" hold as it means $\mathbb{Q} \models$ "$p \leq_{\rm pr} p_\zeta$" which means $\mathbb{Q} \models$ "$p = p$ and $p_\zeta \leq q_\zeta$" which holds.

Second, $\mathbb{Q} \models$ "$p'_\zeta \leq q_\zeta$" $\Rightarrow \mathbb{Q} \models$ "$p_\zeta \leq_{\rm apr} q_\zeta$" is also trivial.

Third, $\mathbb{Q} \models$ "$p'_\zeta \leq_{\rm pr} q_\zeta$" means $p = p'_\zeta = q_\zeta$, so trivially $\mathbb{Q} \models p_\zeta \leq_{\rm pr} q_\zeta$.

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[5]Saharon recheck when Extender chooses weird q's

To prove that this is a winning strategy, for any play $\langle (p'_\zeta, q_\zeta, p_\zeta) : \zeta < \omega_1\rangle$ in which the interpolator uses this strategy, clauses $(\alpha), (\beta)$ hold by clause (c) of the assumption. As for clause (γ) it assumes $p_\varepsilon \le_{\rm pr} q_\varepsilon \wedge p_\zeta \le_{\rm pr} q_\zeta$ but then $q_\varepsilon = p \wedge q_\zeta = p$ so q_ε is a common $\le_{\rm pr}$-bound as required.

What about "strong$^+$"? See Definition 3.1(1). Clause (a) there says $\varepsilon < \zeta \Rightarrow p_\varepsilon \le_{\rm pr} p'_\zeta$ and it holds as the strategy guarantees $p_\varepsilon = p = p'_\zeta$. For clause (b) there it suffices to prove for $\varepsilon < \zeta$ from B that there is r such that (substituting the equalities we know)

- (i) $p \le r$
- (ii) $q_\varepsilon \le r$
- (iii) r is a lub of p, q_ε
- (iv) if $p = q_\varepsilon$ then $p = r$.

So $r := q_\varepsilon$ is as required. $\square_{3.2}$

Claim 3.3. *Let $\mathscr{F}$ be apure c.c.c. full trunk controller iteration of $\bar{\mathscr{F}}$ of length α^{**} and $\bar{\mathbb{Q}}$ be a strongly$^+$ $\mathscr{F}$-psc iteration and $\alpha^* = \ell g(\bar{\mathbb{Q}})$.*
1) If $p \Vdash_{\mathbb{P}_{\alpha^*}}$ "$\underset{\sim}{\tau}_n \in \mathbf{V}$" for $n < \omega$, <u>then</u> we can find q and $\mathscr{I}_n (n < \omega)$ such that: $p \le_{\rm pr} q \in \mathbb{P}_{\alpha^*}$ and for each $n < \omega$ we have $(*)_{q, \mathscr{I}_n, \underset{\sim}{\tau}_n}$, which means:

- (a) *$q \in \mathbb{P}_{\alpha^*}$*
- (b) *$\mathscr{I}_n \subseteq \{r : q \le_{\rm apr} r$ and r forces a value to $\underset{\sim}{\tau}_n\}$*
- (c) *$\mathscr{I}_n$ is countable*
- (d) *$\mathscr{I}_n$ is predense above q.*

2) If $\{p, \bar{\mathbb{Q}}\} \in N \prec (\mathscr{H}(\chi), \in)$ and N is countable and $\langle \underset{\sim}{\tau}(n) : n < \omega\rangle$ lists the $\mathbb{P}_{\alpha^*}$-names of ordinals from N, <u>then</u> we can add

- (e) *q is $(N, \mathbb{P}_{\alpha^*})$-generic.*

Remark 3.4. 1) We can break the claim to two claims; the first saying that for a strong$^+$ $\mathscr{F}$-psc iteration $\bar{\mathbb{Q}}, \mathbb{P}_{\alpha^*}$ is a strong$^+$ $\mathscr{F}$-psc. The second saying that a strong$^+$ $\mathscr{F}$-psc forcing $\mathbb{P}$ satisfies the conclusion of 3.3(1).
2) Concerning $(*)_{q, \mathscr{I}_n, \underset{\sim}{\tau}_n}$, we know by 2.9(3) that for some $q, \langle \mathscr{I}_n : n < \omega\rangle$ clause (a),(c),(d) there holds, as well as

- $(b)^-$ $\mathscr{I}_n \subseteq \{r : q \le r$ and r forces a value to $\underset{\sim}{\tau}_n\}$, <u>but</u> the $q \le_{\rm apr} r$ will be missing.

Proof. Let $\underset{\sim}{\mathbf{St}}_{\alpha,q}$ be a $\mathbb{P}_\alpha$-name of a winning strategy for the Interpolator player in the game for $\underset{\sim}{\mathbb{Q}}_\alpha$ and $q \in \underset{\sim}{\mathbb{Q}}_\alpha$ as guaranteed in the Definition of "strongly$^+$".

By the proof of 2.12 we can combine them to a winning strategy $\mathbf{St}$ for $\mathbb{P}_\alpha$ and p. Now we simulate a play of the game such that the Interpolator player plays according to $\mathbf{St}$ and the Extender play as in the proof of 2.9 for $\langle \underset{\sim}{\tau}_n : n < \omega\rangle$. Let the play produced be $\langle (p'_\zeta, q_\zeta, p_\zeta, m_\zeta, j_\zeta) : \zeta < \omega_1\rangle$. In particular $\langle p_\zeta : \zeta \in [-1, \omega_1)\rangle$ is purely increasing [Saharon: I don't find the reason for purely increasing] and $p_\zeta \le_{\rm apr} q_\zeta$. So for some ζ we have:

- $(*)_0$ there is no $m < \omega$ and q such that
  - (i) $p_\zeta \le q$
  - (ii) if $\varepsilon < \zeta \wedge m = m_\varepsilon$ then q, q_ζ are incompatible.

Let

$$\mathscr{U}^\zeta_m = \mathscr{U}_m = \{\varepsilon < \zeta : m_\varepsilon = m \text{ and } p_\zeta, q_\varepsilon \text{ are compatible}\}.$$

For each $m < \omega, \varepsilon \in \mathscr{U}_m$ let r_ε be a lub of p_ζ, q_ε as guaranteed to exist by clause (b) of Definition 3.1.

So

$(*)_1$ $\varepsilon \in \mathscr{U}_m \Rightarrow$ "$p_\zeta \le_{\rm apr} r_\varepsilon$".

[Why? By the same clause (b) of Definition 3.1, i.e. r_ε has same domain as p_ζ, for all but finitely many coordinates r is a pure extension of p_ζ, and for the exceptional coordinate r is an apure extension, really for all.]

$(*)_2$ $\mathscr{I}_n := \{r_\varepsilon : \varepsilon \in \mathscr{U}_m\}$ is predense above p_ζ.

[Why? If not, there is $q \in \mathbb{P}_\alpha$ above p_ζ which for every $\varepsilon \in \mathscr{U}_m$ is incompatible with r_ε hence with q_ε (if not let $q \le q', q_\varepsilon \le q'$, but $p_\zeta \le q \le q'$ so q' has to be above r_ε as a lub of p_ζ, q_ε, contradiction). So q is above p_ζ but incompatible with every q_ε when $\varepsilon < \zeta \wedge m_\varepsilon = m$. But this contradicts the choice of ζ.]

So $p_\zeta, \langle \mathscr{I}_n : n < \omega\rangle$ are as required.

2) Like 2.9(2). $\square_{3.3}$

Definition 3.5. Let $\mathscr{F}$ be an iterated trunk controller and $\mathbb{Q}$ an $\mathscr{F}$-psc iteration of length α. For $\beta \le \alpha$ let $\mathbb{P}_\alpha \models p \le_{{\rm pr},\beta} q$ mean that:

- (a) $\mathbb{P}_\alpha \models$ "$p \le q$"
- (b) if $\gamma \in {\rm Dom}(p) \backslash \beta$ then $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$p(\gamma) \le_{\rm pr} q(\gamma)$".

Claim 3.6. *If in Definition 3.6 we have $\beta \le \alpha$ and $\mathbb{P}_\alpha \models$ "$p \le_{{\rm pr},\beta} q$" <u>then</u> for some non-limit $\eta \le \beta$ we have $\mathbb{P}_\alpha \models$ "$p \le_{{\rm pr},\eta} q$".*

Proof. If $\beta = 0$ or β is a successor ordinal, choose $\eta = \beta$. If β is a limit ordinal, take $\eta = ({\rm Dom}(p) \cap \beta) + 1$. $\square$

Definition 3.7. 1) A forcing notion $\mathbb{Q}$ has (θ, σ)-pure decidability if:
if $p \in \mathbb{Q}$ and $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in \theta$", <u>then</u> for some $A \subseteq \theta, |A| < \sigma$ and q we have $p \le_{\rm pr} q \in \mathbb{Q}$ and $q \Vdash$ "$\underset{\sim}{\tau} \in A$".
2) We write "θ-pure decidability" for "(θ,θ)-pure decidability".

Claim 3.8. *1) Assume $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration and:*

- (a) *$\beta^* \le \alpha^* = \ell g(\bar{\mathbb{Q}})$ and $\beta \in [\beta^*, \alpha^*)$ implies $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ has pure $(2,2)$-decidability", see Definition 3.7*
- (b) *$\underset{\sim}{\mathbf{t}}$ is a $\mathbb{P}_{\alpha^*}$-name, $\Vdash_{\mathbb{P}_{\alpha^*}}$ "$\underset{\sim}{\mathbf{t}} \in \{0,1\}$"*
- (c) *$(*)_{p,\mathscr{I},\underset{\sim}{\mathbf{t}}}$ from 3.3(1) holds for some $\mathscr{I}$*
- (d) *each $\underset{\sim}{\mathbb{Q}}_\beta$ (for $\beta \in [\beta^*,\alpha^*)$) satisfies $p' \le^{\mathbb{Q}_\beta}_{\rm pr} p'' \Rightarrow {\rm val}^{\underset{\sim}{\mathbb{Q}}_\beta}(p') = {\rm val}^{\underset{\sim}{\mathbb{Q}}_\beta}(p'')$; a natural sufficient condition for this is for every $\beta \in [\beta^*,\alpha^*)$ $\le^{\mathscr{F}_\beta}_{\rm pr}$ is the equality on $\mathscr{F}_\beta$.*

<u>Then</u> there are $\underset{\sim}{\mathbf{t}}', q'$ such that:

- (α) $p \le_{{\rm pr},\beta^*} q'$

(β) $q' \restriction \beta^* = p \restriction \beta^*$

(γ) $\underset{\sim}{\mathbf{t}}'$ *is a* $\mathbb{P}_{\beta^*}$*-name*

(δ) $q' \Vdash_{\mathbb{P}_{\alpha^*}}$ "$\underset{\sim}{\mathbf{t}}' = \underset{\sim}{\mathbf{t}}$".

2) Similarly for pure $(\aleph_0, 2)$*-decidability.*

Remark 3.9. Use in 5.16. Saharon check.

Proof. 1) Case 1: $\mathrm{Dom}(p) \subseteq \beta^*$.
Trivial.

Case 2: $\mathrm{Dom}(p) \backslash \beta^* \neq \emptyset$.

For each $\beta \in [\beta^*, \alpha^*)$ we define $\mathbb{P}_\beta$-names $\underset{\sim}{\mathbf{t}}^0_\beta, \underset{\sim}{\mathbf{t}}^1_\beta, \underset{\sim}{q}_\beta$ (and later $\underset{\sim}{\mathbf{t}}^2_\beta$) as follows. Let $\mathrm{Dom}^+(p) = \mathrm{Dom}(p) \cup \{\sup\{\gamma + 1 : \gamma \in \mathrm{Dom}(p)\}\}$. Let $G_\beta \subseteq \mathbb{P}_\beta$ be generic over $\mathbf{V}$.

Possibility A: There are $q \in \mathbb{P}_{\alpha^*}/G_\beta$ such that $p \leq_{\mathrm{pr},\beta} q, \mathrm{Dom}(q) = \mathrm{Dom}(p)$ and $q \Vdash_{\mathbb{P}_{\alpha^*}/G_\beta}$ "$\underset{\sim}{\mathbf{t}} = \ell$".

Let $\underset{\sim}{\mathbf{t}}^1_\beta[G_\beta]$ be 1 and $\underset{\sim}{\mathbf{t}}^0_\beta[G_\beta] = \ell$ and let $\underset{\sim}{q}_\beta[G_\beta]$ be $(p \restriction \beta) \cup (q \restriction [\beta, \alpha^*])$.

Possibility B: Not possibility A.

Let $\underset{\sim}{\mathbf{t}}^0_\beta[G_\beta] = 0, \underset{\sim}{\mathbf{t}}^1_\beta[G_\beta] = 0$ and $\underset{\sim}{q}_\beta[G_\beta] = p$.

In the first possibility, note that we have demanded $\mathrm{Dom}(q) = \mathrm{Dom}(p)$; in the second possibility this holds automatically.

Let $\underset{\sim}{\mathbf{t}}^2_\beta[G_\beta] \in \{0,1\}$ be 1 iff $\underset{\sim}{\mathbf{t}}^1_\beta[G_\beta] = 1$ and for no $\gamma \in \beta \cap \mathrm{Dom}(p) \backslash \beta^*$ do we have $\underset{\sim}{\mathbf{t}}^1_\gamma[G_\beta \cap \mathbb{P}_\gamma] = 1$, clearly also $\underset{\sim}{\mathbf{t}}^2_\beta$ is a $\mathbb{P}_\beta$-name of a number $\in \{0,1\}$.

Now note

$\boxtimes$ $\Vdash_{\mathbb{P}_{\alpha^*}}$ "there is one and only one $\beta \in \mathrm{Dom}(p) \backslash \beta^*$ such that $\underset{\sim}{\mathbf{t}}^2_\beta[\underset{\sim}{G}_{\alpha^*} \cap \mathbb{P}_\beta] = 1$ call it $\underset{\sim}{\beta}$".

[Why? First the "at most one" follows by the definition of $\underset{\sim}{\mathbf{t}}^2_\beta$. Second, for the "at least one" it is enough that $p \Vdash_{\mathbb{P}_{\alpha^*}}$ "for some $\beta \in \mathrm{Dom}(p), \underset{\sim}{\mathbf{t}}^1_\beta = 1$". Now we separate the proof to two cases. By clause (c) we have $(*)_{p,\mathscr{I},\underset{\sim}{\mathbf{t}}}$ from 3.3(1), so if $G_{\alpha^*} \subseteq \mathbb{P}_{\alpha^*}$ is generic over $\mathbf{V}$ and $p \in G_\beta$ then some $q \in \mathscr{I}$ belongs to G_β so by clause (b) from 3.3 we have $p \leq_{\mathrm{apr}} q$ and q forces a value of $\underset{\sim}{\mathbf{t}}$. This means that $\underset{\sim}{\mathbf{t}}^1_{\gamma(*)}[G_{\alpha^*}] = 1$ for $\gamma(*) = \sup\{\gamma + 1 : \gamma \in \mathrm{Dom}(p)\}$.]

So we can define $q \in \mathbb{P}_{\alpha^*}$ as follows $\mathrm{Dom}(q) = \mathrm{Dom}(p)$ and for $\gamma \in \mathrm{Dom}(p) : q(\gamma)$ is a $\mathbb{P}_\beta$-name, so let $G_\gamma \subseteq \mathbb{P}_\gamma$ be generic over $\mathbf{V}$, now if $\gamma \geq \beta^*$ and $\underset{\sim}{\beta}[G_\gamma]$ is well defined and $\leq \gamma$, i.e. for some $\gamma_1 \leq \gamma, \underset{\sim}{\mathbf{t}}^2_{\gamma_1}[G_\gamma \cap \mathbb{P}_{\gamma_1}] = 1$ then $q(\gamma) = \underset{\sim}{q}_{\underset{\sim}{\beta}}(\gamma)$, otherwise $q(\gamma) = p(\gamma)$. (So q in a sense purely extends p but only on a relevant end segment. Next we shall try to make this end-segment as short as possible).

Now for each $\beta \in \mathrm{Dom}(p) \backslash \beta^*$ we define a $\mathbb{P}_\beta$-name $q^{[\beta]} \in \underset{\sim}{\mathbb{Q}}_\beta$, such that:

$\odot$ (α) $q(\beta) \leq_{\mathrm{pr}} \underset{\sim}{q}^{[\beta]}$, i.e. $\Vdash_{\mathbb{P}_\beta}$ "$q(\beta) \leq^{\mathbb{Q}_\beta}_{\mathrm{pr}} q^{[\beta]}$"

(β) $\mathrm{val}^{\underset{\sim}{\mathbb{Q}}_\beta}(\underset{\sim}{q}^{[\beta]}) = \mathrm{val}^{\underset{\sim}{\mathbb{Q}}_\beta}(p(\beta))$; recalling clause (d) of the assumption

(γ) if $\ell \leq 2$, and there are $\underset{\sim}{q}', \underset{\sim}{i}$ such that $\Vdash_{\mathbb{P}_\beta}$ "$q(\beta) \leq^{\mathbb{Q}_\beta}_{\mathrm{pr}} q'$ and $\underset{\sim}{q}' \Vdash_{\underset{\sim}{\mathbb{Q}}_\beta} \underset{\sim}{\mathbf{t}}^\ell_{\beta+1} = \underset{\sim}{i}$" where $\underset{\sim}{i}$ is a $\mathbb{P}_\beta$-name, then the triple $(\underset{\sim}{q}^{[\beta]}, \underset{\sim}{i}^\ell_\beta, \ell)$ satisfies this for some $\mathbb{P}_\beta$-name $\underset{\sim}{i}^\ell_\beta$ of a number < 2, moreover

$(\gamma)'$ $\underset{\sim}{q}, \underset{\sim}{i}^0_\beta, \underset{\sim}{i}^1_\beta$ are such that:

(i) $\underset{\sim}{q}^{[\beta]}$ is a $\mathbb{P}_\beta$-name of a member of $\underset{\sim}{\mathbb{Q}}_\beta$, purely extending $q(\beta)$

(ii) $\underset{\sim}{i}^0_\beta, \underset{\sim}{i}^1_\beta$ are $\mathbb{P}_\beta$-names of number < 3

(iii) in $\mathbf{V}^{\mathbb{P}_\beta}$ either $\underset{\sim}{q}^{[\beta]} \Vdash_{\mathbb{Q}_\beta}$ "$\underset{\sim}{\mathbf{t}}^\ell_{\beta+1} = \underset{\sim}{i}^\ell_\beta$" <u>or</u> $\underset{\sim}{q}^{[\beta]}$ has no $\leq^{\mathbb{Q}_\beta}_{\rm pr}$-extension forcing a value to $\underset{\sim}{\mathbf{t}}^\ell_{\beta+1}$ and $\underset{\sim}{i}^\ell_\beta = 2$.

So as above $\underset{\sim}{\mathbf{t}}^1_\beta[G_{\alpha^*} \cap \mathbb{P}_\beta] = 1$ and let q witness it, so Possibility A holds so there is $q \in \mathbb{P}_{\alpha^*}/(G_{\alpha^*} \cap \mathbb{P}_\beta)$ such that $p \leq_{\rm pr} q, {\rm Dom}(q) = {\rm Dom}(p)$ and $q \Vdash_{\mathbb{P}_{\alpha^*}} / \underset{\sim}{G}_\beta$"$\underset{\sim}{\mathbf{t}} = \ell$". This holds in $\mathbf{V}[G_\beta]$ hence there is $r \in G_\beta$ which forces all this, and without loss of generality $\mathbb{P}_\beta \models$ "$q \restriction \beta \leq r$".

We now define $q^* \in \mathbb{P}_{\alpha^*}$ as follows ${\rm Dom}(q^*) = {\rm Dom}(q), q^* \restriction \beta^* = q \restriction \beta^*$ and $\beta \in {\rm Dom}(q) \backslash \beta^* \Rightarrow q^*(\beta) = q^{[\beta]}(\beta)$, note that ${\rm val}(q^*) = {\rm val}(q)$, and we shall show that q^* is as required, this suffices. For this it suffices to show that $q^* \Vdash$ "$\beta = \beta^*$". Toward this let $G_{\alpha^*} \subseteq \mathbb{P}_{\alpha^*}$ generic over $\mathbf{V}$ satisfying $q^* \in G_{\alpha^*}$, so $\underset{\sim}{\mathbf{t}}[G_{\alpha^*}] = i$ for some $i \in \{0,1\}$, let $G_\beta = G_{\alpha^*} \cap \mathbb{P}_\beta$ for $\beta \leq \alpha^*$.

Now:

$(*)_1$ for some $\beta \in [\beta^*, \alpha^*] \cap {\rm Dom}^+(p)$ we have $\underset{\sim}{\mathbf{t}}^1_\beta[G_{\alpha^*} \cap \mathbb{P}_\beta] = 1$.

[Why? As $p \leq_{\rm pr} q^* \in G_{\alpha^*}$, by assumption (c) necessarily $\mathscr{I} \cap G_{\alpha^*} \neq \emptyset$ so let $r \in \mathscr{I} \cap G_{\alpha^*}$, and so $\beta = \alpha^*$ is as required.]

So let $\beta \in [\beta^*, \alpha^*]$ be minimal such that $\underset{\sim}{\mathbf{t}}^1_\beta[G_{\alpha^*} \cap \mathbb{P}_\beta] = 1$. So β is unique such that $\underset{\sim}{\mathbf{t}}^2_\beta[G_{\alpha^*} \cap P_\beta] = 1$, hence by $\boxtimes$ we have

$(*)_2$ $\beta = \underset{\sim}{\beta}[G_{\alpha^*}]$

$(*)_3$ β cannot be a limit ordinal $> \beta^*$.

[Why? By the finiteness clause in the definition of order in $\mathbb{P}_{\alpha^*}$ by claim **??**.]

$(*)_4$ $\beta = \gamma + 1 > \beta^*$ is impossible.

[Why? If $\gamma \notin {\rm Dom}(p)$ this is trivial by the choice of β, so assume $\gamma \in {\rm Dom}(p)$. Now in $\mathbf{V}[G_{\alpha^*} \cap \mathbb{P}_\gamma]$ the forcing notion $\underset{\sim}{\mathbb{Q}}_\gamma[G_{\alpha^*} \cap \mathbb{P}_\gamma]$ has pure $(2,2)$-decidability hence clearly $\underset{\sim}{q}^{[\gamma]}[G_{\alpha^*} \cap \mathbb{P}_\gamma] \Vdash_{\mathbb{Q}_\gamma}$ "$\underset{\sim}{\mathbf{t}}^\ell_{\gamma+1} = \underset{\sim}{i}^\ell_\gamma[G_{\alpha^*} \cap \mathbb{P}_\gamma]$". Now $\underset{\sim}{\mathbf{t}}^1_{\gamma+1}[G_{\alpha^*} \cap \mathbb{P}_{\gamma+1}] = 1$ by the choice of $\beta = \gamma + 1$, hence $\underset{\sim}{i}^1_\gamma[G_{\alpha^*} \cap \mathbb{P}_\gamma] = 1$. Define q' as follows: $q' \restriction \beta = q^* \restriction \beta, q' \restriction ({\rm Dom}(p) \backslash \beta) = \underset{\sim}{q}_\beta \restriction ({\rm Dom}(p) \backslash \beta)$ it proves γ could serve instead of β, contradiction.]

So $\beta = \beta^*$ and we are easily done by the definition of q^*.

2) Similar to the proof of part (3). $\square_{3.3}$

* * *

Discussion 3.10. 0) (f,g)-bounding as application.

We may consider the following variant of our definitions and claims (we do not mention the cases which trivially do not change). [Why we have not used it in §2? There was a reunion; what it was? not want to have three orders.]

(A) Defining of $\leq^{\mathbb{P}_\alpha}_{\rm apr}$ in iteration. In Definition 1.10$(\delta)(iv)$ we add (those are objects)

$$\mathrm{val}^{\mathbb{Q}_\beta}_{\sim}(p(\beta)) = \mathrm{val}^{\mathbb{Q}_\beta}_{\sim}(q(\beta)).$$

(B) For an $\mathscr{F}$-psc forcing $\mathbb{Q}$ let $p \le^{\mathbb{Q}}_{\mathrm{vpr}} q$ means $(p \le^{\mathbb{Q}}_{\mathrm{pr}} q) \wedge \mathrm{val}^{\mathbb{Q}}(p) = \mathrm{val}^{\mathbb{Q}}(p)$ (vpr stands for very purely).
(C) The associativity lemma 1.20 still works for this iteration.
(D) In $\boxtimes$ of Definition 2.1(1) we strengthen "$p'_\zeta \le_{\mathrm{pr}} q_\zeta \Rightarrow p_\zeta \le_{\mathrm{pr}} q_\zeta$" to "$p'_\zeta \le_{\mathrm{pr}} q_\zeta \Rightarrow p_\zeta \le_{\mathrm{vpr}} q_\zeta$.
(E) The examples in **??**(1),(2) work.
(F) Claim 2.9 is not changed: actually our demands are just stronger.
(G) Lemma 2.12: still true with minor changes in the proof. To show that $(*)_1$ is possible, we have to check that "$p'_\zeta \le_{\mathrm{pr}} q_\zeta \Rightarrow p_\zeta \le_{\mathrm{vpr}} q'_\zeta$ and the same proof gives it.
(H) At last we get a gain: in 3.8 we can omit clause (d) of the assumption.

Discussion 3.11. We may consider replacing stationary $A, B \subseteq \omega_1$ by a subset of $[\omega_1]^2$, so we use:

To clarify "$\mathscr{F}$ has the apure c.c.c." note.

Definition 3.12. We call $(\mathfrak{H}, \le_{\mathfrak{H}})$ a (D,κ)-witness if:

- (a) $\le_{\mathfrak{H}}$ is a partial order of $\mathfrak{H}$
- (b) D a normal filter on a regular uncountable κ
- (c) $\mathfrak{H} \subseteq \mathscr{P}([\kappa]^2)\backslash\{\emptyset\}$
- (d) for any $X \in \mathfrak{H}, E \in D$ and a pressing down function h on E for some $Y \le_{\mathfrak{H}} X$ we have $h \restriction \cup\{\{\alpha,\beta\} : \{\alpha,\beta\} \in Y\}$ is constant and $Y \subseteq [E]^2$.

Definition 3.13. 1) For $(\mathfrak{H}, \le_{\mathfrak{H}})$ a (D,κ)-witness and a forcing notion $\mathbb{Q}$ we say that $\mathbb{Q}$ satisfies the $(\mathfrak{H}, <_{\mathfrak{H}})$-c.c. when: for every κ-sequence $\langle p_\alpha : \alpha < \kappa\rangle$ of members of $\mathbb{Q}$ and $X \in \mathfrak{H}$, there is $Y \le_{\mathfrak{H}} X$ such that $(\alpha,\beta) \in Y \Rightarrow p_\alpha, p_\beta$ are compatible in $\mathbb{Q}$.
2) Similarly "a trunk controller $\mathscr{F}$" satisfies the apure $(\mathfrak{H}, <_{\mathfrak{H}})$-c.c. for $\kappa = \aleph_1$, see (c) below.

Claim 3.14. *The trunk controller $\mathscr{F}$ satisfies the apure $(\mathfrak{H}, \le_{\mathfrak{H}})$-c.c. <u>when</u> the following hold:*

- (a) *$(\mathfrak{H}, <_{\mathfrak{H}})$ is a $(D,\aleph_1)$-witness (or for some normal filter D on ω_1)*
- (b) *$\mathscr{F}$ is fully based on $\langle X_\beta : \beta < \alpha^*\rangle$*
- (c) *if $\beta < \alpha^*, \langle y_\varepsilon : \varepsilon < \omega_1\rangle$ is $\le_{\mathrm{pr}}$-increasing in $\mathscr{F}_\beta, y_\varepsilon \le^{\mathscr{F}_\beta}_{\mathrm{apr}} z_\varepsilon$ <u>then</u> the set $\{(\varepsilon,\zeta) : z_\varepsilon, z_\zeta$ are compatible in $\mathscr{F}_\beta\}$ belongs to $\mathfrak{H}$.*
  Alternatively
- $(c)'$ *$\mathscr{F}_\beta$ satisfies the $(\mathfrak{H}, <_{\mathfrak{H}})$-c.c.*

Proof. Immediate. $\square$

4. Averages by an ultrafilter and restricted non-null trees

Definition 4.1. For Borel subsets $\mathscr{B}, \mathscr{B}_n\,(n < \omega)$ of ${}^\omega 2$ and a filter D on ω, let $\mathscr{B} = \text{ms} - \lim_D \langle \mathscr{B}_n : n < \omega \rangle$ or "$\langle \mathscr{B}_n : n < \omega \rangle$ does D-converge" to $\mathscr{B}$ mean that for every $\varepsilon > 0$ the set $\{n < \omega : \text{Leb}(\mathscr{B} \Delta \mathscr{B}_n) < \varepsilon\}$ belongs to D.

Proposition 4.2. *Assume*

(a) $\mathbf{V}_1 = \mathbf{V}[\mathbf{r}]$ *where* $\mathbf{r}$ *is a random real over* $\mathbf{V}$,

(b) *in* $\mathbf{V}, D$ *is a non-principal ultrafilter on* ω.

Then we can find in $\mathbf{V}_1$ *a non-principal ultrafilter* D_1 *on* ω *extending* D *such that*

$(*)_{\mathbf{r},D,D_1}$ *if in* $\mathbf{V}, \mathscr{B}, \mathscr{B}_n$ *are Borel subsets of* ${}^\omega 2$, *(so* $\mathscr{B}, \langle \mathscr{B}_n : n < \omega \rangle \in \mathbf{V}$*) and* $\langle \mathscr{B}_n : n < \omega \rangle$ *does* D*-converge to* $\mathscr{B}$, *then the following conditions are equivalent in* $\mathbf{V}_1$:

   (i) $\mathbf{r} \in \mathscr{B}$ *(recall,* $\mathscr{B}$ *being a Borel set, is actually a definition of a set and so* $\mathscr{B}$ *is a definition in* $\mathbf{V}$ *of a Borel set, so it defines a Borel set in* $\mathbf{V}[\mathbf{r}]$*)*

   (ii) *the set* $\{n : \mathbf{r} \in \mathscr{B}_n\}$ *belongs to* D_1.

Proof. It suffices to find in $\mathbf{V}_1$ an ultrafilter D_1 over ω containing D' where $D' = D \cup \{\{n : \mathbf{r} \in \mathscr{B} \equiv \mathbf{r} \in \mathscr{B}_n\}$: in $\mathbf{V}$ the sequence $\langle \mathscr{B}_n : n < \omega \rangle$ does D-converge to $\mathscr{B}$ in $\mathbf{V}\}$.

For then D_1 satisfies: if $\langle \mathscr{B}_n : n < \omega \rangle \in \mathbf{V}$ does D-converge to $\mathscr{B} \in \mathbf{V}$ then $(i) \Rightarrow (ii)$ as $D \subseteq D' \subseteq D_1$. Also $\neg(i) \Rightarrow \neg(ii)$ as $\langle {}^\omega 2 \backslash \mathscr{B}_n : n < \omega \rangle \in \mathbf{V}$ does D converge to ${}^\omega 2 \backslash \mathscr{B} \in \mathbf{V}$ and we apply $(i) \Rightarrow (ii)$ for it.

The existence of an ultrafilter D_1 over ω containing D' is equivalent to "any intersection of finitely many members of D' is not empty". As D is closed under finite intersection, clearly it suffices to prove:

⊠ Assume in $\mathbf{V}$ that $m^* < \omega$ and for each $m < m^*$, the sequence $\langle \mathscr{B}_{m,n} : n < \omega \rangle - D$- converges to $\mathscr{B}_m$ and $\mathscr{B} \in D$. Then for some $n \in \mathscr{B}$ in $\mathbf{V}_1$ we have $m < m^* \Rightarrow [\mathbf{r} \in \mathscr{B}_m \equiv \mathbf{r} \in \mathscr{B}_{m,n}]$.

$\square_{4.2}$

Proof. Proof of ⊠

It is enough, given a positive real $\varepsilon > 0$ to find a Borel set $\mathscr{B} = \mathscr{B}_\varepsilon \in \mathbf{V}$ of Lebesgue measure $< \varepsilon$ such that, in $\mathbf{V}$

$(*)$ $\mathbf{r} \in {}^\omega 2 \backslash \mathscr{B} \Rightarrow (\exists n \in \mathscr{B})(\forall m < m^*)[\mathbf{r} \in \mathscr{B}_m \equiv \mathbf{r} \in \mathscr{B}_{m,n}]$.

(Why? As then we can find in $\mathbf{V}$ a sequence $\langle \mathscr{B}_{1,(k+1)} : k < \omega \rangle$, each $\mathscr{B}_{1,(k+1)}$ as above; so $\mathbf{r}$ being random over $\mathbf{V}$ does not belong to $\bigcap\limits_k \mathscr{B}_{1,(k+1)}$ hence for some $k, \mathbf{r} \notin \mathscr{B}_{1,(k+1)}$ and so there is n as required by $(*)$ because $(*)$ holds also in $\mathbf{V}[\mathbf{r}]$ by absoluteness).

Given $\varepsilon > 0$, for $m < m^*$, as $\langle \mathscr{B}_{m,n} : n < \omega \rangle$ does D-converge to $\mathscr{B}_m$ clearly we can find $\mathscr{B}'_m \in D$ such that $(\forall n \in \mathscr{B}'_m)[\text{Leb}(\mathscr{B}_m \Delta \mathscr{B}_{m,n}) < \varepsilon/(m^*+1)]$. Let $\mathscr{B}'' = \bigcap\limits_{m<m^*} \mathscr{B}'_m \cap \mathscr{B}$, so clearly $\mathscr{B}'' \in D$. Now choose any $n \in \mathscr{B}''$, and note that the Borel set $\mathscr{B} = \bigcup\limits_{m<m^*} (\mathscr{B}_m \Delta \mathscr{B}_{m,n})$ will do: clearly $\text{Leb}(\mathscr{B}) < \varepsilon$ and easily n is as required. $\square$

Recall

Remark 4.3. $T^{[\eta]} = \{\nu \in T : \nu \triangleleft \eta \text{ or } \eta \triangleleft \nu \in T\}$.
$\lim(T) = \{\eta \in {}^\omega 2 : (\forall n < \omega)(\eta \restriction n \in T)\}$.

We can do the following more generally, but what we do is enough for our intended example.

Definition 4.4. If $g : \omega \to \omega$ satisfies $n < \omega \Rightarrow g(n) > n$ and is increasing we define $\mathbf{T}_g$ as the family of subtrees T of ${}^{\omega>}2$ such that for every $n < \omega$ and $\eta \in T \cap {}^n 2$ we have

$$(*)_n \quad (1 - 1/n)|T^{[\eta]} \cap {}^{g(n)}2|/2^{g(n)} \le \mathrm{Leb}(\lim(T^{[\eta]})) \le |T^{[\eta]} \cap {}^{g(n)}2|/2^{g(n)}$$

(the second inequality holds automatically), equivalently, for every $m \ge g(n)$

$$(*)_{n,m} \quad (1 - 1/n)|T^{[\eta]} \cap {}^{g(n)}2|/2^{g(n)} \le |T^{[\eta]} \cap {}^m 2|/2^m \le |T^{[\eta]} \cap {}^{g(n)}2|/2^{g(n)}.$$

Definition 4.5. 1) For subtrees T_n of ${}^{\omega>}2$ (for $n < \omega$) and a filter D on ω we say $T = \lim_D \langle T_n : n < \omega\rangle$ if $T = \{\eta \in {}^{\omega>}2 : \{n < \omega : \eta \in T_n\} \in D\} = \{\eta \in {}^{\omega>}2 : \{n < \omega : \eta \in T_n\} \ne \emptyset \mod D\}$ (so if η is undecided, such T does not exist).

Similarly for $T_n \subseteq \mathscr{H}(\aleph_0)$. We may omit D if D is the family of co-bounded subset of ω. Note that $\lim_D\langle T_n : n < \omega\rangle$, if it exists, is uniquely defined and is absolute and if D is an ultrafilter it is always well defined.
2) Let $\mathscr{G}^{\mathbf{V}}$ be $\{g \in \mathbf{V} : g$ is an increasing function from ω to ω and $g(n) > n\}$, let $\mathscr{G}$ vary on subsets of $\mathscr{G}^{\mathbf{V}}$. Let $\mathbf{T}_{\mathscr{G}} = \cup\{\mathbf{T}_g : g \in \mathscr{G}\}$. Let $\mathscr{G}^{\mathbf{V}}_w = \{g \restriction w : g \in \mathscr{G}^{\mathbf{V}}\}$ for $w \subseteq \omega$.

Claim 4.6. *1) For subtrees T_n of ${}^{\omega>}2$ (for $n < \omega$) and an ultrafilter D on $\omega, \lim_D\langle T_n : n < \omega\rangle$ is well defined and it is a subtree of ${}^{\omega>}2$.*
2) If T_n is a subtree of ${}^{\omega>}2$ for $n < \omega, D$ is a filter on ω containing the cofinite sets and $\lim_D\langle T_n : n < \omega\rangle$ is well defined and $n < \omega \Rightarrow T_n \in \mathbf{T}_g$ (as in Definition 4.4), <u>then</u>

(a) *d " $\lim_D\langle T_n : n < \omega\rangle$ belongs to $\mathbf{T}_g$*
(b) *if $T = \lim_D\langle T_n : n < \omega\rangle$ then $\lim(T) = \mathrm{ms} - \lim_D\langle \lim(T_n) : n < \omega\rangle$.*

3) If $D_1 \subseteq D_2$ are filters on ω containing the cofinite sets then

(a) *$T = \lim_{D_1}\langle T_n : n < \omega\rangle$ implies $T = \lim_{D_2}\langle T_n : n < \omega\rangle$*
(b) *if $\mathscr{B} = \mathrm{ms} - \lim_{D_1}\langle \mathscr{B}_n : n < \omega\rangle$ then $\mathscr{B} = \mathrm{ms} - \lim_{D_2}\langle \mathscr{B}_n : n < \omega\rangle$.*

Proof. Easy. $\square_{4.6}$

Definition 4.7. 1) We say $\rho \in {}^\omega 2$ is $(N, \mathbf{T}_{\mathscr{G}}, D)$-continuous or $\mathscr{G}$-continuous over N for D <u>if</u>:

(a) $N \subseteq \mathbf{V}$ a transitive class, a model of ZFC, or $\prec (\mathscr{H}(\chi), \in)$ for some χ; or more generally, a set or a class, which is a model of enough set theory (say ZFC$^-$) and $\mathscr{H}(\aleph_0) \subseteq N, \omega \in N$, with reasonable absoluteness and $D \in N$ is a filter on ω containing the co-finite sets (so $(D \cap N)^{\mathbf{V}}$ is the filter generated in $\mathbf{V}$ by $D \cap N$)
(b) $\mathscr{G} \in N$ (and of course $\mathscr{G}^N \subseteq \mathscr{G}^{\mathbf{V}}$, see Definition 4.5(2)) and

(c) if $m(*) < \omega$ and for each $m < m(*)$ we have $g_m \in \mathscr{G} \cap N$, and $\langle T^m_n : n < \omega\rangle \in N, T^m_n \in N \cap \mathbf{T}_{g_m}, T^m \in N \cap \mathbf{T}_{g_m}$ and $T^m = \lim_D \langle T^m_n : n < \omega\rangle$, <u>then</u> $\rho \in \cap\{\lim(T^m) : m < m(*)\} \Rightarrow \{n : \text{if } m < m(*) \text{ then } \rho \in \lim(T^m_n)\} \neq \emptyset$ mod $(D \cap N)^{\mathbf{V}}$.

2) We define the ideal $\mathrm{Null}_{\mathscr{G},D}$ as the σ-ideal generated by the sets of the form $\{\rho \in {}^\omega 2 : \rho \in \lim(T^m) \text{ for } m < m(*) \text{ but } \{n : \text{if } m < m(*) \text{ then } \rho \in \lim(T^m_n)\} = \emptyset$ mod $D\}$ with $T^m, T^m_n \in \mathbf{T}_{g_m}, T^m = \lim_D \langle T^m_n : n < \omega\rangle$, for some $m < m(*)$ and $\langle (T_m, \langle T^m_n : n < \omega\rangle) : m < m(*)\rangle$ where $m(*) < \omega, g_m \in \mathscr{G}$.
3) We may write the dual ideal instead of the filter, if D is the filter of co-finite subsets of ω, we may omit it. [?? Saharon]

Observation 4.8. *Let $D, \mathscr{G}$ be as above.*
1) If $\mathscr{G}_1 \subseteq \mathscr{G}_2$ and $D_1 \subseteq D_2$ then $\mathrm{Null}_{\mathscr{G}_1,D_1} \subseteq \mathrm{Null}_{\mathscr{G}_2,D_2}$.
2) If $\mathscr{G} = \mathscr{G}^{\mathbf{V}}$ then $\mathrm{Null}_{\mathscr{G},D}$ = the ideal of null subsets of ${}^\omega 2$.

Proof. 1) Assume $\mathscr{B} \in \mathrm{Null}_{\mathscr{G}_1,D_1}$, so then necessarily for some $\langle \mathscr{B}_k : k < \omega\rangle, \langle T^{k,m}, T^{k,m}_n : k < \omega, m < m(k), n < \omega\rangle$ and $\langle g^k_m : k < \omega, m < m(k)\rangle$ we have:

⊛ (a) $g^k_m \in \mathscr{G}_1 \subseteq \mathscr{G}_2$
(b) $T^{k,m}, T^{k,m}_n \in \mathbf{T}_{g^k_m}$
(c) $T^{k,m} = \lim_{D_1} \langle T^{k,m}_n : n < \omega\rangle$, for every $n < n(k)$
(d) $\mathscr{B}_k := \{\rho \in {}^\omega 2 : \rho \in \lim(T^{k,m}) \text{ for } m < m(k) \text{ but } \{n\text{: if } m < m(k) \text{ then } \rho \in \lim(T^{k,m}_n)\} = \emptyset \text{ mod } D_1\}$
(e) $\mathscr{B} \subseteq \cup\{\mathscr{B}_k : k < \omega\}$.

Now if we replace D_1 by D_2 then still $T^{k,m} = \lim_{D_2} \langle T^{k,m}_n : n < \omega\rangle$ and the set $\mathscr{B}_k$ can only increase so clearly $\mathscr{B} \in \mathrm{Null}_{\mathscr{G}_2,D_2}$.
2) Let $\mathscr{B}$ be a Borel subset of ${}^\omega 2$ such that $\mathrm{Leb}(\mathscr{B}) = 1$. So we can find a sequence $\langle T_n : n < \omega\rangle$ such that T_n is a perfect subtree of ${}^{\omega>}2$ such that $\mathrm{Leb}(\lim(T_n)) \geq 1 - 2^{-n}, \lim(T_n) \subseteq \mathscr{B}$ and $T_n \subseteq T_{n+1}$.
So

$(*)_1$ ${}^{\omega>}2 = \lim_{J^{\mathrm{bd}}_\omega} \langle T_n : n < \omega\rangle$

$(*)_2$ define $g \in^\omega \omega$ by $g(n) = \mathrm{Min}\{k : k > n$ and $k > g(n')$ for $n' < n$ and for every $m \leq 2n+2$ and $\eta \in T_m \cap {}^n 2$ we have $(1-1/n)|T^{[\eta]}_m \cap {}^k 2| \leq \mathrm{Leb}(T^{[\eta]}\}$.

Now note that the demand on $k = g(n)$, $(*)_2$ holds also for $m \geq 2n+2$ because: for $\eta \in T_m \cap {}^n 2$ we have

$$\begin{aligned} \mathrm{Leb}(T^{[\eta]}) &\geq 2^{-n} - (1 - \mathrm{Leb}(\lim(T_m)) = 2^{-n} - 2^{-m} \\ &\geq 2^{-n} - 2^{-(2n+2)} \geq (1-1/n) - 2^{-n} \geq (1-1/n)\mathrm{Leb}(\lim T^{[\eta]} \cap {}^n 2) \end{aligned}$$

So $\{T\} \cup \{T_m : m < \omega\} \subseteq \mathbf{T}_g$ and $g \in \mathscr{G}^{\mathbf{V}}$.
So we can conclude then $2^\omega, \langle T_m : m < \omega\rangle$ witness $\omega_2 \backslash \mathscr{B} \in \mathrm{Null}_{\mathscr{G}}$. So the ideal of null subsets of ${}^\omega 2$ is included in $\mathrm{Null}_{\mathscr{G}}$. For the other inclusion let $T^m = \lim_D \langle T^m_n : n < \omega\rangle$ for $m < m(*)$ where $T^m, T^m_n \in \mathbf{T}_g, g \in \mathscr{G}^{\mathbf{V}}$. Easily $\{\rho \in {}^\omega 2 : m < m(*) \Rightarrow \rho \in \lim(T^m)$ but $\{n : \rho \in T^m_n$ for $m < m(*)\}$ is finite$\}$ is a null set. As the ideal of null subsets of ${}^\omega 2$ is a σ-ideal we are done. $\square$

Remark 4.9. 1) Note that the ideal $\mathrm{Null}_{\mathscr{G},D}$ is included in the ideal of null sets.
2) If in Definition 4.7 we have two candidates $D_1 \subseteq D_2$ for D and ρ is $(N, T_{\mathscr{G}}, D_2)$-continuous then ρ is also $(N, T_{\mathscr{G}}, D_1)$-continuous. So for small D's there are more such ρ's relevant to Definition 4.7.

Observation 4.10. *1) Assume $\mathscr{G} \in \mathbf{V}$ is $\neq \emptyset$ and $\mathbf{V}_1$ extends $\mathbf{V}$. If $({}^\omega 2)^{\mathbf{V}}$ is not in the ideal $(\mathrm{Null}_{\mathscr{G}})^{\mathbf{V}_1}$, then there is no $\rho \in ({}^\omega 2)^{\mathbf{V}_1}$ which is a Cohen real over $\mathbf{V}$.*
2) If D is an ultrafilter on ω (in $\mathbf{V}$), then in Definition 4.7(1),(2) the case $m(*) = 1$ suffices.

Proof. 1) Choose $g \in \mathscr{G}$ and choose $\langle m_i : i < \omega\rangle$ by $m_0 = 0, m_{i+1} = 3g(m_i) > m_i$ hence $\prod\limits_{j\geq i+1}(1 - \frac{1}{2^{m_{j+1}-m_j}}) \geq (1 - \frac{1}{m_{i+1}+1})$.

Assume toward a contradiction that $\rho \in {}^\omega 2$ is Cohen over $\mathbf{V}$. On ${}^\alpha 2$ for $\alpha \leq \omega$ let $+$ be the coordinatewise addition mod 2.

In $\mathbf{V}$ we can find a sequence $\langle T_i : i < \omega\rangle$ of subtrees of ${}^{\omega >}2$ such that: $\mathrm{Leb}(\lim(T_i)) \geq 1-1/2^i, {}^{m_i \geq}2 \subseteq T_i$ and $i \leq j < \omega and \eta \in T_j \cap {}^{m_j}2 \Rightarrow (\exists! \nu)(\eta \triangleleft \nu \in {}^{(m_{j+1})}2 and \nu \notin T_i)$. So easily $T_i \in \mathbf{T}_g$ and $\lim(T_i) \subseteq {}^\omega 2$ is nowhere dense and $T =: \lim_{J^{\rm bd}_\omega}\langle T_n : n < \omega\rangle$ is ${}^{\omega >}2$. Now if $\nu \in ({}^\omega 2)^{\mathbf{V}}$ then $\rho + \nu$ is Cohen over $\mathbf{V}$ hence for each $n < \omega$ we have $\rho + \nu \notin \lim(T_n)$ hence $\nu \notin \rho + \lim(T_n)$. So letting $T'_n = \{\nu + \rho \restriction k : \nu \in T_n \cap {}^k 2, k < \omega\}$, still $\lim_{J^{\rm bd}_\omega}\langle T'_n : n < \omega\rangle$ is $T = {}^{\omega >}2$. Therefore for every $\nu \in ({}^\omega 2)^{\mathbf{V}}$ we have $n < \omega \Rightarrow \nu \notin \lim(T'_n)$ but $\nu \in \lim(T)$. So $T, \langle T'_n : n < \omega\rangle$ exemplify that $({}^\omega 2)^{\mathbf{V}}$ is in $\mathrm{Null}_{\mathscr{G}}$, contradiction.
2) Easy to check. $\square_{4.10}$

Conclusion 4.11. Assume

- (a) $\mathbf{V}_1 \supseteq \mathbf{V}$
- (b) $\mathscr{G} \in \mathbf{V}$ so (recalling 4.5(2)) we have $\mathscr{G} \subseteq \mathscr{G}^{\mathbf{V}}$
- (c) in $\mathbf{V}, D$ is a non-principal ultrafilter on ω.
- (d) $\mathbf{r} \in ({}^\omega 2)^{\mathbf{V}_1}$ is $\mathscr{G}$-continuous over $\mathbf{V}$ (recall 4.7(3)).

Then we can find D_1 such that in $\mathbf{V}_1$

- (α) $D_1 (\in \mathbf{V}_1)$ is a non-principal ultrafilter on ω extending D
- (β) if $g \in \mathscr{G}$ and $T, \langle T_n : n < \omega\rangle \in \mathbf{V}, \{T, T_n : n < \omega\} \subseteq \mathbf{T}_g$ $T = \lim_D\langle T_n : n < \omega\rangle$ then
$$\{n : (\mathbf{r} \in \lim(T)) \equiv (\mathbf{r} \in \lim(T_n))\} \in D_1.$$

Remark 4.12. In (d) we mean $\mathscr{G}$-continuous over $\mathbf{V}$ not $(\mathbf{V}, \mathbf{T}_{\mathscr{G}^{\mathbf{V}}}, D)$-continuous over $\mathbf{V}$. If we assume the later we can use 4.2 + 4.6(2).

Proof. To find such D_1, it is enough to prove

$\boxtimes$ assume

$(*)_1$ $m^* < \omega$ and for $m < m^*, g_m \in \mathscr{G}, T^m, T^m_n \in \mathbf{T}_{g_m}$ for $n < \omega$ and $\langle T^m_n : n < \omega\rangle \in \mathbf{V}$ and $T^m = \lim_D\langle T^m_n : m < \omega\rangle \in \mathbf{V}$ and lastly $B \in D$.

Then for some $n \in B$ we have $(\forall m < m^*)(\mathbf{r} \in \lim(T^m) \equiv \mathbf{r} \in \lim(T^m_n))$.

For $m < m^*$ we can find $k_m < \omega$ such that $\mathbf{r} \notin \lim(T^m) \Rightarrow \mathbf{r} \restriction k_m \notin T^m$, let $k = \max\{k_m : m < m^*\} < \omega$ and let $u = \{m < m^* : \mathbf{r} \in \lim(T^m)\}$. For each $m < m^*, m \notin u$ we know that $\mathbf{r} \restriction k \notin T^m$ hence $A_m =: \{n : \mathbf{r} \restriction k \notin T^m_n\} \in D$, and clearly $n \in A_m \Rightarrow \mathbf{r} \restriction k \notin T^m_n$. Let $B_1 = B \cap \bigcap\{A_m : m < m^*, m \notin u\}$, clearly $B_1 \in \mathbf{V}$ and $B_1 \in D$ hence B_1 is infinite. So we can choose by induction on $i < \omega$, a number $n_i \in B_1$ such that $n_i > n_j$ for $j < i$ and $m < m^* \Rightarrow T^m_{n_i} \cap {}^{i \geq} 2 = T^m \cap {}^{i \geq} 2$ moreover we do this in $\mathbf{V}$ (possible as $\mathbf{r} \restriction k \in \mathbf{V}$) so $\langle n_i : i < \omega \rangle \in \mathbf{V}$ and clearly $T^m = \lim \langle T^m_{n_i} : i < \omega \rangle$ for each $m \in u$. By assumption (d), (this is the only place it is used) and the definition of "$\mathbf{r}$ is $\mathscr{G}$-continuous over $\mathbf{V}$", the Borel set

$$\begin{aligned} \mathscr{B} = \{ \quad & \eta \in {}^\omega 2 : \eta \in \cap\{\lim(T^m) : m \in u\} \text{ but} \\ & \{i < \omega : \eta \in \cap\{\lim(T^m_{n_i}) : m \in u\}\} \text{ is finite}\} \end{aligned}$$

satisfies: $\mathbf{r} \notin \mathscr{B}^{\mathbf{V}_1}$. But $\mathbf{r} \in \cap\{\lim(T^m) : m \in u\}$ by the choice of u hence (by the definition of $\mathscr{B}$) for infinitely many i's we have $\mathbf{r} \in \cap\{\lim(T^m_{n_i}) : m \in u\}$. Hence we can choose i such that

(∗) $m \in u \Rightarrow \mathbf{r} \in \lim(T^m_{n_i})$.

Now if $m < m^*, m \notin u$ then $\mathbf{r} \notin \lim(T^m)$ by the choice of u and $\mathbf{r} \notin \lim(T^m_{n_i})$ as $n_i \in B_1 \subseteq A_m$, see above. In particular $n_i \in B$ so n_i is an n as required in $\boxtimes$. $\square_{4.11}$

Claim 4.13. *Assume*

(a) *δ is a limit ordinal*
(b) *$\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ is a $\lessdot$-increasing sequence of forcing notions*
(c) *for $\alpha < \delta$*
$\Vdash_{\mathbb{P}_\alpha}$ "*$\underset{\sim}{D}_\alpha$ is a non-principal ultrafilter on ω*
(d) *if $\alpha < \beta < \delta$ then $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{D}_\alpha \subseteq \underset{\sim}{D}_\beta$"*
(e) *$\underset{\sim}{\mathbf{r}}$ is a $\mathbb{P}_\delta$-name of a real (i.e. $\in {}^\omega 2$)*
(f) *$\mathscr{G} \subseteq \mathscr{G}^{\mathbf{V}}$*
(g) *if $\alpha < \delta$ then $\Vdash_{\mathbb{P}_\delta}$ "$\underset{\sim}{\mathbf{r}}$ is $\mathscr{G}$-continuous over $\mathbf{V}^{\mathbb{P}_\alpha}$".*

<u>Then</u> we can find $\underset{\sim}{D}_\delta$ such that

(α) *$\underset{\sim}{D}_\delta$ is a $\mathbb{P}_\delta$-name of a non-principal ultrafilter over ω*
(β) *if $\alpha < \delta$ then $\Vdash_{\mathbb{P}_\delta}$ "$\underset{\sim}{D}_\alpha \subseteq \underset{\sim}{D}_\delta$"*
(γ) *like (β) of part 4.11 with $\mathbf{V}^{\mathbb{P}_\alpha}, \mathbf{V}^{\mathbb{P}_\delta}$ here standing for $\mathbf{V}, \mathbf{V}_1$ there for any $\alpha < \delta$.*

Proof. Like 4.11. $\square_{4.13}$

5. On iterating $\mathbb{Q}_{\bar{D}}$

Definition 5.1. 1) Let **IF** be the family of $\bar{D} = \langle D_\eta : \eta \in {}^{\omega>}\omega \rangle$ with each D_η a filter on ω containing all the co-finite subsets of ω.
2) **IUF** is the family of $\bar{D} = \langle D_\eta : \eta \in {}^{\omega>}\omega \rangle$ with each D_η a non-principal ultrafilter on ω.

On $\mathbb{Q}_{\bar{D}}$ see [**?**].

Definition 5.2. 1) For $\bar{D} \in$ **IF** we define $\mathbb{Q}_{\bar{D}}$ as follows:

- (α) the set of elements is $Q_{\bar{D}} = \{T : T \subseteq {}^{\omega>}\omega$ is closed under initial segments, is non-empty and for some member $\eta = \mathrm{tr}(T) \in T$ which is increasing, and is called the trunk, we have: $\nu \in T$ and $\ell g(\nu) \le \ell g(\eta) \Rightarrow \nu \trianglelefteq \eta$ and $\eta \trianglelefteq \nu \in T \Rightarrow \{n : \nu\hat{\ }\langle n \rangle \in T\} \in D_\nu\}$
- (β) $\le = \le^{\mathbb{Q}_{\bar{D}}}$ is the inverse of inclusion
- (γ) $\le_{\mathrm{pr}} = \le^{\mathbb{Q}_{\bar{D}}}_{\mathrm{pr}}$ is defined by $T_1 \le_{\mathrm{pr}} T_2 \equiv (T_2 \subseteq T_1$ and $\mathrm{tr}(T_1) = \mathrm{tr}(T_2))$
- (δ) $\le_{\mathrm{apr}} = \{(p,q) : p \le q\}$
- (ε) $\mathrm{val}(T) = \mathrm{tr}(T) \in {}^{\omega>}\omega \subseteq \mathscr{H}(\aleph_0)$.

2) Let $\underset{\sim}{\eta} = \underset{\sim}{\eta}(\mathbb{Q}_{\bar{D}}) = \underset{\sim}{\eta}_{\mathbb{Q}_{\bar{D}}}$ be $\cup\{\mathrm{tr}(p) : p \in \underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}\}$, this is a $\mathbb{Q}_{\bar{D}}$-name of a member of ${}^{\omega}\omega$ (which is increasing).
3) For $p \in Q_{\bar{D}}, \eta \in p$ let $p^{[\eta]} = \{\nu \in p : \nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu\}$; so we have $p \le p^{[\eta]} \in Q_{\bar{D}}, \mathrm{tr}(p^{[\eta]}) \in \{\eta, \mathrm{tr}(p)\}$.
4) We define $\mathbb{Q}'_{\bar{D}}$ similarly except that we change $\le^{\mathbb{Q}'_{\bar{D}}}_{\mathrm{apr}}$ to be $\{(p,q) : p,q \in \mathbb{Q}_{\bar{D}}$ and $q = p^{[\eta]}$ for $\eta = \mathrm{tr}(q)\}$.

Fact 5.3. For $\bar{D} \in$ **IUF** and $\mathscr{F}$ a trunk controller such that the set of elements of $\mathscr{F}$ is ${}^{\omega>}\omega$ and $\eta \le^{\mathscr{F}}_{\mathrm{pr}} x \Leftrightarrow \eta = x$ and $\eta \le^{\mathscr{F}}_{\mathrm{apr}} x \Leftrightarrow \eta \trianglelefteq x (\in {}^{\omega>}\omega)$, we have:

- (a) $\mathbb{Q}_{\bar{D}}$ is a σ-centered, very clear $\mathscr{F}$-forcing,
- (b) $\mathbb{Q}_{\bar{D}}$ is an strong$^+$ $\mathscr{F}$-psc forcing notion hence $\mathbb{Q}_{\bar{D}}$ is purely proper (see Definition 3.1, Claim 3.3(2)),
- (c) from $\underset{\sim}{\eta}[\underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}]$ we can reconstruct $\underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}$ so it is a generic real for $\mathbb{Q}_{\bar{D}}$
- (d) $p \le_{\mathrm{apr}} p^{[\eta]} \in \mathbb{Q}_{\bar{D}}$ for $\eta \in p \in \mathbb{Q}_{\bar{D}}$
- (e) $\mathscr{F}$ is simple.

Proof. Clause (a): $\mathbb{Q}_{\bar{D}}$ is an $\mathscr{F}$-forcing. Just check Definition 1.6.

$\mathbb{Q}_{\bar{D}}$ is very clear: See Definition 1.6(2).

Assume $\mathrm{tr}^{\mathbb{Q}}(p_1) \le^{\mathscr{F}}_{\mathrm{pr}} y$ and $\mathrm{tr}^{\mathbb{Q}}(p_2) \le^{\mathscr{F}}_{\mathrm{pr}} y$ so necessarily $y \in {}^{\omega>}\omega$ and $\mathrm{tr}^{\mathbb{Q}}(p_1) = y = \mathrm{tp}^{\mathbb{Q}}(p_2)$ hence $q := p_1 \cap p_2$ is a subset of ${}^{\omega>}\omega$, closed under initial segments $\mathrm{tr}^{\mathbb{Q}}(q) = y$ and $y \trianglelefteq \eta \in p_\ell \Rightarrow \{n : \eta\hat{\ }\langle n \rangle \in p_\ell\} \in D_\eta$ hence $y \trianglelefteq \eta \in q \Rightarrow \{n : \eta\hat{\ }\langle n \rangle \in p_1\} \in D_\eta \wedge \{n : \eta\hat{\ }\langle n \rangle \in p_2\} \in D_\eta \Rightarrow \{n : \eta\hat{\ }\langle n \rangle \in p_1$ and $\eta\hat{\ }\langle n \rangle \in p_2\} = \{n : \eta\hat{\ }\langle n \rangle \in p_1\} \cap \{n : \eta\hat{\ }\langle n \rangle \in p_2\} \in D_\eta \Rightarrow \{n : \eta\hat{\ }\langle n \rangle \in q\} \in D_\eta$ so because each D_η is a filter on ω clearly $q \in \mathbb{Q}$.

This proves also that "$\mathbb{Q}$ is σ-centered".

Clause (b): To prove "$\mathbb{Q}$ is $\mathscr{F}$-psc", see Definition 2.1 let the strategy of the Interpolator player be to have $p'_\varepsilon = p_\varepsilon = p$.

For a play $\langle (p'_\zeta, q_\zeta, p_\zeta) : \zeta < \omega_i\rangle$ of $\eth_p$, given stationary $B \subseteq \omega_1$, we can find $\eta \in {}^{\omega >}\omega$ such that $A := \{\varepsilon \in B : \mathrm{tr}(q_\varepsilon) = \eta\}$ is stationary. By the proof of σ-centered, for $\varepsilon < \zeta$ from $A, q_\varepsilon, q_\zeta$ are purely compatible. For the "strong$^+$" see Definition 3.1, as clause (a) there ($\varepsilon < \zeta \Rightarrow p_\varepsilon \le_{\mathrm{pr}} p'_\zeta$) holds trivially, we just have to show in addition: for $\varepsilon < \zeta$ from A there is r such that $p_\zeta \le_{\mathrm{apr}} r, q_\varepsilon \le r, r$ is lub of p_ζ, q_ε and $p_\varepsilon \le_{\mathrm{pr}} q_\varepsilon \Rightarrow p_\zeta \le_{\mathrm{pr}} q_\varepsilon$. Let $r := p_\zeta \cap q_\varepsilon = q_\varepsilon$, so clearly r is lub of p_ζ, q_ε and $p_\varepsilon \le_{\mathrm{pr}} q_\varepsilon \Rightarrow \eta = \mathrm{tr}(p_\varepsilon)$ but $p_\zeta = p_\varepsilon$, so we are done.

<u>clause (c),(d)</u>:
Left to the reader.

<u>Clause (e)</u>: ($\mathscr{F}$ is simple, see Definition **??**(1)), holds as its set of elements is ${}^{\omega>}\omega$).
Trivial. $\square_{5.3}$

For completeness we prove the basic properties of $\mathbb{Q}_{\bar D}$.

Claim 5.4. *For $\bar D \in$ **IUF** letting $\mathbb{Q} = \mathbb{Q}_{\bar D}$ we have*
1) $\mathbb{Q}$ has pure 2-decidability, i.e. if $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in \{0,1\}$" then for some $q, p \le_{\mathrm{pr}} q$ and q forces a value to $\underset{\sim}{\tau}$.
2) If $p \in \mathbb{Q}$ and $\mathscr{I} \subseteq \mathbb{Q}$ is dense above p, <u>then</u> for some q we have $p \le_{\mathrm{pr}} q$ and $Y_0 = \{\eta \in p : \mathrm{tr}(p) \triangleleft \eta$ and there is r such that $p^{[\eta]} \le_{\mathrm{pr}} r \in \mathscr{I}\}$ contains a front of q (where being a front means that $\eta \in \lim(q) \Rightarrow (\exists! n)[\eta \restriction n \in Y_0])$ so without loss of generality $\eta \in Y_0 \Rightarrow q^{[\eta]} \in \mathscr{I}$.
3) If $p \in \mathbb{Q}$ and $Y \subseteq p$ satisfies $\eta \in Y \Rightarrow \mathrm{tr}(p) \trianglelefteq \eta$ and $\eta \in Y \wedge \eta \triangleleft \nu \in p \Rightarrow \nu \in Y$, <u>then</u> there is q such that $p \le_{\mathrm{pr}} q$ and: either $q \cap Y = \emptyset$ or there is a function $h : (q \backslash Y) \cup \{\mathrm{tr}(q)\} \to \omega_1$ such that for $\eta \triangleleft \nu$ in $\mathrm{Dom}(h), h(\eta) > h(\nu)$.
4) Let $p \in \mathbb{Q}, \mathscr{I} \subseteq \mathbb{Q}$. <u>Then</u> $\mathscr{I}$ is dense above p (in $\mathbb{Q}$) <u>iff</u> there are $Y, \langle (p_\eta, h_\eta)$: $\mathrm{tr}(p) \trianglelefteq \eta \in p\rangle$ and $\langle q_\eta : \eta \in Y\rangle$ such that:

- (a) *$p^{[\eta]} \le_{\mathrm{pr}} p_\eta \in \mathbb{Q}$ and $Y \subseteq p$ and $\eta \in Y \Rightarrow p^{[\eta]} \le_{\mathrm{pr}} q_\eta \in \mathbb{Q}$*
- (b) *if $\mathrm{tr}(p) \trianglelefteq \eta \in p$ then*
  - (α) *h_η is a function*
  - (β) *$\mathrm{dom}(h_\eta)$ is a subset of $\{\nu : \eta \trianglelefteq \nu \in p_\eta\}$ closed under initial segments*
  - (γ) *range of h_η is $\subseteq \omega_1$*
  - (δ) *h_η decreasing (i.e. $\rho \triangleleft \nu \Rightarrow h(\rho) > h(\nu)$ <u>when</u> $\rho, \nu \in \mathrm{Dom}(h_\eta)$)*
  - (ε) *$\nu \in \mathrm{Dom}(h_\eta)$ and $\nu \notin Y$ then $\{\ell : \nu\hat{}\langle \ell\rangle \in \mathrm{Dom}(h_\eta)\} \ne \emptyset \mod D_\nu$*
  - (ζ) *if $\nu \in Y, \nu \triangleleft \rho \in {}^{\omega>}\omega$ then $\rho \notin \mathrm{Dom}(h_\eta)$*
- (c) *$q_\eta \in \mathscr{I}$ and $\mathrm{tr}(q_\eta) = \eta \in q_\eta$ for $\eta \in Y$.*

Proof. 1) Let $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in \{0,1\}$". Let $Y_0 =: \{\eta \in p : \mathrm{tr}(p) \trianglelefteq \eta$ and there is $q \in \mathbb{Q}$ forcing a value to $\underset{\sim}{\tau}$ such that $p^{[\eta]} \le_{\mathrm{pr}} q\}$ and let $Y =: \{\eta \in p :$ for some $\nu \in Y_0$ we have $\mathrm{tr}(p) \trianglelefteq \nu \trianglelefteq \eta\}$. We apply part (3), (trivially Y is as assumed there) so let $q, p \le_{\mathrm{pr}} q \in \mathbb{Q}$ be as there (and without loss of generality $\eta\hat{}\langle \ell_1, \ell_2\rangle \in q \Rightarrow \ell_1 < \ell_2$). If $q \cap Y = \emptyset$ let r be such that $q \le r$ and r forces a value to $\underset{\sim}{\tau}$; hence $\mathrm{tr}(r) \in q \cap Y$, contradiction. So there is h as there. Stipulate $h(\nu) = -1$ if $\nu \in Y \backslash \{\mathrm{tr}(p)\}$. We prove by induction on $\alpha < \omega_1$ (and $\alpha \ge -1$) that:

$(*)_\alpha$ if $\mathrm{tr}(q) \trianglelefteq \eta \in \mathrm{Dom}(h)$ and $h(\eta) = \alpha$ <u>then</u> there is $r = r_\eta$ such that $q \le r$ and $\mathrm{tr}(q) \trianglelefteq \mathrm{tr}(r_\eta) \trianglelefteq \eta$ and r_η forces a value to $\underset{\sim}{\tau}$.

Now if $\alpha = -1$ then $\eta \in Y$ hence (by the definition of Y) for some ν we have $\mathrm{tr}(q) \trianglelefteq \nu \trianglelefteq \eta$ and $\nu \in Y_0$. Hence (by the definition of Y_0) there is r such that $q^{[\nu]} \le_{\mathrm{pr}} r \in \mathbb{Q}$ and r forces a value to $\underset{\sim}{\tau}$, so r is as required. If $\alpha \geq 0$, for each $\ell < \omega$ such that $\eta\hat{}\langle \ell \rangle \in q$ there are $i_\ell < 2$ and a condition $r_{\eta\hat{}<\ell>} \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} = i_\ell$" as guaranteed by the induction hypothesis, noting $h(\eta\hat{}\langle \ell \rangle)) < h(\eta) = \alpha$. If for some such $\ell, \mathrm{tr}(r_{\eta\hat{}<\ell>}) \trianglelefteq \eta$ we are done, otherwise as D_η is an ultrafilter, for some $i < 2$ we have $A =: \{\ell : i_\ell = i$ and $\eta\hat{}<\ell> \in q$ and $\mathrm{tr}(r_{\eta\hat{}<\ell>}) = \eta\hat{}<\ell>\} \in D_\eta$ and let $r_\eta = \cup\{r_{\eta\hat{}<\ell>} : \ell \in A\}$, clearly r_η, i are as required.

Having carried out the induction, for $\alpha = h(\mathrm{tr}(q)), r_{\mathrm{tr}(q)}$ is as required: it forces a value to $\underset{\sim}{\tau}$ and $\mathrm{tr}(q) \trianglelefteq \mathrm{tr}(r_{\mathrm{tr}(q)}) \trianglelefteq \mathrm{tr}(q)$ we have $\mathrm{tr}(r_{\mathrm{tr}(q)}) = \mathrm{tr}(q)$ hence $q \le_{\mathrm{pr}} r_{\mathrm{tr}(q)}$ but $p \le_{\mathrm{pr}} q$ hence $p \le_{\mathrm{pr}} r_{\mathrm{tr}(q)}$.

2) Let $Y = \{\eta :$ for some ν we have $\mathrm{tr}(p) \trianglelefteq \nu \trianglelefteq \eta \in p$ and $\nu \in Y_0\}$ and $Y' = \{\nu \in Y_0$: there is no $\rho \in Y_0$ such that $\mathrm{tr}(p) \trianglelefteq \rho \triangleleft \nu\}$, clearly Y' is a set of pairwise $\triangleleft$-incompatible sequences. Apply part (3) to p and Y (clearly Y is as required there) and get q as there. If $q \cap Y = \emptyset$ find r such that $q \le r \in \mathscr{I}$, (exists by the density of $\mathscr{I}$ above p) so by our definitions $\mathrm{tr}(r) \in Y_0 \subseteq Y$ and $\mathrm{tr}(r) \in r \subseteq q$ so $q \cap Y \neq \emptyset$, contradiction. So assume $q \cap Y \neq \emptyset$ hence necessarily there is h as there, in part (3), and for every $\eta \in \lim(q)$, as $\langle h(\eta \restriction \ell) : \ell \in [\ell g(\mathrm{tr}(q)), \omega)\rangle$ cannot be a strictly decreasing sequence of ordinals, necessarily for some $\ell \geq \ell g(\mathrm{tr}(q))$ we have $\eta \restriction \ell \notin \mathrm{Dom}(h)$ hence $\eta \restriction \ell \in Y$ hence for some $m \in [\ell g(\mathrm{tr}(q)), \ell]$ we have $\eta \restriction m \in Y_0$ hence for some $k \in [\ell g(\mathrm{tr}(q)), m]$ we have $\eta \restriction k \in Y'$. We have actually proved that $Y' \subseteq Y_0$ is a front of q.

3) Let $Z = \{\eta : \mathrm{tr}(p) \trianglelefteq \eta \in p$ and for $p^{[\eta]} \in \mathbb{Q}$ there are q and h as required in the claim$\}$.

Clearly

$(*)_1$ $Y \subseteq Z \subseteq \{\eta : \mathrm{tr}(p) \trianglelefteq \eta \in p\}$.

[Why? If $\eta \in Y$ use h_η with $\mathrm{Dom}(h_\eta) = \{\nu : \nu \trianglelefteq \eta\}, h_\eta(\nu) = \ell g(\eta) - \ell g(\nu)$.]

$(*)_2$ if $\mathrm{tr}(p) \trianglelefteq \eta \in p$ and $A = \{\ell : \eta\hat{}\langle \ell \rangle \in Z\} \in D_\eta$ then $\eta \in Z$.

[Why? Let the pairs (q_ℓ, h_ℓ) witness $\eta\hat{}\langle \ell \rangle \in Z$ for $\ell \in A$, let $q = \cup\{q_\ell : \ell \in A\}$ and $\alpha^* = \cup\{h_\ell(\eta\hat{}\langle \ell \rangle) + 1 : \ell \in A\}$ and define h: $\mathrm{Dom}(h) = \{\nu : \nu \trianglelefteq \eta\} \cup \bigcup\{\mathrm{Dom}(h_\ell) \backslash \{\nu : \nu \trianglelefteq \eta\} : \ell \in A\}$ and

- $h \restriction (\mathrm{Dom}(h_\ell) \backslash \{\nu : \nu \trianglelefteq \eta\})$ is h_ℓ
- $h(\nu) = \alpha^* + \ell g(\eta) - \ell g(\nu)$ if $\nu \trianglelefteq \eta$.]

If $\mathrm{tr}(p) \in Z$ we get the second possibility in the conclusion. If $\mathrm{tr}(p) \notin Z$, let $q = \{\eta \in p$: there is no $\nu \trianglelefteq \eta$ which belongs to $Z\}$, so $\{\eta : \eta \trianglelefteq \mathrm{tr}(p)\} \subseteq q$ (see Z's definition + present assumption) and q is closed under initial segments (read its definition) and by $(*)_2$ we can prove by induction on $m \geq \ell g(\mathrm{tr}(p))$ that $\eta \in q \cap {}^m\omega$ implies $\{\ell : \eta\hat{}<\ell> \in q\} \in D_\eta$. So clearly $p \le_{\mathrm{pr}} q \in \mathbb{Q}_{\bar{D}}, q \cap Y = \emptyset$ hence q is as required.

4) Let $Y := \{\eta \in p : \mathrm{tr}(p) \trianglelefteq \eta$ and there is $q \in \mathbb{Q}_{\bar{D}}$ such that $p^{[\eta]} \le_{\mathrm{pr}} q$ and $q \in \mathscr{I}\}$.

So we can choose $\langle q_\eta : \eta \in Y\rangle$ such that $\eta \in Y \Rightarrow p^{[\eta]} \le_{\mathrm{pr}} q_\eta \in \mathscr{I}$ hence clauses (a),(c) of part (4) holds. To prove clause (b) assume $\mathrm{tr}(p) \trianglelefteq \eta \in p$. If $\eta \in Y$ we are done, so assume $\eta \notin Y$. We apply part (3) to $p^{[\eta]}$ and $Y_\eta := \{\nu \in p : \eta \trianglelefteq \nu$ and there is $\rho \in Y$ such that $\eta \trianglelefteq \rho \trianglelefteq \nu\}$, this pair satisfies the demands in part (3), so one of the two possibilities there holds. The first one says that there is a $q, p^{[\eta]} \leq_{\rm pr} q$ and $q \cap Y_\eta = \emptyset$, but as $\mathscr{I}$ is dense above p, there is r such that $q \leq r \in \mathscr{I}$ hence $\mathrm{tr}(r) \in Y$ and trivially $\eta = \mathrm{tr}(q) \trianglelefteq \mathrm{tr}(r)$ hence $\mathrm{tr}(r) \in Y_\eta \cap q$ contradiction to "q disjoint to Y_η". Hence the second possibility in part (3) holds, i.e., there are $q, p^{[\eta]} \leq_{\rm pr} q$ and a function h as there (for $p^{[\eta]}, Y_\eta$), and it is required in the second possibility in clause (b). $\square_{5.4}$

The following is natural to note if we are interested in the Borel conjecture. (Of course, this claim does not touch the problem of preserving the property by the later forcings in the iteration we intend to use.) Compare with 5.6.

Claim 5.5. *Assume*

(a) $\bar{D} \in \mathbf{IUF}$
(b) $N \prec (\mathscr{H}(\chi), \in)$ *is countable*, $\bar{D} \in N$
(c) $\rho_m \in {}^\omega 2 \backslash N$ *for* $m < \omega$
(d) $p \in \mathbb{Q}_{\bar{D}} \cap N$.

Then we can find q such that

(α) $p \leq_{\rm pr} q \in \mathbb{Q}_{\bar{D}}$
(β) $q \Vdash$ *"if* $f \in {}^\omega 2$ *and* $f \in N[\underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}]$ *and* $m < \omega$ *then* $(\forall^\infty n)(f \restriction [\underset{\sim}{\eta}(n), \underset{\sim}{\eta}(n+1)) \neq \rho_m \restriction [\underset{\sim}{\eta}(n), \underset{\sim}{\eta}(n+1))$", *recalling* $\underset{\sim}{\eta}$ *is the generic sequence of* $\mathbb{Q}_{\bar{D}}$ *as defined in 5.2(2).*

Proof. As $\mathbb{Q}_{\bar{D}}$ satisfies the c.c.c. necessarily p is $(N, \mathbb{Q}_{\bar{D}})$-generic, hence $p \Vdash N[\underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}] \cap ({}^\omega 2)^{\mathbf{V}} = N \cap ({}^\omega 2)^{\mathbf{V}}$ hence $\rho_m \notin N[\underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}]$ for $m < \omega$.

Let $\langle \underset{\sim}{f}_\ell : \ell < \omega \rangle$ list the $\underset{\sim}{f} \in N$ such that $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{f} \in {}^\omega 2$".

Now by repeated use of 5.4(1) for every $\mathrm{tr}(p) \trianglelefteq \eta \in p$ and $\ell < \omega$ there is a function $f_{\ell,\eta} \in {}^\omega 2$ such that: for every $k < \omega$ there is $q_{\ell,\eta,k} \in \mathbb{Q}_{\bar{D}} \cap N$ such that $p^{[\eta]} \leq_{\rm pr} q_{\ell,\eta,k}$ and $q_{\ell,\eta,k} \Vdash_{\mathbb{Q}_{\bar{D}}}$ "for $n < k$ we have: $f_{\ell,\eta}(n) = \underset{\sim}{f}_\ell(n)$".

Now $q_{\ell,\eta,k} \in N$ and without loss of generality $\langle q_{\ell,\eta,k} : \eta \in p$ and $k < \omega\rangle, \langle f_{\ell,\eta} : \eta \in p\rangle$ belongs to N for each ℓ (but we cannot have $\langle q_{\ell,\eta,k} : \eta \in p, \eta$ and $k < \omega\rangle \in N$). Now for each ℓ, η, k as $f_{\ell,\eta} \in N$ and $\rho_m \notin N$ clearly the set $\{n : f_{\ell,\eta}(n) \neq \rho_m(n)\}$ is infinite so let $k(\ell,\eta,m) = \mathrm{Min}\{k : f_{\ell,\eta}(k) \neq \rho_m(k)$ and $k > \sup(\mathrm{Rang}(\eta))\}$.

Now define q as $\{\eta \in p :$ if $\mathrm{tr}(p) \trianglelefteq \nu \triangleleft \eta \in p, \ell \leq \ell g(\nu)$ and $m \leq \ell g(\nu)$ then $\eta \in q_{\ell,\nu,k(\ell,\nu,m)}$ and $\eta(\ell g(\nu)) > k(\ell,\nu,m)\}$.

The checking is straightforward. $\square_{5.5}$

A closely related claim is

Claim 5.6. *Assume*

(a) $\bar{D} \in \mathbf{IUF}$
(b) $\bar{D} \in N \prec (\mathscr{H}(\chi), \in)$
(c) $\rho \in {}^\omega 2 \backslash N$
(d) $\underset{\sim}{\eta} = \underset{\sim}{\eta}_{\mathbb{Q}_{\bar{D}}}$.

Then

(α) $\Vdash_{\mathbb{Q}_{\bar{D}}}$ *"if* $f \in N[\underset{\sim}{G}_{\mathbb{Q}_{\bar{D}}}]$ *and* $f \in \prod_{n<\omega} {}^{\underset{\sim}{\eta}(n)}2$ *then* $(\forall^\infty n)(\neg f(n) \triangleleft \rho)$",
moreover

(β) $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "*if* $f \in N[G_{\mathbb{Q}_D}]$, f *a function with domain* ω, $f(n) \subseteq {}^{\underset{\sim}{\eta}(n)}2$ *and* $|f(n)| \le \underset{\sim}{} \to \eta(n-1)$ *when* $n \ge 0$, *then* $(\forall^\infty n)(\rho \restriction \underset{\sim}{\eta}(n) \notin f(n))$ *stipulating* $\underset{\sim}{\eta}(-1) = 1$".

Proof. Let $\underset{\sim}{f} \in N$ be such that $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{f}$ is a function with domain ω such that $|\underset{\sim}{f}(n)| \le \underset{\sim}{\eta}(n-1)$ and $\emptyset \ne \underset{\sim}{f}(n) \subseteq {}^{\underset{\sim}{\eta}(n)}2$" and we shall prove that

$$\Vdash_{\mathbb{Q}_{\bar{D}}} \text{“}\rho \restriction \underset{\sim}{\eta}(n) \notin \underset{\sim}{f}(n) \text{ for every } n < \omega \text{ large enough”}.$$

This clearly suffices as clause (β) implies clause (α). For each $n > 0$ and $\nu \in {}^{n+1}\omega$ we can find $q_\nu \in \mathbb{Q}_{\bar{D}}$ and ρ^m_ν for $m < \nu(n-1)$, such that $\mathrm{tr}(q_\nu) = \nu$ and $q_\nu \Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{f}(n) = \{\rho^m_\nu : m < \underset{\sim}{\eta}(n-1)\}$". Note that $q_\nu \Vdash$ "$k < \ell g(\nu) \Rightarrow \underset{\sim}{\eta}(k) = \nu(k)$ in particular for $k = n-1$" and $\Vdash$ "$\underset{\sim}{f}(n) \subseteq {}^{\underset{\sim}{\eta}(n)}2$ and $1 \le |\underset{\sim}{f}(n)| \le \underset{\sim}{\eta}(n-1)$" hence $\rho^\ell_\nu \in {}^{\nu(n)}2$. As $\underset{\sim}{f} \in N$ without loss of generality $\langle (q_\nu, \rho^m_\nu) : m < \nu(n-1)$ and $\nu \in {}^{n+1}\omega, n < \omega\rangle$ belongs to N. Now for each $\nu \in {}^{\omega>}\omega$ and $m < \nu(\ell g(\nu)-1) < k$ we have $\rho^m_{\nu\hat{\ }<k>} \in {}^k2$, so for every $\ell < \omega$ for some $\rho^m_{\nu,\ell} \in {}^\ell 2$ we have $\{k < \omega : \rho^m_{\nu\hat{\ }<k>} \restriction \ell = \rho^m_{\nu,\ell}$ and $k > \ell\}$ belongs to D_ν, and clearly $\rho^m_{\nu,\ell} \triangleleft \rho^m_{\nu,\ell+1}$ and let $\rho^m_{\nu,*} = \bigcup\limits_{\ell<\omega} \rho^m_{\nu,\ell}$ so $\rho^m_{\nu,*} \in N \cap {}^\omega 2$ hence $\rho^m_{\nu,*} \ne \rho$ so for some $\ell(\nu,m) < \omega$ we have $\rho^m_{\nu,*} \restriction \ell(\nu,m) \ne \rho \restriction \ell(\nu,m)$ hence $\{k : (\exists m, \nu(\ell g(\nu)-1))\rho^m_{\nu\hat{\ }<k>} \triangleleft \rho\} = \emptyset \bmod D_\nu$.

Let $p \in \mathbb{Q}_{\bar{D}}$, let us define

$$\begin{aligned} q = \{\nu :\ & \nu \trianglelefteq \mathrm{tr}(p) \text{ or } \mathrm{tr}(p) \triangleleft \nu \in p \text{ and if } k \in [\ell g(\mathrm{tr}(p)), \ell g(\nu)) \\ & \text{and } m < \nu(k-1) \text{ then } \neg(\rho^m_{\nu\restriction(k+1)} \triangleleft \rho)\}. \end{aligned}$$

This is a condition above p forcing $\{n : \rho \restriction \underset{\sim}{\eta}(n) \in \underset{\sim}{f}(n)\}$ is bounded by $\ell g(\mathrm{tr}(p))$, so we are done. $\square_{5.6}$

Claim 5.7. *Assume*

(a) $\bar{D} \in$ **IUF**

(b) D *is a non-principal ultrafilter on* ω

(c) $p \in \mathbb{Q}_{\bar{D}}, I \subseteq p$ *contains a front and is upward closed, and* $(*) \Rightarrow (**)$ *where*

  ($*$) *for* $\nu \in I$ *and* $m < \nu(\ell g(\nu)-1)$ *we have* $D'_{\nu,m} \le_{\mathrm{RK}} D_\nu$, *and, of course,* $D'_{\nu,m}$ *a non-principal ultrafilter on* ω

  ($**$) $D \nsubseteq \cup\{D'_{\nu,m} : \nu \in I$ *and* $m < \nu(\ell g(\nu)-1)\}$.

Then p *forces that in* $\mathbf{V}^{\mathbb{Q}_{\bar{D}}}$ *we have:*

($***$) *if* $w_n \subseteq [\underset{\sim}{\eta}(n), \underset{\sim}{\eta}(n+1)), |w_n| \le \underset{\sim}{\eta}(n)$, *then* $\cup\{w_n : n < \omega\}$ *is disjoint to some member of* D.

Remark 5.8. This is relevant when trying to get no Q-point.

Proof. Without loss of generality $n^* = 0$ (just fix $\underset{\sim}{\eta} \restriction n^*$) and assume

$$\begin{aligned} \Vdash_{\mathbb{Q}_{\bar{D}}} \text{“}\underset{\sim}{\bar{w}} = \ & \langle \underset{\sim}{w}_n : n < \omega\rangle \text{ where for each } n \text{ we have} \\ & \underset{\sim}{w}_n \subseteq [\underset{\sim}{\eta}(n), \underset{\sim}{\eta}(n+1)) \text{ and } |\underset{\sim}{w}_n| \le \underset{\sim}{\eta}(n)\text{”}. \end{aligned}$$

Without loss of generality $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{w}_n \ne \emptyset$". For $\nu \in {}^{n+2}\omega$ let $q_\nu \in \mathbb{Q}_{\bar{D}}$ and $\langle t^m_\nu : m < \nu(n)\rangle$ be such that $\mathrm{tr}(q_\nu) = \nu$ and $q_\nu \Vdash$ "$\underset{\sim}{w}_n = \{t^m_\nu : m < \nu(n)\}$", possibly

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with repetitions. For $\nu \in {}^{n+1}\omega$ and $m < \nu(\ell g(\nu)-1)$, let $D'_{\nu,m} = \{A \subseteq \omega : \{k : t^m_{\nu\char`\^<k>} \in A\} \in D_\nu\}$; clearly $D'_{\nu,m}$ is an ultrafilter on $\omega$ which is $\leq_{\rm RK} D_\nu$ as $\nu(\ell g(\nu)-1) \leq t^m_{\nu\char`\^<k>} < k$.

For $\nu \in {}^{n+1}\omega$ and $m < \nu(n)$ let $D''_{\nu,m}$ be $D'_{\nu,m}$ if $D'_{\nu,m}$ is a non-principal ultrafilter and let $D''_{\nu,m}$ be D_ν if $D'_{\nu,m}$ is a principal ultrafilter on ω. Let $k_{\nu,m} < \omega$ be such that $\{k : t^m_{\nu\char`\^<k>} = k_{\nu,m}\} \in D_{\nu,m}, \{k_{\nu,m}\} \in D'_{\nu,m}$ if $D'_{\nu,m}$ is a principal ultrafilter and $k_{\nu,m} = \nu(m) = \nu(\ell g(\nu)-1)$ otherwise.

By clause (c) of the assumption, as $\langle D''_{\nu,m} : \nu \in I$, and $m < \nu(\ell g(\nu)-1_\rangle$ is a sequence of non-principal ultrafilter on ω, i.e. as in $(*)$ there, there is $A \in D$ which belongs to no $D'_{\nu,m}$ for $\nu \in I, m < \nu(\ell g(\nu)-1), p_0 = p$ and

$$p_1 = \{\eta \in p_0 : \quad \begin{array}{l}\text{if } n+1 < \ell g(\eta) \text{ and } \mathrm{tr}(p), \eta \restriction (n+1)) \in I \\ \text{and } m \leq \nu(n) \text{ then } t_{\nu\restriction(n+1)}, \nu(n+1) \notin A\}.\end{array}$$

Clearly

$(*)$ $p_0 \leq_{\mathbb{Q}_{\bar D}} p_1$.

Let $\langle (\eta_i, M_i) : i < \omega\rangle$ be a sequence of finite sets such that

$\circledast$ (a) $0 = n_0 < n_i < n_{i-1}$

(b) $M_i \subseteq \mathscr{H}(\chi), |M_i| = 2n_i + M_i \cap \omega = n_i$

(c) $(\bar D, \underset{\sim}{\bar w}, p) \in M_0$

(d) if F is a definable n-place function in $(\mathscr{H}(\chi), \in, <_*)$ then for every i large enough, we have $x_1, \dots, x_n \in M_i \Rightarrow F(x_1, \dots, x_n) \in M_{i+1}$.

By the assumption (c) of 5.7 there is a set $u \subseteq \omega$ such that

$\boxtimes$ (a) $\cup\{[n_{i+1}, n_{i+2}) : i \in u\} \in D$

(b) if $\nu \in I$ then $\cup\{[n_i, n_{i+3}) : i \in u\} \notin D_\nu$

(c) if $i < j$ are from u then $i+3 < j$.

Let

$$p_2 := \{\eta \in p_i : \quad \begin{array}{l}\text{if } n < \ell g(\eta) \text{ and } \mathrm{tr}(p_1) \triangleleft \eta \restriction n \in I \\ \text{then } \eta(n) \notin \cup\{[n_i, n_{i+2}) : i \in u\}\}.\end{array}$$

Now p_2 is as required. $\square_{5.7}$

Claim 5.9. *1) Assume*

(a) $\mathbf{V} \subseteq \mathbf{V}_1$

(b) $\bar D \in \mathbf{IUF}^{\mathbf{V}}$.

If $\bar D_1 \in \mathbf{IUF}^{\mathbf{V}_1}$ *and* $\eta \in {}^{\omega>}\omega \Rightarrow D_\eta \subseteq D_{1,\eta}$ <u>*then*</u>

(α) $\mathbb{Q}^{\mathbf{V}}_{\bar D} \subseteq \mathbb{Q}_{\bar D_1}$ *(so* $p \in \mathbb{Q}^{\mathbf{V}}_{\bar D} \Leftrightarrow p \in \mathbb{Q}_{\bar D_1} \cap \mathbf{V}$ *and* $\leq_{\mathbb{Q}^{\mathbf{V}}_{\bar D}} = \leq_{\mathbb{Q}_{\bar D_1}} \restriction \mathbb{Q}^{\mathbf{V}}_{\bar D}$ *and similarly for* $\leq^{\mathbb{Q}_{\bar D}}_{\rm pr}, <^{\mathbb{Q}_{\bar D}}_{\rm apr}$ *and incompatibility)*

(β) *if in* $\mathbf{V}$ *we have "*$\mathscr{I} \subseteq \mathbb{Q}_{\bar D}$ *is predense over* $p \in \mathbb{Q}_{\bar D}$*"* <u>*then*</u> *in* $\mathbf{V}_1$ *we have "*$\mathscr{I} \subseteq \mathbb{Q}^{\mathbf{V}_1}_{\bar D_1}$ *is predense over* p*"*

(γ) *if* $G_1 \subseteq \mathbb{Q}_{\bar D_1}$ *is generic over* $\mathbf{V}_1$, <u>*then*</u> $G =: G_1 \cap \mathbb{Q}^{\mathbf{V}}_{\bar D}$ *is a generic subset of* $\mathbb{Q}^{\mathbf{V}}_{\bar D}$ *over* $\mathbf{V}$.

2) Assume in addition

(c) $\mathscr{G} \in \mathbf{V}$ *and* $\mathscr{G} \subseteq \mathscr{G}^{\mathbf{V}}$

(d) $\mathbf{r} \in ({}^{\omega}2)^{\mathbf{V}_1}$ *is* $\mathscr{G}$*-continuous over* $\mathbf{V}$.

If the pair $(D_{1,\eta}, D_\eta)$ *satisfies clause* $(\alpha), (\beta)$ *from 4.11 (so* $D_{1,\eta}$ *in particular is an ultrafilter on* ω *extending* D_η*) for each* $\eta \in {}^{\omega>}\omega$, *then in* $\mathbf{V}_1$ *we have*
$\Vdash_{\mathbb{Q}_{\bar{D}_1}}$ "$\mathbf{r}$ *is* $\mathscr{G}$*-continuous over* $\mathbf{V}[\underset{\sim}{\eta}(\mathbb{Q}_{\bar{D}_1})]$".

Conclusion 5.10. 1) So if we get $\mathbf{V}_1$ from $\mathbf{V}$ by forcing with $\mathbb{P} \in \mathbf{V}, \underset{\sim}{\bar{D}}_1$ a $\mathbb{P}$-name of $\bar{D}$, as above then $\mathbf{V} \models$ "$\mathbb{Q}_{\bar{D}} \lessdot \mathbb{P} * \mathbb{Q}_{\underset{\sim}{\bar{D}}_1}$".
2) Note that by part (1), if $G_1 \subseteq \mathbb{Q}^{\mathbf{V}_1}_{\underset{\sim}{\bar{D}}_1}$ is generic over $\mathbf{V}$ and $\eta = \underset{\sim}{\eta}_{\mathbb{Q}_{\bar{D}_1}}[G_1]$ then η is a generic real for $\mathbb{Q}_{\bar{D}}$ over $\mathbf{V}$ hence $\mathbf{V}[\eta]$ is a generic extension of $\mathbf{V}$ (for $\mathbb{Q}_{\bar{D}}$).
3) Assume that $\mathbf{V}_1 \subseteq \mathbf{V}_2, \mathbf{V}_\ell \models \bar{D}^{|}ell = \langle D^\ell_\eta : \eta \in {}^{\omega>}\omega\rangle \in \mathbf{IF}$ for $\ell = 1,2$ and $\eta \in {}^{\omega>}\omega \wedge A \subseteq \mathscr{P}(\omega) \Rightarrow (A \in D^1_\eta \equiv A \in D^2_\eta)$ and $\mathbf{V}_1 \models \mathscr{I}$ is a predense subset of $\mathbb{Q}^{\mathbf{V}_1}_{\bar{D}^1}$ (above p) then $\mathbf{V}_2 \models \mathscr{I}$ is a predense subset of $\mathbb{Q}^{\mathbf{V}_2}_{\bar{D}^2}$ (above p^*; not used).

Proof. 1),2) Left to the reader.
3) Clearly $q \in \mathbb{Q}^{\mathbf{V}_1}_{\bar{D}^1} \Leftrightarrow q \in \mathbb{Q}^{\mathbf{V}_2}_{\bar{D}^2}$ and $\leq_{\mathbb{Q}^{\mathbf{V}_1}_{\bar{D}^1}} = \leq_{\mathbb{Q}^{\mathbf{V}_2}_{\bar{D}^2}} \restriction \mathbb{Q}^{\mathbf{V}_1}_{\bar{D}^1}$. So it suffices to prove (without loss of generality every $\eta \in p$ is increasing)

⊛ for $p \in \mathbb{Q}^{\mathbf{V}_1}_{\bar{D}^1}$ and $\mathscr{I} \subseteq^{\mathbf{V}_1}_{\bar{D}^1}$ from $\mathbf{V}_1$ and $\ell \in \{1,2\}$ the following are equivalent

(a) $\mathscr{I}$ is predense above p in $\mathbb{Q}^{\mathbf{V}_\ell}_{\bar{D}^1}$

(b) for every increasing η such that $\mathrm{tr}(p) \leq \eta \in p$ we can find T such that

(α) $T \subseteq {}^{\omega>}\omega$

(β) if $\nu \in T$ then $\eta \trianglelefteq \nu$

(γ) if $\eta \trianglelefteq \rho \trianglelefteq \nu \in T$ then $\rho \in T$

(δ) if $\nu \in T$ then $\{n < \omega : \nu\hat{\ }\langle n\rangle \in T\} \neq \emptyset \mod D^\ell_\nu$

(ε) $\{\nu \in T$: there is $r \in \mathscr{I} \subseteq Q^{\mathbf{V}_\ell}_{\bar{D}^\ell}$ above p with $\mathrm{tr}(\nu) = \nu\}$ contains a front of $T\}$.

As we shall not use it, we do not elaborate. □

Proof. 1) Clause (α) is obvious, clause (β) holds by 5.4(4), and clause (γ) follows (this is done also in [**?**]).
2) By part (1) we can assume that $\mathbf{V}_1 = \mathbf{V}$. So assume that $p \in \mathbb{Q}_{\bar{D}_1}, m^* < \omega$ and for each $m < m^*, g_m \in \mathscr{G}$ and $\underset{\sim}{T}^m, \langle \underset{\sim}{T}^m_n : n < \omega\rangle \in \mathbf{V}$ are $\mathbb{Q}^{\mathbf{V}}_{\bar{D}}$-names hence $\mathbb{Q}_{\bar{D}_1}$-names in $\mathbf{V}_1$ such that:

$(*)_1$ $p \Vdash_{\mathbb{Q}_{\bar{D}_1}}$ "$\underset{\sim}{T}^m, \underset{\sim}{T}^m_n \in \mathbf{T}_{\underset{\sim}{g}_m}$ and $\underset{\sim}{T}^m = \lim\langle \underset{\sim}{T}^m_n : n < \omega\rangle$".

Note that above, $\underset{\sim}{T}^m$ is the limit of $\langle \underset{\sim}{T}^m_n : n < \omega\rangle$ for the co-finite filter on ω. By the definition (4.7) it suffices to prove, for a given $n(**) < \omega$ that for some $n(*) > n(**)$ and q above p (in $\mathbb{Q}_{\bar{D}_1}$), q forces that: $m < m^* \Rightarrow \mathbf{r} \in \lim(\underset{\sim}{T}^m) \equiv \mathbf{r} \in \lim(\underset{\sim}{T}^m_{n(*)})$. By the definition and what we need to prove, as we can replace the name $\langle \underset{\sim}{T}^m_n : n < \omega\rangle$ by a name of an ω-subsequence (which is not necessarily a subsequence of the original sequence of names) without loss of generality

$(*)_2$ $p \Vdash$ "$\underset{\sim}{T}^m \cap {}^{n \geq}2 = \underset{\sim}{T}^m_n \cap {}^{n\geq}2$" for $n < \omega, m < m^*$.

Let $q_0 = \{\eta \in {}^{\omega>}\omega : \eta \text{ increasing}\})$, so $q_0 \in \mathbb{Q}_{\bar{D}}$, now we find $\langle T^m_\eta, T^m_{n,\eta} : \eta \in q_0, n < \omega$ and $m < m^*\rangle$ of course in $\mathbf{V}$ such that:

$(*)_3$ (i) $T^m_\eta, T^m_{n,\eta} \subseteq {}^{\omega>}2$, for $n < \omega, m < m^*$

(ii) for every $\eta \in q_0$ and $k < \omega$ we can find $q^m_{\eta,k}, q^m_{n,\eta,k} \in \mathbb{Q}_{\bar{D}}$ such that:
$q^{[\eta]}_0 \leq_{\rm pr} q^m_{\eta,k}$,
$q^{[\eta]}_0 \leq_{\rm pr} q^m_{n,\eta,k}$,
$q^m_{\eta,k} \Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{T}^m \cap {}^{k\geq}2 = T^m_\eta \cap {}^{k\geq}2$"
$q^m_{n,\eta,k} \Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{T}^m_n \cap {}^{k\geq}2 = T^m_{n,\eta} \cap {}^{k\geq}2$".

Now clearly

$(*)_4$ (i) $T^m_\eta, T^m_{n,\eta} \subseteq {}^{\omega>}2$,

(ii) $T^m_\eta = \lim_{D_\eta}\langle T^m_{\eta\char`\^<k>} : k < \omega\rangle$,

(iii) $T^m_{n,\eta} = \lim_{D_\eta}\langle T^m_{n,\eta\char`\^<k>} : k < \omega\rangle$.

Next note that

$(*)_5$ (a) $T^m_\eta, T^m_{n,\eta}$ belong to $\mathbf{T}_{g_m}$,

(b) $T^m_\eta = \lim\langle T^m_{n,\eta} : n < \omega\rangle$.

[Why does clause (a) hold? Let $T^m_\eta \cap {}^{g_m(\ell)\geq}2 = t$ then $q^m_{\eta,g_m(\ell)}$ forces that $\underset{\sim}{T}^m \cap {}^{g_m(\ell)\geq}2 = t$ but it also forces that $\underset{\sim}{T}^m$ satisfies the condition $(*)_\ell$ from Definition 4.4, hence in fact t satisfies the relevant parts of it, that is $k \leq \ell \Rightarrow (1 - \frac{1}{k})|t \cap {}^{g_m(k)}2|/2^{g(k)} \leq |t \cap Tg_m(\ell)2|$. As this holds for every ℓ clearly T^m_η satisfies $(*)_\ell$ of 4.4 for every ℓ. Similarly for $T^m_{n,\eta}$. Concerning clause (b) there is q satisfying $p^{[\eta]} \leq_{\rm pr} q \in \mathbb{Q}_{\bar{D}}$ forcing $\underset{\sim}{T}^m \cap {}^\ell 2 = t_m, \underset{\sim}{T}^m_n \cap {}^\ell 2 = t_{m,n}$ so by $(*)_2$, if $n \geq \ell$ they are equal. As any two (even finitely many) pure extensions of $p^{[\eta]}$ are compatible, we have $T^m_\eta \cap {}^\ell 2 = t_m, T^m_{n,\eta} \cap {}^\ell 2 = t_{m,n} = t_m$. This is clearly enough.]

Hence by assumption (d) we have for $u \subseteq m^*$ and $\eta \in p$

$(*)^{u,\eta}_6$ $\mathbf{r} \in \bigcap\limits_{m\in u} \lim(T^m_\eta)$ implies that $(\exists^\infty n)(\mathbf{r} \in \bigcap\limits_{m \in u} \lim(T^m_{n,\eta}))$ and moreover $(\forall A \in ([\omega]^{\aleph_0})^{\mathbf{V}})(\exists^\infty n \in A)[\mathbf{r} \in \bigcap\limits_{m\in u} \lim(T^m_{n,\eta})]$.

But if $\mathbf{r} \notin \lim(T^m_\eta)$ then for some $k^* < \omega, \mathbf{r} \restriction k^* \notin T^m_\eta$ hence for some $n^* < \omega$ we have $n^* < n < \omega \Rightarrow \mathbf{r} \restriction k^* \notin T^m_{n,\eta}$ (by $(*)_5(b)$), so we have

$(*)^{m,\eta}_7$ if $\mathbf{r} \notin \lim(T^m_\eta)$ then $(\forall^{<\infty} n)[\mathbf{r} \notin \lim(T^m_{n,\eta})]$.

By $(*)_4$ we have

$(*)_8(i)$ $\mathbf{r} \in \lim(T^m_\eta)$ iff $\mathbf{r} \in \lim(\lim_{D_\eta}\langle T^m_{\eta\char`\^\langle k\rangle} : k < \omega\rangle)$

(ii) $\mathbf{r} \in \lim(T^m_{n,\eta})$ iff $\mathbf{r} \in \lim(\lim_{D_\eta}\langle T^m_{n,\eta\char`\^\langle k\rangle} : k < \omega\rangle)$.

By $(*)_6 + (*)_7$ applied to $\eta = \text{tr}(p)$, we can find $n(*) > n(**)$, see $(*)_7 + (*)_6$, such that $(\forall m < m^*)[\mathbf{r} \in \lim(T^m_{\text{tr}(p)}) \equiv \mathbf{r} \in \lim(T^m_{n(*),\text{tr}(p)})]$. Next let

$$q =: \{\nu \in p : \begin{array}{l} \text{if } \ell g(\text{tr}(p)) \le \ell \le \ell g(\nu) \text{ and } m < m^* \text{ then} \\ (\mathbf{r} \in T^m_{\nu \restriction \ell}) \equiv (\mathbf{r} \in T^m_{n(*),\nu\restriction\ell})\}. \end{array}$$

Now $p \le_{\text{pr}} q \in \mathbb{Q}_{\bar D}$ by $(*)_8$. Lastly, let $q^* =: \{\nu \in q :$ if $\text{tr}(p) \trianglelefteq \nu$, then $\ell g(\text{tr}(p)) \le \ell < \ell g(\nu) \Rightarrow \nu \in q_{\nu\restriction\ell,\ell}\}$.

Does $q^* \Vdash_{\mathbb{Q}_{\bar D_1}}$ "$\mathbf{r} \in \lim(\underset{\sim}{T}^m) \equiv \mathbf{r} \in \lim(\underset{\sim}{T}^m_{n(*)})$"? If not, then for some q^{**} we have $q^* \le q^{**}$ and $q^{**} \Vdash_{\mathbb{Q}_{\bar D_1}}$ "$\mathbf{r} \in \lim(\underset{\sim}{T}^m) \equiv \mathbf{r} \notin \lim(\underset{\sim}{T}^m_{n(*)})$" and moreover, for some k we have $q^{**} \Vdash_{\mathbb{Q}_{\bar D_1}}$ "$\mathbf{r} \restriction k \in \underset{\sim}{T}^m \equiv \mathbf{r} \restriction k \notin \underset{\sim}{T}^m_{n(*)}$". But $q^{**}, q_{\text{tr}(q^{**}),k}, q^m_{n(*),\text{tr}(q^{**}),k}$ are compatible having the same trunk, so let q' be a common upper bound with $\text{tr}(q') = \text{tr}(q^{**})$ and we get a contradiction. $\square$

* * *

Results here are used in the next section; formally we have to specialize them as $\mathbb{Q}_0$ is just j random reals forcing.

For preservation, including "cardinals are not collpased" we use §2 or §3 (really more explicit version).

Hypothesis 5.11.

(a) $\mathbf{V} \models \text{CH}$

(b) $\mathscr{F}$ is a full trunk controller of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle$, each $\mathscr{F}_\alpha$ is as defined in Fact **??** if $\alpha > 0$ and α^* is large enough and

(c) $\mathscr{K}(0)$ is a family whose elements we denote by $\bar R$ and Lim is a function with domain $\mathscr{K}(0)$ such that for each $\bar R \in \mathscr{K}(0), \text{Lim}(\bar R)$ is a c.c.c. forcing notion such that for simplicity two compatible elements has a l.u.b. and $\mathbb{Q}$ is considered as a psc forcing by the identity function as in **??** (so for each member $\bar R$ of $\mathscr{K}(0), \lim(\bar R) \subseteq \mathscr{F}_0$)

(d) $\le_{\mathscr{K}(0)}$ is a partial order on $\mathscr{K}(0)$ such that $\bar R' \le_{\mathscr{K}(0)} \bar R'' \Rightarrow \text{Lim}(\bar R') \lessdot \text{Lim}(\bar R'')$.

Remark 5.12. Recall that "κ-closed" means every increasing sequence of length $< \kappa$ has an upper bound. We say $\mathscr{K}(0)$ is θ-exactly closed if for $\le_{\mathscr{K}(0)}$-increasing sequence $\langle \bar R^i : i < \theta \rangle$ there is $\bar R \in \mathscr{K}(0)$ such that $i < \theta \Rightarrow \bar R^i \le_{\mathscr{K}(0)} \bar R$ and $\text{Lim}(\bar R) = \bigcup\limits_{i<\theta} \text{Lim}(\bar R_i)$.

Definition 5.13. 1) For an ordinal $\alpha > 0$ (assuming $\alpha \le \alpha^*$ recalling 5.11(b)) let $\mathfrak{K}_\alpha$ be the family of $\bar{\mathbb{Q}}$ such that:

(a) $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration of length α

(b) $\mathbb{Q}_0$ is a c.c.c. forcing notion from $\mathfrak{K}(0)$, i.e. it is $\text{Lim}(\bar R), \bar R \in \mathfrak{K}(0)$, in principle $\text{Lim}(\bar R)$ may not determine $\bar R$ uniquely but we shall ignore this writing $\bar R^{\mathbb{Q}_0}$ or $\bar R^{\bar{\mathbb{Q}}}$

(c) if $0 < \beta < \alpha$ then $\underset{\sim}{\mathbb{Q}}_\beta$ is $\underset{\sim}{\mathbb{Q}}_{\underset{\sim}{\bar D}_\beta}$ where $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\bar D}_\beta \in \mathbf{IUF}$" (on $\underset{\sim}{\mathbb{Q}}_{\bar D}$ see 5.2, on **IUF**, see 5.1).

2) Let $\mathfrak{K} = \cup\{\mathfrak{K}_\alpha : \alpha < \alpha^*\}$ and $\mathfrak{K}_{<\alpha} = \cup\{\mathfrak{K}_\beta : \beta < \alpha\}$ and $\mathfrak{K}_{\le\alpha} = \mathfrak{K}_{<\alpha+1}$.

We use $\mathbb{P}_\alpha = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}} \restriction \alpha)$, so e.g. $\mathbb{P}^1_\beta = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}^1 \restriction \beta)$, etc., recalling that: if $\beta < \alpha$ and $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ then $\mathbb{P}_\beta \lessdot \mathbb{P}_\alpha$.

Claim 5.14. *1) $\mathscr{F}$ is apurely c.c.c. full trunk controller iteration.*
2) If $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ then $\bar{\mathbb{Q}}$ is strongly$^+$ $\mathscr{F}$-psc Definition 5.3(2), relying on Definition 2.6(1) and Definition 3.1(1).
3) If $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ then $\bar{\mathbb{Q}}$ satisfies the criterion from 1.18 for "$\aleph_2$-c.c."
4) If $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ then $\mathrm{Lim}(\bar{\mathbb{Q}})$ has (r,i)-pure decidability for $[1, \ell g(\bar{\mathbb{Q}}))$.

Proof. 1) Note that $\mathscr{F}$ is a full trunk control iteration by clause (b) of 5.11.

Version A: Now $\mathscr{F}_0$ is apurely c.c.c. as in 3.2, see clause (c) of 5.11. Also each $\mathscr{F}_{1+\alpha}$ is as in 3.1. So by xxx from §1 we get the desired result. Saharon CHECK!

Version B: Now why is $\mathscr{F}$ apurely c.c.c. (see Definition 2.3)? Let $\langle y_\varepsilon : \varepsilon < \omega_1\rangle$ be $\le^{\mathscr{F}}_{\rm apr}$-increasing and $y_\varepsilon \le^{\mathscr{F}}_{\rm apr} z_\varepsilon$. So

- (a) $\varepsilon < \zeta \wedge 0 \in \mathrm{Dom}(y_\varepsilon) \Rightarrow y_\varepsilon = y_\varepsilon$
- (b) $\langle \mathrm{Dom}(y_\varepsilon) : \varepsilon < \omega_1\rangle$ is increasing
- (c) if $\varepsilon < \zeta$ and $\alpha \in \mathrm{Dom}(y_\varepsilon)\backslash\{0\}$ then $p_\varepsilon(\alpha) = p_\zeta(\alpha)$
- (d) if $\varepsilon < \omega_1$ then $w_\varepsilon = \{\alpha : \alpha \in \mathrm{Dom}(p_\varepsilon)\backslash\{0\}$ and $y_\varepsilon(\alpha) \ne z_\varepsilon(\alpha)\}$ is finite.

So we can find a stationary $S \subseteq \omega_1, \omega_2, z_*$ such that:

- (e) $\varepsilon \in S \Rightarrow |w_\varepsilon| = n_*$
- (f) $\varepsilon \in S \Rightarrow w_\varepsilon \cap \cup\{\mathrm{Dom}(p_\zeta) : \zeta < \varepsilon\} = w_*$
- (g) $z_\varepsilon \restriction w_* = z_*$.

By clause (c) of the Hypothesis 5.11 clearly $\mathscr{F}_0$ satisfies the c.c.c. hence we can find $\varepsilon < \zeta$ from S and $z_0 \in \mathscr{F}_0$ which is above $z_\varepsilon(0), z_\zeta(0)$ when defined.

Now we define $z \in \mathscr{F}$:

- (α) $\mathrm{Dom}(z) = \mathrm{Dom}(z_\varepsilon) \cup \mathrm{Dom}(z_\zeta)$
- (β) $z(\alpha) = z_0$ if $\alpha = 0 \in \mathrm{Dom}(z_\varepsilon) \cup \mathrm{Dom}(z_\zeta)$
- (γ) $z(\alpha) = y_\varepsilon(\alpha) = y_\zeta(\alpha)$ if $\alpha \in \mathrm{Dom}(y_\varepsilon)\backslash w_*\backslash\{0\}$
- (δ) $z(\alpha) = z_\varepsilon(\alpha) = z_\zeta(\alpha)$ if $w_*\backslash\{0\}$
- (ε) $z(\alpha) = z_\zeta(\alpha)$ if $\alpha \in \mathrm{Dom}(z_\zeta)\backslash\mathrm{Dom}(z_\varepsilon)$.

Now z is a common upper bound of z_ε, z_ζ, (noting then on $\mathrm{Dom}(z_\varepsilon)\backslash w_*\backslash\{0\}$ it agrees with z_ε, etc.).

2) We have to show that each $\underset{\sim}{\mathbb{Q}}_\alpha$ is (forced to be) strongly$^+$ $\mathscr{F}_\alpha$-psc.

For $\alpha = 0$: If $\mathrm{val}^{\mathbb{Q}_0}(q_\varepsilon), \mathrm{val}^{\mathbb{Q}_0}(q_\zeta)$ are compatible in $\mathscr{F}$, then trivially p_ε, q_ζ are compatible in $\mathbb{Q}_0$, so have a common upper bound (see clause (c) of Hypothesis 5.11, call it r. Now "$p_\zeta \le_{\rm apr} r$" as $\le^{\mathscr{F}_0}_{\rm apr} = \le^{\mathscr{F}_0}$ and so clause (i) of 3.1(b), also "$q_\varepsilon \le r$" so clause (ii) there holds, "r is a l.u.b. of p_ζ, q_ε" by its choice so clause (iii) there holds and lastly if $p_\varepsilon \le_{\rm pr} q_\varepsilon$ then $p_\varepsilon = q_\varepsilon$ but $p_\varepsilon \le_{\rm pr} p_\zeta$ hence $r = p_\zeta$ is as required.

3) Note that for $\alpha > 0$ any two members of $\underset{\sim}{\mathbb{Q}}$ with the same trunk has a common $\le_{\rm pr}$-upper bound with the same trunk. So the criterion is easy.

4) By 3.6(1). $\square_{5.14}$

Definition 5.15. 1) For $\bar{\mathbb{Q}}_1, \bar{\mathbb{Q}}_2 \in \mathfrak{K}$ let $\bar{\mathbb{Q}}_1 \le_{\mathfrak{K}} \bar{\mathbb{Q}}_2$ if:

(a) $\ell g(\bar{\mathbb{Q}}_1) \le \ell g(\bar{\mathbb{Q}}_2)$ and $\bar{R}^{\bar{\mathbb{Q}}_1} \le_{\mathfrak{K}(0)} \bar{R}^{\bar{\mathbb{Q}}_2}$

(b) for $\beta < \ell g(\bar{\mathbb{Q}}_1)$ we have $\mathbb{P}_{1,\beta} \lessdot \mathbb{P}_{2,\beta}$, i.e. $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1 \restriction \beta) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2 \restriction \beta)$

(c) for $\beta < \ell g(\bar{\mathbb{Q}}_1), \beta \neq 0$ and $\eta \in {}^{\omega >}\omega$ we have $\Vdash_{\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2 \restriction \beta)}$ "$\underset{\sim}{D}_{1,\beta,\eta} \subseteq \underset{\sim}{D}_{2,\beta,\eta}$"

(d) if $\ell g(\bar{\mathbb{Q}}_1) = \beta < \ell g(\bar{\mathbb{Q}}_2)$ <u>then</u> $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1 \restriction \beta) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2 \restriction \beta)$.

Claim 5.16. *Assume $\bar{\mathbb{Q}}_1 \le_{\mathfrak{K}} \bar{\mathbb{Q}}_2$ are from $\mathfrak{K}_\alpha$.*
1) If $\ell g(\bar{\mathbb{Q}}_1)$ is not a limit ordinal <u>then</u>

$$\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2).$$

2) If $\alpha = \ell g(\bar{\mathbb{Q}}_1)$ is a limit ordinal, $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\mathbb{Q}_2), \mathscr{G} \subseteq \mathscr{G}^{\mathbf{V}}, \beta < \alpha, \underset{\sim}{\nu}$ is a $\mathbb{P}_{2,\beta}$-name such that, for every $\gamma \in [\beta, \alpha)$ we have $\Vdash_{\mathbb{P}_{2,\gamma}}$ "$\underset{\sim}{\nu}$ is $\mathscr{G}$-continuous over $\mathbf{V}^{\mathbb{P}_{1,\gamma}}$" and $\mathbb{P}_{1,\alpha} = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathbb{P}_{2,\alpha} = \mathrm{Lim}(\bar{\mathbb{Q}}_2 \restriction \alpha)$, <u>then</u> $\Vdash_{\mathbb{P}_{2,\alpha}}$ "$\underset{\sim}{\nu}$ is $\mathscr{G}$-continuous over $\mathbf{V}^{\mathbb{P}_{1,\alpha}}$".

Proof. 1) Let $\alpha = \ell g(\bar{\mathbb{Q}}_1)$, by 1.14 we know that $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2 \restriction \alpha) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2)$ so it suffices to prove that $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2 \restriction \alpha)$. If $\alpha = \beta + 1$, if $\beta = 0$ use the second phrase of clause (a) of Definition 5.15, so assume $\beta > 0$, by clause (b) of Definition 5.15 we know that $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1 \restriction \beta) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2 \restriction \beta)$ and by clause (c) of Definition 5.11 we can apply 5.9(1) so we are done. If $\alpha = 0$ the statement is trivial and the case α limit was excluded (really $\mathrm{cf}(\alpha) \neq \aleph_0$ suffices.
2) So assume that $m(*) < \omega, g_m \in \mathscr{G}$ for $m < m(*)$ and $\Vdash_{\mathbb{P}_{1,\alpha}}$ "$\underset{\sim}{T}_m, \underset{\sim}{T}^m_m \in \mathbf{T}_{g_m}$ and $\underset{\sim}{T}_m = \lim\langle \underset{\sim}{T}^m_n : n < \omega\rangle$ for $m < m(*)$".
Without loss of generality

$\circledast$ "$\Vdash_{\mathbb{P}_{1,\alpha}} \underset{\sim}{T}_m \cap {}^n 2 = \underset{\sim}{T}_{m,n} \cap {}^n 2$ for $m < m(*), n < \omega$".

[Why? As in an earlier proof, creating appropriate name of a subsequence.]
By 3.3(1), for a dense set of $p \in \mathbb{P}_{1,\alpha}$ we have

$(*)_p$ (a) for every $m, n, k < \omega$, the set $\mathscr{I}_{m,n,k} = \{q : p \le_{\mathrm{apr}} q$ and q forces a value to $\underset{\sim}{T}_m \cap {}^{k\ge}2$ and to $\underset{\sim}{T}^m_n \cap {}^{k\ge}2\}$ is predense above p

(b) if $\gamma \in \mathrm{Dom}(p) \backslash \{\delta\}$ [SAHARON: what for $\gamma - 0$? better avoid] and $y \in \mathscr{F}, \mathrm{Dom}(y) = \mathrm{Dom}(p) \cap [\gamma, \alpha)$ and $\mathrm{tr}(p) \restriction [\gamma, \alpha) \le^{\mathscr{F}}_{\mathrm{apr}} y$ and $m, n < \omega$ then $\underset{\sim}{T}_{\gamma,y,m}, \underset{\sim}{T}_{\gamma,y,m,n}$ are $\mathbb{P}_{1,\gamma}$-names of members of $\mathbf{T}_{g_m}$ such that

$\Vdash_{\mathbb{P}_{1,\gamma}}$ " if there is q satisfying $p \le q, \mathrm{tr}(q) \restriction [\gamma, \alpha) = y$ and $q \restriction \gamma \in \underset{\sim}{G}_{\mathbb{P}_{1,\gamma}}$
<u>then</u> for every k for some r we have $q \le r$,
$\mathrm{tr}(r) \restriction [\gamma, \alpha) = y, r \restriction \gamma \in \underset{\sim}{G}_{\mathbb{P}_{1,\gamma}}$ and $r \Vdash_{\mathbb{P}_{1,\alpha}/\mathbb{P}_{1,\gamma}}$
"$\underset{\sim}{T}_m \cap {}^{k>}2 = \underset{\sim}{T}_{\gamma,y,m}[\underset{\sim}{G}_{\mathbb{P}_{1,\gamma}}] \cap {}^{k\ge}2$
and $\underset{\sim}{T}^m_n \cap {}^{k>}2 = \underset{\sim}{T}_{\gamma,y,m,n}[\underset{\sim}{G}_{\mathbb{P}_{1,\gamma}}] \cap {}^{k\ge}2$".

This is possible as each $\mathbb{Q}_{1+\alpha}$ has pure $(2,2)$-decidability, and so we can apply 3.8 Saharon.
[Why? Recall that each $\underset{\sim}{\mathbb{Q}}_{1+\alpha}$ has pure $(2,2)$-decidability hence claim 3.8 apply.]
So easily

$$\Vdash_{\mathbb{P}_{1,\gamma}} \text{“} \quad \underset{\sim}{T}_{\gamma,y,m} \in \mathscr{T}_{\mathscr{G}} \text{ and } \underset{\sim}{T}_{\gamma,y,m,n} \in \underset{\sim}{T}_{\mathscr{G}} \text{ and } \underset{\sim}{\nu} \in \underset{\sim}{T}_{\gamma,y,m} \text{ and } T_{\gamma,y,m} = \lim\langle \underset{\sim}{T}_{\gamma,y,m,n} : n < \omega\rangle \text{ by } \circledast \text{”.}$$

when $m < m(*), n < \omega, y$ as above. So suppose $n(*) < \omega, q \in \mathbb{P}_{2,\alpha}$ and $q \Vdash_{\mathbb{P}_{2,\alpha}}$ “$\underset{\sim}{\nu} \in \cap\{\lim(\underset{\sim}{T}_m) : m < m(*)\}$ and we shall prove $q \nVdash$ “$\neg(\exists n)(n \geq n(*)) \wedge \underset{\sim}{\nu} \in \cap\{\underset{\sim}{T}_{m,n} : m < m(*)\}$.

Now $\{p \in \mathbb{P}_{1,\alpha} : (*)_p\}$ is dense in $\mathbb{P}_{1,\alpha}$ hence by an assumption also in $\mathbb{P}_{2,\alpha}$. Hence q is compatible with some p such that $(*)_p$, so without loss of generality $\mathbb{P}_{2,\alpha} \models$ “$p \leq q$”. So we can find γ such that:

$(*)$ (a) $0 < \gamma < \alpha$

(b) if $\beta \in \mathrm{Dom}(p), \mathrm{tr}(p)(\beta) \neq \mathrm{tr}(q)(\beta)$ then $\beta < \alpha$.

[Need considerably more! Saharon!]

Let $q \restriction \gamma \in G_{2,\gamma} \subseteq P_{2,\gamma}, G_\gamma$ generic over $\mathbf{V}$ and let $G_{1,\gamma} = G_{2,\gamma} \cap \mathbb{P}_{1,\gamma}$ hence it is a generic subset of $\mathbb{P}_{1,\gamma}$ over $\mathbf{V}$. Let $y = \mathrm{tr}(p){\restriction}[\gamma, \alpha)$.

Now in $\mathbf{V}[G_{1,\gamma}]$ the objects $\underset{\sim}{T}_{\gamma,y,m}[G_{1,\gamma}], \underset{\sim}{T}_{\gamma,y,m,n}[G_{1,\gamma}]$ belongs to $\underset{\sim}{T}_{\mathscr{G}}^{\mathbf{V}[G_{1,\gamma}]}$ and $\underset{\sim}{\nu}[G_{1,\gamma}] = \cap\{\lim(\underset{\sim}{T}_{\gamma,y,m}[G_{1,\gamma}]) : m < m(*)\}$ and $\underset{\sim}{T}_{\gamma,y,m}[G_{1,\gamma}] = \lim\langle \underset{\sim}{T}_{\gamma,y,m,n}[G_{1,\gamma}] : n < \omega\rangle$. As we have assumed that $\Vdash_{\mathbb{P}_{2,\gamma}}$ “$\underset{\sim}{\nu}$ is $\mathscr{G}$-continuous over $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{2,gamma}} \cap \mathbb{P}_{1,\gamma}]$”, it follows that for some $n \in (n(*), \omega)$ we have $\underset{\sim}{\nu}[G_{1,\gamma}] \in \cap\{\underset{\sim}{T}_{\gamma,y,m,n}[G_{1,\gamma}] : m < m(*)\}$.

We continue as in 3.3 [MORE]!! $\square_{5.16}$

Definition 5.17. 1) By induction on $\alpha \geq 1$ we define $\mathfrak{K}^+_\alpha$ as the family of $\bar{\mathbb{Q}}_1 \in \mathfrak{K}^\pm_\alpha = \{\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha : (\forall \beta)(1 \leq \beta < \alpha \rightarrow \bar{\mathbb{Q}} \restriction \beta \in \mathfrak{K}^+_\beta)\}$ such that $\bar{\mathbb{Q}}_1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}_2 \in \mathfrak{K}^\pm_\alpha \Rightarrow \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2)$.
2) Let $\mathfrak{K}^+_{<\beta} = \cup\{\mathfrak{K}^+_\alpha : \alpha < \beta\}$ and $\mathfrak{K}^+ = \mathfrak{K}^+_{<\alpha^*}$.

Remark 5.18. Should we now replace the demand in 5.17(1) by: if $p \in \lim(\bar{\mathbb{Q}}_1)$ and $\mathscr{I} \subseteq \{q : p \leq_{\mathrm{apr}} q\}$ is predense above p in $\lim(\bar{\mathbb{Q}}_1)$ then also in $\mathrm{Lim}(\bar{\mathbb{Q}}_2)$? Saharon!

Observation 5.19. *1) If $\bar{\mathbb{Q}}_1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}_2$ are both from $\mathfrak{K}_{\alpha+1}$ <u>then</u> $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2)$.*
2) $\leq_{\mathfrak{K}}$ is a partial order on $\mathfrak{K}$.
3) If $\bar{\mathbb{Q}}_1, \bar{\mathbb{Q}}_2 \in \mathfrak{K}^+_\alpha$ and $\bar{\mathbb{Q}}_1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}_2$ <u>then</u> $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_2)$.
4) If $\bar{\mathbb{Q}}_1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}_2 \in \mathfrak{K}_\alpha$ and $\mathbb{Q}_{1,0} = \mathbb{Q}_{2,0}$ <u>then</u> $\bar{\mathbb{Q}}_2 = \bar{\mathbb{Q}}_1$.
5) If $\bar{\mathbb{Q}}_i \in \mathfrak{K}^+_\alpha$ for $i < \delta$ is $\leq_{\mathfrak{K}}$-increasing, $\delta < \kappa$ and $\mathfrak{K}(0)$ is κ-closed <u>then</u> there is $\bar{\mathbb{Q}} \in \mathfrak{K}^+_\alpha$ such that $i < \delta \Rightarrow \bar{\mathbb{Q}}_i \leq_{\mathfrak{K}} \bar{\mathbb{Q}}$.
6) If in (5) if $\mathfrak{K}(0)$ is $\mathrm{cf}(\delta)$-exactly closed, $\mathrm{cf}(\delta) > \aleph_0$, then we can add $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}) = \bigcup\limits_{i<\delta} \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_i)$.
7) If $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha, (\forall \beta < \kappa)(|\beta|^{\aleph_0} < \kappa = \mathrm{cf}(\kappa))$ and $\mathfrak{K}(0)$ is κ-closed, each member of $\mathfrak{K}_\alpha(0)$ is of cardinality $< \kappa, \alpha < \kappa, \aleph_0 < \theta = \mathrm{cf}(\theta) < \kappa, \mathfrak{K}(0)$ is exactly θ-closed and $|\mathscr{F}| < \kappa$ <u>then</u> there is $\bar{\mathbb{Q}}' \in \mathfrak{K}^+_\alpha$ such that $\bar{\mathbb{Q}} \leq_{\mathfrak{K}} \bar{\mathbb{Q}}'$ (we can normally bound the cardinality of $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}')$).

Proof. By induction on α, quite straightforward. $\square_{5.19}$

6. On a relative of Borel conjecture with large $\mathfrak{b}$

Hypothesis 6.1.
(*a*) $\mathbf{V} \models CH$
(*b*) $\mathrm{cf}(\lambda) = \lambda > \aleph_2, (\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda), S \subseteq \lambda$ is stationary, $(\forall \delta \in S)(\mathrm{cf}(\delta) > \aleph_0)$
(*c*) $\mathscr{F}$ is as in 5.11(b), in particular a full trunk controller iteration of $\langle \mathscr{F}_\alpha : \alpha < \alpha^* \rangle, \alpha^* > \lambda^+, \mathscr{F}_{1+\alpha}$ is from 5.2, $\mathscr{F}_0 = \mathrm{Random}_\lambda$ (so below and 3.2). As $\mathscr{F}$ is constant we shall write $\mathrm{Lim}(\bar{\mathbb{Q}})$ instead of $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$.

Note that $\lambda > \aleph_2$ is not a real restriction.

We now specify the $\mathfrak{K}$ from §5.

Definition 6.2. 1) Let $\mathfrak{K}(0)$ be the family of $\{\mathrm{Random}_A : A \subseteq \lambda\}$ where $d(A) = \{\omega\alpha + n : \alpha \in A, n < \omega\}$ and Random_A is the family of Borel subsets of ${}^{d(A)}2$ of positive Lebesgue measure. Let $\underset{\sim}{\nu}_\alpha = \cup\{f : f$ a finite function from $[\omega\alpha, \omega\alpha + \omega)$ to $\{0, 1\}$ such that $[f] = \{g \in {}^A2 : f \subseteq g\}$ belongs to the generic$\}$. Let $A(\mathbb{Q}) = A$ if $\mathbb{Q} = \mathrm{Random}_A$ and $A(\bar{\mathbb{Q}}) = A(\mathbb{Q}_0)$. Let $\mathrm{Random}_A \leq_{\mathfrak{K}(0)} \mathrm{Random}_B$ if $A \subseteq B$ hence $\mathrm{Random}_A \lessdot \mathrm{Random}_B$. (So $\leq_{\mathrm{pr}}$ will be just equality, $\leq_{\mathrm{apr}}$ will be the usual order).
2) For $\alpha \geq 1$, let $\mathfrak{K}_\alpha$ be defined as in 5.13.
3) We define for any ordinal α and $\ell < 2$ the class $\mathfrak{K}'_{\ell,\alpha} \subseteq \mathfrak{K}_\alpha$ as the class of $\bar{\mathbb{Q}}$ such that:

(*a*) $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration
(*b*) $\ell g(\bar{\mathbb{Q}}) = \alpha$
(*c*) $\mathbb{Q}_0 \in \mathfrak{K}(0)$ and $A[\mathbb{Q}_0] \in [\lambda]^{<\lambda}$ if $\ell = 0$ and $A[\mathbb{Q}_0] = \lambda$ if $\ell = 1$
(*d*) if $0 < \beta < \alpha$ then $\underset{\sim}{\mathbb{Q}}_\beta = \mathbb{Q}(\underset{\sim}{\bar{D}}_\beta)$ where $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\bar{D}}_\beta = \langle \underset{\sim}{D}_{\beta,\eta} : \eta \in {}^{\omega>}\omega \rangle \in$ **IUF**"
(*e*) $\bar{\mathbb{Q}} \restriction \gamma \in \mathfrak{K}^+_\gamma$ for every $\gamma < \alpha$ where $\mathfrak{K}^+_\gamma$ is defined in 5.17 for our particular case.

3A) If we omit ℓ, we [6] mean $\ell = 0$ when $\alpha < \lambda$ and we mean $\ell = 1$ when $\alpha \geq \lambda$. We let $\mathfrak{K}'_\ell = \cup\{\mathfrak{K}'_{\ell,\alpha} : \alpha$ an ordinal $\leq \alpha^*\}$ and $\mathfrak{K}' = \cup\{\mathfrak{K}'_\alpha : \alpha$ an ordinal $\leq \alpha^*\}$.
4) For $\ell = 0, 1$, we define a partial order $\leq_{\mathfrak{K}'_\ell}$ on $\mathfrak{K}'_\ell$ by:
$\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}'_\ell} \bar{\mathbb{Q}}^2$ <u>iff</u> $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}^2$ (see Definition 5.15) and $[\ell g(\bar{\mathbb{Q}}^1) < \ell g(\bar{\mathbb{Q}}^2) \Rightarrow$ [there is $\bar{\mathbb{Q}}' \in \mathfrak{K}^+_{\ell g(\bar{\mathbb{Q}}^1)}$ such that $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}' \leq_{\mathfrak{K}} \bar{\mathbb{Q}}^2$] and

$(*)$ if γ is the minimal member of $A(\bar{\mathbb{Q}}^2) \backslash A(\bar{\mathbb{Q}}^1)$ <u>then</u> (so if $A(\bar{\mathbb{Q}}^1) = A(\bar{\mathbb{Q}}^2)$ this holds vacuously)
$\Vdash_{\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}^2)}$ "$\underset{\sim}{\nu}_\gamma$ is $\mathscr{G}^{\mathbf{V}}$-continuous over $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}}^1)}$" (see 4.7(1), (3)).

4A) We similarly define the partial order $\leq_{\mathfrak{K}'}$ on $\mathfrak{K}'$.
5) Let $\mathfrak{K}''_\alpha$ be the family of $\bar{\mathbb{Q}} \in \mathfrak{K}'_\alpha$ such that:

[6] Why? As we shall build $\bar{\mathbb{Q}} \in \mathfrak{K}'_\lambda$ such that $\mathrm{Lim}(\bar{\mathbb{Q}})$ make the continuum λ; it is built as the limit of an increasing sequence $\langle \bar{\mathbb{Q}}_\alpha : \alpha < \lambda \rangle, \bar{\mathbb{Q}}_\alpha \in \mathfrak{K}'_\alpha$ and we like that $A(\bar{\mathbb{Q}}_\alpha) \in [\lambda]^{<\lambda}$ for $\alpha < \lambda$ but is λ for $\alpha = \lambda$. We later consider $\bar{\mathbb{Q}}_\alpha$ for $\alpha \in [\lambda, \lambda^+]$, so we write $\alpha \geq \lambda$.

(f) if $\alpha \geq \lambda, \beta \in (0,\alpha]$ and we have $g \in \mathscr{G}^{\mathbf{V}}$ and $T, \langle T_n : n < \omega\rangle \in \mathbf{T}_g^{\mathbf{V}^{\mathbb{P}_\beta}}, T = \lim\langle T_n : n < \omega\rangle$, then for some club[7] E of λ for every $j \in E \cap S$, we have $\underset{\sim}{\nu}_j \in \lim(T) \Rightarrow (\exists^\infty n)\underset{\sim}{\nu}_j \in \lim(T_n)$.

6) Let $\mathfrak{K}'' = \cup\{\mathfrak{K}''_\alpha : \alpha < \alpha^*\}$ and $\leq_{\mathfrak{K}''} = \leq_{\mathfrak{K}'} \restriction \mathfrak{K}''$

Claim 6.3. *1) If $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}_0} \bar{\mathbb{Q}}^2$ then*

(a) *if $p \in \mathrm{Lim}(\bar{\mathbb{Q}}^1)$ then $p \in \mathrm{Lim}(\bar{\mathbb{Q}}^2)$*
(b) *if $x \in \{\mathrm{us}, \mathrm{pr}, \mathrm{apr}\}$ and $p, q \in \mathrm{Lim}(\bar{\mathbb{Q}}^1)$ then $\mathrm{Lim}(\bar{\mathbb{Q}}) \models p \leq_x q$ iff $\mathrm{Lim}(\bar{\mathbb{Q}}^2) \models p \leq_x q$*
(c) *if $p, q \in \mathrm{Lim}(\bar{\mathbb{Q}}^1)$ then p, q are compatible in $\mathrm{Lim}(\bar{\mathbb{Q}}^1)$ iff p, q are compatible in $\mathrm{Lim}(\bar{\mathbb{Q}}^2)$.*

2) Assume β is a limit ordinal, $\bar{\mathbb{Q}}^\ell \in \mathfrak{K}'_\beta$ and $\gamma < \beta \Rightarrow \bar{\mathbb{Q}}^1 \restriction \gamma \leq_{\mathfrak{K}'_\gamma} \bar{\mathbb{Q}}^2 \restriction \gamma$ then $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^2$ (and if additionally $\bar{\mathbb{Q}}^1 \in \mathfrak{K}^+_\beta$ then $\mathrm{Lim}(\bar{\mathbb{Q}}^1) \lessdot \mathrm{Lim}(\bar{\mathbb{Q}}^2)$).

Proof. Should be clear. $\square_{6.3}$

Claim 6.4. *1) The two place relations $\leq_{\mathfrak{K}'_0}, \leq_{\mathfrak{K}''_1}$ are partial orders.*
2) The two place relations $\leq_{\mathfrak{K}'}, \leq_{\mathfrak{K}''}$ are partial orders (on $\mathfrak{K}'_0, \mathfrak{K}''_0$ respectively or $\mathfrak{K}', \mathfrak{K}''$ respectively).
3) Assume

(a) *δ is a limit ordinal*
(b) *$\bar{\mathbb{Q}}^1, \bar{\mathbb{Q}}^2 \in \mathfrak{K}_{0,\delta}$ and $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}^2$*
(c) *$\mathbb{P}^1_\delta = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}^1 \restriction \delta) \lessdot \mathbb{P}^2_\delta = \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}^2 \restriction \delta)$*
(d) *$\alpha < \delta \Rightarrow \bar{\mathbb{Q}}^1 \restriction \alpha \leq_{\mathfrak{K}'_{0,\alpha}} \bar{\mathbb{Q}}^2 \restriction (\alpha + 1)$, see 6.2(4).*

Then $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}'_0} \bar{\mathbb{Q}}^2$.
4) If $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}^2$ and $\mathrm{cf}(\ell g(\bar{\mathbb{Q}}^1)) \neq \aleph_0 < \ell g(\bar{\mathbb{Q}}^2)$ then $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}^1) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}^2 \restriction \alpha)$.
5) In (4), $\bar{\mathbb{Q}}^1 \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}^2$ if the $\Vdash_{\mathrm{Lim}(\bar{\mathbb{Q}}^2)}$ "$\underset{\sim}{\nu}_\gamma$ is $\mathscr{G}$-continuous over $\mathbf{V}[\underset{\sim}{G}_{\mathrm{Lim}(\bar{\mathbb{Q}}^1)}$" when $\gamma = \mathrm{Min}(A(\bar{\mathbb{Q}}^2) \backslash A(\bar{\mathbb{Q}}^1)$.

Proof. 1) Easy.
2) Easy.
3) By 6.2.
4) By 5.9(1). $\square_{6.4}$

Observation 6.5. *1) If $\bar{\mathbb{Q}} \in \mathfrak{K}''_\beta$, then in $\mathbf{V}^{\mathbb{P}_\beta}$ we have: if $\mathscr{B}^*$ is a Borel subset of ${}^\omega 2$ Lebesgue of measure $1, \mathscr{B}^* = \bigcup\limits_{n<\omega} \lim(T_n)$ and $\{T_n : n < \omega\} \subseteq \mathbf{T}_g, g \in \mathscr{G}^{\mathbf{V}}$ then for a club of $j \in S$ we have $(\mathrm{cf}(j) > \aleph_0$ and) $\underset{\sim}{\nu}_j \in \mathscr{B}^*$.*
2) If $\bar{\mathbb{Q}} \in \mathfrak{K}'_\ell$ and $\beta \leq \ell g(\bar{\mathbb{Q}})$, then the $\mathscr{F}$-forcing notion $\mathbb{P}_\beta$ is $\mathscr{F}$-psc and has 2-pure decidability over $\mathbb{Q}_0$, recall Definition 3.7(2).
3) Moreover, (in (2)) $\bar{\mathscr{F}}$ is semi-simple, hence if $p \in \mathbb{P}_\beta, p \Vdash$ "$\underset{\sim}{\tau} \in {}^\omega\mathrm{Ord}$" then for some q we have $p \leq_{\mathrm{pr}} q$ and for each $n, \mathscr{I}_n = \{r : q \leq_{apr} r$ and r forces a value to $\underset{\sim}{\tau}(n)\}$ is predense above q.

[7] from where is E? as $\mathbb{P}_\beta$ satisfies the $\aleph_2$-c.c. and $\lambda \geq \aleph_2$ it does not matter

Proof. 1) Apply clause (e) of Definition 6.2 to $\mathscr{B}^*, \langle \lim(T_n) : n < \omega \rangle$. [Andrzej: how you handle?]
2), 3) By previous theorems. $\square_{6.5}$

Claim 6.6. *0) $\mathfrak{K}'_0 \neq \emptyset$.*
1) If $\alpha_* \leq \alpha < \lambda, \bar{\mathbb{Q}} \in \mathfrak{K}'_{0,\alpha_*}, j \in \lambda \backslash A(\bar{\mathbb{Q}})$ and $A^* \in [\lambda]^\lambda$, <u>then</u> there is $\bar{\mathbb{Q}}' \in \mathfrak{K}'_{0,\alpha}$ such that $\bar{\mathbb{Q}} \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}' \in \mathfrak{K}^+_{0,\alpha}$ and $A(\bar{\mathbb{Q}}') \supseteq A(\bar{\mathbb{Q}}) \cup \{j\}$ but $A(\bar{\mathbb{Q}}') \backslash A(\bar{\mathbb{Q}}) \backslash \{j\} \subseteq A^*$.
2) If $\bar{\mathbb{Q}}^\zeta \in \mathfrak{K}'_{0,\leq\alpha}$ for $\zeta < \delta$ and $\varepsilon < \zeta < \delta \Rightarrow \bar{\mathbb{Q}}^\varepsilon \leq_{\mathfrak{K}'_\alpha} \bar{\mathbb{Q}}^\zeta$ and $\alpha < \lambda$ so $\delta < \lambda$, <u>then</u> for some $\bar{\mathbb{Q}}^\delta \in \mathfrak{K}'_{0,\leq\alpha}$ we have $\zeta < \delta \Rightarrow \bar{\mathbb{Q}}^\zeta \leq_{\mathfrak{K}'_{0,\alpha}} \bar{\mathbb{Q}}^\delta$.
2A) Moreover, if $\beta \in [1,\alpha], \bar{\mathbb{Q}}^* \in \mathfrak{K}_{0,\beta}$ and $\zeta < \delta \Rightarrow \bar{\mathbb{Q}}^\zeta \restriction \beta \leq_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^*$ <u>then</u> we can demand then $\bar{\mathbb{Q}}^* \leq_{\mathfrak{K}'_{0,\alpha}} \bar{\mathbb{Q}}^\delta$.
2B) Moreover if α is limit and $\langle \bar{\mathbb{Q}}^{\beta,*} : \beta \in [1,\alpha) \rangle$ is $\leq_{\mathfrak{K}'_{0,\leq\alpha}}$-increasing and $\bar{\mathbb{Q}}^{\beta,*} \in \mathfrak{K}'_{0,\beta}$ and $\zeta < \delta \wedge \beta \in [1,\alpha) \Rightarrow \bar{\mathbb{Q}}^\zeta \restriction \beta \leq_{\mathfrak{K}'_{\leq\beta}} \bar{\mathbb{Q}}^{\beta,*}$ <u>then</u> we can add in (2): $\beta \in [1,\alpha) \Rightarrow \bar{\mathbb{Q}}^{\beta,*} \leq_{\mathfrak{K}'_{0,\leq\alpha}} \bar{\mathbb{Q}}^\delta$.
3) In part (2), if $\mathrm{cf}(\delta) \geq \aleph_2$ then we can demand $\mathrm{Lim}(\bar{\mathbb{Q}}^\delta) = \cup\{\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta) : \gamma < \delta\}$ and if $[\zeta < \delta \Rightarrow \bar{\mathbb{Q}}^\zeta \in \mathfrak{K}^+_{0,<\alpha}]$ <u>then</u> $\bar{\mathbb{Q}}^\delta \in \mathfrak{K}^+_{0,<\alpha}$.
4) For $\alpha < \lambda$ and $A \in [\lambda]^{<\lambda}$ there is $\bar{\mathbb{Q}} \in \mathfrak{K}'_\alpha$ with $A(\bar{\mathbb{Q}}) \supseteq A$.

Remark 6.7. Note that we have to prove that "$\underset{\sim}{\nu}_\gamma$ is $\mathscr{G}^{\mathbf{V}}$-continuous" is preserved.

Proof. We prove by induction on α all parts simultaneously and for α we use part (2),(3) in the proof of part (1) and we use part (3) in the proof of part (2).
0) Trivial.
1) We can ignore the case $A(\bar{\mathbb{Q}}') \backslash A(\bar{\mathbb{Q}}) \backslash \{j\} \subseteq A^*$ as Random_λ has enough automorphisms.

We choose $\bar{\mathbb{Q}}^\beta$ for $\beta \in [1,\alpha]$ by induction on β such that:

- (i) $\bar{\mathbb{Q}}^\beta \in \mathfrak{K}'_\beta$
- (ii) $\bar{\mathbb{Q}} \restriction \beta \leq_{\mathfrak{K}'_0} \bar{\mathbb{Q}}^\beta$ hence $\mathrm{Lim}(\bar{\mathbb{Q}}_\beta \restriction \beta) \lessdot \mathrm{Lim}(\bar{\mathbb{Q}}^\beta)$
- (iii) $\bar{\mathbb{Q}}^\beta \in \mathfrak{K}^+_\beta$.

<u>Case 1</u>: $\beta = 1$.

We choose $\mathbb{Q}^\beta_0 = \mathrm{Random}_{A(\bar{\mathbb{Q}}) \cup \{j\}}$.

<u>Case 2</u>: $\beta = \beta_* + 1 > 1$.

By 5.9(1) + (2) we can choose $\bar{\mathbb{Q}}'_\beta \in \mathfrak{K}'_{0,\beta}$ such that $\bar{\mathbb{Q}}'_\beta \restriction \beta_* = \bar{\mathbb{Q}}^{\beta_*}$ and $\bar{\mathbb{Q}} \restriction \beta \leq_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}'_\beta$. But maybe $\bar{\mathbb{Q}}'_\beta \notin \mathfrak{K}^+_{0,\beta}$. So we try to choose by induction on $i < \lambda, \bar{\mathbb{Q}}'_{\beta,i}$ such that:

- (i) $\bar{\mathbb{Q}}'_{\beta,i} \restriction \beta_* \in \mathfrak{K}^+_{0,\beta_*}$ and $\bar{\mathbb{Q}}'_{\beta,0} = \bar{\mathbb{Q}}'_\beta = \bar{\mathbb{Q}}^{\beta_*}$
- (ii) $\bar{\mathbb{Q}}'_{\beta,i}$ is $\leq_{\mathfrak{K}_{0,\beta}}$-increasing
- (iii) for each $i, \bar{\mathbb{Q}}'_{\beta,i+1}$ exemplifies $\bar{\mathbb{Q}}'_{\beta,i} \notin \mathfrak{K}^+_\beta$
- (iv) if i is a limit ordinal of cofinality $\geq \aleph_2$ then $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,i} = \cup\{\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,j}) : j < i\}$.

For $i = 0$ no problem, for i limit use the part (2) or part (3) for β by the induction hypothesis. For i successor if we cannot continue, we have succeeded having carried out the induction.

Now $S := \{\alpha < \lambda : \mathrm{cf}(\alpha) = \aleph_2\}$ is a stationary subset of λ as $\lambda > \aleph_2$. For each $\delta \in S$, clearly the statement "$\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta}) \lessdot \mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta+1})$" fails, hence there is a maximal antichain $\mathscr{I}_\delta$ of $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta})$ which is not a maximal antichain of $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta+1})$. But the assumptions of claim 1.8 holds by ? SAHARON!, hence the forcing notion $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta})$ satisfies the $\aleph_2$-c.c. and therefore $|\mathscr{I}_\delta| \le \aleph_1 < \mathrm{cf}(\delta)$. But by clause (iv), $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta})$ is the union of the increasing sequence $\langle \mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,i}) : i < \delta\rangle$, this sequence of forcing notions is increasing in the sense that membership relation, being smaller, and being incompatible are preserved, see 6.3.

Hence for some $\gamma(\delta) < \delta$ we have $\mathscr{I}_\delta \subseteq \mathscr{I}_{\gamma(\delta)}$. By Fodor lemma for some stationary $S_1 \subseteq S$, we have $\delta \in S_1 \Rightarrow \gamma(\delta) = \gamma(*)$. As $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\gamma(*)})$ has cardinality $\le (|\beta| + |A|)^{\aleph_0}$ which is $< \lambda$, clearly for some $\mathscr{I}_*$ the set $S_2 = \{\delta \in S_1 : \mathscr{I}_\delta = \mathscr{I}^*_x\}$ is stationary. So consider $\delta < \delta_2$ from S_2, now $\mathscr{I}_{\delta_1}$ is not a maximal antichain in $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta_1+1})$ hence for some $q \in \mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta_2+1}) \backslash \mathscr{I}_{\delta_1}$, the set $\mathscr{I}_{\delta_1} \cup \{q\}$ is an antichain in $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta_2+1})$ hence in $\mathrm{Lim}(\bar{\mathbb{Q}}'_{\beta,\delta_2})$, contradicting $\mathscr{I}_{\delta_1} = I_*$.

<u>Case 3</u>: β limit.
By part (2).

<u>Case 1</u>: $\alpha = 1$.
Let $\mathbb{Q}^\delta_0 = \mathrm{Random}_{\cup\{A(\bar{\mathbb{Q}}^\zeta):\zeta<\delta\}}$.

<u>Case 2</u>: $\alpha = \gamma + 1$.
First by part (1) (and the induction hypothesis) for γ, we can find $\bar{\mathbb{Q}}^{\delta,*} \in \mathfrak{K}^+_{0,\gamma}$ such that $\zeta < \delta \Rightarrow \bar{\mathbb{Q}}^\zeta \le \bar{\mathbb{Q}}^{\delta,*}$. Let $u_\gamma = \{\zeta < \delta : \gamma < \ell g(\bar{Q}^\zeta)\}$, (if $u_\gamma = 0$ we are done so assume $u_\gamma \ne 0$) so for each $\zeta < \xi$ from u_γ we have $\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta \restriction \gamma) \lessdot \mathrm{Lim}(\bar{\mathbb{Q}}^\xi \restriction \gamma) \lessdot \mathrm{Lim}(\bar{\mathbb{Q}}^{\delta,*})$ hence for each $\eta \in {}^{\omega>}\omega$ we have

$$\Vdash_{\mathrm{Lim}(\bar{\mathbb{Q}}^{\delta,\gamma,*})} \text{“} \begin{array}{l}\langle \underset{\sim}{D}^{\bar{Q},\zeta}_{\eta,\gamma} : \zeta \in u_\gamma\rangle \text{ is an increasing sequence of filters}\\ \text{on } \omega \text{ containing the co-bounded subsets of } \omega\text{”}.\end{array}$$

Hence we can find a $\mathrm{Lim}(\bar{\mathbb{Q}}^{\delta,\gamma,*})$-name of an ultrafilter on ω containing $\cup\{\underset{\sim}{D}^{\bar{\mathbb{Q}}^\zeta}_{\eta,\gamma} : \zeta \in u_\gamma\}$. We now can define $\bar{\mathbb{Q}}^{\delta,\beta} \in \mathfrak{K}'_\beta$ such that $\bar{\mathbb{Q}}^{\delta,\beta} \restriction \gamma = \bar{\mathbb{Q}}^{\delta,*}$ and $\underset{\sim}{D}^{\bar{\mathbb{Q}}^{\delta,\beta}}_\eta$ is as above. This clearly is O.K.

<u>Case 3</u>: α limit.
We choose by induction on $\varepsilon < \omega_2$ a sequence $\langle \bar{\mathbb{Q}}^{\delta,\varepsilon,\beta} : \beta \in [1,\alpha)\rangle$ such that

(*i*) $\bar{\mathbb{Q}}^{\delta,\varepsilon,\beta} \in \mathfrak{K}^+_{0,\beta}$

(*ii*) $\gamma < \beta \Rightarrow \bar{\mathbb{Q}}^{\delta,\varepsilon,\gamma} \le_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^{\delta,\varepsilon,\beta}$

(*iii*) $\zeta < \delta \Rightarrow \bar{\mathbb{Q}}^\zeta \restriction \beta \le_{\mathfrak{K}'_{0,\beta}} \bar{\mathbb{Q}}^{\delta,\varepsilon,\beta}$

(*iv*) if $\zeta < \varepsilon$ and $\gamma \in [\beta,\alpha]$ then $\bar{\mathbb{Q}}^{\delta,\zeta,\gamma} \restriction \beta <_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^{\delta,\varepsilon,\beta}$.

For each ε we do it by induction on $\beta \in [1,\alpha)$, so all parts hold by the induction hypothesis.

For $\beta = 0$ we act as in case 1, for β successor we act as in case (2) and for limit use the induction to find $\bar{\mathbb{Q}}^{\delta,\varepsilon,\beta} \in \mathfrak{K}^+_\beta$ such that $\gamma < \beta \Rightarrow \bar{\mathbb{Q}}^{\delta,\varepsilon,\gamma} \le_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^{\delta,\varepsilon,\beta}$. Now for $\zeta < \varepsilon$ we know that $\bar{\mathbb{Q}}^{\delta,\zeta,\beta} \le_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^{\delta,\varepsilon,\beta}$ by 6.3(2).

3) Without loss of generality either $(\forall \zeta < \delta)(\ell g(\bar{\mathbb{Q}}^\zeta) = \alpha)$ or α is a limit ordinal and $\langle \ell g(\bar{\mathbb{Q}}^\zeta) : \zeta < \delta \rangle$ is increasing with limit α. We choose $\bar{\mathbb{Q}}^{\delta,\beta}$ by induction on $\beta \in [1, \alpha]$ such that

$\circledast$ (i) $\bar{\mathbb{Q}}^{\delta,\beta} \in \mathfrak{K}'_{0,\beta}$

(ii) if $\beta < \alpha$ then $\bar{\mathbb{Q}}^{\delta,\beta} \in \mathfrak{K}^+_{0,\beta}$

(iii) if $\gamma < \beta$ then $\bar{\mathbb{Q}}^{\delta,\gamma} \le_{\mathfrak{K}'_\beta} \bar{\mathbb{Q}}^{\delta,\beta}$; moreover

$(iii)^+$ if $\gamma < \beta$ then $\bar{\mathbb{Q}}^{\delta,\gamma} = \bar{\mathbb{Q}}^{\delta,\beta} \restriction \gamma$

(iv) if $\zeta < \delta$ then $\bar{\mathbb{Q}}^\zeta \restriction \beta \le_{\mathfrak{K}'_{\gamma,\beta}} \bar{\mathbb{Q}}^{\delta,\beta}$

(v) $\mathrm{Lim}(\bar{\mathbb{Q}}^{\delta,\beta}) = \cup\{\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta \restriction \beta) : \zeta < \delta\}$.

Case 1: As in part (2).

Case 2: $\beta < \alpha$.
By the induction hypothesis and the uniqueness of the limit (and preserved by taking "$\restriction \gamma$" for $\gamma < \beta$.

Case 3: $\beta = \alpha$ is limit.
By $(iii)^+$ clearly we can define $\bar{\mathbb{Q}}^{\delta,\beta}$, and easily it is the union. Now it belongs to $\mathfrak{K}^+_{0,\beta}$ by the proof of (1), case (2).

Case 4: $\beta = \alpha = \gamma + 1$.
So $P_\gamma = \mathrm{Lim}(\bar{\mathbb{Q}}^{\delta,\gamma})$ is well defined. We define $\langle \underset{\sim}{D}_{\gamma,\eta} : \eta \in {}^{\omega>}\omega \rangle$ as in the proof of case (2), part (2) and so $\bar{\mathbb{Q}}^{\delta,\beta}$ is well defined. It belongs to $\mathfrak{K}^+_{0,\beta}$ again as in the proof of part (1), case (2).
4) Should be clear. $\square_{6.6}$

So in particular

Conclusion 6.8. For $\alpha < \lambda$, for every $\varepsilon < \lambda$ and $\bar{\mathbb{Q}} \in \mathfrak{K}_{;\le \alpha}$ there is $\bar{\mathbb{Q}}' \in K^+_{\le \alpha}$ such that $\varepsilon \subseteq A(\bar{\mathbb{Q}}')$ and $\bar{\mathbb{Q}}_1 \le_{\mathfrak{K}'_\alpha} \bar{\mathbb{Q}}'$.

Now we turn to $\mathfrak{K}'_{1,\alpha}$.

Claim 6.9. *1) If $\alpha < \lambda, \bar{\mathbb{Q}} \in \mathfrak{K}'_{0,\alpha}$ and $j \in \lambda \backslash A(\bar{\mathbb{Q}})$ <u>then</u> there is $\bar{\mathbb{Q}}'$ such that*

(a) $\bar{\mathbb{Q}} \le_{\mathfrak{K}_{;0,\alpha}} \bar{\mathbb{Q}}' \in \mathfrak{K}^+_{0,\alpha}$

(b) $\Vdash_{\mathrm{Lim}(\bar{\mathbb{Q}}')}$ "$\underset{\sim}{\nu}_j$ is $\mathscr{G}^{\mathbf{V}}$-continuous over $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}})}$.

2) If $\alpha < \lambda, \delta < \lambda$ a limit ordinal $\langle \bar{\mathbb{Q}}^\zeta : \zeta < \delta \rangle$ is a $\le_{\mathfrak{K}'_{0,\le\alpha}}$-increasing sequence of members of $\mathfrak{K}^+_{0,\le\alpha}, \bar{\mathbb{Q}} = \bar{\mathbb{Q}}^0$ and $j \in A(\bar{\mathbb{Q}}^1) \backslash A)\bar{\mathbb{Q}}^0)$ and $\zeta \in [1, \delta) \Rightarrow \Vdash_{\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta)}$ "ν_j is $\mathscr{G}^{\mathbf{V}}$-continuous over $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}})}$" <u>then</u> we can choose $\bar{\mathbb{Q}}^\delta$ such that all the conditions on $\zeta < \delta$ hold $\zeta = \delta$, too.
3) If in part (2), $\mathrm{cf}(\delta) \ge \aleph_2$ then we can add $\mathrm{Lim}(\bar{\mathbb{Q}}^\delta) = \cup\{\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta) : \zeta < \delta\}$.

Proof. This is y (simultaneous) induction on α.
1) Case 1: $\alpha = 1$.
Trivial, as in case 1 of the proof of 6.6(1), respectively.

Case 2: $\alpha = \beta + 1$.
We use 4.11(1).

Case 3: α is a limit ordinal.

We choose $\bar{\mathbb{Q}}^\alpha$ by 6.6(2B) (for a constant sequence). Why is the $\mathscr{G}^{\mathbf{V}}$ continuity preserved?

We just apply **??**(2),

2) Case 1: $\alpha = 1$.

As in 6.6(2).

Case 2: $\alpha = \beta + 1$.

We use 4.11(2).

Case 3: α is a limit ordinal.

As in the proof of (1).

3) No new point. $\square_{6.10}$

Conclusion 6.10. 1) For any ordinal $\alpha < \lambda$ we can find $\langle \bar{\mathbb{Q}}^\zeta : \zeta < \lambda \rangle$ such that:

(a) $\bar{\mathbb{Q}}^\zeta \in \mathfrak{K}^+_{1+\zeta}$

(b) $\varepsilon < \zeta \Rightarrow \bar{\mathbb{Q}}^\varepsilon \leq_{\mathfrak{K}'_\zeta} \bar{\mathbb{Q}}^\zeta$

(c) $A(\bar{\mathbb{Q}}^\zeta) \supseteq \zeta$

(d) if $\zeta < \lambda$ is a limit ordinal of uncountable cofinality then $\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta) = \bigcup\limits_{\varepsilon<\zeta} \mathrm{Lim}(\bar{\mathbb{Q}}^\varepsilon)$

(e) if $\bar{\zeta} < \lambda$ and $\varepsilon_\zeta = \min(\lambda \backslash A(\bar{\mathbb{Q}}^\zeta))$ then $\varepsilon_\zeta \in A(\bar{\mathbb{Q}}^{\zeta+1})$ and $\Vdash_{\mathrm{Lim}(\bar{\mathbb{Q}}^{\zeta+1})}$ "$\underset{\sim}{\nu}_{\varepsilon_\zeta}$ is $\mathscr{G}^{\mathbf{V}}$-continuous over $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta)}$.

2) There is $\bar{\mathbb{Q}}^\lambda \in \mathfrak{K}''_\alpha$ such that $\zeta < \lambda \Rightarrow \bar{\mathbb{Q}}^\zeta \leq_{\mathfrak{K}'_{\leq\lambda}} \bar{\mathbb{Q}}^\lambda$ and $\mathrm{Lim}(\bar{\mathbb{Q}}^\lambda) = \cup\{\mathrm{Lim}(\bar{\mathbb{Q}}^\zeta) : \zeta < \lambda\}$.

3) Let $\alpha \leq \lambda$ and $\mathbb{P} = \mathbb{P}_\alpha$ be $\mathrm{Lim}(\bar{\mathbb{Q}}^\lambda \restriction \varepsilon)$. Then

(a) $\mathbb{P}$ is a proper forcing notion of cardinality λ satisfying the $\aleph_2$-c.c. (so cardinal arithmetic in $\mathbf{V}^{\mathbb{P}}$ should be clear)

(b) if $\mathrm{cf}(\alpha) > \aleph_0$ then $\Vdash_{\mathbb{P}}$ "$\mathfrak{b} = \mathrm{cf}(\alpha) = \mathfrak{d}$"

(c) $\Vdash_{\mathbb{P}}$ "there is a set $\{\nu_\zeta : \zeta < \lambda\} \subseteq {}^\omega 2$ which is not in the $\mathscr{G}^{\mathbf{V}}$-ideal"

(d) the continuum in $\mathbf{V}^{\mathbb{P}}$ is λ.

3) For limit $\zeta < \lambda$ of uncountable cofinality, letting $\mathbb{P}_\zeta = \mathrm{Lim}(\bar{\mathbb{Q}}^\zeta)$, we have

(a) $\mathbb{P}_\zeta$ is a proper forcing notion of cardinality $(|\alpha|+|\zeta|)^{\aleph_0}$ satisfying the $\aleph_2$-c.c.

(b) if $\mathrm{cf}(\alpha) > \aleph_0$ then $\Vdash_{\mathbb{P}}$ "$\mathfrak{b} = \mathrm{cf}(\alpha) = \mathfrak{d}$"

(c) if $A \subseteq \zeta = \sup(A)$ then $\Vdash_{\mathbb{P}_\zeta}$ "the set $\{\nu_i : i \in A\}$ is not in the $\mathscr{G}^{\mathbf{V}}$-ideal"

(d) the continuum in $\mathbf{V}^{\mathbb{P}}$ is $((|\alpha| + |\zeta|)^{\aleph_0})^{\mathbf{V}}$.

Discussion 6.11. : 1) Is there a Cohen reals in $\mathbf{V}^{\mathbb{P}}$ over $\mathbf{V}$? By the way we construct in general, yes, as possibly $\mathbb{P}^0_2 \lessdot \mathbb{P}$ and $\mathbb{P}^0_2$ is $\mathbb{Q}_{\bar{D}^0_1}$ which may add Cohen. To replace $\langle \underset{\sim}{\nu}_i : i < \lambda \rangle$ by say a Sierpinski set in $\mathbf{V}$ we do not know.

2) Similarly, the Borel conjecture may fail.

Proof. Proof of 6.10:

Easy by quoting.

1) We choose $\bar{\mathbb{Q}}^\zeta$ by induction on ζ.

Case 1: $\zeta = 0$ so $1 + \zeta = 1$.

Tivial (or use 6.6(4).

Case 2: ζ is a limit ordinal.

Use 6.6(2).

Case 3: $\zeta = \varepsilon + 1$.

Use 6.9(1).

2) Let $\mathbb{P}$ be $\mathrm{Lim}(\bar{\mathbb{Q}}$ when $\bar{\mathbb{Q}} \in \mathfrak{K}'_{1,\lambda}$ and $\mathrm{Lim}(\bar{Q}) = \cup\{\mathrm{Lim}(\bar{\mathbb{Q}}_\zeta) : \zeta < \lambda\}$, this is O.K. by 6.6(3) as $\mathrm{cf}(\lambda) \geq \aleph_1$.

Now for clause (a): properness holds by 3.3 and $\aleph_2$-c.c. by 1.17. $\square_{6.10}$

Claim 6.12. *1) There is $\bar{\mathbb{Q}} \in \mathfrak{K}''_{\lambda^+}$.*
2) If $\bar{\mathbb{Q}} \in \mathfrak{K}''_\alpha, \alpha \geq \lambda$ and $\underset{\sim}{P} = \mathrm{Lim}(\bar{\mathbb{Q}})$ then

- (a) *$\mathbb{P}$ is a proper forcing notion of cardinality $|\alpha|^{\aleph_0}$ satisfying the $\aleph_2$-c.c.*
- (b) *if $\mathrm{cf}(\alpha) \geq \aleph_0$ then $\Vdash_{\mathbb{P}}$ "$\mathfrak{b} = \mathrm{cf}(\alpha) = \mathfrak{d}$"*
- (c) *$\Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{\nu}_i : i < \lambda\}$ is not in the $\mathscr{G}^{\mathbf{V}}$-null ideal".*

Proof. 1) Using $\diamondsuit_\lambda$, as in [**?**, Ch.IV] (or force by approximations). But now we replace approximations by $\bar{\mathbb{Q}} \in \mathfrak{K}'_\alpha, \alpha < \lambda$ by $\bar{\mathbb{Q}} \in \mathfrak{K}'_u$ for $u \in [\lambda]^{<\lambda}$.

2) Like 6.10. $\square_{6.12}$

7. Continuing [?]

Context 7.1. As in §6.

At present we can deal with this for "$\mathscr{G}^{\mathbf{V}_0}$-continuous" instead of Random. To do it fully we need to make the ultrafilter $\underset{\sim}{D}_{\alpha,\eta}$ Ramsey but we do not know to guarantee this.

Theorem 7.2. *Assume*

- $(*)(i)$ $\kappa \le \theta < \mu < \lambda = \lambda^{<\mu} = 2^\kappa$
- (ii) κ *regular and* $(\forall \alpha < \kappa)(|\alpha|^{\aleph_0} < \kappa)$
- (iii) $\theta = \mathrm{cf}(\theta)$ *and* $(\forall \alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$
- (iv) μ *is a limit cardinal*
- (v) $\mathscr{G} = \mathscr{G}^{\mathbf{V}}$.

Then for some forcing notion $\mathbb{P}$ we have:

- (α) $\mathbb{P}$ *is an $\aleph_2$-c.c. proper forcing notion of cardinality λ*
- (β) *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathrm{cov}(\mathscr{G}$*-continuous ideal* $= \mathrm{Null}_{\mathscr{G}}) = \mu$
- (γ) *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathfrak{b} = \mathfrak{d} = \theta$.

Remark 7.3. 1) We rely on [?], if instead we rely on [?], then we can weaken the assumptions on the cardinals.
2) By observation in $\mathbf{V}^{\mathbb{P}}$ we have: the covering by closed null sets number is also μ so can be $\aleph_\omega$, i.e. "$\mathfrak{d} < \mu$".

Proof. Proof of 7.2

Let $\mathfrak{K}(0)$ be the family of $\bar{\mathbb{Q}} \in \mathfrak{K}^3$ from Definition 2.11 of [?] of length $< \lambda$ ordered naturally: $\bar{\mathbb{Q}}' \le \bar{\mathbb{Q}}''$ iff $\bar{\mathbb{Q}}' = \bar{\mathbb{Q}}'' \restriction \ell g(\bar{\mathbb{Q}}')$, of course, $\bar{\mathbb{Q}}$ stands for the forcing $\mathrm{Lim}(\bar{\mathbb{Q}})$, 5.11(d). Clearly $\bar{\mathbb{Q}}$ is FS iteration, this fixes the choice in 5.11. Sometimes we replace the ordinal $< \lambda$ by a set of ordinals, with obvious meaning. To avoid confusion we use $\bar{R}$ for the FS iteration mentioned above and if $\mathbb{Q}_0$ is such a forcing, i.e. $\mathrm{Lim}(\bar{R})$ we let $\bar{R} = \bar{R}_{\mathbb{Q}_0}$ and $\langle \underset{\sim}{\eta}_\zeta : \zeta < \ell g(\bar{R})\rangle$ for the generic in 5.13 - ??. [REF(18A)]

Clearly $\mathfrak{K}(0)$ is $\mathrm{cf}(\lambda)$-closed, but as $\lambda = \lambda^{<\mu}$ necessarily $\mathrm{cf}(\lambda) > \mu > \theta$. For $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ let $\langle \underset{\sim}{\eta}_i[\bar{\mathbb{Q}}] : 0 < i < \alpha\rangle$ denote the sequence of generic reals, $\underset{\sim}{\eta}_i[\bar{\mathbb{Q}}]$ for $\underset{\sim}{\mathbb{Q}}_i$. So we can find $\bar{\mathbb{Q}}_0 \in \mathfrak{K}'_\theta$. Now by induction on $\zeta < \lambda$ we define $\bar{\mathbb{Q}}_\zeta$ such that:

- (a) $\bar{\mathbb{Q}}_\zeta \in \mathfrak{K}^+_\theta \cap \mathfrak{K}'_\theta$
- (b) $\varepsilon < \zeta \Rightarrow \bar{\mathbb{Q}}_\varepsilon \le_{\mathfrak{K}'} \bar{\mathbb{Q}}_\zeta$ (hence $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_\varepsilon) \lessdot \mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_\zeta)$)
- (c) letting $\bar{R}_{\bar{\mathbb{Q}}_\zeta} \in \mathfrak{K}(0)$ be such that $\mathbb{Q}_{\zeta,0} = \mathrm{Lim}(\bar{R}_{\bar{\mathbb{Q}}_\zeta})$ we have $\ell g(\bar{R}_{\bar{\mathbb{Q}}_\zeta}) = \ell g(\bar{R}_0) + \xi_\zeta$ REF: recalling $\ell g(\bar{R}_0) < \lambda, \xi_\gamma < \lambda$ is increasing with λ
- (d) <u>if</u>
  - (i) $\varepsilon < \lambda, \theta \le \chi = \chi^{\aleph_0} < \mu$
  - (ii) $\bar{\mathbb{Q}}' \in \mathfrak{K}^+_\theta, \bar{\mathbb{Q}}' \le_{\mathfrak{K}'} \bar{\mathbb{Q}}_\varepsilon$
  - (iii) $A \subseteq \ell g(\bar{R}_{\mathbb{Q}_{\varepsilon,0}})$ is of cardinality $\le \chi$

  <u>then</u> for some $\zeta \in [\varepsilon, \lambda)$ we have $\Vdash_{\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_{\zeta+1})}$ "$\underset{\sim}{\eta}_{\ell g(\bar{R}_{\mathbb{Q}_{\zeta,0}})}$, the partial random real of the $\ell g(\bar{R}_{\bar{\mathbb{Q}}_{\zeta,0}})$-iterand of the iteration $\mathbb{Q}_{\zeta,0} \in \mathfrak{K}(0)$ is $\mathscr{G}^{\mathbf{V}}$-continuous over $\mathbf{V}[\langle \underset{\sim}{\eta}_\zeta : \zeta \in A\rangle{}^\frown\langle \underset{\sim}{\eta}_i : i < \theta\rangle]$".

There is no problem for $\zeta = 0$ and ζ limit. For $\zeta = \varepsilon + 1$, let $\gamma = \ell g(\bar{R}_{\mathbb{Q}_{\varepsilon,0}}), \bar{R} = \bar{R}_{\bar{\mathbb{Q}}_{\varepsilon,0}}$. By bookkeeping we are given $\xi \leq \varepsilon$ and $A_\varepsilon \subseteq \ell g(\bar{R}_{\mathbb{Q}_{\xi,0}})$ of cardinality $\leq \chi$. By the Löwenheim-Skolem argument, choose $A^*_\varepsilon \subseteq \gamma$ of cardinality $\leq \chi$ including A_ε, closed enough (in particular, as required in [?], see 2.16 (1),(2) and $\bar{R}_{\bar{\mathbb{Q}}_{\varepsilon,0}} \restriction A^*_\varepsilon \in \mathfrak{K}(0)$) and there is $\bar{\mathbb{Q}}_\varepsilon \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}_\zeta$ such that $\bar{R}_{\mathbb{Q}_{\varepsilon,0}} = \bar{R}_{\mathbb{Q}_{\zeta,0}} \restriction A^*_\varepsilon$ REF: Def. By the bookkeeping we can ensure every A will appear. Let $\underset{\sim}{R}_\gamma = \text{Random}^{\mathbf{V}[\underset{\sim}{\eta}_\beta:\beta\in A^*_\varepsilon]}$ and let $\bar{R}''_\varepsilon$ be $\bar{R}_{\mathbb{Q}'_{\varepsilon,0}}$ when we add $\underset{\sim}{R}_\gamma$, i.e. $\bar{R}_{\bar{\mathbb{Q}}'_{\varepsilon,0}} \leq_{\mathfrak{K}} \ell g(\bar{R}'_\varepsilon) = \ell g(\bar{R}_{\bar{\mathbb{Q}}'_{\varepsilon,0}}) \cup \{\gamma\}$ ([REF(19B)] abusing notation) $(\bar{R}''_\varepsilon)_\gamma = \underset{\sim}{R}_\gamma$ ($\bar{R}''_\varepsilon$ exists by [?]). By ?? we can find $\bar{\mathbb{Q}}'' \in \mathfrak{K}^+ \cap \mathfrak{K}'_\theta$ and $\bar{R}''_\varepsilon <_{\mathfrak{K}(0)} \mathbb{Q}''_0$ which satisfies $\Vdash_{\text{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}'')}$ "$\underset{\sim}{\eta}_\gamma$ is random over $\mathbf{V}^{\text{Lim}(\bar{R}\restriction A)}$".

By renaming [REF(22C)], without loss of generality $A(\mathbb{Q}''_0) \cap A(\bar{\mathbb{Q}}^\varepsilon) = \underset{\sim}{B}^*_\varepsilon$.

Now we can define $\bar{\mathbb{Q}}_\zeta$ by amalgamation, i.e. 7.4 below. Let $\bar{\mathbb{Q}}_\lambda$ by $\bigcup_{\zeta<\lambda} \bar{\mathbb{Q}}_\zeta$ and $\mathbb{P} = \text{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_\lambda) = \bigcup_{\zeta<\lambda} \text{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_\zeta)$. It is as required: $\Vdash_{\mathbb{P}}$ "$\mathfrak{b} = \mathfrak{d} = \theta$" easily, and $\Vdash_{\mathbb{P}}$ "cov() $\geq \mu$" by clause (d) and the bookkeeping concerning the A_ε's.

Lastly, [REF(22D)] $\Vdash_{\mathbb{P}}$ "cov() $\leq \mu$" because $\Vdash_{\mathbb{Q}_{\lambda,0}}$ "cov() $\leq \mu$" by [?] and properties of $\text{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}}_\lambda)/\mathbb{Q}_{\lambda,0}$ [is null, null$_{\mathscr{G}}$ or nul??]. $\square_{7.2}$

Claim 7.4. *Assume*

(a) $\mathbb{Q}_{\ell,0} \in \mathfrak{K}(0)$ *for* $\ell = 0,1,2,3$ *and* $\mathbb{Q}_{0,0} \lessdot \mathbb{Q}_{\ell,0} \lessdot \mathbb{Q}_{3,0}$ *moreover* $\mathbb{Q}_{3,0} = \mathbb{Q}_{1,0} *_{\mathbb{Q}_{0,0}} \mathbb{Q}_{2,0}$
(b) $\bar{\mathbb{Q}}_\ell \in \mathfrak{K}^+_\alpha$ *for* $\ell = 0,1,2$
(c) $\bar{\mathbb{Q}}_0 \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}_1$ *and* $\bar{\mathbb{Q}}_0 \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}_2$.

Then we can find $\bar{\mathbb{Q}}_3 \in \mathfrak{K}^+_\alpha$ *such that* $\bar{\mathbb{Q}}_\ell \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}_3$ *for* $\ell < 3$.

Remark 7.5. 1) How do we get such $\mathbb{Q}_{\ell,0} \in \mathfrak{K}(0)$? By [?, Lemma 2.16].
2) We can replace $\mathbb{Q}_{3,0}$ by $\bar{\mathbb{Q}}_{3,\alpha}$ as the proof. [REF?]

Proof. We choose $\bar{\mathbb{Q}}_3 \restriction \beta$ by induction on $\beta \in [1,\alpha]$, for $\beta = 1$ there is nothing to do. For β limit just use $\bar{\mathbb{Q}}_\ell \restriction \beta \in \mathfrak{K}^+_\alpha$. For $\beta = \gamma + 1$ use 7.6 below [REF: see (19A) + (20C)]; we could have demanded something on how $\bar{\mathbb{Q}}_0 \leq_{\mathfrak{K}'} \bar{\mathbb{Q}}_2$ (i.e. choosing $\bar{A}_\varepsilon$ in the proof of 7.2 but not needed). $\square$

Claim 7.6. *Assume*

(a) $\mathbb{Q}_\ell$ *is a forcing notion for* $\ell \leq 3$
(b) $\mathbb{Q}_0 \lessdot \mathbb{Q}_\ell \lessdot \mathbb{Q}_3$
(c) $\mathbb{Q}_3 = \mathbb{Q}_1 *_{\mathbb{Q}_0} \mathbb{Q}_2$
(d) *for* $\ell = 0,1,2$ *we have* $\underset{\sim}{D}_\ell$ *is a* $\mathbb{Q}_\ell$*-name of an ultrafilter on* ω
(e) *for* $\ell = 1,2$ *we have* $\Vdash_{\mathbb{Q}_\ell}$ "$\underset{\sim}{D}_0 \subseteq \underset{\sim}{D}_\ell$".

Then we can find a $\mathbb{Q}_3$*-name* $\underset{\sim}{D}$ *such that* $\Vdash_{\mathbb{Q}_3}$ "$\underset{\sim}{D}$ *is an ultrafilter on* ω *extending* $\underset{\sim}{D}_1 \cup \underset{\sim}{D}''_2$.

Proof. As in [?, §3]. $\square_{7.6}$

Claim 7.7. *Let* $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ *and for* $\beta \in (0,\alpha)$ *let* $\underset{\sim}{\eta}_\beta$ *be the generic real of* $\mathbb{Q}_\beta$. *Then* $\underset{\sim}{G}_{\text{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})}$ *can be computed from* $\langle \underset{\sim}{G}_{\mathbb{Q}_0}\rangle{}^\frown\langle \underset{\sim}{\eta}_\beta : \beta \in (0,\alpha)\rangle$.

Proof. As usual (or see [?]). $\square$

8. On η is $\mathscr{L}$-big over M

Definition 8.1. 1) Let $\mathbf{T} = \{\mathscr{T} : \mathscr{T} \subseteq {}^{\omega>}\mathscr{H}(\aleph_0), \mathscr{T} \neq \emptyset, \mathscr{T}$ closed under initial segments, no $\triangleleft$-maximal member and $\mathscr{T}_n = \{\eta \in \mathscr{T} : \ell g(\eta) = n\}$ finite for $n < \omega\}$.
2) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ let $\mathbf{R}_{\mathscr{T}_1,\mathscr{T}_2} = \{R : R$ a closed subset of $\lim(\mathscr{T}_1) \times \lim(\mathscr{T}_2)\}$. Similarly for $\mathbf{R}_{\mathscr{T}}$.

We write $\eta R \nu$ instead of $(\eta, \nu) \in R$. We always assume that $\mathscr{T}_1, \mathscr{T}_2$ can be reconstructed from $\bar{R} \in \mathbf{R}_{\mathscr{T}_1,\mathscr{T}_2}$ and write $\mathscr{T}_1[R], \mathscr{T}_2[R]$; similarly for $R \in \mathbf{R}_{\mathscr{T}}$. Let $\mathbf{R}_* = \cup\{\mathbf{R}_{\mathscr{T}_1,\mathscr{T}_2} : \mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}\}$.
3) If R is a closed subset of $\lim(\mathscr{T}_1) \times \lim(\mathscr{T}_2)$ and $k < \omega$ then let $R^{<k>} = \{(\eta \restriction k, \nu \restriction k) : (\eta, \nu) \in R\}$. Similarly for $R \subseteq \lim(\mathscr{T})$.
4) For every $\mathscr{Y} \subseteq \mathbf{Y} =: \{(f, \mathscr{T}) : \mathscr{T} \in \mathbf{T}, f \in \prod\limits_{n<\omega} \mathscr{P}(\mathscr{T}_n)\}$ let $D_{\mathscr{Y}} = \{A \subseteq \omega$: for some $k, m < \omega$ and $(f_\ell, \mathscr{T}_\ell) \in \mathscr{Y}$ and $\nu_\ell \in \lim(\mathscr{T}_\ell)$ for $\ell < k$ we have $A \supseteq \{n : n \geq m$ and $\ell < k \Rightarrow \nu_\ell \restriction n \in f_\ell(n)\}\}$. We say $\mathscr{Y}$ is nontrivial if $\emptyset \notin D_{\mathscr{Y}}$. Let $J_{\mathscr{Y}}$ be the dual ideal.

Definition 8.2. 1) We say D is $(f, \mathscr{T})$-narrow <u>if</u> :

- (i) $f \in \prod\limits_{n<\omega} \mathscr{P}(\mathscr{T}_n)$
- (ii) D is a filter on ω containing the co-finite subsets
- (iii) for every $\nu \in {}^{\omega}2$ the set $\{n < \omega : \nu \restriction n \in f(n)\}$ belongs to D.

2) For $\mathscr{Y} \subseteq \mathbf{Y}$, we say D is $\mathscr{Y}$-narrow if D is $(f, \mathscr{T})$-narrow for every $(f, \mathscr{T}) \in \mathscr{Y}$.
3) $\mathbf{Z} = \{(\eta, R) : \eta \in \lim(\mathscr{T}_2[R]), R \in \mathbf{R}_*\}, \mathbf{Z}_M = \{(\eta, R) \in \mathbf{Z} : R \in M$ and $\eta \in \lim(\mathscr{T}_2[R])\}$.
4) We say that D is (η, R)-<u>big</u> over M if: it is $\{(\eta, R)\}$-big over M (see below).
5) [REF(20B)] We say that η, D is $\mathscr{L}$-big over M if

- (i) M is a set or a class (usually an inner model), D is a filter on ω containing the co-bounded subsets of ω
- (ii) $\mathscr{L} \subseteq \mathbf{Z}_M, R \in \mathbf{R}^M_{\mathscr{T}_1 \mathscr{T}_2}$ where $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T} \in \mathbf{T}^M$
- (iii) $\eta \in \lim(\mathscr{T}_2[R])$ when $\mathscr{L} = \{(\eta, R)\}$ for [8] every $m^* < \omega, \langle \nu_{m,n} : n < \omega\rangle \in M, \nu_m \in M$ for $m < m^*, (\eta_m, R_m) \in \mathscr{L}$ such that $\{\nu_{m,n}, \nu_m : n < \omega\} \subseteq \lim(\mathscr{T}_1[R_m])$, <u>if</u> $m < m^* \Rightarrow \nu_m = \lim_D\langle \nu_{m,n} : n < \omega\rangle$ <u>then</u> $\{n : m < m^* \Rightarrow \nu_m R_m \eta_m \equiv \nu_{m,n} R_m \eta_m\} \neq \emptyset \mod D$.

6) [REF(20A)] We say η is R-big over M if the filter of co-finite subsets of ω is (η, R)-big. We say that η is $\mathscr{L}$-big over M if (η, D) is with D the filter of co-finite subsets of ω.

One Step Claim 8.3. *Assume (all in $\mathbf{V}_2$)*

- (a) $\mathbf{V}_1 \subseteq \mathbf{V}_2$
- (b) *in* $\mathbf{V}_1, D_1$ *is a non principal ultrafilter on* ω
- (c) *for* $\zeta < \zeta^*_1, f_\zeta \in \mathbf{V}_1, \mathbf{T}_\zeta \in \mathbf{T}^{\mathbf{V}_1}$ *[REF: omit]*
- (d) D_1 *is* $\mathscr{Y}$*-narrow,* $\mathscr{Y} \subseteq \mathbf{Y}^{\mathbf{V}_1}$ *(of course,* $\mathscr{Y} \in \mathbf{V}_2$*, but possibly* $\mathscr{Y} \notin \mathbf{V}_1$*)*
- (e) $\mathscr{Z} \subseteq \mathbf{Z}_{\mathbf{V}_1}$

[8] note that if D is an ultrafilter then the case $m^* = 1$ suffices

(f) $(\mathscr{Y},\mathscr{Z})$ *is high over* $\mathbf{V}_1$ *which means (in* $\mathbf{V}_2$*):*
$\mathscr{Y} \subseteq \mathbf{Y}^{\mathbf{V}_1}, \mathscr{Z} \subseteq \mathbf{Z}^{\mathbf{V}_2}_{\mathbf{V}_1}$ *[REF(21B)], and if* $m^* < \omega, (\eta_m, R_m) \in \mathscr{Z}$ *for* $m < m^*, \mathscr{Y}' \subseteq \mathscr{Y}$ *is finite,* $B \in (J_{\mathscr{Y}'})^+$ *and for* $m < m^*, \bar\nu_m = \langle \nu_{m,n} : n \in B\rangle \in \mathbf{V}_1$ *and* $\nu_{m,n}, \nu_m \in \lim(\mathscr{T}_1[R_m])^{\mathbf{V}_1}, \nu_m = \lim\langle \nu_{m,n} : n \in B\rangle$ *then* $\{n \in B\colon$ *if* $m < m^*$ *then* $\nu_{m,n} R_m \eta_m \equiv \nu_m R_m \eta_m\}$ *is infinite.*

Then there is D_2 *such that:*

(α) $D_2\,(\in \mathbf{V}_2)$ *is an ultrafilter on* ω
(β) $D_1 \subseteq D_2$
(γ) D_2 *is* $\mathscr{Y}$*-narrow over* $\mathbf{V}_1$ *[REF(22C): not defined yet]*
(δ) D_2 *is* $\mathscr{Z}$*-big over* $\mathbf{V}_1$.

Remark 8.4. Better if we predetermine $D_2 \cap \mathscr{P}(\omega)^{\mathbf{V}_2}$, good for $\mathfrak{u} = \mathrm{cf}(\alpha^{\mathfrak{q}}) > \aleph_0$. [REF: unclear]

Proof. Proof of 8.3
For $(f,\mathscr{T}) \in \mathscr{Y}$ and $\rho \in \lim(\mathscr{T})^{\mathbf{V}_2}$ let

$$A^1_{f,\rho} = \{n : \rho \restriction n \in f(n)\}.$$

For $(\eta, R) \in \mathscr{Z}$ and $\nu_n, \nu \in \lim(\mathscr{T}_1[R])^{\mathbf{V}_1}$ for $n < \omega$ such that $\bar\nu = \langle \nu_n : n < \omega\rangle \in \mathbf{V}_1$ and $\nu = \lim_D(\bar\nu)$ we let

$$A^2_{\eta,R,\bar\nu,\nu} = \{n : \nu_n R \eta \equiv \nu R \eta\}.$$

So we just need to find an ultrafilter D_2 on ω which extends $D_1 \cup \{A^1_{f,\nu} : \nu \in \lim(\mathscr{T}), (f,\mathscr{T}) \in \mathscr{Y}\} \cup \{A^2_{\eta,R,\bar\nu,\nu} : (\eta,R) \in \mathscr{Z}$ and $\bar\nu, \nu \in \mathbf{V}_1$ are as above$\}$. For this it suffices to prove

($*$) assume $B \in D_1, n^*_1 < \omega, n^*_2 < \omega, A^1_{f_\ell,\rho_\ell}, A^2_{\eta_m,R_m,\bar\nu_m,\nu_m}$ well defined for $\ell < n^*_1, m < n^*_2$ then $B \cap \cap\{A^1_{f_\ell,\rho_\ell} : \ell < n^*_1\} \cap \cap\{A^2_{\eta_m,R_m,\bar\nu_m,\nu_m} : m < n^*_2\}$ is non-empty where $(f_\ell,\mathscr{T}_\ell) \in \mathscr{Y}$ where $\rho_\ell \in \lim(\mathscr{T}_\ell)(\eta_m,R_m) \in \mathscr{Z}$ and $\bar\nu_m, \nu_m$ as usual.

As $\nu_m = \lim_D\langle \nu_{m,n} : n < \omega\rangle$ and $B \in D_1$ we have $B_{m,k} = \{n \in B : \nu_{m,n} \restriction k = \nu_m \restriction k\} \in D_1$ for $m < n^*_2, k < \omega$ hence $B_k = \bigcap\limits_{m<n^*_2} B_{m,k} \in D_1$ and clearly $B_{k+1} \subseteq B_k$, hence $B_k \ne \emptyset \mod D_1$ hence $B_k \notin J^{\mathbf{V}_1}_{\mathscr{Y}^*}$, see Definition 8.1(4) where $\mathscr{Y}^* =: \{(f_\ell,\mathscr{T}_\ell) : \ell < n^*_1\}$.

Clearly it suffices to prove that $B \cap \cap\{A^2_{\eta_m,R_m,\bar\nu_m,\nu_m} : m < n^*_2\}$ is not in $J_{\mathscr{Y}^*}$. By part (2) of Claim 8.5 below there is $B^* \subseteq B$ in $\mathbf{V}_1$ such that $B^* \backslash B_k$ is finite for $k < \omega$ and $B^* \notin J_{\mathscr{Y}^*}$.

So we have:

($*$) (i) $\mathscr{Y}^* \subseteq \mathscr{Y}$ is finite
(ii) $B^* \subseteq \omega$ is from $\mathbf{V}_1, B^* \notin J_{\mathscr{Y}^*}$
(iii) $\nu_m, \nu_{m,n} \in \lim(\mathscr{T}_1[R_m])$ for $m < n^*_2, n < \omega$
(iv) $\nu_m = \lim\langle \nu_{m,n} : n \in B^*\rangle$ for $m < n^*_2$
(v) $\langle \nu_{m,n} : n < \omega\rangle$ and ν_m belong to $\mathbf{V}_1$.

By assumption (f) we are done. $\square_{8.3}$

Claim 8.5. *1) Let $\mathscr{Y} \subseteq \mathbf{Y}$, then the following are equivalent for $B \subseteq \omega$:*

(*i*) *$B \notin J_{\mathscr{Y}}$*

(*ii*) *for every $n^* < \omega, (f_\ell, \mathscr{T}_\ell) \in \mathscr{Y}$ for $\ell < n^*$ and $m_0 < \omega$ there is $m_1 \in (m_0, \omega)$ such that:*

(∗) *if $\nu_\ell \in (\mathscr{T}_\ell)_{m_1}$ for $\ell < n^*$ <u>then</u> for some $n \in B \cap [m_0, m_1)$ we have $(\forall \ell < n^*)(\nu_\ell \restriction n \in f_\ell(n))$.*

2) For $\mathscr{Y} \subseteq \mathbf{Y}$, if $B_n \in J^+_{\mathscr{Y}}, B_{n+1} \subseteq B_n$ <u>then</u> there is $B \in J^+_{\mathscr{Y}}$ such that $n < \omega \Rightarrow B \subseteq^* B_n$. [REF(21A)]
3) If $\mathbf{V}_1 \subseteq \mathbf{V}_2, \mathscr{Y} \subseteq \mathbf{Y}^{\mathbf{V}_1}, \mathscr{Y} \in \mathbf{V}_1, A \in \mathscr{P}(\omega)^{\mathbf{V}_1}$, <u>then</u> $A \in J^{\mathbf{V}_1}_{\mathscr{Y}} \Leftrightarrow A \in J^{\mathbf{V}_2}_{\mathscr{Y}}$.

Proof. Easy.
1) Assume clause (i), i.e. $B \notin J_{\mathscr{Y}}$; to prove clause (ii) assume toward a contradiction that n^* and $\langle (f_\ell, \mathscr{T}_\ell) : \ell < n^* \rangle$ and $m_0 < \omega$ are as there but there is no $m_1 \in (m_0, \omega)$ such that (∗) there holds, so there are $\nu^{m_1}_\ell \in (\mathscr{T}_\ell)_{m_1}$ for $\ell < n^*$ such that $n \in B \cap [m_0, m_1) \Rightarrow (\exists \ell < n^*)(\nu^{m_1}_\ell \restriction n \notin f_\ell(n))$. By König lemma there are $\nu_\ell \in \mathrm{Lim}(\mathscr{T}_\ell)$ for $\ell < n^*$ such that $\forall m < \omega, \exists^\infty m_1 < \omega (m < m_1 and m_0 < m_1$ and $\bigwedge\limits_{\ell < n^*} \nu^{m_1}_\ell \restriction m = \nu_\ell \restriction m)$. [REF: $m_0 < m < m_1$ see (22C)]

Now for each ℓ, by Definition 8.1(4) as $(f_\ell, \mathscr{T}_\ell) \in \mathscr{Y}$, the set $A_\ell =: \{m < \omega : \nu_\ell \restriction m \in f_\ell(m)\} \in D_{\mathscr{Y}}$ hence $A = \bigcap\limits_{\ell < n^*} A_\ell \in D_{\mathscr{Y}}$, but we assume $B \neq \emptyset \mod D_{\mathscr{Y}}$ hence $A \cap B \neq \emptyset \mod D_{\mathscr{Y}}$ so there is $m, m_0 < m \in A \cap Y$. Let $m_1 > m$ be such that $\ell < n^* \Rightarrow \nu^{m_1}_\ell \restriction m = \nu_\ell \restriction m$ and this m_1 contradicts the choice of $\langle \nu^{m_1}_\ell : \ell < n^* \rangle$. So $(i) \Rightarrow (ii)$ indeed. The other direction is even easier.
2) Just use clause (ii) of part (1) as the definition. This is straight.
3) Follows using clause (ii) of part (1). $\square_{8.6}$

The Limit Claim 8.6. *Assume:*

(*a*) *δ a limit ordinal*

(*b*) *$\langle \mathbf{V}_\zeta : \zeta < \delta \rangle$ is an increasing sequence of inner models*

(*c*) *$\mathscr{Y}_\zeta \subseteq \mathbf{Y}^{\mathbf{V}_\zeta}$ is increasing with ζ*

(*d*) *D_ζ is a filter on $\mathscr{P}(\omega)^{\mathbf{V}_\zeta}$, increasing with ζ*

(*e*) *D_ζ is disjoint to $J_{\mathscr{Y}}$ for every finite $\mathscr{Y} \subseteq \mathscr{Y}_\zeta$.*

<u>Then</u> $\cup \{D_\zeta : \zeta < \delta\}$ can be extended to a uniform ultrafilter on ω disjoint to $J_{\cup \{\mathscr{Y}_\zeta : \zeta < \delta\}}$.

Proof. Easy. $\square$

Definition 8.7. 1) Assume $\mathscr{T} \in \mathbf{T}, h \in {}^\omega\omega, \infty = \lim \langle h(n) : n < \omega \rangle$ and $f \in \prod\limits_{n < \omega} \mathscr{P}(\mathscr{T}_{h(n)})$. We say that "$D$ on ω is $(f, h, \mathscr{T})$-narrow" if

(*a*) D is a filter on ω containing the co-bounded subsets

(*b*) for every $\nu \in \lim(\mathscr{T})$, the set $\{n : \nu \restriction h(n) \in f(n)\}$ belongs to D.

2) We say $(f', \mathscr{T}')$ is the translation of $(f, h, \mathscr{T})$ if:

$$\mathscr{T}' = \{\langle \eta \restriction h(m) : m < n\rangle : n < \omega, \eta \in \lim(\mathscr{T})\},$$

$$f'(n) = \{\langle \eta \restriction h(m) : m \leq n\rangle : \eta \in \lim(\mathscr{T}) \text{ and } \eta \restriction h(n) \in f(n)\}.$$

[REF: used??]

Remark 8.8. We may like to have $\mathscr{Y}' \subseteq \mathbf{Y}^{\mathbf{V}_2}$ is this needed? Helpful.
[REF: not clear]

Definition 8.9. 1) Let for a class $M, \mathfrak{Z}_M$ be the set of $(\eta, \bar{R})$ such that:

- (a) $\bar{R} = \langle R_n : n < \omega\rangle \in {}^\omega(\mathbf{R}^M_*), \mathscr{T}_1[R_n] = \mathscr{T}_1[R_0]$ and $\mathscr{T}_2[R_n] = \mathscr{T}_2[R_0]$ and for $\bar{R}$ we let $\mathscr{T}_\ell[\bar{R}] = \mathscr{T}_\ell[R_0]$ for $\ell = 1, 2$
- (b) $\eta \in \lim(\mathscr{T}_2[R_0])$
- (c) η does $\bar{R}$-cover M, which means $(\forall \nu \in \lim(\mathscr{T}_1[\bar{R}])^M)(\exists n < \omega)[\nu R_n \eta]$ [return: context with one R??].

Claim 8.10. *1) Assume (with $\mathbf{V} = \mathbf{V}_2$)*

- (a) $\mathbf{V}_1 \subseteq \mathbf{V}_2 = \mathbf{V}$
- (b) $\mathscr{L} \subseteq \mathbf{Z}_{\mathbf{V}_1}$
- (c) $\bar{D}_2 = \langle D_{2,\eta} : \eta \in {}^{\omega >}\omega\rangle, D_{2,\eta}$ *a non principal ultrafilter on* ω *(all in* $\mathbf{V}_2$*)*
- (d) $\langle D_{1,\eta} : \eta \in {}^{\omega >}\omega\rangle \in \mathbf{V}_1$ *where* $D_{1,\eta} = D_{2,\eta} \cap \mathscr{P}(\omega)^{\mathbf{V}_1}$
- (e) $D_{2,\eta}$ *is* $\mathscr{L}$*-big ultrafilter over* $\mathbf{V}_1$ *for every* $\eta \in {}^{\omega >}\omega$. *[REF:(22B)]*

Then $\Vdash_{\mathbb{Q}_{\bar{D}_2}}$ *"η is R-big over $\mathbf{V}_1[\underset{\sim}{\eta}_{\mathbb{Q}_{\bar{D}_2}}]$". [REF:(22A)($\eta$ double role)]*
2) Assume (a), (c), (d) above and

- $(b)'$ $(\eta, \bar{R}) \in \mathfrak{Z}_{\mathbf{V}_1}$
- $(e)'$ $D_{2,\eta}$ *is* (ρ, R_n)*-big ultrafilter when* $\rho \in {}^n\omega$.

Then $\Vdash_{\mathbb{Q}_{\bar{D}_2}}$ "$(\eta, \bar{R}) \in \mathfrak{Z}_{\mathbf{V}_1[\underset{\sim}{\eta}[Q_{\bar{D}_2}]]}$".

Proof. [Saharon revised: copied from 5.9(2). [See REF(23A); why not prove(2)?]
1) By clause (a) of Definition 8.9 this is a special case of part (2). So assume that $p \in \mathbb{Q}_{\bar{D}_1}, m^* < \omega$ and for each $m < m^*, (\rho_m, R_m) \in \mathscr{L}$ and $\underset{\sim}{\nu}^m, \langle \underset{\sim}{\nu}^m_n : n < \omega\rangle \in \mathbf{V}_1$ are $\mathbb{Q}^{\mathbf{V}}_{\bar{D}_2}$-names hence $\mathbb{Q}_{\bar{D}_1}$-names such that

$(*)_1$ $p \Vdash_{\mathbb{Q}_{\bar{D}_1}}$ "$\underset{\sim}{\nu}^m, \underset{\sim}{\nu}^m_n \in \lim(\mathscr{T}_1[R_m])$ and $\underset{\sim}{\nu}^m_n = \lim\langle \underset{\sim}{\nu}_m : n < \omega\rangle$".

By the definition and what we need to prove, without loss of generality

$(*)_2$ $p \Vdash$ "$\underset{\sim}{\nu}^m \restriction n = \underset{\sim}{\nu}^m_n \restriction n$".

We shall find $p' \geq p$ in $\mathbb{Q}_{\bar{D}_2}$ such that $p' \Vdash$ "$(\underset{\sim}{\nu}^m R_m \rho_m) \equiv (\underset{\sim}{\nu}^m R_m \rho_m)$ for every $m < m^*$; for some $n < \omega$", this suffices (see 4.10(2) REF!); work in $\mathbf{V}_1$. Let $q_0 = ({}^{\omega >}\omega)$, so $q_0 \in \mathbb{Q}_{\bar{D}_1}$, now we find $\langle \nu^m_\eta, \nu^m_{n,\eta} : \eta \in q_0, n < \omega\rangle$ of course in $\mathbf{V}_1$ such that:

$(*)_3$ (i) $\nu^m_\eta, \nu^m_{n,\eta} \in \lim(\mathscr{T}_1[R_m]), m < m^*$

(ii) for every $\eta \in q_0$ and $k < \omega$ we can find $q^m_{\eta,k}, q^m_{n,\eta,k} \in \mathbb{Q}_{\bar{D}_1}$ such that:
$q^{[\eta]}_0 \leq_{\rm pr} q^m_{\eta,k}, q^{[\eta]}_0 \leq_{\rm pr} q^m_{n,\eta,k}$
$q^m_{\eta,k} \Vdash_{\mathbb{Q}_{\bar{D}_1}}$ "$\underset{\sim}{\nu}^m \restriction k = \nu^m_\eta \restriction k$"
$q^m_{n,\eta,k} \Vdash_{\mathbb{Q}_{\bar{D}_1}}$ "$\underset{\sim}{\nu}^m_n \restriction k = \nu^m_{n,\eta} \restriction k$".

Now clearly

$(*)_4$ (i) $\nu^m_\eta = \lim_{D_{1,\eta}} \langle \nu^m_{\eta\hat{\ }<k>} : k < \omega \rangle$
(ii) $\nu^m_{n,\eta} = \lim_{D_{1,\eta}} \langle \nu^m_{n,\eta\hat{\ }<k>} : k < \omega \rangle$.

Next note that

$(*)_5$ $\nu^m_\eta = \lim \langle \nu^m_{n,\eta} : n < \omega \rangle$.

[Why? By $(*)_2$.]

Let $u_\eta = \{m < m^* : \nu^m_\eta R_m \rho_m \text{ holds}\}$.

Now as each R_m is closed (see Definition 8.1(2)) there is $k_\eta < \omega$ such that

$(*)_6$ if $m < m^*, m \notin u_\eta$ and $\nu^m_\eta \restriction k_\eta \triangleleft \nu \in \lim(\mathscr{T}_1[R_m]), \rho_m \restriction k_\eta \triangleleft \rho \in \lim(\mathscr{T}_2[R_m])$ then $\neg(\nu R_m \rho)$.

By $(*)_4(i) + (ii) + (*)_6$ we have

$(*)_7$ (i) $\neg(\nu^m_\eta R_m \rho_m)$ implies $\{k < \omega : \nu^m_{\eta\hat{\ }\langle k\rangle} R_m \rho_m\} \notin D_{1,\eta}$
(ii) $\neg(\nu^m_{n,\eta} R_m \rho_m)$ implies $\{k < \omega : \nu^m_{n,\eta\hat{\ }\langle k\rangle} R_m \rho_m\} \notin D_{1,\eta}$.

By the assumption (e) on $D_{2,\eta}$ and $(*)_7(i) + (ii)$ we have

$(*)_8$ (i) $\nu^m_\eta R_m \rho_m$ iff $\{k < \omega : \nu^m_{\eta\hat{\ }\langle k\rangle} R_m \rho_m\} \in D_{2,\eta}$
(ii) $\nu^m_{n,m} R_m \rho_m$ iff $\{k : \nu^m_{n,\eta\hat{\ }\langle k\rangle} R_m \rho_m \text{ holds}\} \in D_{2,\eta}$.

By $(*)_5$ applied to $\eta = \text{tr}(p)$, as the cofinite filter is $\mathscr{L}$-big over $\mathbf{V}_1$ (which is a consequence of assumption (e) [REF] we can find $n(*) < \omega$ such that $(\forall m < m^*)[(\nu^m_\eta R_m \rho_m) \equiv (\nu^m_{n(*),\eta} R_m \rho_m)]$. Next let

$$p^* =: \{\nu \in p : \text{ if } \ell g(\text{tr}(p)) \leq \ell \leq \ell g(\nu) \text{ and } m < m^* \text{ then } \nu^m_{\nu \restriction \ell} R_m \rho_m \equiv \nu^m_{n(*),\nu\restriction \ell} R_m \rho_m\}.$$

Now $p \leq_{\rm pr} p^* \in \mathbb{Q}_{\bar{D}_2}$ by $(*)_8$. Lastly, let $q^* =: \{\nu \in p^* :$ if $\ell < \ell g(\nu)$, <u>then</u> $\nu \in q^m_{\nu\restriction \ell, k_\eta}$ and $\nu \in q^m_{n(*),\nu\restriction\ell,k_\eta}\}$.

Does $q^* \Vdash_{\mathbb{Q}_{\bar{D}_1}}$ "$(\underset{\sim}{\nu}^m R_m \rho_m) \equiv (\underset{\sim}{\nu}^m_{n(*)} R_m \rho_m)$"? If not, then for some q^{**} we have $q^* \leq q^{**}$ and $q^{**} \Vdash_{\mathbb{Q}_{\bar{D}_2}}$ "$(\underset{\sim}{\nu}^m R_m \rho_m) \equiv \neg(\underset{\sim}{\nu}^m_{n(*)} R_m \rho_m)$"; moreover, without loss of generality for some truth value $\mathbf{t}, q^{**} \Vdash_{\mathbb{Q}_{\bar{D}_2}}$ "$(\underset{\sim}{\nu}^m R_m \rho_m) \equiv \mathbf{t}$ and $(\underset{\sim}{\nu}^n_{n(*)} R_m \rho_m) \equiv \neg\mathbf{t}$" and for some $k^* < \omega$, $q^{**} \Vdash_{\mathbb{Q}_{\bar{D}_2}}$ "$\mathbf{t} = \text{false} \Rightarrow (\forall \nu, \rho)[\underset{\sim}{\nu}^m \restriction k^* \trianglelefteq \nu \in \lim(\mathscr{T}_1[R_m]) and \rho_m \restriction k^* \triangleleft \rho \in \lim(\mathscr{T}_2[R_m]) \rightarrow (\nu,\rho) \notin R_m]$ and $\neg\mathbf{t} = \text{false} \Rightarrow (\forall \nu, \rho)[\underset{\sim}{\nu}^m_{n(*)} \restriction k^* \triangleleft \nu \in \lim(\mathscr{T}_1[R_m]) and \rho_m \restriction k^* \triangleleft \rho \in \lim(\mathscr{T}_2[R_m]) \rightarrow (\nu,\rho) \notin R_m]$".

But $q^{**}, q^m_{\text{tr}(q^{**}),k^*}, q^m_{n(*),\text{tr}(q^{**}),k^*}$ for $m < m^*$ are compatible having the same trunk, so let q' be a common upper bound with $\text{tr}(q') = \text{tr}(q^{**})$ and we get a contradiction.

2) Return! ?? [REF:(24B)]. Left to the reader. $\square_{8.10}$

Definition 8.11. 1) For $g \in \mathscr{G}$ let

$$\mathscr{T}[g] = {}^{\omega>}2$$

$$\mathscr{T}_2[g] = \{\langle T \cap {}^{g(\ell)}2 : \ell < n\rangle : T \in \mathbf{T}_g, n < \omega\},$$

so $\eta \in \lim(\mathscr{T}_2[g])$ can be identified with

$$T = T[\eta] \in \mathbf{T}_g : \eta = \eta_T = \big\langle\langle T \cap {}^{\ell \geq}2 : \ell < n\rangle : n < \omega\big\rangle$$

$$R_g = \{(\eta, \nu) : \nu \in {}^{\omega>}2, \eta \in \lim(\mathscr{T}_2[g]) \text{ and } \nu \in \lim(T[\eta])\}.$$

2) Let $\bar{w}^* = \langle w^*_k : k < \omega\rangle$ list with no repetition $\cup\{\mathscr{P}({}^n2)\backslash\{\emptyset\} : n < \omega\}$ such that $w^*_\ell \in {}^{\text{les}[w^*]}2, \ell_1 < \ell_2 \Rightarrow {}^{\text{les}}[w^*_{\ell_1}] \leq [w^*_{\ell_2}]$ and $\ell_1 < \ell_2 and \text{les}[w^*_{\ell_1}] = \text{les}[w^*_{\ell_2}] \Rightarrow$ the $<_{\text{lex}}$-first ρ such that $\rho \in w^*_{\ell_1} \equiv \rho \notin w^*_{\ell_2}$ satisfies $\rho \in w^*_\ell$. Let it be defined as $\mathscr{T}^0 = {}^\omega 2$ and $h^*(\ell) = n[w^*_\ell]$. [REF: (25B): (a) not clear, (b) where used?]

The following shows that the "$\mathscr{G}$-continuous" treated in §4-§6 fits our present framework.

Claim 8.12. *1) Assume $\mathbf{V}_1 \subseteq \mathbf{V}_2, g \in \mathscr{G}^{\mathbf{V}_1}, \mathbf{r} \in ({}^\omega 2)^{\mathbf{V}_2}$, then we have: $\mathbf{r}$ is $\{g\}$-continuous over $\mathbf{V}_1$ iff $\mathbf{r}$ is R_g-big over $\mathbf{V}_1$.*
2) Assume:

- (a) $\mathbf{V}_1 \subseteq \mathbf{V}_2 = \mathbf{V}$
- (b) $\mathscr{Z} \subseteq \{(\mathbf{r}, R_g) : g \in \mathscr{G}^{\mathbf{V}_1}, \mathbf{r} \in ({}^\omega 2)^{\mathbf{V}_2}$ *and* $\mathbf{r}$ *is* R_g*-big over* $\mathbf{V}_1\}$.

Then $\mathscr{Y} = \emptyset$ and $\mathscr{Z}$ are as required in 8.3, i.e. $(\mathscr{Y}, \mathscr{Z})$ is high over $\mathbf{V}_1$ (i.e. clause (f) there).
3) Assume (a), (b) as in (2) and

- (c) *$\mathscr{Y}$ has the form $\{(\bar{w}, h^*, \mathscr{Y}^0_*)\}$. [REF: (24A)]*

Then $(\mathscr{Y}, \mathscr{Z})$ are as required in 8.3.

Proof. (Or use $\mathscr{Z}$ with R^1, see later [REF??]).
1) Compare the definitions 4.7(1) + (3) and 8.2(4),(5),(6) check.
2) So assume $m^* < \omega$ and $(\eta_m, R_m) \in \mathscr{L}$ for $m < m^*$ and $B \subseteq \omega$ is infinite and $\nu_m, \nu_{m,n} \in \lim(\mathscr{T}_1[R_m]), \langle\nu_{m,n} : n < \omega\rangle \in \mathbf{V}_1$ and $\nu_m = \lim\langle\nu_{m,n} : n \in B\rangle$ for $m < m^*$. We should prove that $\{n \in B : \text{if } m < m^* \text{ then } \nu_m R_m \eta_m \equiv \nu_{m,n} R_m \eta_m\}$ is infinite. [REF (25A): what if $\langle\eta_m : m < \omega\rangle$ are pairwise distinct]
3) Left to the reader. $\square_{8.12}$

9. Refinements of §1 - §3

SAHARON - what is **??**?

Definition 9.1. We define the $(\bar{\alpha}, X, \zeta)$-standard trunk controller $\mathscr{F} = \mathscr{F}_{\bar{\alpha},\zeta}[X]$ in $\mathbf{V}$ by induction on $\zeta \le \omega_1$, where $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon \le \zeta \rangle, \alpha_\varepsilon$ an ordinal and X is a trunk controller, but we may write $\bar{\alpha}'$ with $\bar{\alpha}' \restriction (\zeta + 1) = \bar{\alpha}$ instead of $\bar{\alpha}$: the $(\bar{\alpha}, X, \zeta)$-standard trunk controller $\mathscr{F}$ in $\mathbf{V}$ is:

- (a) the set of elements is the set of functions f from a countable subset of α_ζ into $X \cup \bigcup \{\mathscr{F}_{\bar{\alpha},\varepsilon}[X] : \varepsilon < \zeta\}$, abusing notation we assume that $\langle X \rangle \hat{} \langle \mathscr{F}_{\bar{\alpha},\varepsilon}[X] : \varepsilon < \zeta \rangle$ is an increasing sequence of structures and for $\varepsilon = 0$ we stipulate $\mathscr{F}_{\bar{\alpha},\varepsilon-1}[X] = X$
- (b) $f_1 \le_{\rm pr} f_2$ iff $\mathrm{Dom}(f_1) \subseteq \mathrm{Dom}(f_2)$ and $\beta \in \mathrm{Dom}(f_1) \Rightarrow \bigvee\limits_{\varepsilon \in [-1,\zeta)} \mathscr{F}_{\bar{\alpha},\varepsilon}[X] \models f_1(\beta) \le_{\rm pr} f_2(\beta)$
- (c) $f_1 \le f_2$ iff
  - (i) $\mathrm{Dom}(f_1) \subseteq \mathrm{Dom}(f_2)$
  - (ii) $\beta \in \mathrm{Dom}(f_1) \Rightarrow \bigvee\limits_{\varepsilon \in [-1,\zeta)} \mathscr{F}_{\bar{\alpha},\varepsilon}[X] \models f_1(\beta) \le f_2(\beta)$
  - (iii) the set $\{\beta \in \mathrm{Dom}(f_1) : \bigvee\limits_{\varepsilon \in [-1,\zeta)} \mathscr{F}_{\bar{\alpha},\varepsilon}[X] \models f_1(\beta) \le f_2(\beta) \wedge \neg [f_1(\beta) \le_{\rm pr} f_2(\beta)]\}$ is finite
- (d) $f_1 \le_{\rm apr} f_2$ iff
  - (i) $f_1 \le f_2$ (this in fact follows by the later clauses)
  - (ii) $\mathrm{Dom}(f_1) = \mathrm{Dom}(f_2)$
  - (iii) for all but finitely many $\beta \in \mathrm{Dom}(f_1)$ we have $f_1(\beta) = f_2(\beta)$ and for the rest $\bigvee\limits_{\varepsilon \in [-1,\zeta)} \mathscr{F}_{\bar{\alpha},\varepsilon}[X] \models f_1(\beta) \le_{\rm apr} f_2(\beta)$.

Claim 9.2. *1) We say $\mathscr{F}$ is transparent if $p_0 \le_{\rm pr} p_1$ and $p_0 \le_{\rm pr} p_2 \Rightarrow (\exists p_3)(p_1 \le_{\rm pr} p_3 and p_2 \le_{\rm pr} p_3)$.*
2) For every trunk controller X and ζ and $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon \le \zeta \rangle, \mathscr{F}_{\bar{\alpha},\zeta}[X]$, is a well defined trunk controller, simple if X is simple.

Definition 9.3. 1) $\mathbb{Q}$, an $\mathscr{F}$-forcing is very clear (as an $\mathscr{F}$-forcing) or a very clear $\mathscr{F}$-forcing if:

- $(*)$ if $p_0, p_1 \in \mathbb{Q}$ and $\mathrm{val}^{\mathbb{Q}}(p_0), \mathrm{val}^{\mathbb{Q}}(p_1)$ have a common $\le^{\mathscr{F}}_{\rm pr}$-upper bound y then for some $q \in \mathbb{Q}$ we have $p_0 \le_{\rm pr} q, p_1 \le_{\rm pr} q$ and $\mathrm{val}^{\mathbb{Q}}(q) = y$.

2) $\mathbb{Q}$ is basic when: if $p_0 \le p_2$ then for some p_1 we have $p_0 \le_{\rm pr} p_1 \le_{\rm apr} p_2$ and $\mathrm{val}^{\mathbb{Q}}(p_1) = \mathrm{interval}^{\mathscr{F}}(p_0, p_2)$.
3) Let $\mathbb{Q}$ be an $\mathscr{F}$-forcing, it is straight, or $\mathscr{F}$-straight when: if $p_1 \le_{\rm apr} q_1, p_1 \le_{\rm pr} p_2$ and p_2, q_1 are compatible, then there is q_2 such that $q_1 \le_{\rm pr} q_2, p_2 \le_{\rm apr} q_2$ which is a $\le^{\mathbb{Q}}$-lub of p_2, q_1 and $\mathrm{val}^{\mathbb{Q}}(q_2)$ can be computed from $\langle \mathrm{val}^{\mathbb{Q}}(p_1), \mathrm{val}^{\mathbb{Q}}(p_2), \mathrm{val}^{\mathbb{Q}}(q_1) \rangle$, and we stipulate that this computation is a function which is part of the trunk controller. We call it $\mathrm{amal}_{\mathscr{F}}(-,-,-)$ (the point is that when we iterate over $\mathbf{V}$ this function will be in $\mathbf{V}$). If p_2, q_1 are incompatible, we use $q_2 = q_1$. [Used in 2.8, **??**].

4) An $\mathscr{F}$-forcing $\mathbb{Q}$ is called pseudo clear or pseudo $\mathscr{F}$-clear when: if $p \le_{\rm pr} p_1, p \le_{\rm pr} p_2$ and p_1, p_2 are $\le_{\rm pr}$-compatible then they have a common $\le_{\rm pr}$-upper bound q with $\mathrm{val}^{\mathbb{Q}}(q)$ computable (see (3)) from $\langle \mathrm{val}^{\mathbb{Q}}(p), \mathrm{val}^{\mathbb{Q}}(p_1), \mathrm{val}^{\mathbb{Q}}(p_2)\rangle$ and we denote it by $\mathrm{pramal}(-,-,-)$. [Used in 2.8, **??**; the difference from part (1) is the assumption of compatibility, and the val of the common upper bound is not any $\le^{\mathscr{F}}_{\rm pr}$-common upper bound of $\mathrm{val}(p_0), \mathrm{val}(p_1)$ but a specific one].
5) An $\mathscr{F}$ forcing $\mathbb{Q}$ is weakly clear when:

(1) If $p_0, p_1 \in \mathbb{Q}$ and $\mathrm{val}^{\mathbb{Q}}(p_0), \mathrm{val}^{\mathbb{Q}}(p_1)$ are $\le_{\rm pr}$-compatible in $\mathscr{F}$, <u>then</u> p_0, p_1 are $\le_{\rm pr}$-compatible.

6) We say $\mathbb{Q}$ is transparent [9] (or $\mathscr{F}$-transparent) when: if $p_0 \le_{pr} p_1, p_0 \le_{\rm pr} p_2, \mathrm{val}^{\mathbb{Q}}(p_1) \le_{\rm pr} y_3, \mathrm{val}^{\mathbb{Q}}(p_2) \le_{\rm pr} y_3 \in \mathscr{F}$, <u>then</u> there is $p_3 \in \mathbb{Q}$ such that $p_1 \le_{\rm pr} p_3, p_2 \le_{\rm pr} p_3$ and $\mathrm{val}^{\mathbb{Q}}(p_3) = y_3$.

Definition 9.4. 1) $\mathscr{F}$ is a trunk controller with inter when it may also have a function inter $=$ inter^F such that: if $\mathscr{F} \models$ "$p_0 \le p_2$" then $\mathrm{inter}(p_0, p_2) \in \mathscr{F}$ is well defined and $\mathscr{F} \models$ "$p_0 \le_{\rm pr} \mathrm{inter}(p_0, p_2) \le_{\rm apr} p_2$". $\mathscr{F}$ is a trunk[3] controller.
If we write "trunk" we mean it does not matter which case we use.
2) For $\mathscr{F}$ is a trunk controller with inter we say $\mathbb{Q}$ is a $\mathscr{F}$-forcing notions if (a)-(e) of Definition 1.6 and

(*e*) if $p_0 \le p_2$ then for some p_1 we have $p_0 \le_{\rm pr} p_1 \le_{\rm apr} p_2$ and $\mathrm{val}^{\mathbb{Q}}(p_1) = \mathrm{inter}^{\mathscr{F}}(\mathrm{val})^{\mathbb{Q}}(p_0), \mathrm{val}^{\mathbb{Q}}(p_2))$.

Claim 9.5. *1) For an $\mathscr{F}$-forcing $\mathbb{Q}$: very clear implies clear and implies weakly clear.*
*2) Assume $\mathbb{Q}$ is a $\mathscr{F}$-forcing, $\mathbb{Q}$ is weakly clear (**??**(5)), and $\mathscr{F}$ is semi-simple, <u>then</u> $\mathbb{Q}$ and even $(\mathbb{Q}, \le_{\rm pr})$ satisfies the $\aleph_2$-c.c.*
3) Assume $\mathbb{Q}$ is an $\mathscr{F}$-forcing, $\mathbb{Q}$ is weakly clear and $\mathscr{F}$ is simple, <u>then</u> $\mathbb{Q}$ and even $(\mathbb{Q}, \le_{\rm pr})$ satisfies the regressive $S^{\aleph_2}_{\aleph_1}$-c.c.

Remark 9.6. No harm demanding

(*c*) $\underset{\sim}{\mathbb{Q}}_0$ satisfies the c.c.c. and $\le^{\mathbb{Q}_0}_{\rm pr}$ is equality, $\le^{\mathbb{Q}_0}_{\rm apr}$ is $\le^{\mathbb{Q}_0}$ and $\mathrm{val}^{\mathbb{Q}_0}$ is constantly 0.

Definition 9.7. Adding the adjective "semi" (in 2.1(1) hence in (1) of 2.6) means that in clause (β) we just ask for some $\varepsilon < \zeta$ in B, the conditions q_ε, q_ζ are compatible in $\mathbb{Q}$; so we call the games semi-$\eth_p$, semi-$\eth'_p$. In Definition 2.5 and adding "semi" means, $\mathbb{Q}_0$ satisfies the semi version, each $\underset{\sim}{\mathbb{Q}}_{1+\alpha}$ the regular one.

Claim 9.8. *1) If in Definition 2.1, $\mathbb{Q}$ is bare $\mathscr{F}$-psc forcing and is $\mathscr{F}$-clear or at least straight (see Definition **??**) <u>then</u> $\mathbb{Q}$ is $\mathscr{F}$-psc (as witnessed by some $\mathbf{H}$, so $\mathbf{H}$ is redundant), similarly with $\mathscr{P}$.*

Proof. Straightforward. In part (1), the "$\mathbb{Q}$ is a clear $\mathscr{F}$-forcing" is used for clause (γ) in 2.1(1); for using "straight" note that in clause (γ) also $p_\varepsilon \le_{\rm pr} q_\varepsilon$ holds by the demands in $2.1(\gamma)$, as well as $p_\varepsilon \le_{\rm apr} q_\varepsilon$ by $\boxtimes$ of 2.1(1). $\square_{2.8}$

[9]This simplifies quite a number of definitions below. Of course, instead of for every y_3 it is enough to have one such $y_3 = y_3(\mathrm{val}^{\mathbb{Q}}(p_\ell) : \ell < 3)$, this function being part of $\mathscr{F}$

Definition 9.9. Let $\mathscr{F}$ be a trunk controller.
1) $\mathscr{F}$ satisfies the psc if:

- $(*)$ if $\mathscr{F} \models$ "$x \le_{\rm pr} y_\varepsilon \le_{\rm apr} z_\varepsilon$" for $\varepsilon < \omega_1$ and $A \subseteq \omega_1$ is stationary then for some stationary $B \subseteq A$ we have:
  if $\varepsilon < \zeta$ are from B then z_ε, z_ζ has a common upper bound z such that:
  - (α) $z = {\rm glue}_{\mathscr{F},\varepsilon,\zeta}(z_\varepsilon, z_\zeta, y_\varepsilon, y_\zeta)$
  - (β) if $y_\varepsilon \le_{\rm pr} z_\varepsilon, y_\zeta \le_{\rm pr} z_\zeta$ then $z_\zeta \le_{\rm pr} z, z_\varepsilon \le_{\rm pr} z$.

2) $\mathscr{F}$ satisfies the almost-psc if:

- $(*)$ if $\mathscr{F} \models$ "$x \le_{\rm pr} y_\varepsilon \le_{\rm apr} z_\varepsilon$" for $\varepsilon < \omega_1$, <u>then</u> for some $\varepsilon < \zeta < \omega_1, z_\varepsilon, z_\zeta$ has a common upper bound z such that $(\alpha) + (\beta)$ above holds.

3) In parts (1), (2) we add the adjective "continuous" if in $(*)$ we add $\varepsilon < \zeta < \omega_1 \Rightarrow \mathscr{F} \models$ "$y_\varepsilon \le_{\rm pr} y_\zeta$". We add Knaster if we replace stationary by unbounded (this is alternative to semi[7], but not for iteration!)
4) We add the adjective "finished" if we omit clause (γ) in 2.1(1) and its variants.
5) We say that an $\mathscr{F}$-forcing $\mathbb{Q}$ is [Knaster] explicitly [almost] $\mathscr{F}$-psc forcing <u>if</u>

- (a) $\mathbb{Q}$ is an $\mathscr{F}$-forcing
- (b) $\mathscr{F}$ satisfies the [Knaster][almost] psc
- (c) if $z = {\rm glue}_{\mathscr{F},\varepsilon,\zeta}(z', z'', y', y'')$ and $q', q'', p', p'' \in \mathbb{Q}$ and $p' \le_{\rm pr} p'', p' \le_{\rm apr} q', p'' \le_{\rm apr} q'', y' = {\rm val}^{\mathbb{Q}}(p'), y'' = {\rm val}^{\mathbb{Q}}(p''), z' = {\rm val}^{\mathbb{Q}}(q'), z'' = {\rm val}^{\mathbb{Q}}(q'')$, <u>then</u> there is $q \in \mathbb{Q}$ such that:
  - (α) ${\rm val}^{\mathbb{Q}}(q) = {\rm glue}_{\mathscr{F}}(z, z', z'', y', y'')$
  - (β) if $p' \le_{\rm pr} q', p'' \le_{\rm pr} q''$ then $q'' \le_{\rm pr} q$.

6) We say $\bar{\mathbb{Q}}$ is a Knaster explicit [almost] $\mathscr{F}$-psc iteration if:

- (a) $\mathscr{F}$ is a trunk controller, fully based on some $\alpha' \ge \ell g(\bar{\mathbb{Q}})$
- (b) $\bar{\mathbb{Q}}$ is an $\mathscr{F}$-iteration
- (c) $\mathscr{F}^{[0]}$ satisfies the Knaster [semi] psc; $\aleph_1$-complete
- (d) $\mathscr{F}^{[1+\beta]}$ satisfies the [Knaster] psc when $1 + \beta < \ell g(\bar{Q}); \aleph_1$-complete
- (e) $\mathbb{Q}_0$ is explicitly [Knaster][semi] $\mathscr{F}^{[0]}$-psc,$\le_{\rm pr}$ $-\aleph_1$-complete
- (f) $\mathbb{Q}_{1+\beta}$ is (forced to be) an explicitly [Knaster] $\mathscr{F}$-psc forcing $\le_{\rm pr}$ $-\aleph_1$-complete.

7) We (in (5), (6)) add continuous if so are the $\mathscr{F}$'s.

Claim 9.10. *1) Assume*

- (a) *$\mathscr{F}$ is a [semi]-psc trunk controller*
- (b) *$\mathbb{Q}$ is an $\mathscr{F}$-forcing notion.*

<u>Then</u> $\mathbb{Q}$ is a [semi] $\mathscr{F}$-psc forcing notion.
2) Assume

- (a) *$\mathscr{F}$ is a continuous [semi]-psc trunk controller*
- (b) *$\mathbb{Q}$ is a straight $\mathscr{F}$-forcing notion*
- (c) *$(\mathbb{Q}, \le_{pr})$ is $\aleph_1$-complete.*

Then $\mathbb{Q}$ *is a [semi] strong* $\mathscr{F}$*-psc forcing notion.*

Claim 9.11. *Assume* $\bar{\mathbb{Q}}$ *is an explicitly [Knaster/semi]* $\mathscr{F}$*-psc iteration.*
1) If $\beta \le \ell g(\bar{\mathbb{Q}})$*, then* $\bar{\mathbb{Q}} \restriction \beta$ *is an explicitly [Knaster/semi]* $\mathscr{F}$*-psc iteration.*
2) If the trunk controller $\mathscr{F}$ *is fully based on* $\langle \mathscr{F}^\beta : \beta < \alpha^* \rangle$ *each* $F^{[1+\beta]}$ *satisfies [Knaster]-psc and* $\mathscr{F}^{[0]}$ *satisfies [Knaster/semi]-psc, then* $\mathscr{F}$ *satisfies [Knaster/semi]-psc.*
3) If $\mathbb{Q}$ *is explicitly [semi]* $\mathscr{F}$*-psc forcing, then* $\mathbb{Q}$ *is strongly [semi]* $\mathscr{F}$*-psc forcing.*

Proof. Should be clear. □

Claim 9.12. *If* $\mathbb{Q}$ *is explicitly [Knaster/semi]* $\mathscr{F}$*-psc, then* $\mathbb{Q}$ *is purely proper.*

Proof. Same as 2.9 only easier (in some cases by **??**(3)). □

Claim 9.13. *FILL.*

Proof. 1) Assume not and let $\bar{\mathbf{H}}$ be a witness for "$\bar{\mathbb{Q}}$ is $\mathscr{F}$-psc" hence by 2.12 some $\mathbf{H}$ witnesses $\mathbb{P}_{\alpha^*}$ is $\mathscr{F}$-pcf. So simulate a play of the game $\partial_p = \partial_{p,\mathbb{P}_{\alpha^*},\mathbf{H}}$, where the interpolator plays using a fixed winning strategy whereas the extender chooses q_ζ and $n_\zeta \le \omega$ such that:

- (α) $p'_\zeta \le q_\zeta$ (see notation in 2.1(1)) (i.e. a legal move)
- (β) $n_\zeta \le \omega$ is the minimal n such that $\{q_\varepsilon : \varepsilon < \zeta, n_\varepsilon = n$, and q_ε forces a value of $\underset{\sim}{\tau}(n)\}$ is not predense over p^*_ζ
- (γ) if $n_\zeta < \omega, q_\zeta$ forces a value to $\underset{\sim}{\tau}(n_\zeta)$, call it j_ζ
- (δ) if $n_\zeta < \omega$ then q_ζ is incompatible with q_ε if $\varepsilon < \zeta and n_\varepsilon = n_\zeta$.

Now

$\boxtimes$ for some $\zeta, n_\zeta = \omega$.

Why? Otherwise the extendor can choose q_ζ for every $\zeta < \omega_1$, and $n_\zeta = n^*$ for every $\zeta \in [\zeta^*, \omega_1)$ for some ζ^*; in the end $\zeta^* < \varepsilon < \zeta \Rightarrow q_\varepsilon, q_\zeta$ are incompatible but the interpolator has to win the play (as he has used his winning strategy), contradicting clauses (β) of 2.1(1).

So necessarily for some $\zeta < \omega_1, n_\zeta = \omega$. Let p^* be: p if $\zeta = 0, p_{\zeta-1}$ if ζ is a successor ordinal and p'_ζ if ζ is a limit ordinal, so $p \le_{\rm pr} p^*$ by the definition of the game. For each $\varepsilon < \zeta$ let q'_ε be a $\le$-lub of q_ε, p^*, exists as $\mathbb{Q}$ is straight, so $p^* \le_{\rm apr} q'_\varepsilon$. Let $\mathscr{I}_n = \{q'_\varepsilon : \varepsilon < \zeta$ and $n_\varepsilon = n\}$ and we shall show that $p^*, \mathscr{I}_n$ are as required (p^* standing for q). Now clauses (a), (b), (c) are obvious, toward clause (d) assume $n < \omega, p^* \le q$ and q is incompatible with all members of $\mathscr{I}_n$ and let $\zeta_n = \mathrm{Min}\{\zeta : n_\zeta > n\}$, so q could not have been a good candidate for q_{ζ_n} hence is compatible with some $q_\varepsilon, n_\varepsilon = n$. So by the choice of q'_ε clearly $q'_\varepsilon \le q$ and $q'_\varepsilon \in \mathscr{I}_n$, contradiction. □

10. §T

Recall ([?, ?]).

Definition 10.1. 1) For an ordinal $\alpha(*), \mathscr{F} = \mathscr{F}_{\alpha(*)}$ is the α-th standard trunk controller (we let $\alpha(\mathscr{F}) = \alpha$) so

(a) $f \in \mathscr{F}$ <u>iff</u> f is a function from some countable $u \subset \alpha$ into ${}^{\omega>}\omega$

(b) $\le_x = \le_x^{\mathscr{F}}$ is the following partial order on $\mathscr{F}_\alpha$

  (α) $f \le_{\rm pr} g$ <u>iff</u> $f = g \restriction {\rm Dom}(f)$

  (β) $f \le_{\rm apr} g$ <u>iff</u> ${\rm Dom}(f) = {\rm Dom}(g)$
  $(\forall \beta \in {\rm Dom}(f))(f(\beta) \trianglelefteq g(\beta))$
  $(\exists^{<\aleph_0} \beta \in {\rm Dom}(f))(f(\beta) \ne g(\beta))$

  (γ) $f \le_{\rm us} g$ iff $(\exists h)(f \le_{\rm pr} h \le_{\rm apr} g)$.

2) We define also $\le_x = \le_x^{\mathscr{F}_\alpha}$ for $x = {\rm qr}, {\rm aqr}$: we let

(α) $f \le_{\rm qr} g$ iff for some $h, h \le_{\rm apr} f \wedge h \le g$

(β) $f \le_{\rm aqr} g$ iff ${\rm Dom}(f) = {\rm Dom}(g) \wedge f \le_{\rm qr} g$.

3) For $f \in \mathscr{F}$ let R_f be $(\{g : f \le_{\rm aqr} g\}, \le_{\rm apr})$ and let $R'_f = (\{g : f \le_{\rm apr} g\}, \le_{\rm apr})$.

Observation 10.2. *1) $\le_{\rm qr}$ is a partial order on $\mathscr{F}_\alpha$.*
2) $\le_{\rm aqr}$ is an equivalence relation, the R_f's are the equivalence classes.

Definition 10.3. Let $\mathscr{F}$ be a standard trunk controller. For $f \in \mathscr{F}_\alpha$ let

(a) ${\rm aext}(f) = \{g : f \le_{\rm aqr} g\}$ and ${\rm ext}(f) = \{g : f \le_{\rm apr} g\}$

(b) ${\rm pos}(f) = \{\bar\eta : \bar\eta = \langle \eta_\beta : \beta \in {\rm Dom}(f)\rangle$ and $f(\alpha) \triangleleft \eta_\alpha \in {}^\omega\omega$ for $\alpha \in {\rm Dom}(f)$

(c) ${\rm apos}(f) = \cup\{{\rm pos}(g) : g \le_{\rm aqr} f\}$

(d) ${\rm dst}(f) = \{\mathscr{I} : \mathscr{I} \subseteq \{g : f \le_{\rm aqr} g\}$ and for every $g \in {\rm aext}(f)$ and $\bar\eta \in {\rm pos}(g)$ there is h such that $g \le_{\rm ap} h \wedge \bar\eta \in {\rm pos}(h)$
  [${\rm dst}(f)$ is the family of "dense subsets" of $R_f = {\rm aext}(f)$, so if ${\rm Dom}(f) = \{\beta\}$ it means that we have fronts $\mathscr{I}_n$ of ${}^{\omega>}\omega$ for $n < \omega$ such that $(\forall \eta \in \mathscr{I}_n)(\forall \nu \in \mathscr{I}_n)(\neg \nu \trianglelefteq \eta)$ and $\mathscr{I}$ includes (the copy of) $\cup\{\mathscr{I}_n : n < \omega\}$ that is $\mathscr{I} \supseteq \{\{(\beta, \nu)\} : \nu \in \mathscr{I}_n, n < \omega\}$]

(e) if $\mathscr{I} \in {\rm dst}(f)$ and $S \subseteq {\rm aext}(f)$ we say S is decidable by $\mathscr{I}$ when: if $\bar\eta \in {\rm apos}(f)$, then for some $g \in {\rm axnt}(f)$ and truth value $\mathbf{t}$ we have $\bar\eta \in {\rm pos}(g)$ and: for every $h, g \le_{\rm apr} h \in \mathscr{I} \wedge \bar\nu \in {\rm pos}(h) \Rightarrow ((h \in S) \equiv \mathbf{t})$
  [this says in an appropriate sense that for a dense set of open subsets u of ${\rm apos}(f), u \cap \mathscr{I}$ is included in S or disjoint to $\bar S$]

(f) for $\mathscr{I}_1, \mathscr{I}_2 \in {\rm dst}(f)$, let $\mathscr{I}_1 \le_* \mathscr{I}_2$ means that for every $\bar\eta \in {\rm apos}(f)$ there is $g \in {\rm axnt}(f)$ such that $\bar\eta \in {\rm pos}(g) \wedge (\forall h)(g \le_{\rm apr} h \in \mathscr{I}_2 \Rightarrow h \in \mathscr{I}_1)$
  [this says that on a dense open set $\mathscr{I}_2 \subseteq I_1$]

(g) let ${\rm DEC}(f) = \{\mathscr{D} : \mathscr{D} \subseteq {\rm dst}(f), \mathscr{D}$ is $<^*$-downward closed, $\mathscr{D}$ is $(\le_*, \aleph_1)$-directed and for every $S \subseteq {\rm apos}(f)$ is decidable by some $\mathscr{I} \in D\}$.
  [those $\mathscr{D}$'s are like P-points].

Definition 10.4. Assume $f_1 \le_{\rm qr} f_2$.
1) Let ${\rm prj}_{f_1,f_2} : {\rm apos}(f_2) \to {\rm apos}(f_1)$ be ${\rm prf}(g_2) = g_2 \restriction {\rm Dom}(f_1)$.
2) For $\mathscr{I}_2 \in {\rm dst}(f_2)$, let ${\rm prj}_{f_1,f_2}(\mathscr{I}_2) = \{{\rm prj}_{f_1,f_2}(g) : g \in \mathscr{I}\}$.
3) For $\mathscr{D}_2 \in {\rm DEC}(f_2)$, let ${\rm prj}_{f_1,f_2}(\mathscr{D}_2) = \{{\rm prj}_{f_2,f_2}(\mathscr{I}) : I \in \mathscr{D}_2\}$.

Alernatively

Definition 10.5. 1) For $u \in [\alpha(*)] \leq \aleph_0$ let

$$\text{flsq}(u) = \{\bar{D} : \bar{D} = \langle D_{\alpha,\eta} : \alpha \in u, \eta \in {}^{\omega>}\omega\rangle, D_{\alpha,\eta} \text{ an ultrafilter on } \omega\}$$

Let $\text{flsq}(f) = \text{flsq}(\text{Dom}(f))$.
2) Let $\mathscr{F}^1_{\alpha(*)} = \{(u,D) : u \in [\alpha(*)]^{\leq\aleph_0}$ and $\bar{D} \in \text{flsq}(u)\}$.
3) For $f \in \mathscr{F}_{\alpha(*)}$ and $\bar{D} \in \text{flsq}(f)$ let $\text{dst}(f,\bar{D})$ is the set of $\mathscr{I} \subseteq \{g : f \leq_{\text{apr}} g\}$ such that

$(*)_{g,\bar{D},\mathscr{I}}$ in the following game the first player wins (i.e., has a winning strategy)

- (a) a play lasts at most ω moves
- (b) before the n-th move $g_n \in \text{pos}(g)$ is chosen
- (c) $g_0 = g$
- (d) in the n-thmove, the first player chooses.

Observation 10.6. *Assume $f_1 \leq_{\text{qr}} f_2$.*
1) prj_{f_1,f_2} is a function from $\text{apos}(f_2)$ onto $\text{apos}(f_1)$.
2) prj_{f_1,f_2} maps $\text{dst}(f_2)$ onto $\text{dst}(f_1)$.
3) If $\mathscr{I} \subseteq \mathscr{J}$ are from $\text{dst}(f_2)$ <u>then</u> $\mathscr{I} \subseteq \mathscr{J} \Rightarrow \text{prj}_{f_2,f_1}(\mathscr{I}) \subseteq \text{prj}_{f_2,f_1}(\mathscr{J})$ and $\mathscr{I} \leq_* \mathscr{J} \Rightarrow \text{prj}_{f_2,f_2}(\mathscr{I}) \subseteq \text{prj}_{f_2,f_1}(\mathscr{J})$ provided that FILL prj_{f_1,f_2} maps every member of $\text{DEC}(f_2)$ to a member of $\text{DEC}(f_1)$.

Proof. FILL. $\square$

Claim 10.7. *Assume $f \in \mathscr{F}_{\alpha(*)}$.*
1) If $\mathscr{I}_n \in \text{dst}(f)$ and $\mathscr{I}_n \leq_* \mathscr{I}_{n+1}$ for $n < \omega$ <u>then</u> there is $\mathscr{I} \in \text{dst}(f)$ satisfying $n < \omega \Rightarrow \mathscr{I}_n \leq_* \mathscr{I}$.
2) Assume $\langle u_n : n < \omega\rangle$ is an increasing sequence of finite sets with union $\text{Dom}(f)$ and $\mathscr{I} \subseteq \text{apos}(f)$. We have $\mathscr{I} \in \text{dst}(f)$ iff $\mathscr{I}$ has a $\bar{u}$-witness (see Definition below).
3) Similarly for $\mathscr{I}_1 \leq_* \mathscr{I}_2$.

Definition 10.8. Assume $\bar{u} = \langle u_n : n < \omega\rangle$ is an increasing sequence of finite sets with union $\text{Dom}(f)$ and $\mathscr{I} \subseteq \text{apos}(f)$.
1) We say that $(\mathbf{n}, \dot{\zeta})$ is a $\bar{u}$-witness for $\mathscr{I}$ when

- (a) $\mathbf{n}$ is a function from $\text{apos}(f)$ to ω
- (b) if $g_1 \leq_{\text{apr}} g_2$ are from $\text{apos}(f)$ then $\mathbf{n}(g_1) \leq \mathbf{n}(g_2)$
- (c) $\dot{\zeta}$ is a function from $\text{apos}(f)$ to ω_1
- (d) if $\dot{\zeta}(g) = 0$ then $g \in \mathscr{I}$
- (e) if $g_1 <_{\text{apr}} g_2$ are from $\text{apos}(f)$ and $\dot{\zeta}(g_1) > \dot{\zeta}(g_2) = 0$ then $\mathbf{n}(g_1) < \mathbf{n}(g_2)$
- (f) if $g_1 \in \text{apos}(f), \dot{\zeta}(g_1) > 0, g_1 <_{\text{apr}} g_2, u_{\mathbf{n}(g_1)} = \{\beta \in \text{Dom}(f) : g_1(n) \neq g_2(n)\}$ and $\neg(\exists g \in \mathscr{I})(g_1 \leq_{\text{apr}} g \leq_{\text{apr}} q_2)$ then $\dot{\zeta}(g_1) > \dot{\zeta}(g_2)$.

2) Assume $\mathscr{I}_1, \mathscr{I}_2 \leq \text{apos}(f)$. We say that $(\mathbf{n}, \dot{\zeta})$ is a $\bar{u}$-witness that $\mathscr{I}_1 \leq_* \mathscr{I}_2$ when

$(a) - (c)$ as in part (1)

(*d*) if $\dot{\zeta}(g) = 0$ then $(\forall h)(g \le_{\rm ap} h \in \mathscr{I}_2 \Rightarrow h \in \mathscr{I}_1)$
(*e*) if $g_1 \le_{\rm apr} g_2$ are from $\mathrm{apos}(f)$ then $\dot{\zeta}(g_1) \ge \dot{\zeta}(g_2)$
(*f*) as above.

Proof. Proof of 10.7:
1) By parts (2) + (3) and diagonalization.
2), 3) Straight. $\square_{10.7}$

Claim 10.9. *Assume*

(*a*) $f_n, f \in \mathscr{F}_{\alpha(*)}$
(*b*) $f_n \le_{\rm qr} f_{n+1} \le_{\rm qr} f$ *for* $n < \omega$
(*c*) $\mathscr{I}_n \in \mathrm{dst}(f_n)$
(*d*) $\mathscr{I}_n \le_* \mathrm{prj}_{f_n, f_{n+1}}(\mathscr{I}_{n+1})$ *for* $n < \omega$.

Then for some $\mathscr{I} \in \mathrm{dst}(f_n)$ *we have* $n < \omega \Rightarrow \mathscr{I}_n \le_* \mathrm{prj}_{f_n, f}(\mathscr{I})$.

Definition 10.10. $\bar{\mathscr{D}}$ is a full $\mathscr{F}_{\alpha(*)}$-choice if

(*a*) $\bar{\mathscr{D}} = \langle \mathscr{D}_f : f \in \mathscr{F}_{\alpha(*)} \rangle$
(*b*) $\mathscr{D}_f \in \mathrm{DEC}(f)$
(*c*) if $f_1 \le_{\rm qr} f_2$ then $D_1 \approx \mathrm{prj}(D_2)$.

Definition 10.11. 1) The forcing notion $\mathbb{R}^*_{\alpha(*)}$ is defined as follows

(*a*) $p \in \mathbb{R}^*_{\alpha(*)}$ iff $p = (f, \mathscr{I}) = (f^p, \mathscr{I}^p), f \in \mathscr{F}_{\alpha(*)}, \mathscr{I} \in \mathrm{dst}(f)$
(*b*) $\mathbb{R}^*_{\alpha(*)} \models p \le q$ iff $f^p \le_{\rm qr} f^q \wedge \underset{\sim}{\mathscr{I}}^p \le_* \mathrm{prj}_{f^p, f^q}(\underset{\sim}{\mathscr{I}}^q)$.

2) We define a $\mathbb{R}^*_{\alpha(*)}$-name, $\underset{\sim}{\bar{\mathscr{D}}} = \langle \underset{\sim}{\mathscr{D}}_f : f \in \mathscr{F}_{\alpha(*)} \rangle, \underset{\sim}{\mathscr{D}}_f = \{\mathscr{I} \subseteq \mathrm{apos}(f) : \mathscr{I} \le_* \pi_{f, f^p}(\mathscr{I}^p)$ for some $p \in \underset{\sim}{\mathbf{G}}\}$.

Claim 10.12. *Assume CH.*
1) $R^*_{<\alpha(*)}$ *is a* $\aleph_2$*-c.c.,* $\aleph_1$*-complete forcing notions.*
2) $\Vdash_{\mathbb{R}^*_{\alpha(*)}} \underset{\sim}{\bar{\mathscr{D}}}$ *a full* $\mathscr{F}_{\alpha(*)}$*-choice.*

Proof. Should be clear. $\square$

11. §U

SAHARON: What is **??**? See Glossary

Definition 11.1. 1) We say $\mathfrak{x}$ is an $\alpha(*)$-parameter when it consists of the following objects satisfying the following conditions

(a) $\bar{\underset{\sim}{\mathscr{D}}} = \langle \mathscr{D}_f : f \in \mathscr{F}_{\alpha(*)} \rangle$ is a full $\mathscr{F}_{\alpha(*)}$-choice
(b) $\bar{D} = \langle D_{\alpha,\eta} : \alpha < \alpha(*), \eta \in {}^{\omega >}\omega \rangle$ such that $D_{\alpha,\eta}$ is a non-principal ultrafilter on ω.

2) Let $\mathfrak{K}_{\mathfrak{x}}$ be the class of $\bar{\mathbb{Q}}$ such that

(a) $\bar{\mathbb{Q}}$ is as in [**?**, 5.11], but also $\mathbb{Q}_0$ is $\mathbb{Q}_{\bar{D}_0}$
(b) $\underset{\sim}{D}_{\beta,\eta}$, a $\mathbb{P}_\beta$-name is defined as follows: for $\mathbf{G}_\beta$ a subset of $\mathbb{P}_\beta$ generic over $\mathbf{U}, \underset{\sim}{D}_{\beta,\eta}[\mathbf{G}_\beta]$ is the set of $\underset{\sim}{A}[\mathbf{G}_\beta]$ such that for some $\underset{\sim}{A}, p \in \mathbf{G}_\beta$ and $\bar{J} = \langle \mathscr{J}_n : n < \omega \rangle$ we have
(c) $\underset{\sim}{A}$ is a $\mathbb{P}_\beta$-name
(d) $\mathscr{J}_n$ is a subset of $\{q : p \le_{\rm ap} q \in \mathbb{P}_\beta\}$ predense over p if $q \in \mathscr{J}_n$ then q forces a value
(e) if for some $\mathscr{I} \in \mathscr{D}^{\mathfrak{x}}_{{\rm tr}(p)}$ for every $g \in \mathscr{I}$ we have for some $B \in D^{\mathfrak{x}}_{\alpha,\eta}$ we have: sn
   (∗) if $p \le_{\rm apr} q \in \mathbb{P}_\alpha, {\rm tr}(q) = g$ and $n < \omega$ <u>then</u> for some $r, q \le_{\rm apr} r \in \mathbb{P}_\alpha, {\rm tr}(r) = g$ we have $r \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{A} \cap n = B \cap n$".

Claim 11.2. *1) For every $\alpha(*) \in$ Ord and $\alpha(*)$-parameter $\mathfrak{x}$ there is one and only one $\bar{\mathbb{Q}} \in \mathfrak{K}_{\mathfrak{x}}$.*

Proof. We prove this by induction on $\alpha(*)$. □

<u>Case 1</u>: $\alpha(*) = 0$.
Trivial

<u>Case 2</u>: $\alpha(*)$ is a limit ordinal.
This holds by [**?**, xxx].

<u>Case 3</u>: $\alpha(*) = \alpha + 1$.
Clearly $\mathfrak{y} = \mathfrak{x} \restriction \alpha$ is an α-parameter hence $\bar{\mathbb{Q}}^\alpha = \langle P_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ is well defined as well as $\mathbb{P}_\alpha = {\rm Lim}(\bar{\mathbb{Q}}^\alpha)$.

Now for every $\eta \in {}^{\omega >}\omega, \bar{\underset{\sim}{\mathscr{D}}}_{\alpha,\eta}$ is a well defined P_α-name of a subset of $\mathscr{P}(\omega)$. Now by [**?**, xxx] it suffices to prove

⊠ $\bar{\underset{\sim}{\mathscr{D}}}_{\alpha,\eta}$ is a $\mathscr{P}_\alpha$-name of a non-principal on ω.

For this it suffices to prove $(*)_1 + (*)_2$ where

$(*)_1$ if $\underset{\sim}{A}$ is a $\mathbb{P}_\alpha$-name of a subset of ω then for a dense open set of $p \in \mathbb{P}_\alpha, p \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{A} \in \underset{\sim}{D}^{\mathfrak{x}}_{\alpha,eta}$" or $p \Vdash_{\mathbb{P}_\alpha}$ "$\omega \backslash \underset{\sim}{A} \in D^{\mathfrak{x}}_{\alpha,\eta}$".

[Why $(*)_1$ holds? Let $p_0 \in \mathbb{P}_\alpha$ and $\underset{\sim}{A}_q = \underset{\sim}{A}, \underset{\sim}{A}_0 = \omega \backslash \underset{\sim}{A}$. By [**?**, xxx] there is $p, p_0 \leq p_0 \in \mathbb{P}_\alpha$ such that $(p, \underset{\sim}{A})$ satisfies clause (b) of Definition 11.1(2), $S_\ell := \{g \in \text{apos}(\text{tr}(p))$: if there is $q, p \leq_{\text{apr}} q_r, g = \text{tr}(q)$ then $(g, \underset{\sim}{A}_\ell)$ are as in (c) of Definition 11.1(2)$\}$.

As $D^{\mathfrak{x}}_{\text{tr}(p)} \in \text{DEC}(\text{tr}(p))$ clearly $(p, \underset{\sim}{A}_\ell)$ is as in **??**(2) for some $\ell \in \{0, 1\}$ so we are done.]

$(*)_2$ if $n < \omega$ and for $\ell < n, \underset{\sim}{A}_\ell$ is a $\mathbb{P}_\alpha$-name of a subset of $\omega, \underset{\sim}{A}_\ell[G_\alpha] \in D$ then $\bigcap\limits_{\ell < n} \underset{\sim}{A}_\ell[\mathbf{G}_\alpha] \neq \emptyset$.

[Why? For each $\ell < n$ there is $(p_\ell, \mathbf{I}_\ell), p_\ell \in \mathbf{G}_\beta$ witnessing $\underset{\sim}{A}[\mathbf{G}_\alpha] \in \underset{\sim}{D}_{\alpha,\eta}[\mathbf{G}]$ as in **??**(2) so $\mathscr{I}_\ell \in D_{\text{tr}(p_\ell)}$. As $\mathbf{G}$ is directed, there is $p \in \mathbf{G}$ such that $p_\ell \leq_{\mathbb{P}_\alpha} p$ for $\ell < n$, so clearly $\mathscr{F}_\alpha \models \text{tr}(p_\ell) \leq_{\text{ap}} \text{tr}(p)$.

By the assumption $\bar{\mathscr{D}}^{\mathfrak{x}}$, see clause (a) of Definition 11.1, for some $\mathscr{I} \in D_{\text{tr}}$ such that $\ell < n \Rightarrow \mathscr{I}_\ell \leq_* \text{prj}_{\text{tr}(p_\ell),\text{tr}(p)}(\mathscr{I})$. Hence by [**?**, zzz] there is $q, p \leq_{\text{ap}} q$ such that $\ell < n \Rightarrow \text{prj}_{\text{tr}(p_\ell),\text{tr}(p)}(\text{tr}(q)) \in I_\ell$ hence there is B_ℓ as in clause (c) of Definition **??**(2). So $B_0, \ldots, B_{n-1} \in D^{\mathfrak{x}}_{\alpha,et}$ hence $B := \cap\{B_\ell : \ell < n\}$ belongs to the the ultrafilter $D^{\mathfrak{x}}_{\alpha,\eta}$ from $\mathbf{V}$ and let $k(n) \in B$. So for each $\ell < n$ there is q_ℓ such that $p_\ell \leq_{\text{apr}} q_\ell, \text{tr}(q_\ell) = \text{tr}(g) \restriction \text{Dom}(p_\ell), q_\ell \Vdash k(*) \in \underset{\sim}{A}_\ell$. By $\{q\} \cup \{q_\ell : \ell < n\}$ easily has a common upper bound say r so $c \vdash$ "$k \in \cup\{\underset{\sim}{A}_\ell : \ell < n\}$ hence this intersection is non-empty", so also $(*)_2$ holds.]

Claim 11.3. *Let $\bar{\mathbb{Q}} \in \mathfrak{K}_{\mathfrak{x}}, \mathfrak{x}$ a $\alpha(*)$-parameter.*
1) For $\alpha < \alpha(\mathfrak{x})$ and $\eta \in {}^{\omega>}\omega$ if $D^{\mathfrak{x}}_{\alpha,\eta}$ is a P-point, then $\Vdash_{\mathbb{P}^{\mathfrak{x}}_\alpha}$ "$\underset{\sim}{D}^{\mathfrak{x}}_{\alpha,\eta}$ is a P-point.
2) Moreover $\Vdash_{\mathbb{P}^{\mathfrak{x}}_\alpha}$ if $\underset{\sim}{A}_n \in \underset{\sim}{D}^x_{\alpha,n}$ for $n < \omega$ then we can find $v_n \in [A_n]^{f_m(n)}$ such that $\cup\{v_n : n < \omega\} \in D^{\mathfrak{x}}_{\alpha,n}$ if $D^{\mathfrak{x}}_{\alpha,n}$ satisfies this for y where

$(*)$ *$f, g, h, g \in {}^\omega(\omega \backslash \{0\}), h \leq g \leq f, \langle g(n) : n < \omega\rangle, \langle h(n) : n < \omega\rangle$ converge to infinity and $(\forall n)[f(n) \supseteq g(n), h(n)]$.*

Proof. Proof of 11.3
1) Let $p \Vdash$ "$\underset{\sim}{A}_n \in \underset{\sim}{D}^{\mathfrak{x}}_{\alpha,\eta}$" for $n < \omega$. Possibly increasing p without loss of generality each $(p, \underset{\sim}{A}_n)$ is as in **??**(2)(b) for each n. Hence by the proof of 11.2 there is $\mathscr{I}_n \in D^{\mathfrak{x}}_{\text{tr}(p)}$ such that $(p, \underset{\sim}{A}_r, \mathscr{I})$ is as in **??**(2)(c). Without loss of generality $\mathscr{I}_n \subseteq \mathscr{I}_{n+1}$ (see **??**) and there is $\mathscr{I} \in \mathscr{D}^{\mathfrak{x}}_{\text{tr}(p)}$ such that $n < \omega \Rightarrow I_n \leq_* I$ and let $(\mathbf{n}, \zeta)$ $\bar{u}$-witness $\mathscr{I}$ (for some $\bar{u}$). For each $q, p \leq_{\text{ap}} q$ let $B_{q,n} = \{k < \omega$: there is r such that $p \leq_{\text{pr}} q \wedge p \leq_{\text{apr}} q \wedge q \Vdash$ "$k \in \underset{\sim}{A}_n$"$\}$. For each $g \in \mathscr{I}$ let $\mathbf{m}(g) = \max\{m$: if $n \leq \mathbf{n}(g)$ and $n < m$ then $B_{q,n} \in D^{\mathfrak{x}}_{\alpha,\eta}\}$.

Clearly

$$p \Vdash \text{“} \begin{array}{l} \text{for every } m \text{ there is } g \in \text{pos}(tr(p)) \text{ such that} \\ \langle \underset{\sim}{\eta}_\beta : \beta \in \text{Dom}(p)\rangle \in \text{ real}(g) \text{ and} \\ g \leq_{\text{pr}} h \in \mathscr{I} \Rightarrow \mathbf{m}(g) \geq m\text{”}. \end{array}$$

Now we define $\underset{\sim}{A}$ a $\mathbb{P}$-name:

$$\underset{\sim}{A} = \{k < \omega : \begin{array}{l} \text{there is } g \in \mathscr{I} \text{ for which } \langle \underset{\sim}{\eta}_\beta : \beta \in \text{Dom}(p)\rangle \in \text{ real}(g) \\ \text{and there is } g \in \underset{\sim}{G}_{\mathbb{P}_{\alpha(\mathfrak{x})}} \text{ such that } \text{tr}(q) = g, \\ q \Vdash \text{“}k \in \cap\{A_\ell : \ell < boldm(g)\}\text{”}. \end{array}$$

Cearly $\underset{\sim}{A}$ is a $\mathbb{P}^{\mathfrak{x}}_{\alpha(\mathfrak{x})}$-name of a subset of ω and $(p, \mathscr{I})$ witness $\underset{\sim}{A} \in \underset{\sim}{D}^{\mathfrak{x}}_{\alpha,\eta}$ (after minor doctoring). $\square_{11.3}$

Definition 11.4. The $\alpha(*)$-parameter is called Ramsey if for any sequence $\langle(\alpha_\ell, \eta_\ell) : \ell < \omega\rangle$ of members of $\alpha(*) \times {}^{\omega>}\omega$ (possibly with repetitions) in the following game the ultrafilter player has no winning strategy: in the nth play,

the challenger player chooses $A_{n,\ell} \in D^{\mathfrak{x}}_{\alpha_\ell,\eta_\ell}$ for $\ell < n$ the chooser chooses $k_{n,\ell} \in A_{n,\ell}$ for $\ell < n$.

In the end the chooser wins if $\ell < \omega \Rightarrow \{k_{n,\ell} : n \leq \ell < \omega\} \in D_{\alpha_\ell,\eta_\ell}$.

Observation 11.5. *1) If $\mathfrak{x}$ is an $\alpha(*)$-parameter, D a Ramsey ultrafilter on ω and $(\alpha, \eta) \in \alpha(\mathfrak{x}) \times {}^{\omega>}\omega \Rightarrow D^{\mathfrak{x}}_{\alpha,\eta} = D$, then $\mathfrak{x}$ is a Ramsey ultrafilter.*
2) If we force $\langle \bar{D}^{\mathfrak{x}}_{\alpha,\eta} : \alpha < \alpha(\mathfrak{x}), \eta \in {}^{\omega>}\omega\}$ by countable approximations then it is Ramsey.

Claim 11.6. *1) If $\mathfrak{x}$ is a Ramsey $\alpha(*)$-parameter then the forcing notion $\mathbb{P}^{\mathfrak{x}}$ has the Laver property, i.e.*

Definition 11.7. A forcing notion $\mathbb{P}$ has the Laver property when: $f, g \in {}^\omega(\omega\backslash\{0\}), f \leq g$ and $\langle g(n) : n < \omega\rangle$ goes to infinity then P has the (f, g)-bounding property.

Proof. Proof of 11.6
1) Let f, g be as in **??** or **??** and assume $p \in \mathbb{P}^{\mathfrak{x}}, p \Vdash$ "$\underset{\sim}{\eta} \in \prod\limits_{n<\omega} f(n)$". Let $k_n = \text{Min}\{k : (\forall m < n)(k_m < k_n)$ and $m \geq n \Rightarrow g(n) \geq 2^n\}$. We can find $q, p \leq q$ such that $\underset{\sim}{\eta}$ be read purely above r whenever $q \leq_{\text{apr}} r$. For each $g \in \{\text{tr}(r) : q \leq_{\text{apr}} r\}$ let $\underset{\sim}{\eta}_g \in \prod\limits_{n<\omega} f(n)$ be such that for every n for some r, we have $q \leq_{\text{apr}} r \wedge \text{tr}(r) = g \wedge r \Vdash$ "$\underset{\sim}{\eta} \restriction n = \eta_g \restriction n$".

By claim [?, xxx] without loss of generality

(∗) if $\text{tr}(q) \leq_{\text{apr}} g, \beta \in \text{Dom}(q)$ and we let for $k < \omega, h_{g,\beta,k}$ be $g \restriction (\text{Dom}(q)\backslash\{\beta\}) \cup \{\langle\beta, g(\beta) \wedge < k >\rangle$ such that $\eta_g = \lim_{D_{\eta,\beta,g}} \langle \eta_{h_{g,\beta,k}} : k < \omega\rangle$.

Let $\langle(\alpha_\ell, \eta_\ell) : \ell < \omega\rangle$ list the pairs (α, η) such that $\alpha \in \text{Dom}(q), \text{tr}(q) \leq \eta_\ell$ and $\text{tr}(q, \alpha_\ell) \trianglelefteq \eta \triangleleft \eta_\ell \Rightarrow (\alpha_\ell, \eta) \in \{(\alpha_i, \eta_i) : i < \ell\}$.

We simulate a play of the game from **??** (or **??**) for $\langle(\alpha_\ell, \eta_\ell) : \ell < \omega\rangle$ such that the chosen player preserves

⊠ after $n(*)$ moves, for every $n < \omega$ the following set has at most $g(m)$ members

$$
\begin{aligned}
t_{n(*),m}\{\eta_g \restriction m : \quad & \text{tr}(q) \leq_{\text{apr}} g \text{ and} \\
& (\forall \beta \in \text{Dom}(q))(\forall i)(g(\beta) \neq \text{tr}(q(\beta)) \Rightarrow \\
& \wedge \ell g(\text{tr}(q)) \leq i < \ell g(g(\beta)) \Rightarrow \\
& (\exists \ell < n)[(\beta, g(\beta) \restriction i) = (\alpha_\ell, \eta_\ell) \wedge (g(\beta))(i) \in \{k_{n,\ell} : n < n(*)\}.
\end{aligned}
$$

By ⊛ there is no problem to carry this. Now define r such that:

⊙ $q \leq_{\text{pr}} r, q \leq_{\text{apr}} r$ if $\beta \in \text{Dom}(q), \text{tr}(q(\beta)) \trianglelefteq \eta \in {}^{\omega>}\omega$ then $r \restriction \beta \Vdash_{\mathbb{P}}$ "if $\eta \in T^{r(\beta)}$ and $(\beta, \eta) = (q_{\ell(*)}, \eta_{\ell(*)})$ then $\eta \in T^{q(\beta)}$ and

$$
\{k : \eta^\frown < k > \in T^{r(\beta)}\} = \{k : \quad \eta^\frown < k > \in T^{q(\beta)} \text{ and } k \in \{k_{n,\ell(*)} : \ell(*) < n < \omega\}\}.
$$

Clearly there is such r and $r \Vdash$ "$\underset{\sim}{\eta} \restriction m \in \bigcup\limits_m \{t_{n,m} : m < \omega\}$ and $\langle t_{n,m} : n < \omega\rangle$ is increasing hence with union $\leq g(m)$ members so we are done. $\square_{11.6}$

Remark 11.8. 1) This is enough to answer yes. Juday problem.

For CUN(ZFC + $\mathfrak{d}$ large and even $\mathfrak{b}$ large + BC) we need more.

2) So here $\Vdash_{\mathbb{P}^{\mathfrak{r}}_{\beta(*)}}$ "$\underset{\sim}{f} \in {}^{\omega}(\omega \backslash \{0\})$ in increasing with $f(n) > g(n) := n$".

* * *

An alternative is

Definition 11.9. 1) $\bar{\mathscr{J}}$ is a witness for $\mathscr{I} \in \mathrm{dsl}(f)$ which means that

- (a) $\bar{\mathscr{J}} = \langle \mathscr{J}_n : n < \omega\rangle$
- (b) $\mathscr{J}_n \in \mathrm{nac}(f)$, i.e., is a strong antichain of R_f, see below
- (c) $\mathscr{J}_{n+1}$ is above $\mathscr{J}_n$, i.e. $(\forall g_2 \in \mathscr{I}_{n+1})(\exists g_1 \in \mathscr{I}_n)(g_1 <_{\mathrm{apr}} g_2)$
- (d) for every $g \in \mathrm{ext}(f)$ for some (every large enough) n and every $g' \in \mathscr{J}$ is above g or is incompatible with g in $(\mathrm{ext}(f), \leq_{\mathrm{apr}})$
- (e) $\mathscr{J}_n \subseteq \mathscr{I}$ for $n < \omega$.

2) $\bar{\mathscr{J}}$ is a witness for $\mathscr{I} \in \mathrm{adst}(f)$ iff (a),(c) above and

- $(b)'$ $\mathscr{J}_n \in \mathrm{anac}(f)$, see below
- $(d)'$ like (d) using $\mathrm{aext}(f)$.

3) $\mathscr{J} \in \mathrm{ac}(f)$ when

- (a) $\mathscr{J}$ is an antichain of R_f
- (b) for every $\bar{\eta} \in \mathrm{apos}(f)$ there is $g \in \mathscr{J}$ such that $\bar{\eta} \in \mathrm{pos}(g)$ (by (a), g is unique).

4) We define $\mathscr{J} \in \mathrm{aac}(f)$ similarly using R'_f.
5) We say $\mathscr{J} \in \mathrm{nac}(f)$ is decisive for $g \in ...$?

Observation 11.10. *Let $f \in \mathscr{I}$.*
0) $\mathrm{dsf}(f) \subseteq \mathrm{adst}(f), \mathrm{ac}(f) \subseteq \mathrm{aac}(f)$.
1) If $\mathscr{I} \in \mathrm{anac}(f)$ and for each $g \in \mathscr{I}, J_g \in \mathrm{ac}(g)$ then $\mathscr{J} =: \cup\{\mathscr{J}_g : g \in \mathscr{I}\} \in \mathrm{acc}(f)$.
2) If $\mathscr{I} \in \mathrm{nac}(f)$ and for each $g \in \mathscr{I}$, $\mathscr{J} \in \mathrm{nac}(g)$ then $\mathscr{J} =: \cup\{\mathscr{J}_g : g \in \mathscr{I}\} \in \mathrm{nac}(f)$.
3) If $\mathscr{J} \in \mathrm{anac}(f)$ and $g \in \mathrm{aext}(f)$ then for some $g' \in \mathscr{J}$ in $(\mathrm{aext}(f), \leq_{\mathrm{apr}})$ we have g, g' are compatible.
4) Similarly to (d) for $\mathrm{nac}(f), \mathrm{ext}(f)$.

Claim 11.11. *1) $\mathscr{I} \in \mathrm{dst}(f)$ iff there is a witness $\bar{\mathscr{J}}$ for $\mathscr{I} \in \mathrm{dst}(f)$.*
2) $\mathscr{I} \in \mathrm{dst}(f)$ iff there is a witness $\bar{\mathscr{J}}$ for $\mathscr{I} \in \mathrm{ads}(f)$.

Proof. <u>the "if" direction</u>:

Let $\bar{\eta} \in \mathrm{pos}(f)$ and $g \in \mathrm{ext}(f)$. Choose n as in clause (d) of Definition 11.9(1). As $\mathscr{J}_n \in \mathrm{ac}(f)$, see Definition 11.9(3), necessarily there is $g' \in \mathscr{J}_n$ such that $\bar{\eta} \in \mathrm{pos}(g')$. As $\bar{\eta} \in \mathrm{pos}(g) \cap \mathrm{pos}(g')$ by 11.10(x), g, g' cannot be incompatible in $(\mathrm{ext}(f), \leq_{\mathrm{apr}}$, hence by the choice of n necessarily $g <_{\mathrm{apr}} g'$.

The "only if" direction:
First

$\boxtimes$ if $f^* \in \mathrm{ext}(f)$ then there is $\mathscr{J} \in \mathrm{nac}(f^*)$, $\mathscr{J} \subseteq \mathscr{J}$.

[Why? Let

$$Y_1 =: \{g \in \mathrm{ext}(f^*) : \text{ there is } \mathscr{J} \in \mathrm{nac}(g^*), \mathscr{J} \subseteq \mathscr{I}\}$$

$$Y_2 =: \{g \in Y_1 : \text{ there is no } g' \in Y_1 \text{ such that } f <_{\mathrm{apr}} g' <_{\mathrm{apr}} g\}$$

$\square$

12. Glossary

§1 Trunk Controllers

1.1 (Definition) Trunk controllers, standard (trunk control), $\aleph_1$-complete based, fully based, $\mathscr{F}^{[\beta]}$, trivial, transparent

1.2 (Definition) (A trunk controller $\mathscr{F}$ is) simple (= purely regressively $\aleph_2$-c.c. on S^2_1), semi-simple (= $\aleph_2$-c.c. for pure extensions), (semi) simply based (all are simple except the first is semi simple)

1.6 (Definition) $\mathscr{F}$-forcing

?? (Definition) (An $\mathscr{F}$-forcing $\mathbb{Q}$ is) clear (help put together extensions), basic, straight (help put together $p_1 \leq_{\rm apr} q_1, p_1 \leq_{\rm pr} p_2$, (used in 3.3(1)) and transparent

1.10 (Definition) $\mathscr{F}$-iteration

1.13 (Claim) $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$ is a $\mathscr{F}$-forcing

1.14 (Claim) in a $\mathscr{F}$-iteration, $\beta < \gamma \Rightarrow \mathbb{P}_\beta \lessdot \mathbb{P}_\gamma$ naturally

1.15 (Claim) Preservation of clear (+ variant) and straight

1.8 (Claim) Simplicity of $\mathscr{F}+$ clarity of the $\mathscr{F}$-forcing $\mathbb{Q}$, impure/pure $\aleph_2$-c.c. (+ variants)

1.19 (Claim) Existence of $\mathscr{F}$-iteration

1.20 (Claim) Associativity (of $\mathscr{F}$-iterations)

?? (Discussion)

3.7 (Definition) Pure decidability

§2 Being $\mathscr{F}$-pseudo c.c.c. ($\mathscr{F}$-psc) is preserved by $\mathscr{F}$-iterations

2.1 (Definition) $\mathbb{Q}$ is $\mathscr{F}$-psc, $(\mathscr{F},\mathscr{P})$-psc, clear, straight; consider $(\mathscr{F},\mathscr{P})$

2.5 (Definition) $\mathbb{Q}$ is $\mathscr{F}$-psc iteration as witnessed by $\bar{\mathbf{H}}$, is essentially ... (except Q_0), semi-simple $\mathscr{F}$-psc strong

2.6 (Definition) $\mathbb{Q}$ is strong $\mathscr{F}$, psc, $\bar{\mathbb{Q}}$ is strong

2.8 (Claim) Sufficient conditions for ps

?? (Definition) $\mathscr{F}$ is psc, strongly psc; continuous, Knaster, explicit, semi [??]

?? (Claim) Implications

?? (Claim) Basic fact on the explicit version

2.9 (Claim) $\mathscr{F}$-psc implies pure $(\infty,\aleph_1)$-decidability for $\aleph_0$ anmes; the strong version and the explicit version implies purely proper

2.10 (Remark) On "straight" and on stationary $\mathscr{S} \subseteq [\lambda]^{\aleph_0}$

2.11 (Remark) On $\aleph_1$-completeness

2.12 (Lemma) Preserving psc under $\mathscr{F}$-iteration

?? (Claim) Preserving the explicit version under $\mathscr{F}$-iteration

3.12 (Definition) $(\mathfrak{H}, <_{\mathfrak{H}})$ is a c.c.c. witness

?? (Remark)

5.19 (Observation) Collect the properties, not used.

§6 On a relative of Borel conjecture with large $\mathfrak{b}$

6.1 (Hypothesis)

6.2 (Definition) (1) We fix $\mathfrak{K}(0)$,the candidates for first forcing in the iterations as adding λ-randoms and define $A(\mathbb{Q}_0)$.
(2) $\mathfrak{K}_\alpha$ in this context.
(3) $\mathfrak{K}'_\alpha, \mathfrak{K}'_{\ell,\alpha}$; mainly $\bar{\mathbb{Q}} \restriction \beta \in \mathfrak{K}^+_\beta$ for $\beta < \alpha, \bar{\mathbb{Q}} \in \mathfrak{K}'_\alpha$ but for $\alpha \geq \lambda, \mathbb{Q}_0$ has all randoms and they look internally a Sierpinski such that if $\alpha < \lambda$ has $< \lambda$ randoms.
(4) $\leq_{\mathfrak{K}'_{\ell,\alpha}}, \leq_{\mathfrak{K}'_\alpha}$ (the first new random is $\mathscr{G}^{\mathbf{V}}$-continuous over the (smaller forcing).
(5) $\mathfrak{K}''_\alpha$ (mainly $\subseteq \mathfrak{K}^+_\alpha$).

6.5 (Observation) (1) The Sierpinski-ness of the randoms.
(2) Essentially $\mathscr{F}$-psc with 2-pure decidability over Q_0.
(3) Semi-simple + .

6.6 (Claim) Existence of extensions and appear bound of increasing sequences for $\mathfrak{K}'_{\leq\alpha}, \mathfrak{K}''_{\leq\alpha}$.

6.10 (Conclusion) We get $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$ with $\alpha < \lambda$, using λ or less of the randoms manipulating $\mathfrak{b}, \mathfrak{d}$, covering number for $\text{null}_{\mathscr{G}}$ (??)

6.12 (Claim) Similar to 6.10, for $\alpha = \lambda^+$.

§7 Continuing [**?**]

7.2 (Theorem) We find a forcing as in [**?**] replacing the null ideal by $\text{Null}_{\mathscr{G}}$ but with $\mathfrak{b} = \mathfrak{d}$ is quite small, e.g. in $\mathbf{V}^{\mathbb{P}}, \text{cov}(\text{Null}_{\mathscr{G}}) = \aleph_\omega, \mathfrak{b} = \mathfrak{d} = \aleph_2$.

7.3 (Remark) Connection to [**?**].

7.4 (Claim) Amalgamation in $\mathfrak{K}^+_\alpha$ or above an amalgamation in $\mathfrak{K}(0)$.

7.6 (Claim) The fact needed for the induction step in 5.19 putting two ultrafilters together over amalgamated forcings.

7.7 (Claim) The generic real for $\bar{\mathbb{Q}} \in \mathfrak{K}'_{\ell,\alpha}$ are enough.

§8 On "η is $\mathscr{L}$-big over M"

8.1 (Definition) (1) We define $\mathbf{T}$, the set of finitary trees $\subseteq {}^{\omega>}\mathscr{H}(\aleph_0)$.
(2) $\mathbf{R}_{\mathscr{T}_1,\mathscr{T}_2}$ is the set of closed subsets of $\text{Lim}(\mathscr{T}_2)$; also $\mathbf{R}_{\mathscr{T}}, \mathbf{R}_* \subseteq \lim(\mathscr{T}_1[R]) \times \lim(\mathscr{T}_2[R])$. [CHECK??]
(3) $R^{<k>}$.
(4) $\mathbf{Y}$ is a set of $(f, \mathscr{T}), D_Y$ for $\mathscr{Y} \subseteq \mathbf{Y}$.

8.2 (Definition) (1) D is $(f, \mathscr{T})$-narrow for $f \in \Pi\{\mathscr{P}(\mathscr{T}_n) : n < \omega\}$.
(2) D is $\mathscr{Y}$-narrow.
(3) $\mathbf{Z}_M$ set of (η, R) with $R \in M, \eta \in \lim(\mathscr{T}_2[R])$.
(4) D is (η, R)-big over M.
(5) D is $\mathscr{L}$-big over M.
(6) η is R-big or $\mathscr{L}$-big over M.

8.3 (Claim) Sufficient condition for extending the ultrafilters $D_1 \in \mathbf{V}_1$ to $D_2 \in \mathbf{V}_2$ which is $\mathscr{L}$-big, $\mathscr{Y}$-narrow over $\mathbf{V}_1$.

8.5 (Claim) Equivalent condition to $B \notin J_{\mathscr{Y}}$, the narrowness ideal for $\mathscr{Y} \subseteq \mathbf{Y}$.

8.6 (Claim) Limit of $\mathscr{Y}_\zeta$-narrow filters which is $\bigcup\limits_\zeta \mathscr{Y}_\zeta$-narrow.

8.7 (Definition) $(f, h, \mathscr{T})$-narrow (return??) [releveling $\mathscr{T}$]

8.9 (Definition) $\mathfrak{Z}_\mu$ set of $(\eta, \bar{R})$ (?) [return??]

8.10 (Claim) (1) Extending $\bar{D}$ to preserve such that $\eta \in \mathbf{V}_2$ is R-big on $\mathbf{V}_1[\eta_{\mathbb{Q}_{\bar{D}}}]$.
(2) Similarly for $\mathfrak{Z}$.

8.11 (Definition) (1) Tree of subsets for $\mathbf{T}_g$.
(2) More notation on trees.

8.12 (Claim) (1) $\{g\}$-continuous and R_g-bigness equivalent.
(2) Sufficient conditions for 8.3 for $\mathscr{Y} = \emptyset$.
(3) Similarly $\mathscr{Y} \neq \emptyset$ using 8.11(2).

****

4.4 $\mathbf{T}_g$

4.5 $T = \lim_D \langle T_n : n < \omega \rangle, \mathscr{G} \subseteq \mathscr{G}^{\mathbf{V}}, \mathscr{B} = \text{ms} - \lim_D \langle \mathscr{B}_n : n < \omega \rangle$

8.6 ρ is $\mathscr{G}$-continuous over N (for D), $\text{Null}_{\mathscr{G},D}$ **************************

Moved from pgs.5-10:

Move from pg. 34,35,36:

* * *

Moved from pg.8,9:

2) If $\mathscr{F}$ is a basic trunk controller, i.e., $\text{inter}^{\mathscr{F}}$ is well defined, then "$\mathbb{Q}$ is a basic $\mathscr{F}$-forcing" is defined similarly adding

- $(f)_1$ if $p_0 \le p_2$ then for some p_1 we have $p_0 \le_{\text{pr}} p_1 \le_{\text{apr}} p_2$ and $\text{val}(p_1) = \text{inter}_{\mathscr{F}}(\text{val}(p_0), \text{val}(p_2))$.

3) If $\mathscr{F}$ is a straight trunk controller, we say $\mathbb{Q}$ is a straight $\mathscr{F}$-forcing notion, (a)-(e) above and

- $(f)_2$ if $\mathbb{Q} \models p_0 \le_{\text{pr}} p_2, p_0 \le_{\text{apr}} p_1$ and p_1, p_2 are [10] compatible$^+$ which means $(\exists p_3)(p_1 \le_{\text{pr}} p \wedge p_2 \le_{\text{apr}} p_3)$, then for some p_3 we have
  - (α) $\mathbb{Q} \models p_1 \le_{\text{pr}} p_3 \wedge p_2 \le_{\text{apr}} p_3$
  - (β) if also $\mathbb{Q} \models p_0 \le_{\text{pr}} p_1$ then $\mathbb{Q} \models p_2 \le_{\text{pr}} p_3$
  - (γ) $\text{val}^{\mathbb{Q}}(p_3) = \text{amal}^{\mathscr{F}}(\text{val}^{\mathbb{Q}}(p_0), \text{val}^{\mathbb{Q}}(p_1), \text{val}^{\mathbb{Q}}(p_2))$
  - (δ) p_3 is a $\le_{\text{us}}$-lub of p_1, p_2 in $\mathbb{Q}$.

[10]this may seem unnatural but note that it is not satisfied by $\mathbb{Q}_{\bar{D}}$.

* * *

Moved from pg.11:
Clause (f): Assume $p_0 \le p_2$ and we shall define p_1. Let $\mathrm{Dom}(p_1) = \mathrm{Dom}(p_2)$ and let the finite $w \subseteq \mathrm{Dom}(f_1)$ be as in $(\gamma)(iii)$ of 1.10(b), and we choose $p_1(\alpha)$ for $\alpha \in \mathrm{Dom}(p_1)$ a follows. If $\alpha \in \mathrm{Dom}(p_2) \backslash \mathrm{Dom}(p_0)$ we let $p_1(\alpha) = p_2(\alpha)$, and if $\alpha \in \mathrm{Dom}(p_0)$ and $\alpha \notin w$ we let $p_1(\alpha) = p_2(\alpha)$. If $\alpha \in w$ by Definition 1.6(f), we know that $p_2 \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$p_0(\alpha) \le p_2(\alpha)$" hence $(\exists p)[p_0(\alpha) \le^{\mathbb{Q}_\alpha}_{\mathrm{pr}} p \le^{\mathbb{Q}_\alpha}_{\mathrm{apr}} p_2(\alpha) and \mathrm{val}^{\mathbb{Q}_\beta}(p) = \mathrm{inter}_{\mathscr{F}[\alpha]}(\mathrm{val}^{\mathbb{Q}_\alpha}(p_0(\alpha)), \mathrm{val}^{\mathbb{Q}_\alpha}(p_2(\alpha)))]$ and choose $p(\alpha)$ a $\mathbb{P}_\alpha$-name of such p. Now check.

* * *

Moved from pg.18:
1) [?] If $\mathbb{Q}$ is straight (see 1.6(3)) we can add in 2.9(1):

- (c) $\mathscr{F}_{q,\underset{\sim}{\tau}_n} = \{r : q \le_{\mathrm{apr}} r$ and r forces a value to $\underset{\sim}{\tau}_n\}$ is predense over q for each n; see §3. [Here?]

* * *

Moved from pg.4:

Definition 12.1. 1) Let $S \subseteq [\lambda]^{\aleph_0}$ be stationary. We say that $\mathbb{Q}$ is purely $(S, \mathscr{F})$-proper when if $N \prec (\mathscr{H}(\chi), \in)$ is countable, $N \cap \lambda \in S, \mathbb{Q}, \mathscr{F} \in N, p \in \mathbb{Q} \cap N$ then there is q such that:

- (a) $p \le_{\mathrm{pr}} q \in \mathbb{Q}$
- (b) q is $(N, \mathbb{Q})$-generic
- (c) $\mathrm{val}^{\mathbb{Q}}(q) = \mathrm{prop}_{\mathscr{F}}(p, N \cap S)$, as usual $\mathrm{prop}_{\mathscr{F}}$ is considered part of $\mathscr{F}$.

2) We omit S when $S = [\lambda]^{\aleph_0}$.

* * *

Moved from pg.5:

Definition 12.2. 1) A forcing notion $\mathbb{Q}$ has (θ, σ)-pure decidability if:
if $p \in \mathbb{Q}$ and $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in \theta$", then for some $A \subseteq \theta, |A| < \sigma$ and q we have $p \le_{\mathrm{pr}} q \in \mathbb{Q}$ and $q \Vdash$ "$\underset{\sim}{\tau} \in A$".
2) We write "θ-pure decidability" for "(θ, θ)-pure decidability".

* * *

Definition from Definition 2.1, moved from pg.6:
2) We say $\mathscr{F}$ is a straight? trunk controller if $\mathscr{F}$ is a trunk controller expanded by a three-place function $\mathrm{amal}^{\mathscr{F}}$ such that: $\mathrm{amal}^{\mathscr{F}}(f_0, f_1, f_2)$ is well defined when $f_0 \le_{\mathrm{apr}} f_1, f_0 \le_{\mathrm{pr}} f_2$ and $f_3 = \mathrm{amal}^{\mathscr{F}}(f_0, f_1, f_2)$ satisfies $f_1 \le_{\mathrm{pr}} f_0, f_2 \le_{\mathrm{apr}} f_3$ and $f_0 = f_1 \Rightarrow f = f_2$.

* * *

Moved from Definition 2.1, pg.15:
5) We say $\bar{\mathbb{Q}}$ is $\mathscr{F}$-straight if each $\underset{\sim}{\mathbb{Q}}_\beta$ is $\mathscr{F}^{[\beta]}$-straight.
6) We say $\bar{\mathbb{Q}}$ is semi-straight (or semi $\mathscr{F}$-straight) if each $\mathbb{Q}_{1+\beta}$ is $\mathscr{F}^{[\beta]}$-straight. [Used?]

Remark 12.3. Note that 3.3(1) speaks actually on any semi straight $\mathscr{F}$-psc forcing $\mathbb{P}$. See 3.3.

* * *

Moved from pg.22:

Claim 12.4. *Assume $\bar{\mathbb{Q}}$ is a [Knaster] explicit [semi] $\mathscr{F}$-psc iteration. Then $\mathrm{Lim}_{\mathscr{F}}(\bar{\mathbb{Q}})$ satisfies the [Knaster/semi] explicit $\mathscr{F}$-psc.*

Proof. Similar to 2.12. $\square$

Old proof of 3.3,pg.40:
1) We use 2.9(3) to get q and $\mathscr{I}^q_{\underset{\sim}{\tau}}$ for $\underset{\sim}{\tau} \in N$ a $\mathbb{P}_{\alpha^*}$-name of a member of $\mathbf{V}$. Now as $\mathscr{F}$ is straight for each $r \in \mathscr{I}^q_{\underset{\sim}{\tau}}$ we can choose a $r^+ \in \mathbb{Q}$, a lub of p_ε, r in $\mathbb{Q}$ and let $\mathscr{I} = \{r^+ : r \in \mathscr{I}^q_{\underset{\sim}{\tau}}\}$ is as required.
2) Essentially the same proof.

* * *

Moved from pg.42-43:
We can also generalize the preservation theorems.

Claim 12.5. *1) Assume*

(a) *$\bar{\mathbb{Q}}, \mathscr{F}$ are as in 3.3*
(b) *each $\underset{\sim}{\mathbb{Q}}_\beta$ is purely proper*
(c) *$(D, R, <)$ is a fine covering* [11] *model in the sense of* [?, Ch.VI,Definition 1.2]
(d) *$\Vdash_{\mathbb{P}_\beta}$ "$\mathbb{Q}_\beta$ is purely $(D, R, <)$-preserving".*

Then $\mathbb{P}_\beta$ is purely $(D, R, <)$-preserving and purely proper.
2) Similarly for $\bar{\mathbb{Q}}$ a Knaster explicitly semi $\mathscr{F}$-psc iteration.

Proof. See [?, Ch.VI,1.13A,p.270] or see AP here. Andrzej: in the end; why purely?
(a) as was in §3 is 3.3...relevant. $\square$

Moved from pg.52:
From 5.3 proof of clause (a):

$\mathbb{Q}_{\bar{D}}$ straight: So assume $p_0 \le_{\rm pr} p_2$ and $p_0 \le_{\rm apr} p_1$ and p_1, p_2 are compatible so $p = p_2^{[\mathrm{tr}(p_1)]}$ is as required.

$\mathbb{Q}_{\bar{D}}$ basic: If $p_0 \le p_2$ let $\mathrm{inter}(p_0, p_2) = \{\nu : \nu \in p_0$ and if $\mathrm{tr}(p_2) \trianglelefteq \nu$ then $\nu \in p_2\}$.

[11] can use also the weaker version there

Old remark to 5.6 or see 5.5,moved from pg.59:

Instead of $\bar{D} \in N \prec (\mathscr{H}(\chi), \in)$ it is enough to have assumptions like [?].

FILL or drop see [?], [?].

Moved from pgs.63,64,65 (from §5):

Remark 12.6. (was after 5.17)
1) The following is intended to help mainly in chain conditions, but at present not used. Alternatively, to 5.17, $\mathfrak{K}^+_\alpha$ is the family of $\bar{\mathbb{Q}}_1 \in \mathfrak{K}_\alpha$ such that if $\bar{\mathbb{Q}}_2 \in \mathfrak{K}_\alpha$ and $\bar{\mathbb{Q}}_1 \leq_{\mathfrak{K}} \bar{\mathbb{Q}}_2$, then for every $\mathfrak{p}_2 \in \mathrm{cr}(\bar{\mathbb{Q}}_2)$ (see Definition **??**, see below) there is $\mathfrak{p}_1 \in \mathrm{cr}(\bar{\mathbb{Q}}_1)$, strongly isomorphic to $\mathfrak{p}_2$ say as witnessed by the h such that

- $(*)$ if $\beta \in w[\mathfrak{p}] \cup \{\gamma[\mathfrak{p}]\} \backslash \{0\}$ is minimal and $r_1 \in \mathscr{P}_\beta[\mathfrak{p}]$ and $r_2 = h(r_1)$ then r_2, r_1, r has a common upper bound in $\mathrm{Lim}(\bar{\mathbb{Q}})$ [older are compatible in $\mathbb{Q}_0$] whenever $r_1 \restriction \{0\}, r_2 \restriction \{0\} \leq r \in \mathbb{P}_1$ older $r_1 \leq r \in \mathbb{Q}_0$.

Remark **??** use Definition **??** below (though anyhow we do not use it).

Definition 12.7. Let $\bar{\mathbb{Q}} \in \mathfrak{K}_\alpha$.
1) We define $\mathrm{cr}(\bar{\mathbb{Q}})$ as the family of objects $\mathfrak{p}$ consisting of:

- (a) $w[\mathfrak{p}] \in [\alpha]^{\leq \aleph_0}$ and let $\gamma(\mathfrak{p}) = \cup\{\beta + 1 : \beta \in w[\mathfrak{p}]\}$
- (b) $\mathrm{val}(\mathfrak{p}) \in \mathscr{F}, \mathrm{Dom}(\mathrm{val}(\mathfrak{p})) = w[\mathfrak{p}]$
- (c) for $\beta \in w[\mathfrak{p}] \cup \{\gamma[\mathfrak{p}]\} \backslash \{0\}$, a countable subset $\mathscr{P}_\beta[\mathfrak{p}]$ of $\{q : q \in \mathbb{P}_\beta, \mathrm{Dom}(q) = w(\mathfrak{p}) \cap \beta$ and $\mathscr{F} \models \mathrm{val}(\mathfrak{p}) \restriction q \leq^{\mathscr{F}}_{\mathrm{apr}} \mathrm{val}(q)\}$, so for the minimal such β we get a subset of $\mathbb{Q}_0$
- (d) for $\beta \in w[\mathfrak{p}] \cup \{\gamma[\mathfrak{p}]\}$, a countable family $\tau_\beta[\mathfrak{p}]$ of $\mathbb{P}_\beta$-names $\underset{\sim}{\tau}$ of a member of {true,false} and for each $\underset{\sim}{\tau} \in \tau_\beta[\mathfrak{p}]$ we have a set $\mathscr{I}_{\underset{\sim}{\tau}}[\mathfrak{p}] \subseteq \mathscr{P}_\beta[\mathfrak{p}]$ such that $q \in \mathscr{I}_{\underset{\sim}{\tau}}[\mathfrak{p}] \Rightarrow q$ forces a value to $\underset{\sim}{\tau}$
- (e) for $\beta \in w[\mathfrak{p}] \cup \{\gamma[\mathfrak{p}]\}$ and $\underset{\sim}{\tau} \in \tau_\beta[\mathfrak{p}]$ the set $\mathscr{I}_{\underset{\sim}{\tau}}[\mathfrak{p}]$ is a predense subset of $\{q \in \mathbb{P}_\beta : \mathrm{val}(q) \in \mathscr{P}_\beta[\mathfrak{p}]\}$
- (f) if $\beta \in w[\mathfrak{p}] \cup \{\gamma[\mathfrak{p}]\} \backslash \{0\}$ and $q \in \mathscr{P}_\beta[\mathfrak{p}]$ and $\gamma \in w[\mathfrak{p}] \cap \beta$ and $\eta \in {}^{\omega >}\omega$ then each "truth value($\eta \in q(\gamma)$)" belongs to $\tau_\gamma[\mathfrak{p}]$. [So $\mathfrak{p}$ involves a countable subset of $\mathbb{Q}_0$.]

2) We say $\mathfrak{p}_1, \mathfrak{p}_2$ (which are in $\mathrm{cr}(\bar{\mathbb{Q}})$, or more generally $\mathfrak{p}_\ell \in \mathrm{cr}(\bar{\mathbb{Q}}^\ell)$ for $\ell = 1, 2$ with the obvious changes) are strongly isomorphic as witnessed by the function h if:

- (a) $w[\mathfrak{p}_1] = w[\mathfrak{p}_2]$ and $\mathrm{val}(\mathfrak{p}_1) = \mathrm{val}(p_2)$
- (b) $h \restriction \mathscr{P}_\beta[\mathfrak{p}_1]$ is a 1-to-1 mapping from $\mathscr{P}_\beta[\mathfrak{p}_1]$ onto $\mathscr{P}_\beta[\mathfrak{p}_2]$
- (c) $h \restriction \tau_\beta[\mathfrak{p}]$ is a one-to-one mapping from $\tau_\beta[\mathfrak{p}_1]$ onto $\tau_\beta[\mathfrak{p}_2]$
- (d) for $q \in \mathscr{P}_\beta[\mathfrak{p}_1]$ we have $\mathrm{val}(h(q)) = \mathrm{val}(q)$
- (e) if $\underset{\sim}{\tau} \in \tau_\beta[\mathfrak{p}_1]$ then h maps $\mathscr{I}_{\underset{\sim}{\tau}}[\mathfrak{p}_1]$ onto $\mathscr{I}_{h(\underset{\sim}{\tau})}[\mathfrak{p}_2]$ that is $\mathscr{I}_{h(\underset{\sim}{\tau})}[\mathfrak{p}_2] = \{h(q) : q \in \mathscr{I}_{\underset{\sim}{\tau}}[\mathfrak{p}_1]\}$
- (f) if $\mathbf{t}$ is a truth value, $\underset{\sim}{\tau} \in \tau_\beta[\mathfrak{p}_1]$ and $q \in \mathscr{I}_{\underset{\sim}{\tau}}[\mathfrak{p}_1]$ and $q \Vdash$ "$\underset{\sim}{\tau} = \mathbf{t}$" then $h(q) \Vdash$ "$h(\underset{\sim}{\tau}) = \mathbf{t}$".

3) We omit the "strongly" if we replace in part (2) clause (a) by:

$(a)'$ h is an order preserving map from $w[\mathfrak{p}_1]$ onto $w[\mathfrak{p}_2]$ and such that $0 \in w[\mathfrak{p}_1] \Leftrightarrow 0 \in w[\mathfrak{p}_2]$.

Definition 12.8. We define "$\bar{\mathbb{Q}}$ is a semi $\mathscr{F}$-psc iteration as witnessed by $\bar{\mathbf{H}}$ similarly Definition 2.5 but

$(a)'$ $\mathscr{F}$ is a trunk control full iteration of length $\alpha^* \geq \ell g(\bar{\mathbb{Q}})$ but $\mathscr{F}$ is trivial, i.e. has one element (and still)

$(b)'$ $\bar{\mathbb{Q}}$ is a trunk control $\mathscr{F}$-iteration but $\mathbb{Q}_0$ is just a forcing i.e. $\leq^{\mathbb{Q}_0}_{\rm apr}$ is equality as $\leq^{\mathbb{Q}_0}_{\rm pr} = \leq^{\mathbb{Q}_0}_{\rm us}$

$(c)'_1$ for every $\beta < \ell g(\bar{\mathbb{Q}})$ but $\beta \neq 0$ we have

$$\Vdash_{\mathbb{P}_\beta} \text{" } \begin{array}{l} \underset{\sim}{\mathbb{Q}}_\beta \text{ is an } (\mathscr{F}^{[\beta]}, \mathbf{V}) - \text{psc forcing notion} \\ \text{as witnessed by } \mathbf{H}'_\beta \text{ and } \bar{\mathbf{H}} = \langle \mathbf{H}_\beta : \beta < \ell g(\bar{\mathbb{Q}}), \beta \neq 0 \rangle \\ \text{ is an object so } \mathbf{H}_\beta \text{ is not a } \mathbb{P}_\beta\text{-name} \end{array}$$

$(c)'_2$ $\mathbb{Q}_0$ is proper.

Claim 12.9. *1) If $\bar{\mathbb{Q}}$ is a semi $\mathscr{F}$-psc iteration then $\mathrm{Lim}(\bar{\mathbb{Q}})$ satisfies the conclusion of 2.9(1).*
2) $\mathrm{Lim}(\bar{\mathbb{Q}})$ is purely proper.
3) If $\bar{\mathbb{Q}}$ is strong (?? i.e. $\beta \in [1, \ell g(\bar{\mathbb{Q}}) = \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is a strong $\mathscr{F}^{[\beta]}$-psc", see 2.9(2). Saharon see §3.

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at