On $\triangleleft^*$-maximality

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Abstract

This paper investigates a connection between the semantic notion provided by the ordering $\triangleleft^*$ among theories in model theory and the syntactic (N)SOP$_n$ hierarchy of Shelah. It introduces two properties which are natural extensions of this hierarchy, called SOP$_2$ and SOP$_1$. It is shown here that SOP$_3$ implies SOP$_2$ implies SOP$_1$. In [Sh 500] it was shown that SOP$_3$ implies $\triangleleft^*$-maximality and we prove here that $\triangleleft^*$-maximality in a model of GCH implies a property called SOP$_2''$. It has been subsequently shown by Shelah and Usvyatsov that SOP$_2''$ and SOP$_2$ are equivalent, so obtaining an implication between $\triangleleft^*$-maximality and SOP$_2$. It is not known if SOP$_2$ and SOP$_3$ are equivalent.

Together with the known results about the connection between the (N)SOP$_n$ hierarchy and the existence of universal models in the absence of GCH, the paper provides a step toward the classification of unstable theories without the strict order property. [1]

Changes from the published version:

In the published version of this paper it is claimed that witnesses to being SOP$_1$ can be chosen to be highly indiscernible, and this is justified by a certain notion of 1-fbti. The definition of this notion (Definition 2.10) has a typo in a crucial place, and in addition Claim 2.11 for $t = 2$ is incorrect and for $t = 1$ the proof is incomplete. In this version we clarify these statements and proofs by introducting a new notion of indescernibility 3-fbti. The corrected statement is that witnesses to being SOP$_1$ can be chosen to be 3-fbti.

That there are inconsistencies in the notions we used in the original paper was first observed by Lynn Scowl (September 2008), Byunghan Kim (May 2009) and Enrique Casanovas and Martin Ziegler (July 2010). Whilst a Ph.D. student at UEA in 2008, Mark Wong also observed some incosistencies and made partial progress in rectifying them. There is a paper by Kim and Kim (to appear in APAL as of March 2011) which gives a different notion of 1-fbti and shows that witnesses can be chosen with that kind of indiscernibility.

0 Introduction

This paper investigates a connection between the ordering $\triangleleft^*$ among theories in model theory and the (N)SOP$_n$ hierarchy of Shelah and as such provides a step toward the classification of unstable theories without the strict order property. The thesis we pursue is that the syntactic property SOP$_2$ is closely

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related to the semantic property of being maximal in the $\triangleleft^*$-order. We shall now give the relevant definitions and explain the motivation behind the paper as well as noting our main results. For the purpose of this introductory discussion we shall limit ourselves to countable (complete first order) theories.

The following order among theories was introduced and investigated by Keisler in [Ke].

Definition 0.1 (1) For any cardinal λ, the Keisler order $\lessdot_\lambda$ among theories is defined as follows: $T_0 \lessdot_\lambda T_1$ if whenever $M_l (l < 2)$ is a model of T_0, T_1 respectively and $\mathcal{D}$ is a regular ultrafilter over λ, then the λ^+-compactness of $M_1^\lambda/\mathcal{D}$ implies the λ^+-compactness of $M_0^\lambda/\mathcal{D}$.
(2) We say $T_0 \lessdot T_1$ if for all λ we have $T_0 \lessdot_\lambda T_1$.

The relevance of this order to the project of classifying unstable theories without strict order property lies in the two following theorems of Shelah (note that the second one implies the first).

Theorem 0.2 (Shelah [Sh c], VI4.3) Any (countable) theory with the strict order property is $\lessdot$-maximal.

As stated in [Sh c], pg xiv, Ch VI of [Sh c] gives a rather complete picture of Keisler's order and to complete it we should know more about unstable theories without the strict order property. Paper [Sh 500] started a classification of such theories by introducing the hierarchy SOP_n for $n \geq 3$ and in particular it is stated there that being maximal in the Keisler order is not a characterisation of theories with the strict order property,

Theorem 0.3 (Shelah [Sh 500], see also [ShUs 844]) Any theory with SOP_3 is $\lessdot$-maximal.

Details of the proof are given in [ShUs 844]. Precise definitions of properties SOP_n for $n \geq 3$ will be repeated below in §2 but for the moment we note that it was proved in [Sh 500] that for $n \geq 3$

$$\text{strict order property} \implies SOP_{n+1} \implies SOP_n \implies \text{not simple}$$

and that all the implications are irreversible. One may now wonder if having SOP_3 is a characterisation of theories that are maximal in the Keisler order, giving us a semantic equivalent to the syntactic notion of SOP_3. This would be consistent with what is known about this order, see the Introduction to Ch VI of [Sh c]. This question remains open but instead one may attempt to give a characterisation of SOP_3 or SOP_n in terms of some other similarly defined order. This is suggested by [Sh 500] which in fact gives a theorem stronger than 0.3, namely

Theorem 0.4 (Shelah [Sh 500], see also [ShUs 844]) Any theory with SOP_3 is $\triangleleft^*$-maximal.

The definition of this order will be recalled in §1 where we shall also prove that being $\triangleleft^*$-maximal implies being maximal in the Keisler order. Given this fact one may now ask if being $\triangleleft^*$-maximal characterises theories with SOP_3. To test this claim it is natural to investigate a prototypical example of an NSOP_3 theory that is still not simple, which is T^*_{feq}. In §1 we shall recall the definition of this theory and show that in fact it is not $\triangleleft^*$-maximal, as it is consistently strictly below the theory of a dense linear order with no first or last element (all we need for the consistency is a partial GCH assumption).

This naturally leads to the question of the possibility of refining the distinction between simplicity and SOP_3. Definition of the SOP_n hierarchy from [Sh 500] does not immediately give way to such a refinement as SOP_n is roughly speaking, defined in terms of omitting loops of size n. However in §2 we introduce two properties SOP_2 and SOP_1 that in fact satisfy

$$\mathrm{SOP}_3 \implies \mathrm{SOP}_2 \implies \mathrm{SOP}_1 \implies \text{ not simple.}$$

We then ask if these properties in any way characterise the maximality in $\triangleleft^*$. To this end in §3 we prove that any theory that is $\triangleleft^*$-maximal in a model of a sufficient amount of GCH must satisfy a syntactic property SOP''_2. Together with a subsequent result of Shelah and Usvyatsov in [ShUs 844] that proved that SOP''_2 is equivalent to SOP_2 we hence obtain that $\triangleleft^*$-maximality in any model of a sufficiently rich fragment of GCH implies SOP_2. (See §3 for the definition of SOP''_2 and the exact reference from [ShUs 844]). To summarise, our main result, appearing as Corollary 3.9(1) below is

Theorem 0.5 Suppose that T is a theory that is $\triangleleft^*$-maximal in some universe of set theory in which $2^\lambda = \lambda^+$ holds for all large enough regular λ. Then T has SOP_2.

Several questions remain open. The main one of course is if SOP_2 is actually equivalent to $\triangleleft^*$-maximality. Recall from the discussion above that we know that SOP_3 implies $\triangleleft^*$-maximality. It is not known if SOP_3 and SOP_2 are actually equivalent. We also note that Shelah and Usvyatsov have proved in [ShUs 844] a local version of the implication $\mathrm{SOP}_2 \implies \triangleleft^*$ $-$maximality, see §3 for a more detailed discussion.

A burning question also is that we in fact do not know almost anything about the reverse of other implications in the (consistent) diagram

$$\mathrm{SOP}_3 \implies \triangleleft^* -\text{maximality} \implies \mathrm{SOP}_2 \implies \mathrm{SOP}_1 \implies \text{ not simple,}$$

apart that not all of them may be equivalences, as T^*_{feq} is not simple but is NSOP_3. In fact [ShUs 844] proves that T^*_{feq} is not even SOP_1.

Before laying down the organisation of the paper let us also mention the connection of the SOP_n hierarchy with another semantic property, which is the possibility of having a universal model at λ in some universe of set theory where a sufficient amount of GCH fails (under GCH every countable first order theory has a universal model in every uncountable cardinal). The connection between this property and unstable theories without the strict order property has been investigated in a series of papers, notably in [KjSh 409] where it is proved that if GCH fails sufficiently then there are no universal dense linear orders. It was proved in [Sh 500] that SOP_4 is already sufficient for such a negative universality result. The question of universality is interesting also for classes that are not elementary classes of models of a first order theory, for example for classes without amalgamation the most interesting case is the strong limit singular μ of cofinality $\aleph_0$. In [GrSh 174] it is proved that for such μ and $\lambda < \mu$ a strongly compact cardinal the class of models of any $L_{\lambda,\mu}$-theory of cardinality $< \mu$ admits a universal model of cardinality μ. A rather detailed description of what is known about the connection of unstable theories without the strict order property and the universality problem may be found in the introduction to [DjSh 710].

The paper is organised as follows. In the first section we investigate the theory $T^*_{\rm feq}$. This is simply the model completion of the theory of infinitely many parametrised equivalence relations. We show that under a partial GCH assumption, this theory is not maximal with respect to $\triangleleft^*_\lambda$, as it is strictly below the theory of a dense linear order. In the second section of the paper we extend Shelah's NSOP_n hierarchy by introducing two further properties SOP_1 and SOP_2, and we show that their names are justified by their position in the hierarchy. Namely $\mathrm{SOP}_3 \implies \mathrm{SOP}_2 \implies \mathrm{SOP}_1$. Furthermore, SOP_1 theories are not simple. The last section of the paper contains the main result showing that $\triangleleft^*$-maximality in a model of a sufficiently rich fragment of GCH implies SOP''_2, and hence SOP_2 by Shelah-Usvyatsov.

The following conventions will be used in the paper.

Convention 0.6 Unless specified otherwise, a "theory" stands for a first order complete theory. An unattributed T stands for a theory. We use $\tau(T)$ to denote the vocabulary of a theory T, and $\mathcal{L}(T)$ to denote the set of formulae of T.

By $\mathfrak{C} = \mathfrak{C}_T$ we denote a $\bar\kappa$-saturated model of T, for a large enough regular cardinal $\bar\kappa$ and we assume that any models of T that we mention are elementary submodels of $\mathfrak{C}$.

λ, μ, κ stand for infinite cardinals.

1 On the order $\triangleleft^*_\lambda$

Definition 1.1 (1) For (first order complete) theories T_0 and T we say that

$$\bar\varphi = \langle \varphi_R(\bar x_R) : R \text{ a predicate of } \tau(T_0) \text{ or a function symbol of } \tau(T_0) \text{ or } =\rangle,$$

(where we have $\bar x_R = (x_0, \ldots x_{n(R)-1})$), *interprets* T_0 in T, or that $\bar\varphi$ *is an interpretation of* T_0 in T, or that

$$T \vdash \text{“}\bar\varphi \text{ is a model of } T_0\text{”},$$

<u>if</u> each $\varphi_R(\bar x_R) \in \mathcal{L}(T)$, and for any $M \models T$, the model $M^{[\bar\varphi]}$ described below is a model of T_0. Here, $N = M^{[\bar\varphi]}$ is a $\tau(T_0)$ model, whose set of elements

is $\{a : M \models \varphi_=(a,a)\}$ (so $M^{[\bar\varphi]} \subseteq M$) and $R^N = \{\bar a : M \models \varphi_R[\bar a]\}$ for a predicate R of T_0.

For any function symbol f of $\tau(T_0)$ we have that $N \models$ "$f(\bar a) = b$" iff $M \models \varphi_f(\bar a, b)$, while

$$M \models \text{“}\varphi_f(\bar a, b) = \varphi_f(\bar a, c) \implies b = c\text{”}$$

for all $\bar a, b, c$.
(2) We say that the interpretation $\bar\varphi$ is *trivial* if $\varphi_R(\bar x_R) = R(\bar x_R)$ for all $R \in \tau(T_0)$, so $M^{[\bar\varphi]} = M \restriction \tau(T_0)$, for any model M of T.

(The last clause in Definition 1.1(1) shows that we can in fact restrict ourselves to vocabularies without function or constant symbols.)

We use the notion of interpretations to define a certain relation among theories. This relation was introduced by S. Shelah in [Sh 500], section §2 and one can see [ShUs 844] for a more detailed exposition. The reason we are interested in this ordering is Shelah's Theorem 0.3 quoted in the Introduction and we shall now start developing methods for the proof of our main result 3.9.

Definition 1.2 For (complete first order) theories T_0, T_1 we define:

(1) A triple $(T, \bar\varphi_0, \bar\varphi_1)$ is called a (T_0, T_1)-*superior* iff T is a theory and $\bar\varphi_l$ is an interpretation of T_l in T, for $l < 2$.

(2) For a cardinal κ, a (T_0, T_1)-superior $(T, \bar\varphi_0, \bar\varphi_1)$ is called κ-*relevant* iff $|T| < \kappa$.

(3) For regular cardinals λ, μ we say $T_0 \vartriangleleft^*_{\lambda,\mu} T_1$ if there is a $\min(\mu, \lambda)$-relevant (T_0, T_1)-superior triple $(T, \bar\varphi_0, \bar\varphi_1)$ such that in every model M of T in which $M^{[\bar\varphi_1]}$ is μ-saturated, the model $M^{[\bar\varphi_0]}$ is λ-saturated. If this happens, we call the triple a witness for $T_0 \vartriangleleft^*_{\lambda,\mu} T_1$.

(4) We say that $T_0 \vartriangleleft^*_{\lambda,\mu} T_1$ *over* θ if $\theta \le \lambda, \theta \le \mu$ and $T_0 \vartriangleleft^*_{\lambda,\mu} T_1$ as witnessed by a $(T, \bar\varphi_0, \bar\varphi_1)$ with $|T| < \theta$.

(5) If $\lambda = \mu$, we write $\vartriangleleft^*_\lambda$ in place of $\vartriangleleft^*_{\lambda,\mu}$.

(6) We say that $T_1 \lhd^* T_2$ iff $T_1 \lhd^*_\lambda T_2$ holds for all large enough regular λ.

(6) T^* is $\lhd^*_\lambda$-maximal iff $T \lhd^*_\lambda T^*$ holds for all T. The notion of $\lhd^*$-maximality is defined analogously.

(7) We say $T_0 \lhd^*_{\lambda,\neq} T_1$ iff $T_0 \lhd^*_\lambda T_1$ but $\neg(T_1 \lhd^*_\lambda T_0)$.

Although in this paper we do not consider this in its own right, it is natural to define the local versions of the $\lhd^*$-relation. This is used by Shelah and Usvyatsov in [ShUs 844] to obtain their local converse to the implication $\lhd^*$-maximality $\implies$ SOP_2, see §3 for more discussion on this.

Definition 1.3 Relations $\lhd^{*,l}_{\lambda,\mu}$ and $\lhd^{*,l}_{\lambda}$ are the local versions of $\lhd^*_{\lambda,\mu}$ and $\lhd^*_\lambda$ respectively, where by a local version we mean that in the definition of the relations, only types of the form

$$p \subseteq \{\pm\vartheta(x,\bar{a}) : \bar{a} \in {}^{\lg(\bar{y})}M\}$$

for some fixed $\vartheta(x,\bar{y})$ are considered.

Observation 1.4 (0) If $T_0 \lhd^*_{\lambda,\mu} T_1$ and $l < 2$, <u>then</u> there is a witness $(T,\bar{\varphi}^0,\bar{\varphi}^1)$ such that $\bar{\varphi}^l$ is trivial, hence $T_l \subseteq T$.
(1) $\lhd^*_\lambda$ is a partial order among theories (note that $T \lhd^*_\lambda T$ for every complete T of size $< \lambda$, and that the strict inequality is written as $T_1 \lhd^*_{\lambda,\neq} T_2$).
(2) If $T_0 \lhd^*_{\lambda,\mu} T_1$ over θ and $T_1 \lhd^*_{\mu,\kappa} T_2$ over θ, <u>then</u> $T_0 \lhd^*_{\lambda,\kappa} T_2$ over θ.

[Why? (0) Trivial.

(1) Suppose that $T_l \lhd^*_\lambda T_{l+1}$ for $l < 2$ over θ, as exemplified by $(T^*,\bar{\varphi}_0,\bar{\varphi}_1)$ and $(T^{**},\bar{\psi}_1,\bar{\psi}_2)$ respectively. Without loss of generality, $\bar{\varphi}_1$ is trivial (apply part (0)), so as T^* is complete we have $T_1 \subseteq T^*$. Similarly, without loss of generality, $\bar{\psi}_1$ is trivial and so, as T^{**} is complete, we have $T_1 \subseteq T^{**}$. As T_1 is complete, without loss of generality, T^* and T^{**} agree on the common part of their vocabularies, and hence by Robinson Consistency Criterion, $T \stackrel{\text{def}}{=} T^* \cup T^{**}$ is consistent. Also $|T^*| + |T^{**}| < \theta$, hence $|T| < \theta$. Clearly T interprets T_0, T_1, T_2 by $\bar{\varphi}_0$, $\bar{\varphi}_1 = \bar{\psi}_1$ and $\bar{\psi}_2$ respectively and T is complete. We now show that the triple $(T,\bar{\varphi}_0,\bar{\psi}_2)$ is a (T_0,T_2)-superior which witnesses $T_0 \lhd^*_\lambda T_2$ over θ. So suppose that M is a model of T in which

$M^{[\bar\psi_2]}$ is λ-saturated. As $(T^{**}, \bar\psi_1, \bar\psi_2)$ witnesses $T_1 \triangleleft^*_\lambda T_2$, we can conclude that $M^{[\bar\varphi_1]} = M^{[\bar\psi_1]}$ is λ-saturated. We can argue similarly that $M^{[\bar\varphi_0]}$ is λ-saturated.

(2) is proved similarly to (1).]

In this section we consider an example of a theory which is a prototypical example of an NSOP_3 theory that is not simple (see [Sh 457]). It is the model completion of the theory of infinitely many (independent) parametrised equivalence relations, formally defined below. We shall prove that for λ such that $\lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+$, this theory is strictly $\triangleleft^*_{\lambda^+}$-below the theory of a dense linear order with no first or last element.

Definition 1.5 (1) T_{feq} is the following theory in $\{P, Q, E, R, F\}$

(a) Predicates P and Q are unary and disjoint, and $(\forall x)\,[P(x) \vee Q(x)]$,

(b) E is an equivalence relation on Q,

(c) R is a binary relation on $Q \times P$ such that

$$[x\, R\, z \;\&\; y\, R\, z \;\&\; x\, E\, y] \implies x = y.$$

(so R picks for each $z \in P$ (at most one) representative of any E-equivalence class).

(d) F is a (partial) binary function from $Q \times P$ to Q, which satisfies

$$F(x, z) \in Q \;\&\; F(x, z)\, R\, z \;\&\; x\, E\, F(x, z).$$

(so for $x \in Q$ and $z \in P$, the function F picks the representative of the E-equivalence class of x which is in the relation R with z).

(2) T^+_{feq} is T_{feq} with the requirement that F is total.
(3) For $n < \omega$, we let T^n_{feq} be T^+_{feq} enriched by the sentence saying that over any n elements, any (not necessarily complete) quantifier free type consisting of basic (atomic and negations of the atomic) formulae with no direct contradictions, is realised.

Note 1.6 One may easily check that every model of T_{feq} can be extended to a model of T^+_{feq} and that T^+_{feq} has the amalgamation property and the joint embedding property. This theory also has a model completion, which can be constructed directly, and which we denote by T^*_{feq}. It follows that T^*_{feq} is a complete theory with infinite models, in which F is a full function.

Remark 1.7 Notice that T_{feq} has been defined somewhat differently than in [Sh 457, §1], but the difference is non-essential, as the following Claim 1.8 shows that the two theories have the same model completion. This claim also shows the origin of the name "infinitely many independent equivalence relations" for T^*_{feq}.

Claim 1.8 Let T be the theory defined (in [Sh457, 1]) by

(a) T has unary predicates P and Q and a three place relation E writen as $y\,E_x z$,

(b) the universe of any model of T is a disjoint union of P and Q,

(c) $y\,E_x z \implies P(x)\ \&\ Q(y), Q(z)$,

(d) for any fixed $x \in P$ the relation E_x is an equivalence relation on Q.

Then T^*_{feq} is the model completion of T.

Proof of the Claim. Let M be a model of T_{feq}, we shall extend M to a model of T as follows. Each E-equivalence class $e = a/E$ gives rise to an equivalence relation E_e on P given by:

$$z_1 E_e z_2 \text{ iff } z_1, z_2 \in P \text{ and } F(a, z_1) = F(a, z_2).$$

This definition does not depend on a, just on a/E. Let P^N and Q^N be Q^M and P^M respectively. Define $y\,E^N_x z$ iff $y\,E_e z$ where $e = x/E^M$. Clearly N is a model of T.

Now suppose that we have a model M of T and we shall extend it to a model N of T_{feq}. Let P^N and Q^N be Q^M and P^M respectively. Define $x\,E^N x'$ iff for every y, z we have $y\,E_x z$ iff $y\,E_{x'} z$. Choose a representative of each E-equivalence class and for any $z \in Q^N$ and such a representative x let

$F(x,z)=x$. Then for $x'\in Q^N$ which has not been chosen as a representative of any equivalence class, find x which has been chosen as its representative and define $F(x',z)=F(x,z)$ for all $z\in P^N$.

This shows that $T_{\rm feq}$ and T are cotheories ([ChKe], 3.5.6(2)). Being the model completion of $T_{\rm feq}$, $T^*_{\rm feq}$ is its cotheory, and hence a cotheory of T. Hence $T^*_{\rm feq}$ is a model companion of T. In order to prove that it is the model completion of T it suffices to show that T has the amalgamation property ([ChKe], 3.5.18) which is easily seen directly. $\bigstar_{1.8}$

Observation 1.9 $T^*_{\rm feq}$ has elimination of quantifiers and for any n, any model of $T^*_{\rm feq}$ is a model of $T^n_{\rm feq}$.

Notation 1.10 $T_{\rm ord}$ stands for the theory of a dense linear order with no first or last element.

The following convention will make the notation used in this section simpler.

Convention 1.11 Whenever considering $(T_{\rm ord},T^*_{\rm feq})$-superiors $(T,\bar\varphi,\bar\psi)$ we shall abuse the notation and assume $\bar\varphi=(I,<_0)$ and $\bar\psi=(P,Q,E,R,F)$. In such a case we may also write P^M in place of $P^{M[\bar\psi]}$ etc., and we may simply say that T is a $(T_{\rm ord},T^*_{\rm feq})$-superior.

We intend to prove that for λ satisfying $\lambda^{<\lambda}$ and $2^\lambda=\lambda^+$ the theory $T^*_{\rm feq}$ is strictly $\lhd^*_{\lambda^+}$-below $T_{\rm ord}$ (Theorem 1.17 below). This will be done by a diagonalisation argument where for a given λ-relevant $(T_{\rm ord},T^*_{\rm feq})$-superior T we inductively construct a model of T that is saturated for $T^*_{\rm feq}$ but not for $T_{\rm ord}$. Main Claim 1.13 provides one step in the required induction. In Stage A of its proof we use the elimination of quantifiers in $T^*_{\rm feq}$ to reduce the situation to $T_{\rm feq}$-types of four prescribed kinds, and then we show that we may in fact work only with three of them. Stage B contains the main point of the proof, which is the construction of a certain tree of models and embeddings. Once this is done in Stage C we use the analysis from Stage A to show that the $T^*_{\rm feq}$-type defined by the union of the embeddings is consistent. In Stage D we take $N\prec\mathfrak{C}$ of size λ that realises this type and show that

N must omit most of the Dedekind cuts induced by the tree of embeddings, and that most of these cuts are not definable over N. After an application of an appropriate automorphism of $\mathfrak{C}$ this finishes the proof of the Main Claim. The proof of the theorem then follows by induction. The cardinal arithmetic assumptions are used in Stage D and in the inductive proof of the theorem.

Definition 1.12 For a λ-relevant $(T_{\rm ord}, T^*_{\rm feq})$- superior T, *the statement*

$$*[M, \bar{a}, \bar{b}] = *[M, \bar{a}, \bar{b}, T, \lambda]$$

means:

(i) M is a model of T of size λ,

(ii) $\bar{a} = \langle a_i : i < \lambda\rangle$, $\bar{b} = \langle b_i : i < \lambda\rangle$, are sequences of elements of $I^{M^{[\bar{\varphi}]}}$ such that

$$i < j < \lambda \implies a_i <_0 a_j <_0 b_j <_0 b_i,$$

(iii) there is no $x \in M^{[\bar{\varphi}]}$ such that for all i we have $a_i <_0 x <_0 b_i$,

(iv) the Dedekind cut $\{x : \bigvee_{i<\lambda} x <_0 a_i\}$ is not definable by any formula of $\mathcal{L}(M)$ with parameters in M.

Main Claim 1.13 Assume $\lambda^{<\lambda} = \lambda$ and $(T, \bar{\varphi}, \bar{\psi})$ is a λ-relevant $(T_{\rm ord}, T^*_{\rm feq})$-superior. Further assume that $*[M, \bar{a}, \bar{b}]$ holds, and $p = p(z)$ is a (consistent) $T^*_{\rm feq}$-type over $M^{[\bar{\psi}]}$. Then there is $N \models T$ with $M \prec N$, such that $p(z)$ is realised in $N^{[\bar{\psi}]}$ and $*[N, \bar{a}, \bar{b}]$ holds.

Proof of the Main Claim.

Stage A. Without loss of generality, p is complete in the $T^*_{\rm feq}$-language over $M^{[\bar{\psi}]}$. (By Convention 1.11, we can consider p to be a type over M (rather than $M^{[\bar{\psi}]}$). We shall use this Convention throughout the proof). If p is realised in M, our conclusion follows by taking $N = M$, so let us assume that this is not the case. Using the elimination of quantifiers for $T^*_{\rm feq}$, we can without loss of generality assume that $p(z)$ consists of quantifier free formulae with parameters in M. This means that one of the following four cases must happen:

Case 1. (This will be the main case) $p(z)$ implies that $z \in P$ and it determines which elements of Q^M are R-connected to z. Hence for some function $f : Q^M \to Q^M$ which respects E, i.e.

$$a\,E\,b \implies f(a) = f(b),$$

and

$$f(a) \in a/E^M;$$

we have

$$p(z) = \{P(z)\} \cup \{\, b\,R\,z : \, b \in \mathrm{Rang}(f)\}$$

and no $a \in P^M$ satisfies p.

Case 1A. Like Case 1, but f is a partial function and

$$\begin{aligned} p(z) = \{P(z)\} \cup \{\, f(b)\,R\,z : \, b \in \mathrm{Dom}(f)\} \\ \cup \{\, \neg(b\,R\,z) : \, (b/E^M \cap \mathrm{Rang}(f)) = \emptyset\}. \end{aligned}$$

(This Case will be reduced to Cases 1-3 in Subclaim 1.15).

Case 2. $p(z)$ determines that $z \in Q$ and that it is E-equivalent to some $a^* \in Q^M$, but not equal to any "old" element. Note that in this case if b^* realises $p(z)$, we cannot have $b^* Rc$ for any $c \in P^M$, as this would imply $F(a^*, c) = b^* \in M^{[\bar{\psi}]}$ (and we have assumed that p is not realised in $M^{[\bar{\psi}]}$). Hence, the complete M-information is given by

$$p(z) = \{Q(z)\} \cup \{a^*\,E\,z\} \cup \{a \neq z : \, a \in a^*/E^M\}.$$

Case 3. $p(z)$ determines that $z \in Q$, but it has a different E-equivalence class than any of the elements of Q^M. As p is complete, it must determine for which $c \in P^M$ we have $z\,R\,c$, and for which $c, d \in P^M$ we have $F(z, c) = F(z, d)$. Hence, for some $f : P^M \to \{\text{yes, no}\}$ and some $\mathcal{E}$, an equivalence relation on P^M such that $c\mathcal{E}d \implies f(c) = f(d)$, we have

$$\begin{aligned} p = \{Q(z)\} \cup \{\neg(a\ E\ z) : \, a \in Q^M\} \cup \{(z\ R\ b)^{f(b)} : \, b \in P^M\} \\ \cup \{(F(z, c) = F(z, d))^{\mathrm{if} c\mathcal{E}d} : \, c, d \in P^M\}. \end{aligned}$$

In the above description, we have used

Notation 1.14 For a formula ϑ we let $\vartheta^{\mathrm{yes}} \equiv \vartheta$ and $\vartheta^{\mathrm{no}} \equiv \neg\vartheta$.

Subclaim 1.15 It suffices to deal with Cases 1,2,3, ignoring the Case 1A.

Proof of the Subclaim. Suppose that we are in the Case 1A. Let

$$\{d_i/E^M : i < i^* \leq \lambda\}$$

list the d/E^M for $d \in Q^M$ such that $d' \in d/E^M \implies \neg(d' \, R \, z) \in p(z)$. We choose by induction on $i \leq i^*$ a pair (M_i, c_i) such that

(a) $M_0 = M$, $||M_i|| = \lambda$,

(b) $\langle M_i : i \leq i^* \rangle$ is an increasing continuous elementary chain,

(c) $*[M_i, \bar{a}, \bar{b}]$

(d) $c_i \in (d_i/E^{M_{i+1}}) \setminus M_i$, for $i < i^*$.

For i limit or $i = 0$, the choice is trivial. For the situation when i is a successor, we use Case 2.

Let $\langle c_i/E^{M_{i^*}} : i \in [i^*, i^{**}) \rangle$ list without repetitions the $c/E^{M_{i^*}}$ which are disjoint to M. Note that $|i^{**}| \leq \lambda$. Let

$$p^+(z) \stackrel{\text{def}}{=} p(z) \cup \{c_i \, R \, z : i < i^{**}\}.$$

Then $p^+(z)$ is a complete type (for $M_{i^*}^{[\bar{\psi}]}$), and $*[M_{i^*}, \bar{a}, \bar{b}]$ holds by (c). If $p^+(z)$ is realised in M_{i^*}, we can let $N = M_{i^*}$ and we are done. Otherwise, $p^+(z)$ is not realised in M_{i^*} and is a type of the form required in Case 1, so we can proceed to deal with it using the assumptions on Case 1. $\bigstar_{1.15}$

Stage B. Let us assume that p is a type as in one of the Cases 1,2 or 3, which we can do by Subclaim 1.15. We shall define $\langle M_\alpha : \alpha < \lambda \rangle$, an $\prec$-increasing continuous sequence of elementary submodels of M, each of size $< \lambda$, and with union M, such that:

(a) In Case 1, each M_α is closed under f,

(b) In Case 2, $a^* \in M_0$,

(c) For every $\alpha < \lambda$,

$$(M_\alpha, \{(a_j, b_j) : j < \lambda\} \cap M_\alpha) \prec (M, \{(a_j, b_j) : j < \lambda\}),$$

Hence, for some club C of λ consisting of limit ordinals δ, we have that for all $\delta \in C$,

$$a_j \in M_\delta \iff b_j \in M_\delta \iff j < \delta,$$
$$(\forall c \in I^{M_\delta})(\exists j < \delta)\,[c <_0 a_j \vee b_j <_0 c].$$

Let $C = \{\delta_i : i < \lambda\}$ be an increasing continuous enumeration.

Now we come to the **main point** of the proof.

By induction on $i = lg(\eta) < \lambda$ we shall choose $\bar{h} = \langle h_\eta : \eta \in {}^{\lambda>}2\rangle$, a sequence such that

(α) h_η is an elementary embedding of $M_{\delta_{lg(\eta)}}$ into $\mathfrak{C}_T$, whose range will be denoted by N_η.

(β) $\nu \triangleleft \eta \implies h_\nu \subseteq h_\eta$.

(γ) If $\eta_l \in {}^{\lambda>}2$ for $l = 0, 1$ and $\eta_0 \cap \eta_1 = \eta$, <u>then</u>:

- (i) $N_{\eta_0} \cap N_{\eta_1} = N_\eta$.
- (ii) In addition, if $a_l \in Q^{N_{\eta_l}}$ for $l = 0, 1$ and $a_0\, E^{\mathfrak{C}_T} a_1$, <u>then</u> for some $a \in Q^{N_\eta}$ we have $a_l E^{\mathfrak{C}_T} a$ for $l = 0, 1$. (Equivalently, if $a_l \in Q^{N_{\eta_l}}$ and $\neg(\exists a \in N_\eta)(\wedge_{l<2} a_l\, E\, a)$, then $\neg(a_0\, E\, a_1)$).

(δ) $\models$ "$h_{\eta\frown\langle 0\rangle}(b_{\delta_{lg(\eta)}}) <_0 h_{\eta\frown\langle 1\rangle}(a_{\delta_{lg(\eta)}})$" (see Convention 1.11 on $<_0$).

Note that the requirement of h_η being elementary and onto N_η in particular implies that

(δ') If for some $l < 2$ and $\eta \in {}^{\lambda>}2$ we have $a \in N_{\eta\frown\langle l\rangle} \setminus N_\eta$ and $b \in N_\eta$, <u>then</u> $aE^{\mathfrak{C}_T}b$ iff $a = h_{\eta\frown\langle l\rangle}(a')$ for some a' such that $a'E^{\mathfrak{C}_T}h_\eta^{-1}(b)$.

We now describe the inductive choice of h_η for $\eta \in {}^{\lambda>}2$, the induction being on $i = lg(\eta)$. Let $h_{\langle\rangle} = \mathrm{id}_{M_0}$. If i is a limit ordinal, we just let $h_\eta \stackrel{\text{def}}{=} \bigcup_{j<lg(\eta)} h_{\eta\restriction j}$. Hence, the point is to handle the successor case.

Fixing $i < \lambda$, let $\langle \eta_{i,\alpha} : \alpha < \alpha^* \leq \lambda\rangle$ list ${}^{i+1}2$, in such a manner that $\eta_{i,2\alpha} \restriction i = \eta_{i,2\alpha+1} \restriction i$ and $\eta_{i,2\alpha+l}(i) = l$ for $l < 2$ (we are using the assumption $\lambda^{<\lambda} = \lambda$). Now we choose $h_{\eta_{i,2\alpha+l}}$ by induction on α. Hence, coming to α, let us denote by η_l the sequence $\eta_{i,2\alpha+l}$, and let $\eta_0 \cap \eta_1 = \eta$ (so $\eta_0 \restriction i = \eta_1 \restriction i = \eta$). Let $M_{\delta_{i+1}} \setminus M_{\delta_i} = \{d^i_j : j < j^*_i\}$, so that $d^i_0 = a_{\delta_i}$ and $d^i_1 = b_{\delta_i}$. We consider the type Γ, which is the union of

(a)

$$\Gamma_0 \stackrel{\text{def}}{=} \left\{ \begin{array}{r} \varphi(x^0_{j_0}, \ldots, x^0_{j_{n-1}}; h_\eta(\bar{c})) : n < \omega \ \&\ \bar{c} \subseteq M_{\delta_i} \ \&\ j_0, \ldots j_{n-1} < j^*_i \ \& \\ M_{\delta_{i+1}} \models \varphi(d^i_{j_0}, \ldots, d^i_{j_{n-1}}; \bar{c}) \end{array} \right\},$$

(taking care of one "side" (for η_0 or η_1) of the part (α) above)

(b) Γ_1, defined analogously to Γ_0, but with $x^0_{j_0}, \ldots, x^0_{j_{n-1}}$ replaced everywhere by $x^1_{j_0}, \ldots, x^1_{j_{n-1}}$,

(taking care of the remaining "side" of (α) above, interchanging η_0 and η_1)

(c) $\Gamma_2 = \{(x^0_0, x^0_1)_I \cap (x^1_0, x^1_1)_I = \emptyset\}$,

(this says that the above intervals in $<_0$ are disjoint, which after the right choice of $h_{\eta_0}(d^i_j) =$ a realisation of x^0_j or $h_{\eta_0}(d^i_j) =$ a realisation of x^1_j $(j < 2)$, and similarly for h_{η_1}, will take care of the part (δ) above.)

(d) $\Gamma_3 = \Gamma^0_3 \cup \Gamma^1_3$, where for $l < 2$

$$\Gamma^l_3 = \left\{ x^l_j \neq c : \ j < j^*_i, c \in \cup\{\mathrm{Rang}(h_\rho) : \ h_\rho \text{ already defined}\} \right\}.$$

(e)

$$\Gamma_4 = \{x^0_{j_1} \neq x^1_{j_2} : \ j_0, j_1 < j^*_i\}$$

((d)+ (e) are taking care of (γ) above, part (i)).

(f)

$$\Gamma_5 = \left\{ \begin{array}{l} \neg(x^0_{j_0} E x^1_{j_1}) : \text{if } j_0, j_1 < j^*_i \\ \qquad\qquad\qquad \text{but there is no } a \in M_{\delta_i} \text{ with } [d^i_{j_0} Ea \ \ \vee \ \ d^i_{j_1} Ea] \end{array} \right\}.$$

(together with Γ_6 below, taking care of part (γ)(ii), see below. Note that the type is defined using $\vee$ rather than $\wedge$, but this will turn out to be sufficient.)

(g) $\Gamma_6 = \Gamma^0_6 \cup \Gamma^1_6$, where

$$\Gamma^l_6 = \left\{ \begin{array}{l} \neg(x^l_j Eb) : j < j^*_i \text{ and } b \text{ is an element of the set} \\ \qquad\qquad \cup \{\mathrm{Rang}(h_\rho) : \ h_\rho \text{ already defined and } \neg(\exists c \in N_\eta)[b\, E\, c]\} \end{array} \right\}.$$

First note that requiring $\Gamma_5 \cup \Gamma_6$ throughout the construction indeed guarantees that (γ)(ii) can be satisfied. Namely, suppose that the realisations of $x^0_{j_0}$ and $x^1_{j_1}$ are E-equivalent. Then by Γ_5 we must have that for some $l < 2$ and $a \in M_{\delta_i}$ we have that $d^i j_l\, Ea$. By transitivity then the realisation of $x^{1-l}_{j_{1-l}}$ would have to be E-equivalent to $h_\eta(a)$, which might only be precluded by $d^i_{j_{1-l}}$ being E equivalent to some c such that a and c are not E-equivalent. This cannot happen by Γ_6.

We conclude that, if Γ is consistent, as $\mathfrak{C}$ is $\bar{\kappa}$-saturated, the functions h_{η_l} can be defined. Namely, for a realisation $\{c^l_j : j < j^*_i, l < 2\}$ of Γ, we can define $g_l(d^i_j) = c^l_j$, and then we let $h_{\eta_0} = g_0$ if $c^0_1 <_0 c^1_0$, and g_1 otherwise. We let $h_{\eta_1} = g_{1-l}$ if $h_{\eta_0} = g_l$. This guarantees that (δ) above is satisfied.

Let us then show that Γ is consistent. Suppose for contradiction that this is not so, so we can find a finite $\Gamma' \subseteq \Gamma$ which is inconsistent. Let $\{j_0, \dots, j_{n-1}\}$ be an increasing enumeration of a set including all $j < j^*_i$ such that x^l_j is mentioned in Γ' for some $l < 2$ and let $\bar{d} = \langle d^i_{j_0}, \dots d^i_{j_{n-1}} \rangle$. Without loss of generality, 0 and 1 appear in the list $\{j_0, \dots, j_{n-1}\}$ and hence $j_0 = 0$ while $j_1 = 1$. By closing under conjunctions and increasing Γ' (retaining that $\Gamma' \subseteq \Gamma$ is finite) if necessary, we may assume that for some formula $\sigma(x_0, \dots x_{n-1}; \bar{c}) \in \mathrm{tp}(\bar{d}/M_{\delta_i})$, we have

$$\Gamma' \cap \Gamma_l = \{\vartheta_l(x^l_{j_0}, \dots x^l_{j_{n-1}}; h_\eta(\bar{c}))\}$$

for $l < 2$, where $\vartheta_l(x^l_{j_0}, \dots x^l_{j_{n-1}}; h_\eta(\bar{c}))$ is the formula obtained from σ by replacing x_k by $x^l_{j_k}$ and $\bar{c}$ by $h_\eta(\bar{c})$.

Let ϑ_2 be the formula comprising Γ_2 and $\vartheta^l_3(\bar{x}^l; \bar{c}^l_3) = \bigwedge(\Gamma^l_3 \cap \Gamma')$, while $\vartheta_4 = \bigwedge(\Gamma_4 \cap \Gamma')$ and $\vartheta_5 = \bigwedge(\Gamma_5 \cap \Gamma')$. Let $\vartheta_3 = \vartheta^0_3 \wedge \vartheta^1_3$ and $\vartheta = \bigwedge_{k<6, k\neq 2,3} \vartheta_k$. Without loss of generality, ϑ includes statements $x^l_0 \neq \dots \neq x^l_{n-1}$ and $x^l_0 <_0 x^l_1$ for $l < 2$. We may also assume that $(x^0_0, x^0_1)_I \cap (x^1_0, x^1_1)_I = \emptyset$ is included in Γ'. The choice of n may be assumed to have been such that for some $c^l_0, \dots c^l_{n-1}$ (for $l < 2$) from $\bigcup\{\mathrm{Rang}(h_\rho) : h_\rho \text{ already defined}\}$, we have

$$\Gamma' \cap \Gamma_3 = \{x^l_{j_m} \neq c^l_k : l < 2, k < n, m < n\},$$

and finally that

$$\Gamma' \cap \Gamma_5 = \{\neg(x^0_{j_k} E x^1_{j_m}) : k, m < n \,\&\, \neg(\exists a \in M_{\delta_i})[d^i_{j_k} Ea \;\vee\; d^i_{j_m} Ea]\}.$$

By extending h_η to an automorphism $\hat{h}_\eta$ of $\mathfrak{C}$, and applying $(\hat{h}_\eta)^{-1}$, we may assume that $h_\eta = \mathrm{id}_{M_{\delta_i}}$. We can also assume that no c^l_k is an element of M_{δ_i}, as otherwise the relevant inequalities can be absorbed by σ.

We shall use the following general

Fact 1.16 Suppose that $N \prec \mathfrak{C}$ and $\bar{e} \in {}^m\mathfrak{C}$ is disjoint from N, while $N \subseteq A$.

Then

$$r(\bar{x}) \stackrel{\mathrm{def}}{=} \mathrm{tp}(\bar{e}, N) \cup \{x_k \neq d : d \in A \setminus N, k < m\} \cup \{\neg(x_k E d) : d \in A \ \&\ (d/E) \cap N = \emptyset, k < m\},$$

is consistent (and in fact, every finite subset of it is realised in N).

Proof of the Fact. Otherwise, there is a finite $r'(\bar{x}) \subseteq r(\bar{x})$ which is inconsistent. Without loss of generality, $r'(\bar{x})$ is the union of sets of the following form (we have a representative type of the sets for each clause)

- $\{\varrho(\bar{x}, \bar{c})\}$ for some $\bar{c} \subseteq N$ and ϱ such that $\models \varrho[\bar{e}, \bar{c}]$.
- $\{x_k \neq \hat{c}^s_k : k < m\}$ for some $\hat{c}^s_0, \ldots \hat{c}^s_{m-1} \in A \setminus N$ and $s < s^* < \omega$,
- $\{\neg(x_k \, E \, \hat{d}^t_k) : k < m\}$ for some $\hat{d}^t_0, \ldots \hat{d}^t_{m-1} \in A \setminus N$ and $t < t^* < \omega$ such that $(\hat{d}^t_k/E) \cap N = \emptyset$.

By the elementarity of N, there is $\bar{e}' \in N$ with $N \models \varrho[\bar{e}', \bar{c}]$. By the choice of the rest of the formulae in $\bar{r}'(\bar{x})$, we see that $\bar{e}'$ satisfies them as well, which is a contradiction. $\bigstar_{1.16}$

Let $\bar{x}^l = (x^l_0, \ldots, x^l_{n-1})$ for $l < 2$. Let

$$\Phi_0 \stackrel{\mathrm{def}}{=} \{\varphi(\bar{x}^0) : \varphi(x^0_{j_0}, \ldots, x^0_{j_{n-1}}) \in \Gamma' \cap (\Gamma_0 \cup \Gamma^0_3 \cup \Gamma^0_6)\}.$$

Applying the last phrase in the above Fact to $\Phi_0(\bar{x}^0)$, the model M_{δ_i} and $\bar{d}$, we obtain a sequence $\bar{e}^0 = (e^0_0, \ldots e^0_{n-1}) \in M_{\delta_i}$ which realises $\Phi_0(\bar{x}^0)$. For k, m such that $\neg(x^0_{j_k} E x^1_{j_m}) \in \Gamma_5$ we have $\neg(\exists a \in M_{\delta_i})(aEd^i_{j_k} \vee aEd^i_{j_m})$. So

$$\neg(x_k E e^0_m) \wedge \neg(e^0_m \, E \, x_m) \in \mathrm{tp}(\bar{d}/M_{\delta_i}).$$

Let now

$$\Phi_1(\bar{x}^1) = \{\neg(x^1_k \, E \, e^0_m) \wedge \neg(e^0_k \, E \, x^1_m) : \neg(x^0_{j_k} \, E \, x^1_{j_m}) \in \Gamma_5\}$$
$$\cup\{x^1_k \neq e^0_m : k, m < n\} \cup \{\varphi(\bar{x}^1) : \varphi(x^1_{j_0}, \ldots, x^1_{j_{n-1}}) \in \Gamma' \cap (\Gamma_1 \cup \Gamma^1_3 \cup \Gamma^1_6)\}.$$

$\Phi_1(\bar{x}^1)$ is a finite set to which we can apply the last phrase of Fact 1.16. In this way we find $\bar{e}^1 = (e^1_0, \ldots e^1_{n-1}) \in M_{\delta_i}$ realising $\Phi_1(\bar{x}^1)$. Now we show that $\bar{e}^0 \frown \bar{e}^1$ realises $\Gamma' \setminus \Gamma_2$. So suppose $\neg(x^0_{j_k} E x^1_{j_m}) \in \Gamma' \cap \Gamma_5$, then $\neg(x^1_k \, E \, e^0_m) \in \Phi_1$, hence $\neg(e^1_k E e^0_m)$. Also $\wedge_{k,m<n}(e^1_k \neq e^0_m)$ holds, by the choice of Φ_1, so $\bar{e}^0 \frown \bar{e}^1$ realises $\Gamma' \cap \Gamma_4$. Now we need to deal with Γ_2. Let

$$\mathcal{D} \stackrel{\text{def}}{=} \{(\bar{u}^0, \bar{u}^1) : (\bar{u}^0, \bar{u}^1) \text{ satisfies } \vartheta\}.$$

So $\mathcal{D}$ is first order definable with parameters in M_{δ_i} and we have just shown that $\mathcal{D} \cap M_{\delta_i} \neq \emptyset$. Also if $\bar{e}^0 \frown \bar{e}^1 \subseteq M_{\delta_i}$ satisfies ϑ, it necessarily realises $\Gamma' \setminus \Gamma_2$ (as no $c^l_k \in M_{\delta_i}$, see the definition). As Γ' is presumed to be inconsistent, no $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \cap M_{\delta_i}$ can realise Γ', i.e. satisfy ϑ_2, and hence for no $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \cap M_{\delta_i}$ are the intervals $(u^0_0, u^0_1)_I$ and $(u^1_0, u^1_1)_I$ disjoint. Now we claim that if $(\bar{u}^0, \bar{u}^1) \in M_{\delta_i} \cap \mathcal{D}$, then $(u^0_0, u^0_1)_I \cap (a_{\delta_i}, b_{\delta_i})_I \neq \emptyset$.

Indeed, suppose otherwise, say $u^0_1 <_0 d^i_0 = a_{\delta_i}$, so $u^0_1 <_0 x_0 \in \mathrm{tp}(\bar{d}/M_{\delta_i})$. Arguing as above, with $\bar{u}^0$ in place of $\bar{e}^0$ and $\Phi_1(\bar{x}) \cup \{u^0_1 <_0 x^1_0\}$ in place of $\Phi_1(\bar{x}^1)$, we can find $\bar{u} \in M_{\delta_i}$ satisfying $(u^0_1 <_0 u_0)$ and such that $(\bar{u}^0, \bar{u})$ satisfies ϑ. So $(\bar{u}^0, \bar{u}) \in \mathcal{D} \cap M_{\delta_i}$ and the intervals $(u^0_0, u^0_1)_I$ and $(u_0, u_1)_I$ are disjoint, a contradiction. A similar contradiction can be derived from the assumption $b_{\delta_i} = d^{\delta_i}_1 <_0 u^0_0$. Now note that $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \implies (\bar{u}^1, \bar{u}^0) \in \mathcal{D}$, so if $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \cap M_{\delta_i}$ we also have $(u^1_0, u^1_1)_I \cap (a_{\delta_i}, b_{\delta_i})_I \neq \emptyset$.

By the choice of C, there is no $x \in M_{\delta_i}$ with $d^{\delta_i}_0 \leq_0 x \leq_0 d^{\delta_i}_1$, hence

$$\text{if } (\bar{u}^0, \bar{u}^1) \in M_{\delta_i} \cap \mathcal{D} \text{ and } l < 2, \text{ we have } u^l_0 <_0 d^{\delta_i}_0 <_0 d^{\delta_i}_1 <_0 u^l_1. \qquad (*)$$

Let $\sigma^*(\bar{y})$ be $\exists \bar{x}((\bar{x}, \bar{y}) \in \mathcal{D})$. Hence if

$$\varrho_0(z) = (\exists \bar{y})[\sigma^*(\bar{y}) \wedge z \leq_0 y_0]$$

and

$$\varrho_1(z) = (\exists \bar{y})[\sigma^*(\bar{y}) \wedge y_1 \leq_0 z],$$

then

$$M_{\delta_i} \models (\forall z_0, z_1)[\varrho_0(z_0) \wedge \varrho_1(z_1) \implies z_0 <_0 z_1].$$

Of course, this holds in $\mathfrak{C}$ as well, so

(a) $\varrho_0(z)$ defines an initial segment of I,

(b) $\varrho_1(z)$ defines an end segment of I,

(c) the segments defined by $\varrho_0(z)$ and $\varrho_1(z)$ are disjoint,

(d) $\varrho_0(M_{\delta_i}) \cup \varrho_1(M_{\delta_i}) = I \cap M_{\delta_i}$.

[Why? Note that $(\bar{e}^0, \bar{d}) \in \mathcal{D}$. Hence $\sigma^*(\bar{d})$ holds. As for every $a \in I \cap M_{\delta_i}$ we have $a <_I a_{\delta_i}$ or $a >_I b_{\delta_i}$, the conclusions follows.]

(e) $\varrho_0(a_{\delta_i})$ and $\varrho_1(b_{\delta_i})$ hold.

[Why? Again because $\sigma^*(\bar{d})$ holds.]

The above arguments show that $\{x : (\exists \bar{y})[(\sigma^*(\bar{y}) \wedge x <_0 y_0)]\}$ defines the Dedekind cut $\{x : x <_0 a_{\delta_i}\}$ over M_{δ_i}, which contradicts the choice of C and the fact that the Dedekind cut induced by $(\bar{a}, \bar{b})$ is not definable (which is a part of the definition of $*[M, \bar{a}, \bar{b}]$).

Stage C. Now we have shown that the trees $\langle N_\eta : \eta \in {}^{\lambda>}2\rangle$, $\langle h_\eta : \eta \in {}^{\lambda>}2\rangle$ of models and embeddings can be defined as required, and we consider

$$p^* \stackrel{\text{def}}{=} \cup_{\eta \in {}^{\lambda>}2} h_\eta(p \restriction M_{\delta_{\lg(\eta)}}).$$

We shall show that p^* is finitely satisfiable, hence satisfiable. Let $\Gamma' \subseteq p^*$ be finite. Recalling the analysis of p from Stage A, we consider each of the cases by which p could have been defined (ignoring Case 1A, as justified by Subclaim 1.15.)

<u>Case 1</u>. In this case there is a function $f : Q^M \to Q^M$ respecting E, with $aEf(a)$ for all $a \in Q^M$, and without loss of generality there are some $\eta_0, \ldots \eta_{m-1} \in {}^{\lambda>}2$ and $\{b^j_i : j < m, i < n_j\} \subseteq \mathrm{Rang}(f)$ such that

$$\Gamma' = \{P(z)\} \cup \bigcup_{j<m} \{h_{\eta_j}(b^j_i) R\, z : i < n_j\},$$

and for each j we have $\{b^j_i : i < n_j\} \subseteq M_{\delta_{\lg(\eta_j)}}$. Let $n \stackrel{\text{def}}{=} \Sigma_{j<m} n_j$, hence Γ' is a quantifier free (partial) type over n variables in $\mathfrak{C}^{[\bar{\psi}]}$. By Observation 1.9, we only need to check that in Γ' there are no direct contradictions with the axioms of T^+_{feq}.

The only possibility for such a contradiction is that for some j_0, j_1 and $b_i^{j_0}$, $b_k^{j_1}$ we have

$$h_{\eta_{j_0}}(b_i^{j_0}) \neq h_{\eta_{j_1}}(b_k^{j_1}) \wedge h_{\eta_{j_0}}(b_i^{j_0}) E\, h_{\eta_{j_1}}(b_k^{j_1})$$

and $h_{j_0}(b_i^{j_0}) R\, z, h_{j_1}(b_k^{j_1}) R\, z \in \Gamma'$. In such a case, any c which would realise Γ' would contradict part (c) of the definition of $T^+_{\rm feq}$. Suppose that $b_i^{j_0}, b_k^{j_1}$ and η_0, η_1 are as above. Let $\eta_l \stackrel{\rm def}{=} \eta_{j_l}$ for $l < 2$ and let $\eta = \eta_0 \cap \eta_1$. By part (γ)(ii) in the definition of $\bar{h}$, there is $\hat{b} \in N_\eta$ such that $h_{\eta_0}(b_i^{j_0}) E\, \hat{b}$ and $h_{\eta_1}(b_k^{j_1}) E\, \hat{b}$. For some $b \in M_{\delta_{\lg(\eta)}}$ we have

$$h_{\eta_0}(b) = h_{\eta_1}(b) = h_\eta(b) = \hat{b},$$

so applying the elementarity of the maps, we obtain

$$b_i^{j_0} E\, b E\, b_k^{j_1}.$$

On the other hand, by the definition of p^* we have $b_i^{j_0} R\, z \in p(z)$ and $b_k^{j_1} R\, z \in p(z)$. By the the demands on p this implies that $b_i^{j_0} = b_k^{j_1} \notin M_{\delta_{\lg(\eta)}}$ and $f(b) = b_i^{j_0}$, contradicting the fact that $M_{\delta_{\lg(\eta)}}$ is closed under f.

Case 2. For a fixed $a^* \in M_0$ we have

$$p(z) = \{Q(z)\} \cup \{a^* E z\} \cup \{z \neq c : c \in a^*/E^M\},$$

so without loss of generality

$$\Gamma' = \{Q(z)\} \cup \{a^*\, E\, z\} \cup \{z \neq h_{\eta_j}(c_j) : j < m\}$$

for some $c_0, \ldots, c_{m-1} \in a^*/E^M$ and $\eta_0, \ldots, \eta_{m-1} \in {}^{\lambda >}2$, as $h_{\langle\rangle} = {\rm id}_{M_0}$. As a^*/E is infinite in any model of $T^*_{\rm feq}$, the set Γ' is consistent.

Case 3. We may assume that for some equivalence relation $\mathcal{E}$ on P^M, a function f from P^M into $\{{\rm yes}, {\rm no}\}$, sequences $\eta_0, \ldots \eta_{n-1} \in {}^{\lambda >}2$, and $\{a_i^k : i < m, k < n\} \subseteq Q^M$ and $\{b_i^k, c_i^k, d_i^k : i < m, k < n\} \subseteq P^M$ we have $e_1\, \mathcal{E}\, e_2 \implies f(e_1) = f(e_2)$ and

$$\begin{aligned}\Gamma'(z) = &\{Q(z)\} \cup \bigcup_{k<n} \{\neg(h_{\eta_k}(a_i^k) E z) : i < m\} \cup \bigcup_{k<n} \{(z R\, h_{\eta_k}(b_i^k))^{f(b_i^k)} : i < m\} \\ &\cup \bigcup_{k<n} \{[F(z, h_{\eta_k}(c_i^k)) = F(z, h_{\eta_k}(d_i^k))]^{{\rm if}\, c_i^k \mathcal{E} d_i^k} : i < m\}.\end{aligned}$$

We could have a contradiction if for some k_1, k_2, i_1, i_2 we had $f(b^{k_1}_{i_1})$ =yes, $f(b^{k_2}_{i_2})$ =no, but $h_{\eta_{k_1}}(b^{k_1}_{i_1}) = h_{\eta_{k_2}}(b^{k_2}_{i_2})$, which cannot happen by γ(i) and the fact that each h_η is 1-1. Another possibility is that for some $b^{k_1}_{i_1}, b^{k_2}_{i_2}$ we have $f(b^{k_1}_{i_1}) = f(b^{k_2}_{i_2})$=yes, but $h_{\eta_{k_1}}(b^{k_1}_{i_1}) \neq h_{\eta_{k_2}}(b^{k_2}_{i_2})$ while $h_{\eta_{k_1}}(b^{k_1}_{i_1})\, E\, h_{\eta_{k_2}}(b^{k_2}_{i_2})$. To see that this cannot happen, we distinguish various possibilities for $b^{k_1}_{i_1}, b^{k_2}_{i_2}$ and use part (γ)(ii) in the choice of $\bar{h}$.

Yet another possible source of contradiction could come from a similar consideration involving the last clause in the definition of $\Gamma'(z)$, which cannot happen for similar reasons.

Stage D. Now we can conclude, using $\lambda = \lambda^{<\lambda}$ and $|T| < \lambda$, that there is a model $N^* \prec \mathfrak{C}$ of size λ with $\bigcup_{\eta \in {}^{\lambda >}2} N_\eta \subseteq N^*$, such that p^* is realised in N^*. For $\nu \in {}^\lambda 2$, let $h_\nu \stackrel{\text{def}}{=} \cup_{i<\lambda} h_{\nu \restriction i}$, and let $N_\nu \stackrel{\text{def}}{=} \text{Rang}(h_\nu)$, while $p_\nu \stackrel{\text{def}}{=} h_\nu(p)$.

For such ν, let

$$q_\nu(x) \stackrel{\text{def}}{=} \{I(x)\} \cup \{h_\nu(a_i) <_0 x <_0 h_\nu(b_i) : i < \lambda\}.$$

Hence we have that for $\nu \neq \rho$ from ${}^\lambda 2$, the types q_ν and q_ρ are contradictory, by (δ) above. As $||N^*|| + |L(T)| \leq \lambda$, there are only $\leq \lambda$ definable Dedekind cuts of $<_0$ over N^*, and only $\leq \lambda$ types q_ν are realised in N^*. Hence there is $\nu \in {}^\lambda 2$ (actually 2^λ many) such that the Dedekind cut $\{x : \vee_{i<\lambda} x <_0 h_\nu(a_i)\}$ is not definable over N^* and q_ν is not realised in N^*. So N^* omits q_ν and realises p_ν. We let $N = h(N^*)$, where h is an automorphism of $\mathfrak{C}$ extending h_ν^{-1}. $\bigstar_{1.13}$

Theorem 1.17 Assume that $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$.
(1) For any λ-relevant $(T_{\text{ord}}, T^*_{\text{feq}})$ -superior $(T, \bar{\varphi}, \bar{\psi})$, the theory T has a model M^* of cardinality λ^+ such that

(i) $\bar{\varphi}^{M^*}$ is not λ^+-saturated,

(ii) $\bar{\psi}^{M^*}$ is λ^+-saturated.

(2) We can strengthen the claims in (i) and (ii) to include any interpretations of a dense linear order and T^*_{feq}-respectively in M^*, even with parameters.

Proof. We prove (1), and (2) is proved similarly. Using the Main Claim 1.13, we can construct M^* of size λ^+, by an $\prec$-increasing continuous sequence $\langle M^*_i : i \leq \lambda^+\rangle$, with $||M^*_i|| = \lambda$ satisfying $*[M_i, \bar{a}, \bar{b}]$ for each $i \leq \lambda^+$, and letting $M^* = M_{\lambda^+}$. The Main Claim 1.13 is used in the successor steps. To assure that M^* is λ^+-saturated for $T^*_{\rm feq}$, we use the assumption $2^\lambda = \lambda^+$, to do the bookkeeping of all $T^*_{\rm feq}$-types involved. $\bigstar_{1.17}$

Conclusion 1.18 Under the assumptions of Theorem 1.17, the theory $T^*_{\rm feq}$ is $\triangleleft^*_{\lambda^+}$ strictly below the theory of a dense linear order with no first or last elements.

[Why? It is below by Shelah's Theorem 0.4 above.]

We recall that our motivation for studying $\triangleleft^*$ is to try to characterise SOP_3 (or SOP_2) theories by the $\triangleleft^*$-maximality. As we explained in the Introduction this has origins in the connection between the maximality in the Keisler order and having the strict order property, so we should show here what is the connection between the maximality in Keisler's order and the maximality in the order $\triangleleft^*$. The following Claim 1.19 does that for countable theories.

Claim 1.19 Suppose that T is a countable theory that is $\triangleleft^*_{\lambda^*}$-maximal. Then it is maximal in the Keisler order $\lessdot_\lambda$.

Proof of the Claim. Suppose otherwise and let T_1 be a theory that is $\triangleleft^*$-maximal but not maximal in the Keisler order $\lessdot_\lambda$. In particular we have $T_{\rm ord} \triangleleft^*_\lambda T_1$, so there is a λ-relevant $(T_{\rm ord}, T_1)$-superior triple $(T, \bar{\varphi}_0, \bar{\varphi}_1)$-exemplifying this. By Observation 1.4(0) we may assume that the interpretation $\bar{\varphi}_1$ is trivial, so $T_1 \subseteq T$- for simplicity.

Since T is not maximal in the Keisler order $\lessdot_\lambda$, by [Sh c] 4.2 (1) there is a regular ultrafilter $\mathcal{D}$ which is not good and a model M of T such that $M^\lambda/\mathcal{D}$ is nevertheless λ^+-compact. We can extend M to a model N of T and consider $N^* = N^\lambda/\mathcal{D}$. This is a model of T and by the Extension Theorem for ultrafilters we have that $[N^*]^{\bar{\varphi}_1} = M^\lambda/\mathcal{D}$, so it is λ^+-compact and hence it is λ^+-saturated. Again by the Extension Theorem we have that $[N^*]^{\bar{\varphi}_1} = (N^{\bar{\varphi}_1})^\lambda/\mathcal{D}$. Now on the one hand we have by the $\triangleleft^*_{\lambda^*}$-maximality of

T_1 that $(N^{\bar{\varphi}_1})^\lambda/\mathcal{D}$ must be λ^+-saturated, hence λ^+-compact. But on the other hand $(N^{\bar{\varphi}_1})^\lambda/\mathcal{D}$ cannot be λ^+-compact because $\mathcal{D}$ is not a good ultrafilter and $T_{\rm ord}$ is maximal in the Keisler order, contradicting [Sh c] 4.2 (1). $\bigstar_{1.19}$

2 On the properties SOP_2 and SOP_1

In his paper [Sh 500], S. Shelah investigated a hierarchy of properties unstable theories without strong order property may have. This hierarchy is named NSOP_n for $3 \leq n < \omega$, where the acronym NSOP stands for "not strong order property". The negation of NSOP_n is denoted by SOP_n. It was shown in [Sh 500] that $\mathrm{SOP}_{n+1} \implies \mathrm{SOP}_n$, that the implication is strict and that SOP_3 theories are not simple. In this section we investigate two further notions, which with the intention of furthering the above hierarchy, we name SOP_2 and SOP_1. The original definitions of SOP_n for $n \geq 3$ do not immediately lend themselves to extending the hierarchy for $n = 1, 2$, but the properties we define nevertheless fulfill that role. In section 3, a connection between this hierarchy and $\triangleleft^*_\lambda$-maximality will be established.

Recall from [Sh 500] one of the equivalent definitions of SOP_3. (The equivalence is established in Claim 2.19 of [Sh 500]).

Definition 2.1 (1) A (complete) theory T has SOP_3 iff there is an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ and formulae $\varphi(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{y})$ such that

(a) $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ is contradictory,

(b) for some sequence $\langle \bar{b}_j : j < \omega \rangle$ we have

$$i \leq j \implies \models \varphi[\bar{b}_j, \bar{a}_i] \text{ and } i > j \implies \models \psi[\bar{b}_j, \bar{a}_i],$$

(c) for $i < j$, the set $\{\varphi(\bar{x}, \bar{a}_j), \psi(\bar{x}, \bar{a}_i)\}$ is contradictory.

(2) NSOP_3 stands for the negation of SOP_3.

Definition 2.2 (1) T has SOP_2 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this property in $\mathfrak{C} = \mathfrak{C}_T$, which means:

There are $\bar{a}_\eta \in \mathfrak{C}$ for $\eta \in {}^{\omega>}2$ such that

(a) for every $\rho \in {}^{\omega}2$, the set $\{\varphi(\bar{x}, \bar{a}_{\rho \restriction n}) : n < \omega\}$ is consistent,

(b) if $\eta, \nu \in {}^{\omega>}2$ are incomparable, $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$ is inconsistent.

(2) T has SOP_1 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this in $\mathfrak{C}$, which means:

There are $\bar{a}_\eta \in \mathfrak{C}$, for $\eta \in {}^{\omega>}2$ such that:

(a) for $\rho \in {}^{\omega}2$ the set $\{\varphi(\bar{x}, \bar{a}_{\rho \restriction n}) : n < \omega\}$ is consistent.

(b) if $\nu \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega>}2$, <u>then</u> $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu \frown \langle 1 \rangle})\}$ is inconsistent.

(3) NSOP_2 and NSOP_1 are the negations of SOP_2 and SOP_1 respectively.

The following Claim establishes the relative position of the properties introduced in Definition 2.2 within the (N)SOP hierarchy.

Claim 2.3 For any complete first order theory T, we have

$$\mathrm{SOP}_3 \implies \mathrm{SOP}_2 \implies \mathrm{SOP}_1.$$

Proof of the Claim. Suppose that T is SOP_3, as exemplified by $\langle \bar{a}_i : i < \omega \rangle$, $\langle \bar{b}_j : j < \omega \rangle$ and formulae $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ (see Definition 2.1), and we shall show that T satisfies SOP_2. We define

$$\vartheta(\bar{x}, \bar{y}^0 \frown \bar{y}^1) \equiv \varphi(\bar{x}, \bar{y}^0) \wedge \psi(\bar{x}, \bar{y}^1), \text{ where } lg(\bar{y}^0) = lg(\bar{y}^1).$$

Let us first prove the consistency of

$$\Gamma \stackrel{\text{def}}{=} \begin{array}{c} T \cup \{\neg(\exists \bar{x})[\vartheta(\bar{x}, \bar{y}_\eta) \wedge \vartheta(\bar{x}, \bar{y}_\nu)] : \eta \perp \nu \text{ in } {}^{\omega>}2\} \cup \\ \cup \bigcup_{n<\omega} \{(\exists \bar{x})[\bigwedge_{k \le n} \vartheta(\bar{x}, \bar{y}_{\eta \restriction k})] : \eta \in {}^{n}2\}. \end{array}$$

Suppose for contradiction that Γ is not consistent, then for some $n < \omega$, the following set is inconsistent:

$$\Gamma' \stackrel{\text{def}}{=} \begin{array}{c} T \cup \{\neg(\exists \bar{x})[\vartheta(\bar{x}, \bar{y}_\eta) \wedge \vartheta(\bar{x}, \bar{y}_\nu)] : \eta, \nu \text{ incomparable in } {}^{n \ge}2\} \\ \cup \{(\exists \bar{x})[\bigwedge_{k \le n} \vartheta(\bar{x}, \bar{y}_{\eta \restriction k})] : \eta \in {}^{n}2\}. \end{array}$$

Fix such n. We pick ordinals $\alpha_\eta, \beta_\eta < \omega$ for $\eta \in {}^{n \ge}2$ so that

(i) $\nu \triangleleft \eta \implies \alpha_\nu < \alpha_\eta < \alpha_\eta + 1 < \beta_\eta < \beta_\nu$,

(ii) $\beta_{\eta\frown\langle 0\rangle} < \alpha_{\eta\frown\langle 1\rangle}$.

For $\eta \in {}^{n\geq}2$ let $\bar{a}^*_\eta \stackrel{\text{def}}{=} \bar{a}_{\alpha_\eta} \frown \bar{a}_{\beta_\eta}$. We show that $\mathfrak{C}$ and $\{\bar{a}^*_\eta : \eta \in {}^{n\geq}2\}$ exemplify that Γ' is consistent. So, if $\eta \in {}^{n\geq}2$ then we have $\bigwedge_{k\leq n} \vartheta[\bar{b}_{\alpha_\eta+1}, \bar{a}^*_{\eta\restriction k}]$ as for every $k \leq n$ we have $\alpha_{\eta\restriction k} < \alpha_\eta + 1$, so $\varphi[\bar{b}_{\alpha_\eta+1}, \bar{a}_{\alpha_{\eta\restriction k}}]$ holds, but also for all $k \leq n$, as $\eta \restriction k \trianglelefteq \eta$, we have $\beta_{\eta\restriction k} > \alpha_\eta + 1$, so $\psi[\bar{b}_{\alpha_\eta+1}, \bar{a}_{\beta_{\eta\restriction k}}]$ holds. Hence $(\exists \bar{x})[\bigwedge_{k\leq n} \vartheta(\bar{x}, \bar{a}^*_{\eta\restriction k})]$. On the other hand, if $\nu \frown \langle l\rangle \trianglelefteq \eta_l$ for $l < 2$, then $\{\vartheta(\bar{x}, \bar{a}^*_{\eta_0}), \vartheta(\bar{x}, \bar{a}^*_{\eta_1})\}$ is contradictory as the conjunction implies $\psi(\bar{x}, \bar{a}_{\beta_{\eta_0}}) \wedge \varphi(\bar{x}, \bar{a}_{\alpha_{\eta_1}})$, which is contradictory by $\beta_{\eta_0} < \alpha_{\eta_1}$ and (c) of Definition 2.1. This shows that Γ' is consistent, hence we have also shown that Γ is consistent.

Having shown that Γ is consistent, we can find witnesses $\{\bar{a}^*_\eta : \eta \in {}^{\omega>}2\}$ in $\mathfrak{C}$ realising Γ. Now we just need to show that $\{\vartheta(\bar{x}, \bar{a}^*_{\eta\restriction n}) : n < \omega\}$ is consistent for every $\eta \in {}^\omega 2$. This follows by the compactness theorem and the definition of Γ. Hence we have shown that $\mathrm{SOP}_3 \implies \mathrm{SOP}_2$.

The second part of the claim is obvious (and the witnesses for SOP_2 can be used for SOP_1 as well). $\bigstar_{2.3}$

Question 2.4 Are the implications from Claim 2.3 reversible?

Claim 2.5 If T satisfies SOP_1, then T is not simple. In fact, if $\varphi(\bar{x}, \bar{y})$ exemplifies SOP_1 of T, then the same formula exemplifies that T has the tree property.

Proof of the Claim. Let $\varphi(\bar{x}, \bar{y})$ and $\{\bar{a}_\eta : \eta \in {}^{\omega>}2\}$ exemplify SOP_1. Then

$$\Gamma_\eta \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{a}_{\eta\frown\langle 0\rangle_n\frown\langle 1\rangle}) : n < \omega\}$$

for $\eta \in {}^{\omega>}2$ consists of pairwise contradictory formulae. (Here $\langle 0\rangle_n$ denotes a sequence consisting of n zeroes.) For $n < \omega$ and $\nu \in {}^n\omega$ let

$$\rho_\nu \stackrel{\text{def}}{=} \langle 0\rangle_{\nu(0)+1} \frown \langle 1\rangle \frown \langle 0\rangle_{\nu(1)+1} \ldots \frown \langle 0\rangle_{\nu(n-1)+1} \frown \langle 1\rangle,$$

so $\rho_\nu \in {}^{\omega>}2$ and $\nu \trianglelefteq \eta \implies \rho_\nu \trianglelefteq \rho_\eta$. For $\nu \in {}^n\omega$ let $\bar{b}_\nu = \bar{a}_{\rho_\nu}$. We observe first that $\{\varphi(\bar{x}, \bar{b}_{\nu\hat{}\langle k\rangle}) : k < \omega\}$ is a set of pairwise contradictory formulae, for $\nu \in {}^{n}\omega$; namely, if $k_0 \neq k_1$, then $\varphi(\bar{x}, \bar{b}_{\nu \frown \langle k_l \rangle})$ for $l < 2$ are two different elements of Γ_{ρ_ν}. On the other hand, $\{\varphi(\bar{x}, \bar{b}_{\nu \restriction n}) : n < \omega\}$ is consistent for every $\nu \in {}^{\omega}\omega$. Hence $\varphi(\bar{x}, \bar{y})$ and $\{\bar{b}_\nu : \nu \in {}^{\omega >}\omega\}$ exemplify that T has the tree property, and so T is not simple. $\bigstar_{2.5}$

This ends the discussion of the properties of SOP_1 and SOP_2 that are directly relevant to the main thesis of the paper-the reader only interested in the connection with the order $\lhd^*$ can now turn directly to §3. The rest of this section however contains some further syntactic developments of these properties which are of interest if one wishes to understand the type theory induced by them. The indescernibility results we have here were recently used by Shelah and Usvyatsov [ShUs 844] to define a rank function on NSOP_1 theories (see Theorem 2.22)).

The definition of when a theory has SOP_1 can be made in another equivalent fashion.

Definition 2.6 Let $\varphi(\bar{x}, \bar{y})$ be a formula of $\mathcal{L}(T)$. We say $\varphi(\bar{x}, \bar{y})$ has SOP_1' iff there is $\langle \bar{a}_\eta : \eta \in {}^{\omega >}2 \rangle$ in $\mathfrak{C}_T$ such that

(a) $\{\varphi(\bar{x}, \bar{a}_{\rho \restriction n})^{\rho(n)} : n < \omega\}$ is consistent for every $\rho \in {}^{\omega}2$, where we use the notation
$$\varphi^l = \begin{cases} \varphi & \text{if } l = 1, \\ \neg\varphi & \text{if } l = 0 \end{cases}$$
for $l < 2$.

(b) If $\nu \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega >}2$, <u>then</u> $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$ is inconsistent.

We say that T has property SOP_1' iff some formula of $\mathcal{L}(T)$ has it.

Claim 2.7 (1) If $\varphi(\bar{x}, \bar{y})$ exemplifies SOP_1 of T <u>then</u> $\varphi(\bar{x}, \bar{y})$ (hence T) has property SOP_1'.

(2) If T has property SOP_1' <u>then</u> T has SOP_1.

Proof of the Claim. (1) Let $\{\bar{a}_\eta : \eta \in {}^{\omega >}2\}$ and $\varphi(\bar{x}, \bar{y})$ exemplify that T has SOP_1. For $\eta \in {}^{\omega >}2$ we define $\bar{b}_\eta \stackrel{\text{def}}{=} \bar{a}_{\eta \frown \langle 1 \rangle}$. We shall show that $\varphi(\bar{x}, \bar{y})$ and $\{\bar{b}_\eta : \eta \in {}^{\omega >}2\}$ exemplify SOP_1'.

Given $\hat{\eta} \in {}^{\omega}2$. Let $\bar{c}$ exemplify that item (a) from Definition 2.2(2) holds for $\hat{\eta}$. Given $n < \omega$, we consider $\varphi[\bar{c}, \bar{b}_{\hat{\eta}\restriction n}]^{\hat{\eta}(n)}$. If $\hat{\eta}(n) = 1$, then, as $\bar{b}_{\hat{\eta}\restriction n} = \bar{a}_{\hat{\eta}\restriction n \frown \langle 1\rangle} = \bar{a}_{\hat{\eta}\restriction(n+1)}$, we have that $\varphi[\bar{c}, \bar{b}_{\hat{\eta}\restriction n}]^{\hat{\eta}(n)} = \varphi[\bar{c}, \bar{a}_{\hat{\eta}\restriction(n+1)}]$ holds. If $\hat{\eta}(n) = 0$, then

$$(\hat{\eta} \restriction n) \frown \langle 0\rangle = \hat{\eta} \restriction (n+1).$$

As $\varphi[\bar{c}, \bar{a}_{\hat{\eta}\restriction(n+1)}]$ holds, by (b) of Definition 2.2(2), we have that $\varphi[\bar{c}, \bar{a}_{\hat{\eta}\restriction n\frown\langle 1\rangle}]$ cannot hold, showing again that, $\varphi[\bar{c}, \bar{b}_{\hat{\eta}\restriction n}]^{\hat{\eta}(n)} = \neg\varphi[\bar{c}, \bar{a}_{\hat{\eta}\restriction n\frown\langle 1\rangle}]$ holds. This shows that $\{\varphi(\bar{x}, \bar{b}_{\hat{\eta}\restriction n})^{\hat{\eta}(n)} : n < \omega\}$ is consistent, as exemplified by $\bar{c}$.

Suppose $\nu \frown \langle 0\rangle \trianglelefteq \eta \in {}^{\omega>}2$ and that $\varphi[\bar{d}, \bar{b}_\eta] \wedge \varphi[\bar{d}, \bar{b}_\nu]$ holds. So both $\varphi[\bar{d}, \bar{a}_{\eta\frown\langle 1\rangle}]$ and $\varphi[\bar{d}, \bar{a}_{\nu\frown\langle 1\rangle}]$ hold. On the other hand, as $\nu \frown \langle 0\rangle \trianglelefteq \eta$, clearly $\nu \frown \langle 0\rangle \trianglelefteq \eta \frown \langle 1\rangle$, and so (b) of Definition 2.2(2) implies that $\{\varphi(\bar{x}, \bar{a}_{\eta\frown\langle 1\rangle}), \varphi(\bar{x}, \bar{a}_{\nu\frown\langle 1\rangle})\}$ is contradictory, a contradiction. Hence the set $\{\varphi(\bar{x}, \bar{b}_\eta), \varphi(\bar{x}, \bar{b}_\nu)\}$ is contradictory

(2) Define first for $\eta \in {}^{\omega\geq}2$ an element $\rho_\eta \in {}^{\omega\geq}2$ by letting

$$\rho_\eta(3k) = \eta(k),$$

$$\rho_\eta(3k+1) = 0,$$

$$\rho_\eta(3k+2) = 1,$$

and if $lg(\eta) = m < \omega$, then $lg(\rho_\eta) = 3m$. Note that for $\eta \in {}^{\omega}2$ and $k < \omega$ we have $\rho_{\eta\restriction k} = \rho_\eta \restriction (3k)$.

Let $\varphi(\bar{x}, \bar{y})$ and $\{\bar{a}_\eta : \eta \in {}^{\omega>}2\}$ exemplify property SOP'_1. We pick $c_0 \neq c_1$ and define for $\eta \in {}^{\omega>}2$

$$\bar{b}_{\eta\frown\langle 1\rangle} \stackrel{\text{def}}{=} \bar{a}_{\rho_\eta} \frown \bar{a}_{\rho_\eta\frown\langle 1\rangle} \frown \langle c_0, c_1\rangle,$$

$$\bar{b}_{\eta\frown\langle 0\rangle} \stackrel{\text{def}}{=} \bar{a}_{\rho_\eta\frown\langle 0,0\rangle} \frown \bar{a}_{\rho_\eta} \frown \langle c_0, c_1\rangle,$$

$$\bar{b}_{\langle\rangle} \stackrel{\text{def}}{=} \langle c_0\rangle_{2n+2},$$

where $\langle c\rangle_k$ stands for the sequence of k entries, each of which is c, and $n = lg(\bar{y})$ in $\varphi(\bar{x}, \bar{y})$. We define

$$\psi(\bar{x}, \bar{z}) \equiv \psi(\bar{x}, \bar{z}^0 \frown \bar{z}^1 \frown w^0 \frown w^1) \equiv [w^0 = w^1] \vee [\varphi(\bar{x}, \bar{z}^0) \wedge \neg\varphi(\bar{x}, \bar{z}^1)],$$

where $\bar{z} = \bar{z}^0 \frown \bar{z}^1 \frown \langle w^0, w^1 \rangle$ and $lg(\bar{z}^0) = lg(\bar{z}^1) = lg(\bar{y})$. We now claim that $\psi(\bar{x}, \bar{z})$ and $\{\bar{b}_\eta : \eta \in {}^{\omega>}2\}$ exemplify that SOP_1 holds for T. Before we start checking this, note that for $\eta \in {}^{\omega>}2$ we have:

- $\bullet_1$ $\psi(\bar{d}, \bar{b}_{\langle\rangle})$ holds for any $\bar{d}$,
- $\bullet_2$ $\psi(\bar{d}, \bar{b}_{\eta \frown \langle 0 \rangle})$ holds iff $\varphi(\bar{d}, \bar{a}_{\rho_\eta \frown \langle 0,0 \rangle}) \wedge \neg\varphi(\bar{d}, \bar{a}_{\rho_\eta})$ holds,
- $\bullet_3$ $\psi(\bar{d}, \bar{b}_{\eta \frown \langle 1 \rangle})$ holds iff $\neg\varphi(\bar{d}, \bar{a}_{\rho_\eta \frown \langle 1 \rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\rho_\eta})$ holds.

Let us verify 2.2(2)(a), so let $\eta \in {}^{\omega}2$. Pick $\bar{c}$ such that $\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction n}]^{\rho_\eta(n)}$ holds for all $n < \omega$. We claim that

$$\psi[\bar{c}, \bar{b}_{\eta \restriction n}] \text{ holds for all } n. \tag{$*$}$$

The proof is by a case analysis of n.

If $\underline{n = 0}$, this is trivially true. If $\underline{n = k+1}$ and $\underline{\eta(k) = 0}$, then we need to verify that $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \restriction k} \frown \langle 0,0 \rangle}]$ holds and $\neg\varphi[\bar{c}, \bar{a}_{\rho_{\eta \restriction k}}]$ holds. We have

$$\rho_{\eta \restriction k} \frown \langle 0, 0 \rangle = \rho_\eta \restriction (3k+2),$$

and $\rho_\eta(3k+2) = 1$. Hence $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \restriction k} \frown \langle 0,0 \rangle}]$ holds by the choice of $\bar{c}$. On the other hand, we have $\rho_{\eta \restriction k} = \rho_\eta \restriction (3k)$, and $\rho_\eta(3k) = \eta(k) = 0$, hence $\neg\varphi[\bar{c}, \bar{a}_{\rho_{\eta \restriction k}}]$ holds.

If $\underline{n = k+1}$ and $\underline{\eta(k) = 1}$, then we need to verify that $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \restriction k}}]$ holds while $\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction (3k) \frown \langle 1 \rangle}]$ does not. As $\rho_{\eta \restriction k} = \rho_\eta \restriction (3k)$, and $\rho_\eta(3k) = \eta(k) = 1$, we have that $\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction k}]$ holds. Note that $\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction (3k+2)}]$ holds as $\rho_\eta(3k+2) = 1$. We also have $(\rho_\eta \restriction (3k+1)) \frown \langle 0 \rangle \trianglelefteq \rho_\eta \restriction (3k+2)$. Hence $\neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction (3k+1)}]$ by part (b) in Definition 2.6. But

$$\neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction (3k+1)}] \equiv \neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \restriction (3k) \frown \langle 1 \rangle}] \equiv \neg\varphi[\bar{c}, \bar{a}_{\rho_{\eta \restriction k} \frown \langle 1 \rangle}]$$

holds, so we are done proving $(*)$.

Let us now verify 2.2(2)(b). So suppose $\nu \frown \langle 0 \rangle \trianglelefteq \eta$ and consider $\{\psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle}), \psi(\bar{x}, \bar{b}_\eta)\}$. Let σ and l be such that $\eta = \sigma \frown \langle l \rangle$.

Case 1. $\nu = \sigma$.

Hence $l = 0$. So $\psi(\bar{x}, \bar{b}_\eta) \implies \neg\varphi(\bar{x}, \bar{a}_{\rho_\nu})$ and $\psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle}) \implies \varphi(\bar{x}, \bar{a}_{\rho_\nu})$, by $\bullet_2$ and $\bullet_3$ respectively, showing that $\{\psi(\bar{x}, \bar{b}_\eta), \psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle})\}$ is inconsistent.

Case 2. $\nu \lhd \sigma$ and $l = 0$.

Hence $\nu \frown \langle 0 \rangle \trianglelefteq \sigma$. Clearly $\rho_\nu \frown \langle 0 \rangle \trianglelefteq \rho_\sigma \frown \langle 0, 0 \rangle$, as

$$\rho_\sigma(lg(\rho_\nu)) = \sigma(lg(\nu)) = 0.$$

We have $\psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle}) \implies \varphi(\bar{x}, \bar{a}_{\rho_\nu})$ by $\bullet_3$ and $\psi(\bar{x}, \bar{b}_\eta) = \psi(\bar{x}, \bar{b}_\sigma \frown \langle 0 \rangle)$ implies $\varphi(\bar{x}, \bar{a}_{\rho_\sigma \frown \langle 0,0 \rangle})$ by $\bullet_2$, while the two formulae being implied are contradictory, by (b) in the definition of SOP_1'.

Case 3. $\nu \lhd \sigma$ and $l = 1$.

Observe that $\psi(\bar{x}, \bar{b}_\eta) \implies \varphi(\bar{x}, \bar{a}_{\rho_\sigma})$ by $\bullet_3$ and $\psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle}) \implies \varphi(\bar{x}, \bar{a}_{\rho_\nu})$. As above, using $\nu \frown \langle 0 \rangle \trianglelefteq \sigma$, we show that the set $\{\varphi(\bar{x}, \bar{a}_{\rho_\nu}), \varphi(\bar{x}, \bar{a}_{\rho_\sigma})\}$ is inconsistent. $\bigstar_{2.7}$

Conclusion 2.8 T has SOP_1 iff T has property SOP_1' from Claim 2.7.

Question 2.9 Is the conclusion of 2.8 true when the theory T is replaced by a formula φ?

Start changes

It turns out that witnesses to being SOP_1 can be chosen to be highly indiscernible.

Definition 2.10 (1) Given an ordinal α and sequences $\bar{\eta}_l = \langle \eta^l_0, \eta^l_1, \ldots, \eta^l_{n_l} \rangle$ for $l = 0, 1$ of members of ${}^{\alpha >}2$, we say that $\bar{\eta}_0 \approx_1 \bar{\eta}_1$ iff

(a) $n_0 = n_1$,

(b) the truth values of

- $\eta^l_k = \langle \rangle$,
- $\eta^l_{k_1} \cap \eta^l_{k_2} \trianglelefteq \eta^l_{k_3} \cap \eta^l_{k_4}$

do not depend on l,

(c) $\eta^l_{k_1} \ntrianglelefteq \eta^l_{k_2} \implies \eta^0_{k_1}(lg(\eta^0_{k_1} \cap \eta^0_{k_2})) = \eta^1_{k_1}(lg(\eta^1_{k_1} \cap \eta^1_{k_2}))$

for $k_1, k_2, k_3, k_4 \leq n_0$.
(2) We say that the sequence $\langle \bar{a}_\eta : \eta \in {}^{\alpha>}2 \rangle$ of elements of $\mathfrak{C}$ (for an ordinal α) is *1-fully binary tree indiscernible (1-fbti)* iff whenever $\bar{\eta}_0 \approx_1 \bar{\eta}_1$ are sequences of elements of ${}^{\alpha>}2$, then

$$\bar{a}_{\bar{\eta}_0} \stackrel{\text{def}}{=} \bar{a}_{\eta^0_0} \frown \ldots \frown \bar{a}_{\eta^0_{n_0}}$$

and the similarly defined $\bar{a}_{\bar{\eta}_1}$, realise the same type in $\mathfrak{C}$.
(3) Suppose that δ is a limit ordinal > 0. Define $h^* = h^*_\delta : {}^{\delta>}2 \to {}^{\delta>}2$ by letting for $\eta \in {}^{\delta>}2$

- $lg(h^*(\eta)) = 2lg(\eta) + 1$,
- $i < lg(h^*(\eta)) \implies h^*(\eta)(2i) = 0, h^*(\eta)(2i+1) = \eta(i)$,
- $h^*(\eta)(2lg(\eta)) = 1$.

For $n < \omega$ and $\bar{\eta} \in {}^{n}({}^{\delta>}2)$ we define $h^*(\bar{\eta}) = \langle h^*(\eta_l) : l < n \rangle$.

We say $\bar{\eta} \approx_2 \bar{\nu}$ iff $h^*(\bar{\eta}) \approx_1 h^*(\bar{\nu})$. We define 2-fbti like 1-fbti but using $\approx_2$ in place of $\approx_1$.

Observation 2.10 A The following can be easily checked:
(1) Let $\bar{\eta}, \bar{\nu} \in {}^{n}({}^{\alpha>}2)$ and let $\bar{\eta}'$ and $\bar{\nu}'$ be the closures of $\bar{\eta}, \bar{\nu}$, respectively, under intersections. Then $\bar{\eta} \approx_1 \bar{\nu}$ iff $\bar{\eta}' \approx_1 \bar{\nu}'$.
(2) If $\langle \bar{a}_{\bar{\eta}} : \eta \in {}^{\delta>}2 \rangle$ is 1-fbti then $\langle \bar{a}_{h^*(\bar{\eta})} : \eta \in {}^{\delta>}2 \rangle$ is 2-fbti.
(3) $h^*(\eta)$ is never $\langle , \rangle$ and $h^*(\eta_0)$ is never a strict initial segment of $h^*(\eta_1)$.

Claim 2.11 If $t \in \{1, 2\}$ and $\langle \bar{b}_\eta : \eta \in {}^{\omega>}2 \rangle$ are of given constant length, and $\delta \geq \omega$ is a (limit for $t = 2$) ordinal, then we can find $\langle \bar{a}_\eta : \eta \in {}^{\delta>}2 \rangle$ such that

(a) $\langle \bar{a}_\eta : \eta \in {}^{\delta>}2 \rangle$ is t-fbti,

(b) if $\bar{\eta} = \langle \eta_m : m < n \rangle$, where each $\eta_m \in {}^{\delta>}2$, is given, then we can find $\nu_m \in {}^{\omega>}2\, (m < n)$ such that with $\bar{\nu} \stackrel{\text{def}}{=} \langle \nu_m : m < n \rangle$, we have $\bar{\nu} \approx_t \bar{\eta}$ and sequences $\bar{a}_{\bar{\eta}}$ and $\bar{b}_{\bar{\nu}}$ realise the same type in $\mathfrak{C}$.

Proof of the Claim.[2] Let us first deal with $t = 1$. By Observation 2.10 A (1) above, we may reduce to checking clause (b) only for tuples $\bar{b}_\eta$ where η is closed under intersections. By Compactness Theorem it suffices to assume $\delta = \omega$. The proof goes through a series of steps through which we obtain increasing degrees of indiscernibility. We shall need some auxiliary definitions. Let α be an infinite ordinal.

Definition 2.12 (1) Given $\bar{\eta} = \langle \eta_0, \ldots, \eta_{k-1} \rangle$, a sequence of elements of ${}^{\alpha>}2$, and an ordinal γ. We define $\bar{\eta}' = \mathrm{cl}_\gamma(\bar{\eta})$ as follows:

$$\bar{\eta}' = \langle \langle \rangle, \eta_0, \eta_0 \restriction \gamma, \eta_1, \eta_1 \restriction \gamma, \eta_0 \cap \eta_1, \eta_2, \eta_2 \restriction \gamma, \eta_0 \cap \eta_2, \eta_1 \cap \eta_2 \ldots \rangle.$$

We also define $u_\gamma[\bar{\eta}] = \{\eta_l \in \bar{\eta} : lg(\eta_l) > \gamma\}$.
(2) We say that $\bar{\eta} \approx_{\gamma,n} \bar{\nu}$ iff $\bar{\eta}' \stackrel{\text{def}}{=} \mathrm{cl}_\gamma(\bar{\eta})$ and $\bar{\nu}' \stackrel{\text{def}}{=} \mathrm{cl}_\gamma(\bar{\nu})$ satisfy

(i) $\bar{\eta}' = \langle \eta'_l : l < m \rangle$ and $\bar{\nu}' = \langle \nu'_l : l < m \rangle$ are both in ${}^m({}^{\alpha>}2)$ for some m,

(ii) for $l < m$ we have $\eta'_l \in {}^{\gamma\geq}2 \iff \nu'_l \in {}^{\gamma\geq}2$, and for such l we have $\eta'_l = \nu'_l$,

(iii) $n \geq |u_\gamma[\bar{\eta}]|$,

(iv) $\eta'_l, \eta'_k \in u_\gamma[\bar{\eta}] \implies [lg(\eta'_l) < lg(\eta'_k) \iff lg(\nu'_l) < lg(\nu'_k)]$,

(v) $\eta'_{l_1} \trianglelefteq \eta'_{l_2} \iff \nu'_{l_1} \trianglelefteq \nu'_{l_2}$, and the same holds for the equality,

(vi) if η'_{l_1} is not an initial segment of η'_{l_2}, then $\eta'_{l_1}(lg(\eta'_{l_1} \cap \eta'_{l_2})) = \nu'_{l_1}(lg(\nu'_{l_1} \cap \nu'_{l_2}))$.

(3) $\langle \bar{a}_\eta : \eta \in {}^{\alpha>}2 \rangle$ is (γ, n)-*indiscernible* iff for every k, for every $\bar{\eta}, \bar{\nu} \in {}^k({}^{\alpha>}2)$ with $\bar{\eta} \approx_{\gamma,n} \bar{\nu}$, the tuples $\bar{a}_{\bar{\eta}}$ and $\bar{a}_{\bar{\nu}}$ realise the same type.
(4) $(\leq \gamma, n)$-*indiscernibility* is the conjunction of (β, n)-indiscernibility for all $\beta \leq \gamma$.
(5) We say that $\langle \bar{a}_\eta : \eta \in {}^{\alpha>}2 \rangle$ is 0-*fbti* iff it is (γ, n)-indiscernible for all γ and n.

[2] Note that the definition of $\approx_1, \approx_2$ has changed from the one given in the published version of this paper, but the following proof is basically the same as the one there.

Note 2.12 A (1) $\mathrm{cl}_0(\bar\eta)$ is simply the closure of $\bar\eta$ under intersections, joined with $\langle\rangle$ in appropriate places.
(2) $\bar\eta \approx_{\gamma,n} \bar\nu$ iff $\mathrm{cl}_\gamma(\bar\eta) \approx_{\gamma,n} \mathrm{cl}_\gamma(\bar\nu)$.

Subclaim 2.13 If $\bar a_\eta \in {}^k\mathfrak{C}$ for $\eta \in {}^{\omega>}2$ are tuples of constant length and closed under intersections, <u>then</u>

for any $\alpha \geq \omega$ we can find $\bar a' = \langle \bar a'_\eta : \eta \in {}^{\alpha>}2\rangle$ such that

(x) $\bar a'$ is 0-fbti,

(xx) for every m and a finite set Δ of formulae, we can find $h : {}^{m\geq}2 \to {}^{\omega>}2$ such that

(α) $\langle \bar a'_\eta : \eta \in {}^{m\geq}2\rangle$ and $\langle \bar a_{h(\eta)} : \eta \in {}^{m\geq}2\rangle$ realise the same Δ-type,

(β) h satisfies $h(\eta)\hat{}\langle l\rangle \trianglelefteq h(\eta\hat{}\langle l\rangle)$ for $\eta \in {}^{m>}2$ and $l < 2$, and

$$lg(\eta_1) = lg(\eta_2) \implies lg(h(\eta_1)) = lg(h(\eta_2)).$$

Proof of the Subclaim. By Compactness Theorem it suffices to work with $\alpha = \omega$.

Let $(*)_{\gamma,n}$ be the conjunction of the statement $(x)_{\gamma,n}$ given by

$$\bar a' \text{ is } (\leq \gamma, n)\text{-indiscernible},$$

and (xx) above. We prove by induction on n and then γ that for any $\gamma \leq \omega$ we can find $\bar a'$ for which $(*)_{\gamma,n}$ holds.

<u>$n = 0$</u>. We use $\bar a'_\eta = \bar a_\eta$.

<u>$n+1$</u>. By induction on $\gamma \leq \omega$, we prove that there is $\bar a'$ for which $(*)_{\gamma,n+1} + (*)_{\omega,n} + (xx)$ holds.

<u>$\gamma < \omega$</u>.

Without loss of generality, the sequence $\langle \bar a_\eta : \eta \in {}^{\omega>}2\rangle$ is $(\leq \omega, n)$-indiscernible, as (xx) as a relation between $\langle \bar a_\eta : \eta \in {}^{\omega>}2\rangle$ and $\langle \bar a'_\eta : \eta \in {}^{\omega>}2\rangle$ is transitive. Suppose we are given $\bar\eta^*, \bar\nu^*$ satisfying $\bar\eta^* \approx_{\gamma,n+1} \bar\nu^*$. By Note 2.1.2 A, we may assume $\bar\eta^*, \bar\nu^*$ to be the same as their cl_γ closures and the same will hold for any $\bar\eta, \bar\nu$ that we mention in this context.

If $|u_\gamma[\bar{\eta^*}]| \le n$, the conclusion follows by the assumptions. We shall assume $|u_\gamma[\bar{\eta^*}]| > n$, so $|u_\gamma[\bar{\eta^*}]| = n+1$. Moreover, if $\min(u_\gamma[\bar{\eta^*}]) = \min(u_\gamma[\bar{\nu^*}])$ and for any l with $lg(\eta^*_l) = \min(u_\gamma[\bar{\eta^*}])$ we have $\eta^*_l = \nu^*_l$, then using $(x)_{\min(u_\gamma[\bar{\eta^*}]),n}$, we get that $\bar{a}_{\bar{\eta}^*}$ and $\bar{a}_{\bar{\nu}^*}$ realise the same type. By the same argument, fixing a finite set Δ of formulae, for every $\bar{\eta}$, we get that the $\mathrm{tp}_\Delta(\bar{a}_{\bar{\eta}})$ depends just on the

$$\bar{\eta}/\approx_{\gamma,n+1} \stackrel{\text{def}}{=} \Upsilon \text{ and } \{\eta_l : l < lg(\bar{\eta})\} \cap {}^{\min(u_\gamma[\bar{\eta}])}2 = \{\eta_l : l \in v^\Upsilon\}$$

for some $v^\Upsilon \subseteq lg(\bar{\eta})$. Let us define $F^0_{\Upsilon,\Delta}$ by $F^0_{\Upsilon,\Delta}(\langle \eta_l : l \in v^\Upsilon \rangle) = \mathrm{tp}_\Delta(\bar{a}_{\bar{\eta}})$. By the closure properties of $\bar{\eta}$ and the definition of $\approx_{\gamma,n+1}$, we get that for $l_1 \ne l_2 \in v^\Upsilon$ the truth value of $\eta_{l_1} \restriction (\gamma+1) = \eta_{l_2} \restriction (\gamma+1)$ depends only on Υ. We can hence replace v^Υ by a set $v^\Upsilon_* \subseteq v^\Upsilon$ such that $\langle \eta_l : l \in v^\Upsilon_* \rangle$ are the representatives under the equality of the restrictions to $\gamma+1$.

As we have fixed Δ, there is a finite set A of Υs that can be used as representatives for the values of $F^0_{\Upsilon,\Delta}$. Let r be the size of the range of $F^0_{\Upsilon,\Delta}$. Let $k^* = 2^{\gamma+1}$ (so finite) and let $\{\mu^*_k : k < k^*\}$ list ${}^{\gamma+1}2$. We define a function $F_{\Upsilon,\Delta}$ on ${}^{k^*}({}^{\omega>}2)$ by letting

$$F_{\Upsilon,\Delta}(x_0,\ldots,x_k,\ldots)_{k<k^*} \stackrel{\text{def}}{=} F^0_{\Upsilon,\Delta}(\langle \eta_l : l \in v^\Upsilon_* \rangle),$$
$$\text{where } \eta_l \restriction (\gamma+1) = \mu^*_k \implies \eta_l = \mu^*_k \frown x_k.$$

Define a function F with arity k^* so that $F((\ldots,x_k,\ldots)_{k<k^*})$ is defined iff for some $m < \omega$ we have $\{x_k : k < k^*\} \subseteq {}^m 2$ and then

$$F((\ldots,x_k,\ldots)_{k<k^*}) = \langle F_{\Upsilon,\Delta}((\ldots,x_k,\ldots)_{k<k^*}) : \Upsilon \in A \rangle.$$

Therefore F is a function from $\bigcup_{m<\omega} \prod_{k<k^*} \mathrm{lev}_m({}^{\omega>}2)$ into a set of size r. We recall the following definition and restatement of the Halpern-Lauchli theorem [HaLa], due to Laver and Pincus and presented in [PiHa].

Definition 2.13 A (1) A tree S is *strongly embedded* in a tree T if there is a strictly increasing embedding f^* of S as a suborder of T such that

- any nonmaximal node in $f(S)$ has the same number of immediate successors in T and in $f(S)$, and

- all nodes on any common level of S are mapped by f to a common level of T.

  (2) A nonempty subtree of ${}^{\omega>}\omega$ is *well-behaved* if it is finitely branching and has no maximal nodes (hence it has ω levels).

Halpern-Lauchli theorem Let $r, d < \omega$. Suppose that $\langle T_i : i < d\rangle$ are well-behaved trees and that c is a colouring of $\bigcup_{n<\omega} \prod_{i<d} \mathrm{lev}_n(T_i)$ into r colours. Then there are f^*, $\langle S_i : i < d\rangle$ and $\langle h_i : i < d\rangle$ such that

- $f^* : \omega \to \omega$ is a strictly increasing function,
- each S_i is a well-behaved tree,
- h_i is a strong embedding of $S_i \to T_i$,
- for each $n < \omega$ and $i < d$, the common height in T_i of elements of h_i"$\mathrm{lev}_n(S_i)$ is $f^*(n)$, and
- $\bigcup_{n\in u} \prod_{i<d} h_i$"$\mathrm{lev}_n(S_i)$ is c-monochromatic.

Moreover, in the case that all T_i are the same tree, we can assume that all h_i are contained in a common function h.

Therefore we can apply the Halpern-Lauchli theorem to F. We get a sequence $\langle S_k : k < k^*\rangle$ of well-behaved trees exemplify the conclusion of the Halpern-Lauchli theorem with $h_k = h \restriction S_k$ and $f^*(n) = \mathrm{ht}[h$"$\mathrm{lev}_n(S_k)]$. Since the only well-behaved subtree of ${}^{\omega>}2$ is ${}^{\omega>}2$ itself, we can conclude that there is $h : {}^{\omega>}2 \to {}^{\omega>}2$ such that

- $h \restriction {}^{\gamma\geq}2$ is the identity,
- $lg(h(\eta))$ depends just on $lg(\eta)$ (not on η),
- $h(\eta) \frown \langle l\rangle \lhd h(\eta \frown \langle l\rangle)$ for $l = 0, 1$,
- for some c we have that for all $m < \omega$

$$\{\eta_k : k < k^*\} \subseteq {}^m2 \implies F((h(\eta_0), h(\eta_1), \ldots, h(\eta_k), \ldots)_{k<k^*}) = c.$$

Let $\bar{a}'_\eta$ for $\eta \in {}^{\omega>}2$ be defined to be $\bar{a}_\eta$ if $\eta \in {}^{\gamma>}2$, and otherwise $\bar{a}_{h(\nu)}$ for the unique ν such that $\eta \restriction (\gamma+1) = \mu^*_k$ and $\eta = \mu^*_k \frown \nu$.

We have obtained the desired conclusion, but localized to Δ. The induction step ends by an application of the compactness theorem.

$\underline{\gamma = \omega}$ This is vacuously true.

Having carried the induction, the conclusion of the Subclaim follows from $\bigwedge_n (*)_{0,n}$. $\bigstar_{2.13}$

Now we go *back to the proof of the Claim.* Given $\langle \bar{b}_\eta : \eta \in {}^{\omega>}2\rangle$ as in the assumptions, by the Subclaim we can assume that they are 0-fbti. We choose by induction on n a function $h_n : {}^{n\geq}2 \to {}^{\omega>}2$ as follows. Let $h_0(\langle\rangle) = \langle\rangle$. If h_n is defined, let

$$k_n \stackrel{\text{def}}{=} \max\{lg(h_n(\eta)) + 1 : \eta \in {}^{n\geq}2\}$$

and let

$$h_{n+1}(\langle\rangle) = \langle\rangle, \quad h_{n+1}(\langle 1\rangle\hat{}\nu) = \langle 1\rangle\hat{}h_n(\nu), \quad h_{n+1}(\langle 0\rangle\hat{}\nu) = \langle 0,\ldots,0\rangle\hat{}h_n(\nu),$$

where the sequence of 0s in the last part of the definition has length k_n. The point of the definition of h_n is that if $\bar{\eta}^l = \langle \eta^l_0, \ldots, \eta^l_{n_l}\rangle$ for $l = 0,1$ are given and $n^* = lg(\mathrm{cl}_0(\bar{\eta}^0))$, then

$$\bar{\eta}^0 \approx_1 \bar{\eta}^1 \implies \langle h_{n^*}(\eta^0_0), \ldots, h_{n^*}(\eta^0_{n_0})\rangle \approx_{0,n^*} \langle h_{n^*}(\eta^1_0), \ldots, h_{n^*}(\eta^1_{n_1})\rangle.$$

To check this, we verify the six relevant items of the definition of $\approx_{0,n^*}$.

(i) Follows because $n_0 = n_1$ by the definition of $\approx_1$.

(ii) If $h_{n^*}(\eta^0_i) \cap h_{n^*}(\eta^0_j) = \langle\rangle$ then $\eta^0_i \cap \eta^0_j = \langle\rangle$ so $\eta^1_i \cap \eta^1_j = \langle\rangle$ by the definition of $\approx_1$, and hence $h_{n^*}(\eta^1_i) \cap h_{n^*}(\eta^1_j) = \langle\rangle$. The opposite implication holds by symmetry.

(iii) Follows by the definition of n^*.

(iv) Suppose

$$0 < lg(h_{n^*}(\eta_i^0) \cap h_{n^*}(\eta_j^0)) < lg(h_{n^*}(\eta_k^0) \cap h_{n^*}(\eta_s^0)).$$

Let $m \leq n^*$ be the first such that

$$0 < lg(h_{n^*}(\eta_i^0 \restriction m) \cap h_{n^*}(\eta_j^0 \restriction m)) < lg(h_{n^*}(\eta_k^0 \restriction m) \cap h_{n^*}(\eta_s^0 \restriction m)).$$

Clearly, $m > 0$. To simplify the notation, let us assume that $m = n^*$. Let $\eta_t^0 = \langle l_t \rangle \frown \nu_t^0$ for $t \in \{i, j, k, s\}$ and for some $l_t \in \{0, 1\}$ depending on t. The situation we describe can happen iff $l_i = l_j = 1$ and $l_k = l_s = 0$, by the definition of h_n. By the definition of $\approx_1$ this can be recognised by the $\approx_1$-type of $\bar{\eta}^0$.

(v), (vi) Follow because the corresponding properties are preserved by h_n.

Fix an $n < \omega$ and define $\bar{a}_\eta = \bar{b}_{h_n(\eta)}$ for $\eta \in {}^{n\geq}2$. By the above argument it follows that $\langle \bar{a}_\eta : \eta \in {}^{n\geq}2 \rangle$ are 1-fbti. As n was arbitrary, we can finish by compactness.

For $t = 2$, we use exactly the same proof. $\bigstar_{2.11}$

The following Theorem 2.15 will finally tell us that witnesses for SOP_1 can be chosen with a certain degree of indiscernability. We need to introduce a new notion of indiscernability:

Definition 2.14 (1) Given an ordinal α and sequences $\bar{\eta}_l = \langle \eta_0^l, \eta_1^l, \ldots, \eta_{n_l}^l \rangle$ for $l = 0, 1$ of members of ${}^{\alpha>}2$, we say that $\bar{\eta}_0 \approx_3 \bar{\eta}_1$ iff

(a) $n_0 = n_1$,

(b) the truth values of

- $\eta_k^l = \langle \rangle$,
- $\eta_{k_1}^l \cap \eta_{k_2}^l \trianglelefteq \eta_{k_3}^l \cap \eta_{k_4}^l$

do not depend on l,

(c) $\eta_{k_1}^l \ntrianglelefteq \eta_{k_2}^l \implies \eta_{k_1}^0(lg(\eta_{k_1}^0 \cap \eta_{k_2}^0)) = \eta_{k_1}^1(lg(\eta_{k_1}^1 \cap \eta_{k_2}^1)),$

(d) $\eta^l_{k_1} \ntrianglelefteq \eta^l_{k_2} \implies \eta^0_{k_1} = (\eta^0_{k_1} \cap \eta^0_{k_2}) \frown \langle 1 \rangle$ iff $\eta^1_{k_1} = (\eta^1_{k_1} \cap \eta^1_{k_2}) \frown \langle 1 \rangle$.

for $k_1, k_2, k_3, k_4 \le n_0$.
(2) We say that the sequence $\langle \bar{a}_\eta : \eta \in {}^{\alpha>}2 \rangle$ of elements of $\mathfrak{C}$ (for an ordinal α) is *3-fully binary tree indiscernible (3-fbti)* iff whenever $\bar{\eta}_0 \approx_1 \bar{\eta}_1$ are sequences of elements of ${}^{\alpha>}2$, then

$$\bar{a}_{\bar{\eta}_0} \stackrel{\text{def}}{=} \bar{a}_{\eta^0_0} \frown \ldots \frown \bar{a}_{\eta^0_{n_0}}$$

and the similarly defined $\bar{a}_{\bar{\eta}_1}$, realise the same type in $\mathfrak{C}$.

Theorem 2.15 Suppose that T has SOP_1 as witnessed by $\varphi(\bar{x}, \bar{y})$ and a sequence $\bar{a} = \langle \bar{a}_\eta : \eta \in {}^{\omega>}2 \rangle$. Then there is $\bar{d} = \langle \bar{d}_\eta : \eta \in {}^{\omega>}2 \rangle$ exemplifying that $\varphi(\bar{x}, \bar{y})$ has SOP_1 and $\langle \bar{d}_\eta : \eta \in {}^{\omega>}2 \setminus \{\langle\rangle\} \rangle$ is 3-fbti.

Proof. Let $k^* = lg(\bar{y})$. First define $\bar{b}_\eta$ for $\eta \in {}^{\omega>}2$ by $\bar{b}_\eta = \bar{a}_{\eta \frown \langle 0 \rangle} \frown \bar{a}_{\eta \frown \langle 1 \rangle}$. Let for any $\bar{z}_0, \bar{z}_1$ of length k^* and $l \in \{0, 1\}$, $\psi_l(\bar{x}, \bar{z}_0 \frown \bar{z}_1) \equiv \varphi(\bar{x}, \bar{z}_l)$. Now we use Claim 2.11 applied to $\langle \bar{b}_\eta : \eta \in {}^{\omega>}2 \rangle$. Therefore we can find $\bar{c} = \langle \bar{c}_\eta : \eta \in {}^{\omega>}2 \rangle$ such that

(a) $\bar{c}$ is 1-fbti,

(b) for any finite n and $\bar{\eta} \in {}^n({}^{\omega>}2)$ there is $\bar{\nu} \in {}^n({}^{\omega>}2)$ such that $\bar{\nu} \approx_1 \bar{\eta}$ and $\bar{b}_\eta$ and $\bar{c}_\nu$ realise the same type in $\mathfrak{C}$.

Let $\bar{d}_\eta$ for $\eta \in {}^{k^*}({}^{\omega>}2)$ be defined by induction on the length of η so that $\bar{d}_{\eta \frown \langle 0 \rangle} \frown \bar{d}_{\eta \frown \langle 1 \rangle} = \bar{c}_\eta$ and $\bar{d}_{\langle\rangle} = \bar{c}_{\langle\rangle}$. This is possible by the choice of $\bar{b}$ and $\bar{c}$.

Claim 2.16 If $\nu \frown \langle 0 \rangle \trianglelefteq \eta$ then $\varphi(\bar{x}, \bar{d}_\eta)$ and $\varphi(\bar{x}, \bar{d}_{\nu \frown \langle 1 \rangle})$ are incompatible.

Proof of the Claim. Let $\eta = \rho \frown \langle l \rangle$ for some $l \in \{0, 1\}$. Consider $\{\psi_l(\bar{x}, \bar{c}_\rho), \psi_1(\bar{x}, \bar{c}_\nu)\}$, we claim that this set is inconsistent. We know that

$$\psi_l(\bar{x}, \bar{c}_\rho) \equiv \varphi(\bar{x}, \bar{d}_{\rho \frown \langle l \rangle}) \equiv \varphi(\bar{x}, \bar{d}_\eta), \quad \psi_1(\bar{x}, \bar{c}_\nu) \equiv \varphi(\bar{x}, \bar{d}_{\nu \frown \langle 1 \rangle}).$$

By the 1-fbti property of $\bar{c}$ and the choice of $\bar{c}$ with respect to $\bar{b}$ it suffices to check that $\{\psi_l(\bar{x}, \bar{b}_\rho), \psi_1(\bar{x}, \bar{b}_\nu)\}$ is inconsistent. This means that $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu \frown \langle 1 \rangle}\}$ is inconsistent, which is true by the choice of $\bar{a}$. $\bigstar_{2.16}$

Claim 2.17 For any $\rho \in {}^{\omega}2$, $\{\varphi(\bar{x}, \bar{d}_{\rho\restriction n}) : n < \omega\}$ is consistent.

Proof of the Claim. It suffices to show that for any

$$\langle\rangle \lhd \eta_0 \lhd \eta_1 \lhd \ldots \eta_k$$

the set $\{\varphi(\bar{x}, \bar{d}_{\eta_{l+1}\restriction lg(\eta_l)\frown\eta_{l+1}(lg(\eta_l))}) : l > k\} \cup \{\varphi(\bar{x}, \bar{d}_{\langle\rangle})\}$ is consistent. This means $\{\psi_{\eta_{l+1}(lg(\eta_l))}(\bar{x}, \bar{c}_{\eta_{l+1}\restriction lg(\eta_l)}) : l < k\} \cup \{\varphi(\bar{x}, \bar{c}_{\langle\rangle})\}$ is consistent. By the choice of $\bar{b}$ and $\bar{c}$ this is to say $\{\psi_{\eta_{l+1}(lg(\eta_l))}(\bar{x}, \bar{b}_{\eta_{l+1}\restriction lg(\eta_l)}) : l < k\} \cup \{\varphi(\bar{x}, \bar{a}_{\langle\rangle})\}$ or $\{\varphi(\bar{x}, \bar{a}_{\eta_{l+1}\restriction lg(\eta_l)}) : l < k\} \cup \{\varphi(\bar{x}, \bar{a}_{\langle\rangle})\}$ is consistent, but this is true by the choice of $\bar{a}$. $\bigstar_{2.17}$

Claim 2.18 $\langle d_\eta : \eta \in {}^{\omega}2 \setminus \{0\}\rangle$ is 3-fbti.

Proof of the Claim. Suppose that $\bar{\eta}_0 \approx_3 \bar{\eta}_1$ and consider $\bar{d}_{\bar{\eta}_0}$ and $\bar{d}_{\bar{\eta}_1}$. For each η^k_l let ν^l_k be such that $\eta^l_k = \nu^l_k \frown \langle m^l_k\rangle$ for some $m^l_k \in \{0,1\}$ and let $\bar{\nu}_0, \bar{\nu}_1$ be defined from $\nu^l_k (l \in \{0,1\}, k < lg(\bar{\eta}_0))$. Then $\bar{\eta}_0 \approx_3 \bar{\eta}_1 \implies \bar{\nu}_0 \approx_3 \bar{\nu}_1$, hence $\bar{c}_{\bar{\nu}_0}$ and $\bar{c}_{\bar{\nu}_1}$ realise the same type, which implies that $\bar{d}_{\bar{\eta}_0}$ and $\bar{d}_{\bar{\eta}_1}$ do. $\bigstar_{2.18}$

$\bigstar_{2.15}$

End changes.

As we mentioned before, it would be really interesting to know if SOP_2 and SOP_1 are equivalent. A step towards understanding this question is provided by the next claim which shows that in the case of theories which are SOP_1 and NSOP_2, the witnesses to being SOP_1 can be chosen to be particularly nice. *note a change here to 3-fbti from the old version*

Claim 2.19 Suppose that $\varphi(\bar{x}, \bar{y})$ satisfies SOP_1, but for no n does the formula $\varphi_n(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1}) \equiv \wedge_{k<n}\varphi(\bar{x}, \bar{y}_k)$ satisfy SOP_2. Then there are witnesses $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2\rangle$ for $\varphi(\bar{x}, \bar{y})$ satisfying SOP_1 which in addition satisfy:

(c) if $X \subseteq {}^{\omega>}2$, and there are no $\eta, \nu \in X$ such that $\eta \frown \langle 0\rangle \trianglelefteq \nu$, then $\{\varphi(\bar{x}, \bar{a}_\eta) : \eta \in X\}$ is consistent.

(d) $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2\rangle$ is 3-fbti.

(In particular, such a formula and witnesses can be found for any theory satisfying SOP_1 and NSOP_2.)

Proof of the Claim. We shall be using the following colouring theorem, for which we could not find a specific reference and so we include a proof of it.

Lemma 2.20 Suppose $\mathrm{cf}(\kappa) = \kappa$ and we colour ${}^{\kappa>}2$ by $\theta < \kappa$ colours. Then there is an embedding $h : {}^{\omega>}2 \to {}^{\kappa>}2$ such that $h(\eta)\hat{\ }\langle l\rangle \trianglelefteq h(\eta\hat{\ }\langle l\rangle)$ and $\mathrm{Rang}(h)$ is monochromatic.

Proof of the Lemma. Let c be a colouring as in the assumptions and let $\{a_i : i < \theta\}$ list $\mathrm{Rang}(c)$. We claim that there is $\nu^* \in {}^{\kappa>}2$ and $j < \theta$ such that for every $\nu \in {}^{\kappa>}2$ satisfying $\nu^* \trianglelefteq \nu$ there is $\rho \in {}^{\kappa>}2$ with $\nu \trianglelefteq \rho$ and $c(\rho) = j$. For otherwise, we can choose by induction on $i \leq \theta$ a member $\eta_i \in {}^{\kappa>}2$ with $i < j \implies \eta_i \trianglelefteq \eta_j$ such that for no $\rho \in {}^{\kappa}2$ do we have $\eta_{i+1} \trianglelefteq \rho$ and $c(\rho) = i$, using $\theta < \mathrm{cf}(\kappa)$. As $\theta < \kappa$, we obtain a contradiction.

Having found such ν^*, j we define $h(\eta)$ for $\eta \in {}^{n}2$ by induction on $n < \omega$. For $n = 0$ we choose $h(\langle\rangle)$ to satisfy $\nu^* \trianglelefteq h(\langle\rangle)$ and $c(h(\langle\rangle)) = j$, which is possible by the choice of ν^* and j. For $n+1$, for any $\eta \in {}^{n+1}2$ we choose for $l = 0, 1$ a member $h(\eta \frown \langle l\rangle)$ of ${}^{\kappa>}2$ which is above $h(\eta) \frown \langle l\rangle$ and on which c is j, which again is possible by the choice of ν^* and j. $\bigstar_{2.20}$

Let $\varphi(\bar{x}, \bar{y})$ be a SOP_1 formula which is not SOP_2, and moreover assume that for no n does the formula φ_n defined as above satisfy SOP_2. By Theorem 2.15, we can find witnesses $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2\rangle$ which are 3-fbti. By the compactness theorem, we can assume that we have a 1-fbti sequence $\langle \bar{a}_\eta : \eta \in {}^{\omega_1>}2\rangle$ with the properties corresponding to (a) and (b) of Definition 2.2(2), namely

(a) for every $\eta \in {}^{\omega_1}2$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction\alpha}) : \alpha < \omega_1\}$ is consistent,

(b) if $\nu \frown \langle 0\rangle \trianglelefteq \eta \in {}^{\omega_1>}2$, then $\{\varphi(\bar{x}, \bar{a}_{\nu\frown\langle 1\rangle}), \varphi(\bar{x}, \bar{a}_\eta)\}$ is inconsistent.

We shall now attempt to choose ν_η and w_η for $\eta \in {}^{\omega_1>}2$, by induction on $lg(\eta) = \alpha < \omega_1$ so that:

(i) $\nu_\eta \in {}^{\omega_1>}2$,

(ii) $\beta < \alpha \implies \nu_{\eta\restriction\beta} \vartriangleleft \nu_\eta$,

(iii) $\beta < \alpha \implies \nu_\eta(lg(\nu_{\eta\restriction\beta})) = \eta(\beta)$,

(iv) $w_\eta \subseteq {}^{\omega_1>}2$ is finite and $\nu \in w_\eta \implies lg(\nu) < lg(\nu_\eta)$,

(v) if $lg(\eta)$ is a limit ordinal > 0, then $w_\eta = \emptyset$,

(vi) if $\eta \in {}^{\beta}2$ and $l < 2$, then $w_{\eta\frown\langle l\rangle} \subseteq \{\rho \in {}^{\omega_1>}2 : \nu_\eta \frown \langle l\rangle \trianglelefteq \rho\}$ and $\max\{lg(\rho) : \rho \in w_{\eta\frown\langle l\rangle}\} < lg(\nu_{\eta\frown\langle l\rangle})$,

(vii) for each η there is $\rho^* = \rho^*_\eta$ such that

(α) $\nu_\eta \vartriangleleft \rho^* \in {}^{\omega_1}2$,

(β) $|\{\alpha < \omega_1 : \rho^*(\alpha) = 1\}| = \aleph_1$,

(γ) letting

$$p_\eta(\bar{x}) \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_{\eta\restriction\gamma} \text{ for some } \gamma \leq lg(\eta)\},$$

we have that for all large enough β^*, the set

$$p_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\rho^*\restriction\beta}) : \beta > \beta^* \ \wedge \ \rho^*(\beta) = 1\}$$

is consistent,

(viii) $p_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta\frown\langle 0\rangle} \cup w_{\eta\frown\langle 1\rangle}\}$ is inconsistent.

Before proceeding, we make several remarks about this definition. Firstly, requirements (vii) and (viii) taken together imply that for each $\eta \in {}^{\omega_1>}2$ we have that $w_{\eta\frown\langle 0\rangle} \cup w_{\eta\frown\langle 1\rangle} \neq \emptyset$. Secondly, the definition of $w_{\eta\frown\langle l\rangle}$ for $l \in \{0, 1\}$ implies that

$$\wedge_{l=0,1} \rho_l \in w_{\eta\frown\langle l\rangle} \implies \rho_0 \perp \rho_1.$$

Thirdly, in (vii), any ρ^* which satisfies that $\nu_\eta \vartriangleleft \rho^*$ and $|\{\gamma : \rho^*(\gamma) = 1\}| = \aleph_1$ can be chosen as ρ^*_η, by indiscernibility.

Now let us assume that a choice as above is possible, and we have made it. Hence for each $\eta \in {}^{\omega_1>}2$ there is a finite $q_\eta \subseteq p_\eta$ such that

$$q_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta\frown\langle 0\rangle} \cup w_{\eta\frown\langle 1\rangle}\} \tag{$*$}$$

is inconsistent. Notice that there are q and $\eta^* \in {}^{\omega_1}2$ such that

$$(\forall \eta_1)[\eta^* \trianglelefteq \eta_1 \in {}^{\omega_1>}2 \implies (\exists \eta_2 \in {}^{\omega_1>}2)(\eta_1 \trianglelefteq \eta_2 \ \wedge \ q_{\eta_2} = q)].$$

Namely, otherwise, we would have the following: each p_η is countable, hence for every η there is $g(\eta)$ with $\eta \triangleleft g(\eta) \in {}^{\omega_1>}2$ and

$$g(\eta) \trianglelefteq \eta_1 \implies q_{\eta_1} \not\subseteq p_\eta.$$

Let $\eta_0 \stackrel{\text{def}}{=} \langle\rangle$, and for $n < \omega$ let $\eta_{n+1} = g(\eta_n)$. Let $\eta \stackrel{\text{def}}{=} \cup_{n<\omega}\eta_n$, hence $p_\eta = \cup_{n<\omega} p_{\eta_n}$ (as $w_\eta = \emptyset$), and so $q_\eta \subseteq p_{\eta_n}$ for some n, a contradiction.

Having found such q, η^*, by renaming and using Lemma 2.20, we can assume that $\eta^* \stackrel{\text{def}}{=} \langle\rangle$ and that for all $\eta \in {}^{\omega}2$ we have $q_\eta = p_{\langle\rangle} = q$ (as $\eta \trianglelefteq \nu \implies p_\eta \subseteq p_\nu$). For $\eta \in {}^{\omega>}2$ let $\bar\tau_\eta$ list w_η. Without loss of generality, by thinning and renaming, we have that for all η_1, η_2,

$$\langle \nu_{\eta_1} \rangle \frown \bar\tau_{\eta_1 \frown \langle 0 \rangle} \frown \bar\tau_{\eta_1 \frown \langle 1 \rangle} \approx_1 \langle \nu_{\eta_2} \rangle \frown \bar\tau_{\eta_2 \frown \langle 0 \rangle} \frown \bar\tau_{\eta_2 \frown \langle 1 \rangle}.$$

Similarly to the proof of Claim 2.7, we can define a formula $\psi(\bar x, \bar y)$ and $\{\bar b_\eta : \eta \in {}^{\omega>}2\}$ such that

$$\psi(\bar x, \bar b_\eta) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_\eta\}.$$

We claim that $\psi(\bar x, \bar y)$ and $\langle \bar b_\eta : \eta \in {}^{\omega>}2\rangle$ exemplify SOP_2 of T, which is then a contradiction (noting that ψ is a formula of the form φ_n for some n, where φ_n was defined in the statement of the Claim). We check the two properties from Definition 2.2(1).

To see (a), let $\eta \in {}^{\omega}2$ be given. We have that p_η is consistent, and $q \subseteq p_\eta$. For $n < \omega$, we have

$$\psi(\bar x, \bar b_{\eta \restriction n}) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_{\eta \restriction n}\}.$$

As this is a conjunction of a set of formulae each of which is from p_η, we have that $\{\psi(\bar x, \bar b_{\eta \restriction n}) : n < \omega\}$ is consistent. To check (b), suppose $\eta \perp \nu \in {}^{\omega>}2$. Let n be such that $\eta \restriction n = \nu \restriction n$ but $\eta(n) \neq \nu(n)$. Hence

$$\psi(\bar x, \bar b_\eta) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_{\eta \restriction n \frown \eta(n)}\}$$

and

$$\psi(\bar{x},\bar{b}_\nu) \equiv \bigwedge q \wedge \bigwedge\{\varphi(\bar{x},\bar{a}_\rho):\ \rho \in w_{\eta\restriction n\frown\nu(n)}\},$$

so taken together, the two are contradictory by the choice of q.

We conclude that the choice of ν_η and w_η cannot be carried throughout $\eta \in {}^{\omega_1>}2$. So, there is $\alpha < \omega_1$ and $\eta \in {}^{\alpha}2$ such that $\nu_\eta, w_{\eta\frown\langle l\rangle}, \nu_{\eta\frown\langle l\rangle}$ for $l<2$ cannot be chosen, and α is the first ordinal for which there is such η. Let $\nu^0_\eta \in {}^{\omega_1>}2 \rhd \cup_{\beta<\alpha}\nu_{\eta\restriction\beta} \frown \langle\eta(\alpha-1)\rangle$ if the latter part is defined, otherwise let $\nu^0_\eta \rhd \cup_{\beta<\alpha}\nu_{\eta\restriction\beta}$. This choice of $\nu_\eta=\rho$ for any $\rho \trianglerighteq \nu^0_\eta$ with $\rho \in {}^{\omega_1}2$ satisfies items (i)-(iii) above. We conclude that $w_{\eta\frown\langle l\rangle}$ for $l<2$ using any $\rho \trianglerighteq \nu^0_\eta$ with $\rho \in {}^{\omega_1>}2$ for ν_η could not have been chosen, and examine why this is so. Note that p_η is already defined. Let

$$\Theta \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (\rho,\gamma,w): & \nu^0_\eta \lhd \rho \in {}^{\omega_1}2, \\ & lg(\nu^0_\eta) \le \gamma < \omega_1, \\ & (\exists^{\aleph_1}\beta<\omega_1)(\rho(\beta)=1), \\ & w \subseteq \{\Upsilon \in {}^{\omega_1>}2:\ \rho\restriction\gamma \trianglelefteq \Upsilon\} \text{ is finite and} \\ & \text{for some } \beta_\rho<\omega_1 \text{ the set} \\ & p_\eta \cup \{\varphi(\bar{x},\bar{a}_{\rho\restriction\beta}):\ \rho(\beta)=1\ \&\ \beta\in[\beta_\rho,\omega_1)\} \cup \\ & \cup\{\varphi(\bar{x},\bar{a}_\Upsilon):\ \Upsilon \in w\} \\ & \text{is consistent} \end{array} \right\}.$$

We make several **observations**:

(0) If $(\rho,\gamma,w)\in\Theta$ and $w\subseteq w'$ with w' finite and $w'\setminus w$ is contained in $\{\rho\restriction\beta:\ \beta_\rho\le\beta\ \wedge\ \rho(\beta)=1\}$, <u>then</u> $(\rho,\gamma,w')\in\Theta$.

[This is obvious.]

(1) If $(\rho_l,\gamma,w_l)\in\Theta$ and for some $\sigma\in{}^{\omega_1>}2$ with $\nu^0_\eta\trianglelefteq\sigma$ we have $\sigma\frown\langle l\rangle\lhd\rho_l\restriction\gamma$ for $l<2$, while ρ_0 and ρ_1 are eventually equal, <u>then</u> $(\rho_l, lg(\sigma), w_0\cup w_1)\in\Theta$.

[Why? We have $w_l\subseteq\{\Upsilon\in{}^{\omega_1>}2:\ \rho_l\restriction\gamma\trianglelefteq\Upsilon\}$ is finite, so clearly $w_0\cup w_1\subseteq\{\Upsilon\in{}^{\omega_1>}2:\ \sigma\trianglelefteq\Upsilon\}$ is finite. By the assumption, we have that for some $\beta_l<\omega_1$ for $l<2$

$$p_\eta\cup\{\varphi(\bar{x},\bar{a}_{\rho_l\restriction\beta}):\ \beta>\beta_l\ \wedge\ \rho_l(\beta)=1\}\cup\{\varphi(\bar{x},\bar{a}_\Upsilon):\ \Upsilon\in w_l\}$$

is consistent. Suppose that (1) is not true with $l=0$ and let $\beta^*\ge\max\{\beta_0,\beta_1\}$ be such that $\beta^*<\omega_1$ and for $\beta>\beta^*$ the equality $\rho_0(\beta)=\rho_1(\beta)$ holds. Hence we have that

$$p_\eta \cup \{\varphi(\bar{x}, \bar{a}_{\rho_0 \restriction \beta}) : \beta > \beta^* \ \wedge \ \rho_0(\beta) = 1\} \cup \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_0 \cup w_1\}$$

is inconsistent. By increasing w_0 if necessary, (0) implies that

$$p_\eta \cup \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_0 \cup w_1\}$$

is inconsistent. Let $\nu_\eta \stackrel{\text{def}}{=} \sigma$, for $l < 2$ let $w_{\eta \frown \langle l \rangle} = w_l$, and let $\nu_{\eta \frown \langle l \rangle} \stackrel{\text{def}}{=} \rho_l \restriction \beta^*_l$ for a large enough β^*_l so that $\beta^* < \beta^*_l$ and $\max(\{lg(\Upsilon) : \Upsilon \in w_{\eta \frown \langle l \rangle}\}) < \beta^*_l$. This choice shows that we could have chosen $\nu_\eta, w_{\eta \frown \langle l \rangle}$ as required, contradicting the choice of η.]

(2) If $\nu^0_\eta \lhd \rho \in {}^{\omega_1}2$ for some ρ such that there are $\aleph_1$ many $\beta < \omega_1$ with $\rho(\beta) = 1$, and $lg(\nu^0_\eta) \le \gamma < \omega_1$, <u>then</u> $(\rho, \gamma, \emptyset) \in \Theta$.

[Why? By the choice of p_η and the remark about the freedom in the choice of ρ^* that we made earlier.]

Now we use the choice of η to define witnesses to T being SOP_1 which also satisfy the requirements of the Claim. For $\tau \in {}^{\omega >}2$, let $\bar{b}_\tau \stackrel{\text{def}}{=} \bar{a}_{\nu^0_\eta \frown \tau}$. Let us check the required properties. Properties (a),(b) and (d) follow from the choice of $\{\bar{a}_\sigma : \sigma \in {}^{\omega_1 >}2\}$. Let $X^* \subseteq {}^{\omega >}2$ be such that there are no $\sigma, \nu \in X^*$ with $\sigma \frown \langle 0 \rangle \trianglelefteq \nu$, we need to show that $\{\varphi(\bar{x}, \bar{b}_\tau) : \tau \in X^*\}$ is consistent. It suffices to show the same holds when X^* replaced by an arbitrary finite $X \subseteq X^*$. Fix such an X. Clearly, it suffices to show that for some ρ, γ, letting $w = \{\nu^0_\eta \frown \tau : \tau \in X\}$, we have $(\rho, \gamma, w) \in \Theta$.

Let $\rho^* \in {}^{\omega_1}2$ be such that $\nu^0_\eta \lhd \rho^*$ and $\rho^*(\beta) = 1$ for $\aleph_1$ many β. By induction on $n \stackrel{\text{def}}{=} |X|$ we show:

there is $\rho \in {}^{\omega_1}2$ such that for some $\gamma \ge \max\{lg(\sigma) : \sigma \in w\}$, we have $(\rho, \gamma, w) \in \Theta$ and $\beta > \gamma \implies \rho(\beta) = \rho^*(\beta)$, while $\rho(\gamma) = 1$.

<u>$n = 0$</u>. Follows by observation (2) above.

<u>$n = 1$</u>. Let $X = \{\tau\}$ and $\gamma = lg(\tau) + lg(\nu^0_\tau)$. Let $\rho \in {}^{\omega_1}2$ be such that $\rho \restriction \gamma = \nu^0_\eta \frown \tau$, $\rho(\gamma) = 1$ and $\beta > \gamma \implies \rho(\beta) = \rho^*(\beta)$. By observation (2) above, we have that $(\rho, \gamma, \emptyset) \in \Theta$. Then, by observation (0), we have $(\rho, \gamma, w) \in \Theta$.

<u>$n = k + 1 \ge 2$</u>. *Case 1.* w is linearly ordered by $\lhd$.

Let $\tau \in w$ be of maximal length, so clearly $\sigma \in w \setminus \{\tau\} \implies \sigma \frown \langle 1 \rangle \trianglelefteq \tau$. Let $\rho \in {}^{\omega_1}2$ be such that $\tau \frown \langle 1 \rangle \triangleleft \rho$ and $\beta > lg(\tau)$, while $\rho(\beta) = \rho^*(\beta)$. Now continue as in the case $n = 1$.

Case 2. Not Case 1.

Let $\sigma \in {}^{\omega_1 >}2$ be $\triangleleft$-maximal such that $(\forall \tau)(\tau \in w \implies \sigma \trianglelefteq \tau)$. This is well defined, as $w \neq \emptyset$ is finite. Let $w_l \stackrel{\text{def}}{=} \{\tau \in w : \sigma \frown \langle l \rangle \trianglelefteq \tau\}$, so $w_0 \cap w_1 = \emptyset$ but neither of w_0, w_1 is empty. Now we have that $\sigma \notin w$, as otherwise we could choose $\tau \in w_0$ such that $\sigma \frown \langle 0 \rangle \trianglelefteq \tau$, obtaining an easy contradiction with our assumptions on X. Hence $w = w_0 \cup w_1$. We can now use observation (1) and the inductive hypothesis. $\bigstar_{2.19}$

To complete this discussion of the syntactic properties (N)SOP1, 2 we shall quote a result from [ShUs 844] in which the understanding of SOP_1' and the witnesses for SOP_1 developed here was used to show that NSOP_1 theories admit a rank function.

Definition 2.21 Given (partial) types $p(\bar{x}), q(\bar{y})$ and a formula $\varphi(\bar{x}, \bar{y})$. By induction on $n < \omega$ we define when

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y})) \geq n.$$

$\underline{n = 0.}$ This happens iff both $p(\bar{x})$ and $q(\bar{y})$ are consistent.
$\underline{n + 1.}$ The rank is $\geq n + 1$ iff for some $\bar{c}$ realising $q(\bar{y})$ both

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c})\}, q(\bar{y})) \geq n$$

and

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c}))\}) \geq n.$$

If the rank is $\geq n$ for all n then we say it is inifinite, otherwise we say it is finite.

Theorem 2.22 (Shelah-Usvyatsov [ShUs 844]) A theory T is NSOP_1 iff

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(\bar{x} = \bar{x}, \bar{y} = \bar{y}) < \infty$$

for every formula $\varphi(\bar{x}, \bar{y})$.

3 $\triangleleft^*$-maximality revisited

In this section we come back to our main thesis, which is that properties SOP_2 and the maximality in the $\triangleleft^*$-order are closely connected.

Our main proof will use two auxiliary notions. The first is the order $\triangleleft^{**}_\lambda$, which is a version of the $\triangleleft^*_\lambda$-order.

Definition 3.1 (1) For (complete first order theories) T_1, T_2 and a regular cardinal $\lambda > |T_1|, |T_2|$, let $T_1 \triangleleft^{**}_\lambda T_2$ mean:

There is a λ-relevant (T_1, T_2)-superior $(T^*, \bar\varphi, \bar\psi)$ (see Definition 1.2) such that T^* has Skolem functions and if $T^{**} \supseteq T^*$ is complete with $|T^{**}| < \lambda$ <u>then</u>

($\oplus$) there is a model M of T^{**} of size λ and an $M^{[\bar\psi]}$-type p omitted by M such that for every elementary extension N of M of size λ which omits p and a type q (in one variable) over $N^{[\bar\varphi]}$, there is an elementary extension of N of size λ which realises q and omits p.

(2) Let $T_1 \triangleleft^{**} T_2$ mean that $T_1 \triangleleft^{**}_\lambda T_2$ holds for all large enough regular λ.

(3) T_1 is said to be $\triangleleft^{**}_\lambda$-maximal iff there is no T_2 such that $T_1 \triangleleft^{**}_\lambda T_2$. Similarly for $\triangleleft^{**}$.

The connection between this notion and $\triangleleft^*$ is given by the following claim:

Claim 3.2 Suppose that T_1, T_2 are theories and $\lambda > |T_1|, |T_2|$ satisfies $2^\lambda = \lambda^+$. <u>Then</u>

$$T_1 \triangleleft^*_{\lambda^+} T_2 \implies \neg(T_2 \triangleleft^{**}_\lambda T_1).$$

Proof. This statement is just a reformulation of the beginning of the proof of Theorem 1.17. In other words, let $(T, \bar\varphi_1, \bar\varphi_2)$ show that $T_1 \triangleleft^*_{\lambda^+} T_2$. This means that $|T| < \lambda^+$ but since $\lambda^{<\lambda} = \lambda$ and $\lambda > |T_1|, |T_2|$ we may assume that $|T^*| < \lambda$. Namely since there is a consistent theory $T \supseteq \bar\varphi_1 \cup \bar\varphi_2$ in which $\bar\varphi_l$ interprets T_l, and each T_l has size $< \lambda$, there is a consistent theory T' of size $< \lambda$ which does the same. Without loss of generality $T' \subseteq T$. In particular $|\tau(T')| < \lambda$ so by extending T' to a complete subtheory of T and renaming we may assume T' is complete. Any model M of T has a reduct N that is a model of T' and that satisfies $M^{[\bar\varphi]} = N^{[\bar\varphi]}$ and similarly for $\bar\psi$.

Hence $(T', \bar{\varphi}, \bar{\psi})$ is a λ-relevant (T_1, T_2)-superior that exemplifies $T_1 \lhd^*_{\lambda^+} T_2$, so by renaming we may assume $|T| < \lambda$.

Suppose for contradiction that $T_2 \lhd^{**}_\lambda T_1$ and let $(T^*, \bar{\varphi}, \bar{\psi})$ exemplify this. Without loss of generality, $\bar{\varphi}_1 = \bar{\psi}$ and $\bar{\varphi}_1 = \bar{\varphi}$ and the common vocabulary of T and T^* is $\tau(\bar{\varphi}_1) \cup \tau(\bar{\varphi}_2)$. Hence $T^{**} = T \cup T^*$ is consistent by Robinson Consistency Criterium. Without loss of generality T^{**} is complete. Hence let M be a model of T^{**} of size λ and p be a $M^{[\bar{\psi}]}$ type omitted by M exemplifying the definition of $\lhd^{**}_\lambda$. Using the assumption $2^\lambda = \lambda^+$ we can build by induction an elementary extension N of M with $|N| = \lambda^+$, with N omitting p and being $\bar{\varphi}$-saturated. This is a contradiction with the choice of T. $\bigstar_{3.2}$

Corollary 3.3 Suppose that for all large enough regular λ we have $2^\lambda = \lambda^+$. Then any $\lhd^*$-maximal theory is also $\lhd^{**}$-maximal.

Proof. Suppose otherwise and let T exemplify this. Hence for every κ there is regular $\lambda > \kappa$ such that T is not $\lhd^{**}$-maximal and $2^\lambda = \lambda^+$. Hence T is not $\lhd^*_{\lambda^+}$-maximal by Claim 3.2, a contradiction. $\bigstar_{3.3}$

The next notion we need is a syntactic property.

Definition 3.4 Let T be a theory.
(1) For a formula $\sigma(x, \bar{y})$ we say that $\sigma(x, \bar{y})$ has SOP''_2 iff for some [by compactness equivalently all] regular $\lambda > |T|$ there is a sequence

$$\left\langle \bar{e}_{\bar{\eta}} : \bar{\eta} = \langle \eta_0, \ldots \eta_{n^*-1} \rangle, \eta_0 \lhd \eta_1 \lhd \ldots \lhd \eta_{n^*-1} \in {}^{\lambda >}\lambda \text{ and } lg(\eta_i) \text{ a successor} \right\rangle$$

such that

(α) for each $\eta \in {}^\lambda\lambda$, the set

$$\left\{ \begin{array}{c} \sigma(x, \bar{e}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \restriction (\alpha_0 + 1), \eta \restriction (\alpha_1 + 1), \ldots \eta \restriction (\alpha_{n^*-1} + 1) \rangle \\ \text{and } \alpha_0 < \alpha_1 < \ldots \alpha_{n^*-1} < \lambda \end{array} \right\}$$

is consistent

(β) for every large enough m, $\underline{\text{if}}$ $g : {}^{n^* \geq}m \to {}^{\lambda >}\lambda$ satisfies

$$\rho \lhd \nu \implies g(\rho) \lhd g(\nu)$$

and

$$\rho \in {}^{n\geq}m \implies lg(g(\rho)) \text{ is a successor,}$$

while for $l < n^* - 1$

$$(g(\rho)) \frown \langle l \rangle \trianglelefteq g(\rho \frown \langle l \rangle),$$

$\underline{\text{then}}$

$$\{\sigma(x, \bar{e}_{\langle g(\rho\restriction 1), g(\rho\restriction 2), \ldots g(\rho)\rangle}) : \rho \in {}^{n^*}m\}$$

is inconsistent. Here $n^* = lg(\bar{y})$ in $\sigma(x, \bar{y})$.

(2) T is said to have SOP_2'' iff some $\sigma(x, \bar{y})$ exemplifies it.

Our theorem 3.6 is phrased in terms of SOP_2''. Answering a question from an earlier version of this paper Shelah and Usvyatsov proved in [ShUs 844] the following Theorem 3.5, which then can be used together with theorem 3.6 to prove Corollary 3.9 which states that $\lhd^*$-maximality implies SOP_2.

Theorem 3.5 (Shelah-Usvyatsov [ShUs 844]) For any theory T, T has SOP_2 iff it has SOP_2''.

Main Theorem 3.6 For any theory T and regular cardinal $\lambda > |T|$, $\underline{\text{if}}$ T is $\lhd^{**}_\lambda$-maximal $\underline{\text{then}}$ T has SOP_2''.

Proof. Let T be a given theory and let $\lambda = \mathrm{cf}(\lambda) > |T|$. We shall assume that T is $\lhd^{**}_\lambda$-maximal and prove that T has SOP_2''. To make the reading of the proof easier we shall break it into stages.

Stage A. Let $T^n_{\mathrm{tree}} \stackrel{\mathrm{def}}{=} \mathrm{Th}({}^{n\geq}2, <_{\mathrm{tr}})$ for $n < \omega$, where $<=<_{\mathrm{tr}}$ stands for the relation of "being an initial segment of", and let $T_{\mathrm{tree}} \stackrel{\mathrm{def}}{=} \lim\langle T^n_{\mathrm{tree}} : n < \omega\rangle$, that is to say the set of all ψ which are in T^n_{tree} for all large enough n. In order to use our assumptions at a later point, let us fix a theory T^* which is a λ-relevant (T_{tree}, T)-superior with Skolem functions (such a T^* is easily seen to exist), and let $\bar{\varphi}$, $\bar{\psi}$ be the interpretations of T_{tree} and T in T^*, respectively. We can without loss of generality, by renaming if necessary, assume that $\mathcal{L}(T) \subseteq \mathcal{L}(T^*)$, so the interpretation $\bar{\psi}$ is trivial.

As $|T|, |T^*| < \lambda$, we can find $A \subseteq \lambda$ which codes T and T^*. Working in $\mathbf{L}[A]$, we shall define a model M of T^* of size λ as follows. Let

$$\begin{aligned}\Gamma \stackrel{\text{def}}{=} T^* &\cup \{\varphi_=(x_\eta, x_\eta) : \eta \in {}^{\lambda>}\lambda\}\\ &\cup \{x_\eta <_\varphi x_\nu : \eta \triangleleft \nu \in {}^{\lambda>}\lambda\}\\ &\{\neg(x_\eta <_\varphi x_\nu) : \neg(\eta \triangleleft \nu) \text{ for } \eta, \nu \in {}^{\lambda>}\lambda)\}.\end{aligned}$$

By a compactness argument and the fact that $\bar{\varphi}$ interprets T_{tree} in T^*, we see that Γ is consistent. Let M be a model of Γ of size $\lambda = \lambda^{<\lambda}$ (as we are in $\mathbf{L}[A]$). For $\eta \in {}^{\lambda>}\lambda$ let a_η be the realisation of x_η in M. For $\eta \in {}^{\lambda}\lambda$, let

$$p_\eta(x) \stackrel{\text{def}}{=} \{a_{\eta\restriction\alpha} <_\varphi x : \alpha < \lambda\}$$

By the choice of M and the compactness argument it follows that each p_η is a (consistent) type. Note that for $\eta_0 \neq \eta_1 \in {}^{\lambda}\lambda$, types p_{η_0} and p_{η_1} are contradictory. Let

$$p'_\eta(x) = \{a <_\varphi x : \text{for some } \alpha < \lambda, a <_\varphi a_{\eta\restriction\alpha}\}.$$

By the axioms of T_{tree}, we have that p_η and p'_η are equivalent. Now we observe that by the size of M there is $\eta^* \in {}^{\lambda}\lambda$ such that the type p'_{η^*} is omitted in M, and p'_{η^*} is not definable in M, i.e. for no formula $\vartheta(y, \bar{z})$ and $\bar{c} \subseteq M$ do we have: for $a \in M$, the following are equivalent: $[a <_\varphi x] \in p'_{\eta^*}$ and $M \models \vartheta[a, \bar{c}]$. Let $p \stackrel{\text{def}}{=} p'_{\eta^*}$ for such a fixed η^*. For $\alpha < \lambda$, let $a_\alpha \stackrel{\text{def}}{=} a_{\eta^*\restriction\alpha}$. We now go back to V and make an observation about M.

Subclaim 3.7 T_{tree} satisfies the following property:

for any formula $\vartheta(x, \bar{y})$ we have that $T_{\text{tree}} \vdash \sigma = \sigma(\vartheta)$, where

$$\begin{aligned}\sigma \equiv (\forall \bar{y})[[(\forall x_1, x_2))\vartheta(x_1, \bar{y}) \wedge \vartheta(x_2, \bar{y}) \implies x_1 \leq_{\text{tr}} x_2 \vee x_2 \leq_{\text{tr}} x_1)]\\ \implies (\exists z)(\forall x)(\vartheta(x, \bar{y}) \implies x \leq_{\text{tr}} z)].\end{aligned}$$

Proof of the Subclaim. Let $\vartheta(x, \bar{y})$ be given. By the definition of T_{tree} we only need to show that $T^n_{\text{tree}} \vdash \sigma$ for all large enough n, which is obvious as for every n the tree ${}^{n\geq}2$ has the top level. $\bigstar_{3.7}$

Hence the interpretation $\bar{\varphi}$ of $T_{\rm tree}$ in T^* satisfies the same statement claimed about $T_{\rm tree}$. We conclude:

$\otimes$ if $M \prec N$ and p is not realised in N, then there is no $\vartheta(x, \bar{c})$ with $\bar{c} \subseteq N$ such that $\vartheta(a_{\eta^* \restriction \alpha}, \bar{c})$ for all $\alpha < \lambda$ holds and every two elements of N satisfying $\vartheta(x, \bar{c})$ are $<_\varphi$-comparable.

Stage B. We shall choose a filtration $\bar{M} = \langle M_i : i < \lambda \rangle$ of M, and an increasing sequence $\langle \alpha_i : i < \lambda \rangle$, requiring:

(a) $M_i \prec M$ and M_i are $\prec$-increasing continuous of size $< \lambda$, with M being the $\bigcup_{i<\lambda} M_i$,

(b) $a_{\alpha_i} \in M_{i+1} \setminus M_i$.

We may note that the branch induced by $\{a_{\alpha_i} : i < \lambda\}$ is the same as the one induced by $\{a_\alpha : \alpha < \lambda\}$. Hence p is realised in any model in which $p'(x) \stackrel{\rm def}{=} \{a_{\alpha_i} <_\varphi x : i < \lambda\}$ is realised (or even the similarly defined type using any unbounded subset of $\{\alpha_i : i < \lambda\}$). Hence, by renaming, without loss of generality we have $\alpha_i = i$ for all $i < \lambda$.

Stage C. At this point we shall use the $\triangleleft^{**}_\lambda$-maximality of T, which implies that it is not true that $T \triangleleft^{**}_\lambda T_{\rm tree}$. In particular, our T^*, M and p do not exemplify this, hence there is N with $M \prec N$ and $||N|| = \lambda$, such that N omits p, but for some $N^{[\bar{\psi}]}$-type q over N, whenever $N \prec N^+$ and N^+ realises q, also N^+ realises p. By $\otimes$, the branch induced by $\{a_{\eta^*} \restriction \alpha : \alpha < \lambda\}$ is not definable in N, so without loss of generality $N = M$. We can also assume that q is a complete type over $M^{[\bar{\psi}]}$. Let us now use the choice of q to define for each club E of λ a family of formulae associated with it, and to show that each of these families is inconsistent. We use the abbreviation c.d. for "the complete diagram of".

For any club E of λ we define

$$\Gamma_E \stackrel{\rm def}{=} \text{c. d.}(M) \cup q(x) \cup \{\neg(a_i <_\varphi \tau(x, \bar{b})) : i \in E, \tau \text{ a term of } T^*, \bar{b} \subseteq M_i\}.$$

Clearly, for any club E, if Γ_E is consistent then there is a model N in which Γ_E is realised. Identifying any $b \in M$ with its interpretation in N and letting a^* be the interpretation of x from Γ_E, we can assume that N is an elementary extension of M in which q is realised by a^*. As T^* has Skolem functions, we have $M \prec N$. Let N_1 be the submodel of N with universe

$$A^* \stackrel{\text{def}}{=} M \cup \bigcup_{i \in E} \{\tau(a^*, \bar{b}) : \bar{b} \subseteq M_i \text{ and } \tau \text{ a term of } T^*\}.$$

Note that the size of N_1 is λ. Clearly, N_1 is closed under the functions of T^*, so $M \subseteq N_1 \subseteq N$. As T^* has Skolem functions, we get that $M \prec N_1 \prec N$. By the third part of the definition of Γ_E, p is omitted in N_1. This is in contradiction with our assumptions, as $a^* \in N_1$ realises $q(x)$.

Hence we can conclude

$$\text{for every club } E \text{ of } \lambda, \text{ the set } \Gamma_E \text{ is inconsistent.}$$

Stage D. Now we start our search for a formula that exemplifies that T has SOP''_2. In the following definitions, we shall use the expression "an almost branch" or the abbreviation a.b. to stand for a set linearly ordered by $<_\varphi$ (but not necessarily closed under $<_\varphi$-initial segments and not necessarily unbounded). Let

$$\Theta^0_{T^*} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} \vartheta(x, y, \bar{z}) : & \text{there is } l = l_\vartheta < \omega \text{ such that} \\ & \text{for every } M^* \models T^*, a \in M^*, \bar{c} \subseteq M^*, \text{ the set} \\ & \vartheta(a, y, \bar{c})^{M^*} \text{ is the union of } \leq l \text{ a.b. in } M^{*[\bar{\varphi}]} \end{array} \right\},$$

and let Θ_{T^*} be the set of all $\vartheta(x, \bar{y}, \bar{z})$ of the form $\bigvee_{j<n} \vartheta_j(x, y_j, \bar{z}_j)$ for some $\vartheta_0, \ldots \vartheta_{n-1} \in \Theta^0_{T^*}$ (where $\bar{y} = \langle y_j : j < n \rangle$ and $\bar{z} = \frown_{j<n} \bar{z}_j$). The formulae in Θ_{T^*} will be called candidates. For every candidate

$$\vartheta(x, \bar{y}, \bar{z}) \equiv \bigvee_{j<n} \vartheta_j(x, y_j, \bar{z}_j)$$

and a $\bar{\psi}$-formula $\sigma(x, \bar{t})$, we consider the following game $\Game_{n,\sigma,\vartheta}$ (whose definition also depends on our fixed p, q and $\bar{M}$), played by two players $\exists$ and $\forall$. The game starts by $\exists$ playing $\bar{b}^0$ from ${}^{\lg(\bar{z}_0)}M$, then $\forall$ playing $\alpha_0 < \lambda$. After that $\exists$ chooses $\beta_0 \in (\alpha_0, \lambda)$ and $\bar{b}^1 \in {}^{\lg(\bar{z}_1)}M$ such that $\bar{b}^0 \in {}^{\lg(\bar{z}_0)}M_{\beta_0}$, after which $\forall$ chooses $\alpha_1 < \lambda$ etc., finishing by $\exists$ choosing $\bar{b}^{n-1} \in {}^{\lg(\bar{z}_{n-1})}M$ and $\forall$

choosing α_{n-1}, while $\exists$ chooses $\beta_{n-1} \in (\alpha_{n-1}, \lambda)$ such that $\bar{b}^{n-1} \in {}^{\lg(\bar{z}_{n-1})} M_{\beta_{n-1}}$. Player $\exists$ wins this game iff for some $\bar{e} \in {}^{\lg(\bar{t})} M$ we have

$$\sigma(x, \bar{e}) \in q \text{ and } M \models (\forall x)[\sigma(x, \bar{e}) \implies \vartheta(x, \langle a_{\beta_0}, \ldots, a_{\beta_{n-1}} \rangle, \widehat{{}_{k<n}} \bar{b}^k)]. \quad (\otimes_1)$$

(Note: the constants a_{β_k} are from the set $\{a_i : i < \lambda\}$ we chose above.) Observe that every sequence $\langle \alpha_0, \ldots \alpha_{n-1} \rangle \in {}^n\lambda$ is an admissible sequence of moves for $\forall$.

We shall show that for some $n \geq 1$ and σ, ϑ, player $\exists$ has a winning strategy in the game $\Game_{n,\sigma,\vartheta}$, where $\vartheta = \bigvee_{j<n} \vartheta_j$ as above. As these are determined games, it suffices to show that for some $n \geq 1$ and σ, ϑ, player $\forall$ does not have a winning strategy. Suppose that this is not the case, arguing in $(\mathcal{H}(\chi), \in, <^*_\chi, \bar{M}, p, q)$, where χ is large enough and $<^*_\chi$ is a fixed well ordering of $\mathcal{H}(\chi)$. Fix for a moment (n, σ, ϑ). Player $\forall$ has a winning strategy in $\Game_{n,\sigma,\vartheta}$, which, replacing the ordinals α_l by constants a_{α_l}, can be represented by a sequence of functions $G^l_{n,\sigma,\vartheta}$ for $l < n$ (in $(\mathcal{H}(\chi), \in, <^*_\chi, \bar{M}, p, q)$), where for $l < n$, if the play up to time l has been $\bar{b}_0, \alpha_0, \beta_0, \ldots, \alpha_{l-1}, \beta_{l-1}, \bar{b}^l$, then $G^l_{n,\sigma,\vartheta}$ applied to this play is a_{α_l} for the α_l in the choice of player $\forall$. We shall assume that these functions are the $<^*$-first which can act in this manner. Using this and elementarity, we notice that for every n, σ, ϑ the values of $G^l_{n,\sigma,\vartheta}$ take place in M, and that

$$E_0 \stackrel{\text{def}}{=} \{\delta < \lambda : (\forall \sigma, \vartheta)(\forall n)(\forall l < n)[M \cap \text{ Skolem}_{(\mathcal{H}(\chi), \in, \bar{M}, G^l_{n,\sigma,\vartheta})}(M_\delta) = M_\delta]\}$$

is a club of λ (as $|T^*|$, $||M_i|| < \lambda$ for all i and $\bar{M}$ is increasing continuous). Let $E \stackrel{\text{def}}{=} \text{acc}(E_0)$. Consider now the set Γ_E. It is contradictory, so there is a finite subset of it which is contradictory. Hence for some $n_0, n_1, n_2 < \omega$ and formulae $\varrho_l(\bar{z}_l)$ $(l < n_0)$ from the c.d.(M), formulae $\sigma_k(x, \bar{e}_k)$ $(k < n_1) \in q(x)$, ordinals $\delta_0 < \ldots < \delta_{n_2-1} \in E$, a sequence $\langle \bar{b}_{j,l} : j < n_2, l < l_j \rangle$ with $\bar{b}_{j,l} \subseteq M_{\delta_j}$ and terms $\langle \tau_{j,l} : j < n_2, l < l_j \rangle$ of T^*, the following is inconsistent:

$$\bigwedge_{l<n_0} \varrho_l(\bar{z}_l) \wedge \bigwedge_{k<n_1} \sigma_k(x, \bar{e}_k) \wedge \bigwedge_{j<n_2, l<l_j} \neg \left(a_{\delta_j} <_\varphi \tau_{j,l}(x, \bar{b}_{j,l}) \right).$$

As ϱ_l come from the c.d.(M) and $q(x)$ is a complete type over $M^{[\bar{\psi}]}$, we may assume that $n_0 = 1$ and $n_1 = 1$. Note that we must have $n_2 \geq 1$ and that there is no loss of generality in assuming that $\bar{b}_{j,l} = \bar{b}_j$ for all $l < l_j$ for $j < n$. We shall omit the subscript 0 from $\varrho, \sigma, \bar{e}$. Let $n = n_2$ and let us define $\vartheta_j(x, y_j, \bar{z}_j)$ for $j < n$ by

$$\vartheta_j(x, y_j, \bar{z}_j) \equiv \bigvee_{l<l_j} y_j <_\varphi \tau_{j,l}(x, \bar{z}_j),$$

and let $\vartheta = \bigvee_{j<n} \vartheta_j$. Note that for each j we have that $\vartheta_j \in \Theta^0_{T^*}$, as $<_\varphi$ is a tree order. Hence ϑ is a candidate, $\sigma(x, \bar{e}) \in q(x)$, and since $M \models \varrho[\bar{d}]$ for some $\bar{d}$ we have

$$M \models (\forall x)[\sigma(x, \bar{e}) \implies \bigvee_{j<n} \vartheta_j(x, a_{\delta_j}, \bar{b}_j)]. \qquad (*)$$

Now we consider the following play of $\Game_{n,\sigma,\vartheta}$. Let $\exists$ choose $\bar{b}_0$. Recall that $\bar{b}_0 \subseteq M_{\delta_0}$. The strategy $G^0_{n,\sigma,\vartheta}$ of $\forall$ yields an ordinal α_0. By the choice of E_0 we have $\alpha_0 < \delta_0$ and $\bar{b}_0 \in M_{\delta_0}$, so we can let $\exists$ choose $\beta_0 = \delta_0$. Let $\exists$ choose $\bar{b}_1$ and then let $\forall$ choose α_1 according to the strategy, etc. At the end of the play, player $\forall$ should have won (as he/she used the supposed winning strategy), but clearly $(*)$ implies that $\exists$ won, a contradiction.

Stage E. We conclude that (for our $\lambda, \bar{M}, p, q$), for some σ, ϑ and $n \geq 1$ the player $\exists$ has a winning strategy in the game $\Game_{n,\sigma,\vartheta}$, call it St. Let us fix $n = n^*, \sigma, \vartheta$, and St and use them to get SOP''_2.

For any $\bar{\alpha} = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in {}^n\lambda$, we can let $\langle \bar{b}^{\bar{\alpha}\restriction k}, \beta^{\bar{\alpha}\restriction(k+1)} : k < n \rangle$ be the sequence of moves that $\exists$ plays by following the winning strategy St in a play in which $\forall$ plays $\bar{\alpha}$, as the dependence is as marked. Let E be a club of λ such that if $k \leq n$ and $\alpha_0 < \ldots < \alpha_{k-1} < \delta \in E$, then $\bar{b}^{\langle \alpha_0, \ldots, \alpha_{k-1} \rangle} \in {}^{\lg(\bar{z}_j)} M_\delta$. (Such a club can be found by a method similar to the one used in Stage D). Renaming the M_i and a_i's, we can without loss of generality assume that $E = \lambda$. For $\bar{\alpha} \in {}^n\lambda$ let $\bar{e}^{\bar{\alpha}}$ be such that:

$$M \models \forall x[\sigma(x, \bar{e}^{\bar{\alpha}}) \implies \bigvee_{j<n} \vartheta_j(x, a_{\beta^{\bar{\alpha}\restriction(j+1)}}, \bar{b}_j^{\bar{\alpha}\restriction(j+1)})].$$

Notice that σ is a formula in the language of T. We shall show that σ, together with a conveniently chosen sequence of $\bar{e}_{\bar{\eta}}$'s, exemplifies SOP''_2. The proof now proceeds similarly to the proof of Main Claim 1.13. Namely

Lemma 3.8 There are sequences

$$\langle N_\eta : \eta \in {}^{\lambda >}\lambda \rangle, \langle h_\eta : \eta \in {}^{\lambda >}\lambda \rangle$$

such that

(i) h_η is an elementary embedding of $M_{lg(\eta)}$ into $\mathfrak{C}_{T^*}$ with range N_η,

(ii) $\nu \trianglelefteq \eta \implies h_\nu \subseteq h_\eta$,

(iii) for $\alpha \neq \beta < \lambda$ and $\eta \in {}^{\lambda >}\lambda$ we have

$$h_{\eta \frown \langle \alpha \rangle}(a_{lg(\eta)}) \perp_\varphi h_{\eta \frown \langle \beta \rangle}(a_{lg(\eta)}),$$

(iv) $N_{\eta_0} \cap N_{\eta_1} = N_{\eta_0 \cap \eta_1}$ for all η_0, η_1.

Proof of the Lemma. This Lemma has the same proof as that of Main Claim 1.13 Stage B. In the notation of that proof, ignore b_{δ_i}. When defining Γ use

$$\Gamma = \cup_{\alpha<\lambda}\Gamma_0^\alpha \cup \cup_{\alpha<\lambda}\Gamma_3^\alpha \cup \Gamma_4 \cup \Gamma_2^+,$$

where $\Gamma_2^+ = \{x_0^\alpha \perp_\varphi x_0^\beta : \alpha \neq \beta < \lambda\}$ and $\Gamma_0^\alpha, \Gamma_3^\alpha$ and Γ_4 are defined as in the proof of Main Claim 1.13, allowing for the replacement of ${}^{\lambda >}2$ by ${}^{\lambda >}\lambda$ by using $\{\bar{x}^\alpha : \alpha < \lambda\}$ in place of $\{\bar{x}^0, \bar{x}^1\}$. Assumptions on $\Gamma_0^\alpha, \Gamma_2^+$ and Γ_3^α are analogous to the ones we made in that proof. Fact 1.16 still holds, except that we drop the last set from the definition of $r(\bar{x})$. The rest of the proof is the same, recalling that the branch induced by $\{a_i : i < \lambda\}$ is undefinable in M. $\bigstar_{3.8}$

Stage F. For $\eta \in {}^\lambda\lambda$, let $h_\eta \stackrel{\text{def}}{=} \cup_{\alpha<\lambda} h_{\eta\restriction\alpha}$. Let $q_\eta \stackrel{\text{def}}{=} h_\eta(q)$, hence each q_η is a consistent type. For $\bar{\eta} = \langle \eta_0, \ldots, \eta_{n-1} \rangle$ and $\eta_0 \lhd \ldots \lhd \eta_{n-1}$ with $lg(\eta_i) = \alpha_i + 1$, let $\bar{e}_{\bar{\eta}} \stackrel{\text{def}}{=} h_{\eta_{n-1}}(\bar{e}^{\langle \alpha_0, \ldots \alpha_{n-1}\rangle})$.

Suppose now that $\eta \in {}^\lambda\lambda$ is given, and consider the set

$$\{\sigma(x, \bar{e}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \restriction (\alpha_0 + 1), \ldots \eta \restriction (\alpha_{n-1} + 1) \rangle \text{ for some } \alpha_0 < \ldots \alpha_{n-1} < \lambda\}.$$

This set is a subset of q_η, and is hence consistent. This proves property (α) from the definition of SOP_2''. For (β), let m be large enough and $g : {}^{n \geq}m \to {}^{\lambda >}\lambda$

be as in the statement of (β). For $\rho \in {}^n m$ let $\bar{e}_{g_\rho} \stackrel{\text{def}}{=} \bar{e}_{\langle g(\rho\restriction 1),\dots g(\rho)\rangle}$ (note that this is always defined). We shall now show that the set

$$\{\sigma(x, \bar{e}_{g_\rho}) : \rho \in {}^n m\}$$

is inconsistent. Suppose otherwise, so let $d \in \mathfrak{C}_{T^*}$ realise it. For each $\rho \in {}^n m$, let $\eta_\rho \in {}^\lambda\lambda \supseteq g(\rho)$ and let $\bar{\alpha}^\rho \stackrel{\text{def}}{=} \langle \alpha^\rho_0, \dots, \alpha^\rho_{n-1}\rangle$ satisfy $lg(g(\rho \restriction k)) = \alpha^\rho_k + 1$ for $k \le n$, so for each $k < n$ we have $g(\rho \restriction (k+1)) = \eta_\rho \restriction (\alpha^\rho_k + 1)$. Now we have that for each $\rho \in {}^n m$

(i) $\sigma(x, \bar{e}_{g_\rho}) \equiv \sigma(x, h_{\eta_\rho \restriction(\alpha^\rho_{n-1}+1)}(\bar{e}^{\bar{\alpha}^\rho})) \in q_{\eta_\rho} \restriction \sigma_{\eta_\rho}(x)$

(ii) $N_{\eta_\rho} \models (\forall x)[\sigma(x, \bar{e}_{g_\rho}) \implies \vartheta(x, \langle h_{\eta_\rho}(a_{\beta_{\bar{\alpha}^\rho\restriction 1}}), \dots h_{\eta_\rho}(a_{\beta_{\bar{\alpha}^\rho}})\rangle, \frown_{j<n} h_{\eta_\rho}(\bar{b}^{\bar{\alpha}^\rho_j}))]$ (hence the same holds in $\mathfrak{C}_{T^*}$),

(iii)

$$\vartheta(x, \langle h_{\eta_\rho}(a_{\beta_{\bar{\alpha}^\rho\restriction 1}}), \dots h_{\eta_\rho}(a_{\beta_{\bar{\alpha}^\rho}})\rangle, \frown_{j<n} h_{\eta_\rho}(\bar{b}^{\bar{\alpha}^\rho_j})) \implies \bigvee_{j<n} \vartheta_j(x, h_{\eta_\rho}(a_{\beta_{\bar{\alpha}^\rho\restriction(j+1)}}), h_{\eta_\rho}(\bar{b}_j^{\bar{\alpha}^\rho\restriction(j+1)}))$$

for our $\vartheta_0, \dots \vartheta_{n-1}$.

For each $\rho \in {}^n m$ let $j(\rho) < n$ be the first such that

$$\vartheta_j(d, h_{\eta_\rho}(a_{\beta_{\bar{\alpha}^\rho\restriction(j+1)}}), h_{\eta_\rho}(\bar{b}_j^{\bar{\alpha}^\rho\restriction(j+1)}))$$

holds. Let $l^* = \max\{l^\vartheta_0, \dots, l^\vartheta_{n-1}\}$.

As m is large enough, there are $\rho_0, \dots, \rho_{l^*} \in {}^n m$ such that $j(\rho_s) = j^*$ for all $s \in \{0, \dots, l^*\}$, while $\rho_s \restriction j^*$ is fixed and $\rho_s(j^*) \ne \rho_t(j^*)$ for $s \ne t \le l^*$. (We use that there is a full ${}^{l^*+1\ge}n$ subtree t^* of ${}^{n\ge}m$ such that for all $\rho \in t^* \cap {}^n m$ we have $j(\rho) = j^*$. Choose ρ_s belonging to t^* and splitting at the level j^*). In particular, $\alpha^{\rho_s}_0 = \alpha_0, \dots, \alpha^{\rho_s}_{j^*-1} = \alpha_{j^*-1}$ is fixed, and so is $h_{\eta_{\rho_s}} \restriction M_{\alpha^*_{j-1}+1}$, but

$$g(\rho_s) \restriction (\alpha_{j^*-1} + 2) \text{ for } s \le l^* \text{ are incomparable in } {}^\lambda\lambda. \qquad (**)$$

Let $\bar{\alpha} \stackrel{\text{def}}{=} \bar{\alpha}^{\rho_0}$.

For each $\rho \in {}^n m$ and $k < n$ we have that $\bar{b}^{\bar{\alpha}^{\rho \restriction (k+1)}} \in M_{\alpha^\rho_{k+1}}$ (by the choice of E), so in particular $\bar{b}^{\bar{\alpha}^{\rho \restriction j^*}} \in M_{\alpha^\rho_{j^*-1}+1}$, and hence $h_{\eta_{\rho_s}}(\bar{b}^{\bar{\alpha}^{\rho \restriction j^*}})$ is a fixed $\bar{b}^*$. By the choice of d and definitions of j^*, l^* and Θ_{T^*}, there are $s \neq t < l_{\vartheta_{j^*}} \leq l^*$ such that $h_{\eta_{\rho_s}}(a_{\beta^{\bar{\alpha}^{\rho_s} \restriction (j^*+1)}})$ and $h_{\eta_{\rho_t}}(a_{\beta^{\bar{\alpha}^{\rho_t} \restriction (j^*+1)}})$ are on the same almost branch. Now note that for all ρ we have

$$a_{\beta^{\bar{\alpha}^{\rho \restriction (j^*+1)}}} \in M_{\beta^{\bar{\alpha}^{\rho \restriction (j^*+1)}}+1} \setminus M_{\beta^{\bar{\alpha}^{\rho \restriction (j^*+1)}}}$$

and $\beta^{\bar{\alpha}^{\rho \restriction (j^*+1)}} > \alpha^\rho_{j^*}$. Hence $h_{\eta_{\rho_s}}(a_{\beta^{\bar{\alpha}^{\rho_s} \restriction (j^*+1)}})$ and $h_{\eta_{\rho_t}}(a_{\beta^{\bar{\alpha}^{\rho_t} \restriction (j^*+1)}})$ are incomparable, by property (iii) in Lemma 3.8, a contradiction. This shows (β) from the definition of SOP_2'', so finishing the proof. $\bigstar_{3.6}$

Putting this together with Corollary 3.3 and Shelah-Usvyatsov theorem 3.5 above we get the following corollary 3.9.

Corollary 3.9 (1) Suppose that T is a theory that is $\triangleleft^*$-maximal in some universe of set theory in which $2^\lambda = \lambda^+$ holds for all large enough regular λ. Then T has SOP_2.
(2) Suppose that T is a theory that is $\triangleleft^*_{\lambda^+}$-maximal in some universe of set theory in which λ is regular and $2^\lambda = \lambda^+$. Then T has SOP_2.

Proof. (1) Let W be a universe of set theory in which $2^\lambda = \lambda^+$ holds for all large enough regular λ and in which T is $\triangleleft^*$-maximal. Hence by Corollary 3.3 T is $\triangleleft^{**}$-maximal in W and hence by Main Theorem 3.6 in W it satisfies SOP_2''. By Shelah-Usvyatsov Theorem 3.5 above T satisfies SOP_2 in W. An application of the Compactness Theorem shows that satisfying SOP_2 is absolute, hence T satisfies SOP_2 in V.
(2) This follows similarly, but more directly, from Main Theorem 3.6 and the Shelah-Usvyatsov Theorem 3.5. $\bigstar_{3.9}$

This section hence provides us with the proof of one side of our thesis that SOP_2 and $\triangleleft^*$-maximality are closely connected. Recall that Shelah proved in [Sh 500] that SOP_3 implies $\triangleleft^*$-maximality. So an important open question (provided that SOP_3 are not actually equivalent, which we still do not know) is

Question 3.10 Does SOP_2 imply $\triangleleft^*$-maximality?

In a partial answer to this question posed in an earlier version of the paper Shelan and Usvyatsov in Theorem 3.12 of [ShUs 844] provided a local positive answer to this question, where by "local" we mean that they proved that any theory with SOP_2 is $\triangleleft^*$ above T_{tree} when only types localised by a certain formula are considered (see Definition 1.3).

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