MODELS OF EXPANSIONS OF $\mathbb{N}$ WITH NO END EXTENSIONS

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ABSTRACT. We deal with models of Peano arithmetic (specifically with a question of Ali Enayat). The methods are from creature forcing. We find an expansion of $\mathbb{N}$ such that its theory has models with no (elementary) end extensions. In fact there is a Borel uncountable set of subsets of $\mathbb{N}$ such that expanding $\mathbb{N}$ by any uncountably many of them suffice. Also we find arithmetically closed $\mathcal{A}$ with no ultrafilter on it with suitable definability demand (related to being Ramsey).

0. INTRODUCTION

Recently, solving a long standing problem on models of Peano arithmetic, (appearing as Problem 7 in the book [**?**]), Ali Enayat proved (and other results as well):

Theorem 0.1. *[See [**?**]] For some arithmetically closed family $\mathcal{A}$ of subsets of ω, the model $\mathbb{N}_{\mathcal{A}} = (\mathbb{N}, A)_{A \in \mathcal{A}}$ has no conservative extension (i.e., one in which the intersection of any definable subset with $\mathbb{N}$ belongs to $\mathcal{A}$).*

Motivated by this result he asked:

Question 0.2. Is there $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that some model of $\mathrm{Th}(\mathbb{N}_{\mathcal{A}})$ has no elementary end extension?

This asks whether the countability demand in the MacDowell-Specker theorem is necessary. This classical theorem says that if T is a theory in a countable vocabulary $\tau = \tau_T$ extending $\tau(\mathbb{N}) = \{0, 1, +, \times\}$ and T contains $\mathrm{PA}(\tau)$, then any model of T has an (elementary) end extension; Gaifman continues this theorem in several ways, e.g., having minimal extensions (see [**?**] on it). The author [**?**] continues it in another way: we do not need addition and multiplication, i.e., any model of T has an elementary end extension when τ is a countable vocabulary, $\{0, <\} \subseteq \tau$, T is a (first order) theory in $\mathbb{L}(\tau)$, T says that $<$ is a linear order with 0 first, every element x has a successor $S(x)$, and all cases of the induction scheme belong to T.

Mills [**?**] prove that there is a countable non-standard model of PA with uncountable vocabulary such that it has no elementary end extension.

We answer the question 0.2 positively in §4, we give a sufficient condition in §2 and deal with a relevant forcing in §3. In fact we get an uncountable Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that if $B_\alpha \in \mathbf{B}$ for $\alpha < \alpha_*$ are pairwise distinct and α_* is uncountable, then $\mathrm{Th}(\mathbb{N}, B_\alpha)_{\alpha < \alpha_*}$ satisfies the conclusion.

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Enayat [?] also asked:

Question 0.3. Can we prove in ZFC that there is an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}$ carries no minimal ultrafilter?

He proved it for the stronger notion of *2-Ramsey ultrafilter*. We hope to deal with the problem later (see [?]); here we prove that there is an arithmetically closed Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that any expansion $\mathbb{N}$ by any uncountably many members of $\mathbf{B}$ has such a property, i.e., the family of definable subsets of $\mathbb{N}^+$ carry no *2.5-Ramsey ultrafilter*.

Note that

- $(*)$ if $N \neq \mathbb{N}$ is a model of PA which has no cofinal minimal extension, then on $\mathrm{StSy}(N)$ there is no minimal ultrafilter, see Definitions 0.6, 0.7(1).

Enayat also asks:

Question 0.4. For a Borel set $\mathcal{A} \subseteq \mathcal{P}(\omega)$:

- (a) does the model $\mathbb{N}_{\mathcal{A}}$ have a conservative end extension? This is what is answered here (in the light of the previous paragraph).
- (b) Suppose further that $\mathcal{A}$ is arithmetically closed. Is $(\mathcal{A} \cap [\omega]^{\aleph_0}, \supseteq)$ a proper forcing notion?

The results here solve 0.4(a) and the second, 0.4(b), is solved in Enayat-Shelah [?].

Enayat suggests that if we succeed to combine an example for "$\mathrm{StSy}(N)$ has no minimal ultrafilter" and Kaufman-Schmerl [?], then we shall solve the "there is N with no cofinal minimal extension" (Problem 2 of [?]).

Note that our claim on the creature forcing gives suitable kinds of Ramsey theorems.

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Notation 0.5.
1. As usual in set theory, ω is the set of natural numbers. Let $\mathrm{pr} : \omega \times \omega \longrightarrow \omega$ be the standard pairing function (i.e., $\mathrm{pr}(n,m) = \binom{n+m}{2} + n$, so one–to–one onto two–place function).
2. Let $\mathcal{A}$ denote a subset of $\mathcal{P}(\omega)$.
3. The Boolean algebra generated by $\mathcal{A} \cup [\omega]^{<\aleph_0}$ will be denoted by $\mathrm{BA}(\mathcal{A})$.
4. Let D denote a non-principal ultrafilter on $\mathcal{A}$. When $\mathcal{A}$ is not a sub-Boolean-Algebra of $\mathcal{P}(\omega)$, this means that $D \subseteq \mathcal{A}$ and there is a unique non-principal ultrafilter D' on the Boolean algebra $\mathrm{BA}(\mathcal{A})$ such that $D = D' \cap \mathcal{A}$. (In 0.7 this extension makes a difference.)
5. Let τ denote a vocabulary extending $\tau_{\mathrm{PA}} = \tau_{\mathbb{N}} = \{0, 1, +, \times, <\}$, usually countable.
6. $\mathrm{PA}_\tau = \mathrm{PA}(\tau)$ is Peano arithmetic for the vocabulary τ.
7. A model N of $\mathrm{PA}(\tau)$ is *ordinary* if $N{\restriction}\tau_{\mathrm{PA}}$ extends $\mathbb{N}$; usually our models will be ordinary.
8. $\varphi(N, \bar{a})$ is $\{b : N \models \varphi[b, \bar{a}]\}$, where $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ and $\bar{a} \in {}^{\ell g(\bar{y})}N$.
9. $\mathrm{Per}(A)$ is the set (or group) of permutations of A.

(10) For sets u, v of ordinals let $\mathrm{OP}_{v,u}$, "the order preserved function from u to v", be defined by:
$\mathrm{OP}_{v,u}(\alpha) = \beta$ if and only if
$\beta \in v$, $\alpha \in u$ and $\mathrm{otp}(v \cap \beta) = \mathrm{otp}(u \cap \alpha)$.

(11) We say that $u, v \subseteq \mathrm{Ord}$ form a Δ–system pair when $\mathrm{otp}(u) = \mathrm{otp}(v)$ and $\mathrm{OP}_{v,u}$ is the identity on $u \cap v$.

Definition 0.6. (1) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we let

$$\text{ar-cl}(\mathcal{A}) = \{B \subseteq \omega : B \text{ is first order definable in } (\mathbb{N}, A_1, \dots, A_n) \text{ for some } n < \omega \text{ and } A_1, \dots, A_n \in \mathcal{A}\}.$$

The set ar-cl($\mathcal{A}$) is called *the arithmetic closure of* $\mathcal{A}$.

(2) For a model N of $\mathrm{PA}(\tau)$ let *the standard system of* N be

$$\mathrm{StSy}(N) = \{\varphi(N', \bar{a}) \cap \mathbb{N} : \varphi(x, \bar{y}) \in \mathbb{L}(\tau) \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}N\}$$

for any ordinary model N' isomorphic to N.

Definition 0.7. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

(1) For $h \in {}^\omega\omega$ let $\mathrm{cd}(h) = \{\mathrm{pr}(n, h(n)) : n < \omega\}$, where pr is the standard pairing function of ω, see 0.5(1).

(2) An ultrafilter D on $\mathcal{A}$, is called *minimal* when:
if $h \in {}^\omega\omega$ and $\mathrm{cd}(h) \in \mathcal{A}$, then for some $X \in D$ we have that $h{\restriction}X$ is either constant or one-to-one.

(3) An ultrafilter D on $\mathcal{A}$ is called *Ramsey* when:
if $k < \omega$ and $h : [\omega]^k \longrightarrow \{0, 1\}$ and $\mathrm{cd}(h) \in \mathcal{A}$, then for some $X \in D$ we have $h{\restriction}[X]^k$ is constant.
Similarly we define k-Ramsey ultrafilters.

(4) D is called *2.5-Ramsey* or *self-definably closed* when:
if $\bar{h} = \langle h_i : i < \omega\rangle$ and $h_i \in {}^\omega(i + 1)$ and $\mathrm{cd}(\bar{h}) = \{\mathrm{pr}(i, \mathrm{pr}(n, h_i(n)) : i < \omega, n < \omega\}$ belongs to $\mathcal{A}$, then for some $g \in {}^\omega\omega$ we have:

$$\mathrm{cd}(g) \in \mathcal{A} \text{ and } (\forall i)[g(i) \le i \wedge \{n < \omega : h_i(n) = g(i)\} \in D];$$

this follows from 3-Ramsey and implies 2-Ramsey.

(5) D is *weakly definably closed* when:
if $\langle A_i : i < \omega\rangle$ is a sequence of subsets of ω and $\{\mathrm{pr}(n, i) : n \in A_i$ and $i < \omega\} \in \mathcal{A}$, then $\{i : A_i \in D\} \in \mathcal{A}$, (follows from 2-Ramsey); Kirby called it "definable"; Enayat uses "iterable".

Definition 0.8. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\mathbb{N}_{\mathcal{A}}$ be $\mathbb{N}$ expanded by a unary relation A for every $A \in \mathcal{A}$, so formally it is a $\tau_{\mathcal{A}}$–model, $\tau_{\mathcal{A}} = \tau_{\mathbb{N}} \cup \{P_A : A \in \mathcal{A}\}$, but below if we use $\mathcal{A} = \{A_t : t \in X\}$, then we actually use $\{P_t : t \in X\}$.

Definition 0.9. Let N be a model of $T \supseteq \mathrm{PA}(\tau)$, $\tau = \tau_T$.

(1) We say that N^+ is an end extension of N when:
  (a) $N \prec N^+$,
  (b) if $a \in N$ and $b \in N^+ \setminus N$, then $N^+ \models a < b$.

(2) We say N^+ is a conservative [end] extension of N whenever (a),(b) hold and
  (c) if $\varphi(x, \bar{y}) \in \mathbb{L}(\tau)$, $\bar{b} \in {}^{\ell g(\bar{y})}(N^+)$, then $\varphi(N^+, \bar{b}) \cap N$ is a definable subset of N.

* * *

Discussion 0.10. We may ask: How is the *creature forcing* relevant? Do we need Roslanowski–Shelah [**?**]?

The creatures (and creatures forcing) we deal with fit [**?**], but instead of CS iteration it suffices for us to use a watered down version of creature iteration. That is here it is enough to define $\mathbb{Q}_u$ for finite $u \subseteq \mathrm{Ord}$ such that:

- $(a)_1$ $\mathbb{Q}_u$ is a creature forcing with generic $\langle \underset{\sim}{t}_\alpha : \alpha \in u\rangle$; this restriction implies that cases irrelevant in full forcing where we have to use countable u, are of interest here; hence we can use creature forcing rather than iterated creature forcing.
- $(a)_2$ In §3, $\mathbb{Q}_u$ is a good enough ${}^\omega\omega$–bounding creature forcing, so we have continuous reading of names.
- $(a)_3$ We are used to do it above a countable models N of ZFC$^-$, and this seems more transparent. But actually asking on the Δ_n–type of the generic over $\mathbb{N}$ suffices. That is, we can, e.g., by Δ_{n+7} formula over $\mathbb{N}$ find, e.g., a condition $p \in \mathbb{Q}_u$ such that any $\bar{t} \in \mathbf{B}_p$, e.g. a branch in the tree its Δ_n-type over $\mathbb{N}$, i.e. the Δ_n-theory of $(\mathbb{N}, \bar{t})$, so t_ℓ acts as a predicate (we can think of $\mathbf{B}_u$ as $\subseteq {}^u({}^\omega 2)$).

Here the construction is by forcing over a countable $N_* \prec (\mathcal{H}(\chi), \in)$. Note that there is no problem to add $\mathcal{A}^* := N_* \cap \mathcal{P}(\omega)$. So we can prove the results for $\mathcal{A} = (\text{countable}) \cup (\text{perfect})$. To improve it to perfect we need to force for PA by induction on n for Σ_n formulas.

- $(a)_4$ Note: for this it is O.K. if in every $p \in \mathbb{Q}_u$ the total number of commitments of the form "ρ is a member of $\varrho_x(i)$" is finite.
- $(b)_1$ We can use $u_n = {}^n 2$, just a notational change, we would like to choose p_n by induction on $n < \omega$ such that:
  - (α) $p_n \in \mathbb{Q}_{u_2}$,
  - (β) p_n is such that for $\bar{t} \in \mathbf{B}_{p_n}$ the Σ_n–theory of $(\mathbb{N}, \bar{t})$ can be read continuously on p,
  - (γ) if $h : {}^n 2 \longrightarrow {}^{n+1}2$ is such that $(\forall \rho \in {}^n 2)(h(\rho){\restriction}n = \rho)$, then $h(p_n) = p_n{\restriction}\mathrm{Rang}(h)$ both defined naturally (can make one duplicating at a time).
- $(b)_2$ In $(b)_1$, the set $\bigcup\{\varrho_x(i) : x \in p\}$ grows from p_n to p_{n+1}, i.e., here we need the major point in the choice of $\mathrm{nor}^0_x(C)$; however we do not need to diagonalize over it as in the proof about $\mathbb{Q}_u$.
- $(c)_1$ However, in §3 we can define full creature iterated forcing, i.e. using countable support; it is of interest but irrelevant here;
- $(c)_2$ but some cases of such creature forcing may look like: look at

$$\mathbf{T}' = \bigcup\{\prod_{k<n}(i+1) : n < \omega\},$$

and the ideal

$$\{A \subseteq \prod_{i<\omega}(i{+}1) : A = \bigcup_{n<\omega} A_n \text{ and } (\forall n < \omega)(\forall \eta \in \mathbf{T}')(\exists \nu \in \mathrm{suc}_{\mathbf{T}'}(\eta))(\forall \eta \in A_n)[\neg(\nu \triangleleft \eta)]\}.$$

- $(c)_3$ In the cases in which $(c)_2$ is relevant, we get a Borel set $\mathbf{B}$ such that $(\mathbb{N}, t)_{t \in \mathbf{B}} \ldots$, but not "for every $\aleph_1$–members of $\mathbf{B}$ we have…".

(d) Actually, what we use are iterated creature forcing, but as we deal only with $\mathbb{Q}_u$, u finite, so here we need not rely on the theory of creature iteration.

1. Models of theories of expansions of $\mathbb{N}$ with no end extensions

Theorem 1.1. (1) *For some $\mathcal{A} \subseteq \mathcal{P}(\omega)$ some model of $\mathrm{Th}(\mathbb{N}_{\mathcal{A}})$ has no end extension.*

(2) *There is an uncountable Borel set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that for any uncountable $\mathcal{A}' \subseteq \mathcal{A}$ the theory $T := \mathrm{Th}(\mathbb{N}_{\mathcal{A}'})$ has a model with no end extension.*

(3) *In fact, any model N of T such that the naturally associated tree (set of levels N, the set of nodes of level $n \in N$ is $({}^n2)^N$) has no undefinable branch is O.K.; such models exist by [**?**].*

(4) *Moreover, without loss of generality, the set of subsets of $\mathbb{N}$ definable in $\mathbb{N}_{\mathcal{A}}$ is Borel.*

The proof is broken to a series of definitions and claims finding a sufficient condition proved in Sections 2, 3. More specifically, Theorem 1.5(b) gives a sufficient condition which is proved in Proposition 3.7.

Definition 1.2. (1) Let sequences $\bar{n}^* = \langle n^*_i : i < \omega\rangle$ and $\bar{k}^* = \langle k^*_i : i < \omega\rangle$ be such that $n^*_0 = 0$, $n^*_i \ll k^*_{i+1} \ll n^*_{i+1}$ for $i < \omega$. We can demand that the ranges of $\bar{n}^*, \bar{k}^*$ are definable in $\mathbb{N}$ even by a bounded formula. In fact, in our computations later we put $n^*_i = \beth(30i+30)$ (for $i > 0$) and $k^*_i = \beth(30i+20)$, where $\beth(0) = 1$, $\beth(i+1) = 2^{\beth(i)}$.

We also let $n_*(i) = n^*_i$.

(2) Let $\mathcal{Y}_\ell = \{\pi : \pi \text{ is a permutation of } {}^{n_*(\ell)}2\}$ and $\mathbf{T}_n = \{\langle \pi_\ell : \ell < n\rangle : \pi_\ell \in \mathcal{Y}_\ell$ for $\ell < n\}$ and $\mathbf{T} = \bigcup\{\mathbf{T}_n : n < \omega\}$.

For $\varkappa \in \mathbf{T}_n$ we keep the convention that $\varkappa = \langle \pi^\varkappa_\ell : \ell < n\rangle$ (unless otherwise stated).

(3) For $\varkappa \in \mathbf{T}$ let $<_\varkappa$ be the following partial order:
  (a) $\mathrm{Dom}(<_\varkappa) = \bigcup\{{}^{n_*(i)}2 : i < \ell g(\varkappa)\}$;
  (b) $\eta <_\varkappa \nu$ if and only if they are from $\mathrm{Dom}(<_\varkappa)$ and for some $i < j$ we have $\eta \in {}^{n_*(i)}2$, $\nu \in {}^{n_*(j)}2$ and $\pi^\varkappa_i(\eta) \triangleleft \pi^\varkappa_j(\nu)$.

Let $t_\varkappa = (\mathrm{Dom}(<_\varkappa), <_\varkappa)$ for $\varkappa \in \mathbf{T}$.

(4) Let $\mathbf{T}_\omega$ be $\lim_\omega(\mathbf{T})$, i.e.,

$$\mathbf{T}_\omega = \{\langle \pi_i : i < \omega\rangle : \pi_i \text{ is a permutation of } {}^{n_*(i)}2 \text{ for } i < \omega\}$$

and for $\varkappa \in \mathbf{T}_\omega$ let $\varkappa{\restriction}n = \langle \pi^\varkappa_i : i < n\rangle$.

We interpret $\varkappa \in \mathbf{T}_\omega$ as the tree $t_\varkappa := (\bigcup_{i<\omega} {}^{n_*(i)}2, <_\varkappa)$, where $<_\varkappa = \bigcup\{<_{\varkappa{\restriction}n} : n < \omega\}$, so $t = t_\varkappa$ is $(\mathrm{Dom}(t), <_t)$.

(5) Let F be a one–to–one function from $\bigcup\{{}^{n_*(i)}2 : i < \omega\}$ onto ω, defined in $\mathbb{N}$ (i.e., the functions $n \mapsto \ell g(F^{-1}(n))$ and $(n,i) \mapsto (F^{-1}(n))(i)$ are definable in $\mathbb{N}$ even by a bounded formula) such that F maps each ${}^{n_*(i)}2$ onto an interval. Then clearly F^{-1} is a one–to–one function from $\mathbb{N}$ onto $\bigcup\{{}^{n_*(i)}2 : i < \omega\}$. If $\bar{n}^*, \bar{k}^*$ are not definable in $\mathbb{N}$ then we mean definable in $(\mathbb{N}, \bar{n}^*, \bar{k}^*)$, considering $\bar{n}^*, \bar{k}^*$ as unary functions.

(6) For $\varkappa \in \mathbf{T}_\omega$ let $<^*_\varkappa$ be $\{(F(\eta), F(\nu)) : \eta <_\varkappa \nu\}$ and $A_\varkappa = \{\mathrm{pr}(n_1, n_2) : n_1 <^*_\varkappa n_2\}$ and let $t^*_\varkappa = (\omega, <^*_\varkappa)$; similarly $t^*_\varkappa$ for $\varkappa \in \mathbf{T}$.

(7) For $\mathbf{S} \subseteq \mathbf{T}_\omega$ let $\mathcal{A}_{\mathbf{S}} = \{A_\varkappa : \varkappa \in \mathbf{S}\}$ and let $\mathbf{A}_{\mathbf{S}}$ be the arithmetic closure of $\mathcal{A}_{\mathbf{S}}$ recalling 0.6(1).

Proposition 1.3. *For $\varkappa \in \mathbf{T}_\omega$, in $(\mathbb{N}, A_\varkappa)$ we can define $<^*_\varkappa$ and*

$$(\mathbb{N}, A_\varkappa) \models \text{“ } <^*_\varkappa \textit{ is a tree with set of levels } \mathbb{N}, \textit{ set of elements } \mathbb{N} \textit{ and each level finite (=bounded in } \mathbb{N}, \textit{ even an interval) ”}.$$

Of course, $t_\varkappa$ and $t^*_\varkappa = (\omega, <^*_\varkappa)$ are isomorphic trees. Note that in $\mathbb{N}$ we can interpret the finite set theory $\mathcal{H}(\aleph_0)$.

Our aim is to construct objects with the following properties.

Definition 1.4. (1) We say $\mathbf{T}^*_\omega$ is *strongly pcd* (perfect cone disjoint) whenever:
$\mathbf{T}^*_\omega$ is a perfect subset of $\mathbf{T}_\omega$ such that:
$\boxtimes^{\rm st}_{\mathbf{T}^*_\omega}$ if $n < \omega$ and $\varkappa_0, \varkappa_1, \ldots, \varkappa_n \in \mathbf{T}^*_\omega$ with no repetitions and for $\ell = 0, 1$, η_ℓ is an ω–branch of $t^*_{\varkappa_\ell}$ which is definable in $(\mathbb{N}, A_{\varkappa_\ell}, A_{\varkappa_2}, \ldots, A_{\varkappa_n})$, then η_0, η_1 belong to disjoint cones (in their respective trees) which means that:
($\boxdot$) for some level n the sets

$$\{a : a \text{ is } <^*_{t_\ell}\text{–above the member of } \eta_\ell \text{ of level } n\} \subseteq \mathbb{N}$$

for $\ell = 0, 1$ are disjoint.

(2) We say $\mathbf{T}^*_\omega$ is *weakly pcd* (perfect cone disjoint) whenever:
$\mathbf{T}^*_\omega$ is a perfect subset of $\mathbf{T}_\omega$ such that:
$\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$ for every n and $\varphi(x, \bar{y}_\ell) \in \mathbb{L}(\tau_{\rm PA} + \{P_0, \ldots, P_n\})$ there is $i(*)$ such that *if*
- $i \in [i(*), \omega)$ and $\varkappa_{m,\ell} \in \mathbf{T}^*_\omega$ for $m \le n$, $\ell = 0, 1$,
- $\varkappa_{0,0} \ne \varkappa_{0,1}$ and
- $\varkappa_{m_1,\ell_1} \restriction i = \varkappa_{m_2,\ell_2} \restriction i$ if and only if $m_1 = m_2$, and
- $P_0, \ldots, P_n$ are unary predicates, $\varphi = \varphi(x, \bar{y}, P_0, \ldots, P_n) \in \mathbb{L}(\tau_{\rm PA} + \{P_0, \ldots, P_n\})$, and $\bar{b}_\ell \in {}^{\ell g(\bar{y})}\mathbb{N}$, $\varphi(x, \bar{b}_\ell, A_{\varkappa_{0,\ell}}, \ldots, A_{\varkappa_{n,\ell}})$ define in $(\mathbb{N}, A_{\varkappa_{0,\ell}}, \ldots, A_{\varkappa_{n,\ell}})$ a branch B_ℓ of $t^*_{\varkappa_{0,\ell}}$ for $\ell = 0, 1$

then the branches B_0, B_1 have disjoint cones (in their respective trees).

(3) Conditions $\otimes^{\rm wk}_{\mathbf{T}^*_\omega}$ and $\otimes^{\rm st}_{\mathbf{T}^*_\omega}$ are defined like $\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$, $\boxtimes^{\rm st}_{\mathbf{T}^*_\omega}$ above replacing "have disjoint cones" (i.e., ($\boxdot$)) by "have bounded intersection", which means that
($\odot$) for some a the sets $\{b \in \eta_0 : b \text{ is of level} > a\}$ and $\{b \in \eta_1 : b \text{ is of level} > a\}$ are disjoint.
Then we define *weakly pbd* and *strongly pbd* (where *pbd* stands for *perfect branch disjoint*) in the same manner as pcd above, replacing $\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$, $\boxtimes^{\rm st}_{\mathbf{T}^*_\omega}$ by $\otimes^{\rm wk}_{\mathbf{T}^*_\omega}$ and $\otimes^{\rm st}_{\mathbf{T}^*_\omega}$, respectively.

(4) Omitting strongly/weakly means weakly.

One may now ask if the existence of pcd/pbd (Definition 1.4) can be proved and if this concept helps us. We shall prove the existence of pbd in Sections 2 and 3, specifically in 3.7. The existence of pcd remains an open question. Below we argue that objects of this kind are usefull to prove Theorem 1.1.

Theorem 1.5. (a) *If $\mathbf{T}^*_\omega$ is a pcd, i.e., it is a perfect subset of $\mathbf{T}_\omega$ satisfying $\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$ from Definition 1.4, then $\mathcal{A} = \mathcal{A}_{\mathbf{T}^*_\omega}$ (see Definition 1.2(7)) is as required in 1.1.*

(b) *Even if $\mathbf{T}^*_\omega$ is a pbd then $\mathcal{A} = \mathcal{A}_{\mathbf{T}^*_\omega}$ is as required in 1.1.*

Proof. (a) We will deal with each part of Theorem 1.1. First we give details for part (3) of 1.1.

For $\varkappa \in \mathbf{T}^*_\omega$ recall

$$A_\varkappa = \{\mathrm{pr}(F(\eta), F(\nu)) : \eta <^*_\varkappa \nu\} \subseteq \mathbb{N}$$

and $\mathcal{A} = \{A_\varkappa : \varkappa \in \mathbf{T}^*_\omega\} \subseteq \mathcal{P}(\omega)$. Assume $\mathcal{A}' \subseteq \mathcal{A}$ is uncountable and let $T = T_{\mathcal{A}'} = \mathrm{Th}(\mathbb{N}_{\mathcal{A}'})$ and $\tau_{\mathcal{A}'}$ be its vocabulary. Then by [?] the theory T has a model M in which definable trees (we are interested just in the case the set of levels being M with the order $<^M$) have no undefinable branches, so, in particular (and this is enough)

if $\varkappa \in \mathcal{A}$, then $(<^*_\varkappa)^M$ has no undefinable branch

(i.e., as in [?], branches mean full branches, "visiting" every level). Note that "the a-th level of $(M, (<^*_\varkappa)^M)$" does not depend on $\varkappa$.

Assume towards contradiction M^+ is an (elementary) end-extension of M and let $b^* \in M^+ \setminus M$. Now consider any $A_\varkappa \in \mathcal{A}$ so $(<^*_\varkappa)^M$ is naturally definable in M and

$M \models$ " for every element a serving as level,
$\langle \{c : b <_\varkappa c\} : b$ is of level a in the tree $t_\varkappa$, i.e. $(M, (<^*_\varkappa)^M)\rangle$
is a partition of $\{x : x$ is of $<^*_\varkappa$-level $> a\}$ to finitely many sets ",

the finite is in the sense of M of course.

As M^+ is an end-extension of M recalling 1.2(5) it follows that the level of b^* in M^+ is above M and b^* defines a branch of $(M, (<^*_\varkappa)^M)$ which we call $\eta_\varkappa = \langle b^\varkappa_a : a \in M\rangle$. That is $b^\varkappa_a$ is the unique member of M of level a such that $M^+ \models$" $b^\varkappa_a \le^*_\varkappa b^*$ ".

By the choice of M the branch $\eta_\varkappa$, i.e., $\{b^\varkappa_a : a \in M\}$ is a definable subset of M, say by $\varphi_\varkappa(x, \bar{d}_\varkappa)$ where $\varphi_\varkappa(x, \bar{y}_\varkappa) \in \mathbb{L}(\tau_{\mathcal{A}'})$ and $\bar{d}_\varkappa \in {}^{\ell g(\bar{y}_\varkappa)}M$. Now by the assumptions on $\mathcal{A}, \mathcal{A}', T$ there are $s_{\varkappa,1}, \dots, s_{\varkappa,n_\varkappa} \in \mathbf{T}^*_\omega \setminus \{\varkappa\}$ with no repetitions, hence $A_{s_{\varkappa,n}} \in \mathcal{A}' \setminus \{A_\varkappa\}$ for $n = 1, \dots, n_\varkappa$, and in $\varphi_\varkappa(x, \bar{y}_\varkappa)$ only $A_{s_{\varkappa,1}}, \dots, A_{s_{\varkappa,n_\varkappa}}$ and $A_\varkappa$ appear (i.e., the predicates $P_{s_{\varkappa,1}}, \dots, P_{s_{\varkappa,n_\varkappa}}, P_\varkappa$ corresponding to them and τ_{PA}, of course). Let $s_{\varkappa,0} = \varkappa$ and we write $\varphi'_\varkappa = \varphi'_\varkappa(x, \bar{y}_\varkappa, \bar{P}_\varkappa)$, where $\bar{P}_\varkappa = \langle P_{s_{\varkappa,\ell}} : \ell \le n_\varkappa\rangle$ and $\varphi'_\varkappa$ has non-logical symbols only from τ_{PA} and so $\varphi'_\varkappa = \varphi''_\varkappa(x, \bar{y}_\varkappa) \in \mathbb{L}(\tau_{\mathrm{PA}} \cup \{P_\ell : \ell \le n_\varkappa\})$, that is $\varphi'_\kappa(x, \bar{y}_\varkappa)$ when we substitute P_ℓ for $P_{s_{\varkappa,\ell}}$ for $\ell \le n_\varkappa$.

For $A_\varkappa \in \mathcal{A}$ let

$$m_\varkappa = \min\{m : s_{\varkappa,\ell}{\restriction}m \text{ for } \ell = 0, \dots, n_\varkappa \text{ are pairwise distinct }\}.$$

Hence for some $\varphi_*(x, \bar{y}_*), n_*, m_*, \bar{s}_*$ the set

$$\mathcal{A}_2 = \{A_\varkappa \in \mathcal{A} : \varphi'_\varkappa = \varphi_*,\ \bar{y}_\varkappa = \bar{y}_*, \text{ so } n_\varkappa = n_*,\ m_\varkappa = m_* \text{ and } \langle s_{\varkappa,\ell}{\restriction}m_* : \ell = 0, \dots, n_*\rangle = \bar{s}_*\}$$

is uncountable. Let $i(*) \ge m_*$ be as guaranteed by $\boxtimes^{\mathrm{wk}}_{\mathbf{T}^*_\omega}$, so for some uncountable $\mathcal{A}_3 \subseteq \mathcal{A}_2$ for some $\bar{s}_{**}$ we have that $\langle s_{\varkappa,\ell}{\restriction}i(*) : \ell = 1, \dots, n_*\rangle = \bar{s}_{**}$ whenever $A_\varkappa \in \mathcal{A}_3$. As $\mathcal{A}$ is uncountable clearly for some $A_{\varkappa_1} \ne A_{\varkappa_2} \in \mathcal{A}$ we have $\{\varkappa_1, \varkappa_2\}$ is disjoint to $\{s_{\varkappa_\ell,m} : m = 1, \dots, n_{\varkappa_\ell}$ and $\ell = 1, 2\}$.

So by $\boxtimes^{\mathrm{wk}}_{\mathbf{T}^*_\omega}$ from Definition 1.4 for some $a \in M$ we have

($\boxdot$) $M \models$" $\{c : b^{\varkappa_1}_a <^*_{\varkappa_1} c\} \cap \{c : b^{\varkappa_2}_a <^*_{\varkappa_2} c\} = \emptyset$ ".

[Why? Because $\mathbb{N}_{\mathcal{A}'} \models$" $(\forall \bar{y}_{\varkappa_1})(\forall \bar{y}_{\varkappa_2})$ [if $\varphi_{\varkappa_\ell}(-, \bar{y}_{\varkappa_\ell})$ define a branch of $t^*_{\varkappa_\ell}$ for $\ell = 1, 2$, then there are x_1, x_2 such that $\varphi_{\varkappa_1}(x_1, \bar{y}_{\varkappa_1}) \wedge \varphi_{\varkappa_2}(x_2, \bar{y}_{\varkappa_2}) \wedge \neg(\exists z)[x_1 \le_{t^*_{\varkappa_1}} z \wedge x_2 \le_{t^*_{\varkappa_2}} z]]$ ".]

But in M^+ the elements b^* belong to both, contradiction to $M \prec M^+$.

Now, parts (2), (3) of 1.1 follow and so does part (1).

(4) See on this [**?**]. Alternatively, when is $\mathcal{B} = \{A \subseteq \mathbb{N} : A$ is definable in $\mathbb{N}_{\mathcal{A}}\}$ Borel? As we can shrink $\mathbf{T}^*_\omega$, without loss of generality there is a function $g \in {}^\omega\omega$ such that for every $f \in {}^\omega\omega$ definable in $\mathbb{N}_{\mathcal{A}}$, we have $f <_{J^{\rm bd}_\omega} g$, i.e., $(\forall^\infty i))(f(i) < g(i))$. This suffices (in fact if we prove 1.4 using forcing notion $\mathbb{Q}_u$, where each $\mathbb{Q}_u$ is ${}^\omega\omega$-bounding this will be true for $\mathbf{T}^*_\omega$ itself and we do this in §3; moreover we have continuous reading for every such f (as a function of $(A_{\varkappa_0}, \ldots, A_{\varkappa_{n-1}})$ for some $\varkappa_0, \ldots, \varkappa_{n-1} \in \mathbf{T}^*_\omega$).

(b) We repeat the proof of (a) above untill the choice of $\{\varkappa_1, \varkappa_2\}$ (right before $(\boxdot)$), but we replace the rest of the arguments for clause (3) of 1.1 by the following. So by $\otimes^{\rm wk}_{\mathbf{T}^*_\omega}$ of Definition 1.4(3), for some $a_* \in M$ we have

$(\odot)$ $M \models$" the sets $\{b^{\varkappa_1}_a : a_* < a\}$, $\{b^{\varkappa_2}_a : a_* < a\}$ are disjoint".

(Remember that all the trees we consider have the same levels.) But in M^+ the element b^* belongs to both definable branches contrary to $M \prec M^+$. $\square$

Theorem 1.6. (1) *If $\mathbf{T}^*_\omega$ is a strong pcd, i.e., it is a perfect subset of $\mathbf{T}_\omega$ satisfying $\boxtimes^{st}_{\mathbf{T}^*_\omega}$ from 1.4, and $\mathcal{A} \subseteq \{A_\varkappa : \varkappa \in \mathbf{T}^*_\omega\}$ is uncountable, then there is no weakly definably closed ultrafilter on* ar-cl$(\mathcal{A})$, *see Definition 0.7(5).*

(2) *Above, we may replace "pcd" with "pbd".*

(3) *Without loss of generality,* ar-cl$(\mathbf{T}^*_\omega)$ *is a Borel set.*

Proof. (1) Assume towards contradiction that a pair $(\mathcal{A}, D)$ forms a counterexample. Let $M = \mathbb{N}_{\mathcal{A}}$ and let M^+ be an $\aleph_2$–saturated elementary extension of M and let $b^* \in M^+$ realizes the type

$$p^* = \{\varphi(x, \bar{a}) : \quad \varphi(x, \bar{y}) \in \mathbb{L}(\tau_M), \bar{a} \in {}^{\ell g(\bar{y})}M \text{ and } \{b \in M : M \models \varphi[b, \bar{a}]\} \text{ includes some member of } D\}.$$

Clearly p^* is a set of formulas over M, finitely satisfiable in M and even a complete type over M.

Now, for every $\varkappa$ such that $A_\varkappa \in \mathcal{A}$ and $i < \omega$ we consider a function $g_{\varkappa,i}$ definable in M as follows:

$(*)_1$ $g_{\varkappa,i}(c)$ is:

- (α) b if c is of $<^*_\varkappa$–level $\geq i$ in $(\mathbb{N}, <_\varkappa)$ and b is of $<^*_\varkappa$–level i and $b \leq^*_\varkappa c$;
- (β) c if c is of $<^*_\varkappa$-level $< i$ in $(\mathbb{N}, <_\varkappa)$.

Clearly $g_{\varkappa,i}$ is definable in $(\mathbb{N}, A_\varkappa)$, the range of $g_{\varkappa,i}$ is finite, so $g_{\varkappa,i} \restriction B_{\varkappa,i}$ is constant for some $B_{\varkappa,i} \in \{g^{-1}_{\varkappa,i}\{x\} : x \in \text{Rang}(g_{\varkappa,i})\} \cap D$. As all co-finite subsets of $\mathbb{N}$ belong to D, also $B_{\varkappa,i}$ cannot be a singleton member of level $\neq i$. Hence for some $b_{\varkappa,i}$ of level i for $<^*_\varkappa$ we have $B_{\varkappa,i} \subseteq \{c : b_{\varkappa,i} \leq^*_\varkappa c\}$. Now moreover for some formula $\varphi_\varkappa(x_0, x_1, x_2) \in \mathbb{L}(\tau_{\rm PA} + P_\varkappa)$, for each $i \in \mathbb{N}$ the formula $\varphi_\varkappa(x_0, x_1, i)$ defines $g_{\varkappa,i}(x_1) = x_1$. By the "weakly definable closed" (see Definition 0.7(5)), $\{b_{\varkappa,i} : i < \omega\}$ is definable in $\mathbb{N}_{\mathcal{A}}$.

Now we continue as in the proof of 1.5.

(2) Similarly.

(3) As in 1.5 (for clause (4) of 1.1). $\square$

2. The (iterated) creature forcing

We continue the previous section, so we use notation as there, see Definitons 1.2 and 1.4. In particular, $n^*_0 = 0$, $n_*(i) = n^*_i = \beth(30i+30)$ (for $i > 0$) and $k^*_i = \beth(30i+20)$. We also set $\ell^*_i = \beth(30i+10)$.

Definition 2.1. For $i < \omega$ and a finite set u of ordinals we define:

(A) OB^u_i is the set of all triples (f,g,e) such that ($\mathrm{Per}(A)$ stands for the set of permutations of A):
   (a) $f, g \in {}^u(\mathrm{Per}({}^{n_*(i)}2))$;
   (b) if $i-1 = j \geq 0$ and $\alpha \in u$, then $(f(\alpha)(\rho))\restriction n^*_j = (g(\alpha)(\rho))\restriction n^*_j$ for all $\rho \in {}^{n_*(i)}2$,
   (c) e is a function with domain u such that for each $\alpha \in u$
$$e(\alpha) : \mathrm{Per}({}^{n_*(i-1)}2) \longrightarrow \mathrm{Per}({}^{n_*(i)}2) \times \mathrm{Per}({}^{n_*(i)}2).$$
   Above, we stipulate $n_*(i-1) = 0$ if $i = 0$. Also, let us note that some triples will never be used, only $\bigcup\{\mathrm{suc}(x) : x \in \mathrm{OB}^u_i\}$ and we should iterate.

(B) For $x \in \mathrm{OB}^u_i$ we let $x = (f_x, g_x, e_x)$ and $i = \mathbf{i}(x)$ and $u = \mathrm{supp}(x)$.

(C) For $x \in \mathrm{OB}^u_i$ we set
$$\mathrm{suc}(x) = \big\{y \in \mathrm{OB}^u_{i+1} : \quad (\forall \rho \in {}^{n_*(i+1)}2)\big(\forall \alpha \in u\big)\big(g_x(\alpha)(\rho\restriction n^*_i) = (f_y(\alpha)(\rho))\restriction n^*_i\big) \text{ and } \big(\forall \alpha \in u\big)\big(e_y(\alpha)(g_x(\alpha)) = (f_y(\alpha), g_y(\alpha))\big)\big\}.$$

(D) For $j \leq \omega$ let
$$\mathbf{S}_{u,j} = \big\{\langle x_\ell : \ell < j\rangle : (\ell < j \Rightarrow x_\ell \in \mathrm{OB}^u_\ell) \text{ and } (\ell + 1 < j \Rightarrow x_{\ell+1} \in \mathrm{suc}(x_\ell))\big\}.$$

(E) $\mathbf{S}_u = \bigcup\{\mathbf{S}_{u,\ell} : \ell < \omega\}$; we consider it a tree, ordered by $\lhd$.

(F) For $x \in \mathrm{OB}^u_i$ and $w \subseteq u$ let $x\restriction w = (f_x\restriction w, g_x\restriction w, e_x\restriction w)$.

(G) For $i \leq \omega$, $w \subseteq u$ and $\bar{x} = \langle x_j : j < i\rangle \in \mathbf{S}_{u,i}$ let $\bar{x} \upharpoonright w = \langle x_j\restriction w : j < i\rangle$ and for $\alpha \in u$ let $\varkappa^\alpha_{\bar{x}} = \langle f_{x_j}(\alpha) : j < i\rangle$.

(H) For $\bar{x} \in \mathbf{S}_{u,\ell}$, $\ell \leq \omega$, and $\alpha \in u$ let $t_{\bar{x},\alpha} = t^\alpha_{\bar{x}}$ be the tree with $\ell g(\bar{x})$ levels, with the i-th level being ${}^{n_*(i)}2$ for $i < \ell g(\bar{x})$ and the order $<_{t_{\bar{x},\alpha}}$ defined by
   $\eta <_{t_{\bar{x},\alpha}} \nu$ if and only if
   for some $i < j < \ell g(\bar{x})$ we have $\eta \in {}^{n_*(i)}2$, $\nu \in {}^{n_*(j)}2$ and $f_{x_i}(\alpha)(\eta) \lhd f_{x_j}(\alpha)(\nu)$.

Since we are interested in getting "bounded branch intersections" we will need the following observation (part (5) is crucial in proving cone disjointness in some situation later).

Proposition 2.2. *Assume $\bar{x} \in \mathbf{S}_u$ and $\alpha \in u$.*

(1) *If $\rho \in {}^{n_*(j)}2$ and $j < \ell g(\bar{x})$, then $\langle g_{x_i}(\alpha)(\rho\restriction n_*(i)) : i \leq j\rangle$ is $\lhd$–increasing noting $g_{x_i}(\alpha)(\rho\restriction n_*(i)) \in {}^{n_*(i)}2$.*

(2) *$\varkappa^\alpha_{\bar{x}} \in \mathbf{T}_{\ell g(\bar{x})}$ and $t_{\varkappa^\alpha_{\bar{x}}} = t^\alpha_{\bar{x}}$, on $t_{\varkappa^\alpha_{\bar{x}}}$ see 1.2(3).*

(3) *If $i < j < \ell g(\bar{x})$ and $\nu \in {}^{n_*(j)}2$, then $(f_{x_j}(\alpha)(\nu))\restriction n^*_i$ depends just on $\bar{x}\restriction(i+1)$, actually just on g_{x_i}, i.e., it is equal to $g_{x_i}(\alpha)(\nu\restriction n^*_i)$.*

(4) *The sequence $\langle g_{x_j}(\alpha), f_{x_j}(\alpha) : j < \ell g(\bar{x})\rangle$ is fully determined by $\langle e_{x_j}(\alpha) : j < \ell g(\bar{x})\rangle$.*

(5) *Assume $\alpha_1 \neq \alpha_2$ are from u and $i < \ell g(\bar{x})$ and $\eta_1, \eta_2 \in {}^{\eta_*(i)}2$ but*
$$(g_{x_i}(\alpha_1))^{-1} \circ f_{x_i}(\alpha_1))(\eta_1) \neq ((g_{x_i}(\alpha_2))^{-1} \circ f_{x_i}(\alpha_2))(\eta_2).$$
Then the sets $\{\rho : \eta_1 <_{t_{\bar{x},\alpha_1}} \rho\}$ and $\{\rho : \eta_2 <_{t_{\bar{x},\alpha_2}} \rho\}$ are disjoint.

Proof. (1), (2), (3) and (4) can be shown by straightforward induction on j.

(5) Assume towards contradiction that

$(*)_1$ $\eta_1 <_{t_{\bar{x},\alpha_1}} \rho$ and $\eta_2 <_{t_{\bar{x},\alpha_2}} \rho$.

So $\rho \in t_{\bar{x},\alpha_2}$ and hence $\rho \in {}^{n_*(j)}2$ for some $j < \ell g(\bar{x})$. Since $\eta_1 <_{t_{\bar{x},\alpha_1}} \rho$, necessarily $i < j < \ell g(\bar{x})$ and by the definition of $<_{t_{\bar{x},\alpha_1}}$ and $<_{t_{\bar{x},\alpha_2}}$:

$(*)_2$ $f_{x_i}(\alpha_1)(\eta_1) \vartriangleleft f_{x_j}(\alpha_1)(\rho)$ and $f_{x_i}(\alpha_2)(\eta_2) \vartriangleleft f_{x_j}(\alpha_2)(\rho)$.

This means that

$(*)_3$ $f_{x_i}(\alpha_1)(\eta_1) = (f_{x_j}(\alpha_1)(\rho))\restriction n_i^*$ and $f_{x_j}(\alpha_2)(\eta_2) = (f_{x_j}(\alpha_2)(\rho))\restriction n_i^*$.

Consequently, by part (3), letting $\rho' = \rho\restriction n_i^*$:

$(*)_4$ $f_{x_i}(\alpha_1)(\eta_1) = g_{x_i}(\alpha_1)(\rho')$ and $f_{x_i}(\alpha_2)(\eta_2) = g_{x_i}(\alpha_2)(\rho')$,

and therefore

$(*)_5$ $((g_{x_i}(\alpha_1))^{-1} \circ f_{x_i}(\alpha_1))(\eta_1) = \rho' = ((g_{x_i}(\alpha_2))^{-1} \circ f_{x_i}(\alpha_2))(\eta_2)$,

contradicting our assumptions. $\square$

Below we may replace the role of D_i^u by $\{\langle (f_{x_j}(\alpha), g_{x_j}(\alpha)) : j < i\rangle : \bar{x} \in \mathbf{S}_{u,i}\}$.

Definition 2.3. For a finite set $u \subseteq \mathrm{Ord}$ and an integer $i < \omega$ we let

(I) (α) $D_i^u = \{(\alpha, g) : \alpha \in u$ and $g \in \mathrm{Per}({}^{n_*(i-1)}2)$ if $i > 0$, $g \in \mathrm{Per}({}^{0}2)$ if $i = 0\}$;

if $\bar{x} \in \mathbf{S}_{u,i}$ and $\alpha \in u$, then stipulate $g_{x_{-1}}(\alpha)$ is the unique $g \in \mathrm{Per}({}^{0}2)$.

(β) pos_i^u is the set of all functions h with domain D_i^u such that $h(\alpha, g)$ is a pair $(h_1(\alpha, g), h_2(\alpha, g))$ satisfying

- $h_1(\alpha, g), h_2(\alpha, g) \in \mathrm{Per}({}^{n_*(i)}2)$, and
- $(h_\ell(\alpha, g)(\rho))\restriction n_*(i-1) = g(\rho\restriction n_*(i-1))$ for $\ell \in \{1, 2\}$, $i > 0$ and $\rho \in {}^{n_*(i)}2$.

Also, for $h \in \mathrm{pos}_i^u$ and $w \subseteq u$ we let $h \upharpoonright w = h\restriction D_i^w$.

(γ) wpos_i^u is the family of all functions $\mathcal{F} : \mathrm{pos}_i^u \longrightarrow [0, 1]$ which are not constantly zero, and

$$\mathrm{vpos}_i^u = \Big\{\mathcal{F} \in \mathrm{wpos}_i^u : \mathrm{range}(\mathcal{F}) \subseteq \big\{\frac{m}{2^{n_*(i)}} : m = 0, 1, \ldots 2^{n_*(i)}\big\}\Big\}.$$

If above we allow the constantly zero function instead of $\mathrm{wpos}_i^u, \mathrm{vpos}_i^u$ we get $\mathrm{ypos}_i^u, \mathrm{xpos}_i^u$, respectively. A set $A \subseteq \mathrm{pos}_i^u$ will be identified with its characteristic function $\chi_A \in \mathrm{vpos}_i^u$.

(δ) For $\mathcal{F} \in \mathrm{wpos}_i^u$ we let

$$\mathrm{set}(\mathcal{F}) = \{h \in \mathrm{pos}_i^u : \mathcal{F}(h) > 0\} \quad \text{and} \quad \|\mathcal{F}\| = \sum\{\mathcal{F}(h) : h \in \mathrm{pos}_i^u\}.$$

If $|\mathrm{pos}_i^u| \geq \|\mathcal{F}\| \cdot (k_i^*)^{3^{k_i^*}-1}$, then we put $\mathrm{nor}_i^0(\mathcal{F}) = 0$; otherwise we let

$$\mathrm{nor}_i^0(\mathcal{F}) = k_i^* - \log_3\Big(\log_{k_i^*}\Big(\frac{k_i^* \cdot |\mathrm{pos}_i^u|}{\|\mathcal{F}\|}\Big)\Big).$$

(ε) For $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{wpos}_i^u$ we let

- $\mathcal{F}_1 \leq \mathcal{F}_2$ if and only if $(\forall h \in \mathrm{pos}_i^u)(\mathcal{F}_1(h) \leq \mathcal{F}_2(h))$;
- $(\mathcal{F}_1 + \mathcal{F}_2)(h) = \mathcal{F}_1(h) + \mathcal{F}_2(h)$ and $(\mathcal{F}_1 \cdot \mathcal{F}_2)(h) = \mathcal{F}_1(h) \cdot \mathcal{F}_2(h)$ for $h \in \mathrm{pos}_i^u$;
- $[\mathcal{F}_1]$ is the function from pos_i^u to $\{\frac{m}{2^{n_*(i)}} : m = 0, 1, \ldots, 2^{n_*(i)}\}$ given by

$$[\mathcal{F}_1](h) = \lfloor \mathcal{F}_1(h) \cdot 2^{n_*(i)} \rfloor \cdot 2^{-n_*(i)} \qquad \text{for } h \in \mathrm{pos}_i^u.$$

(ζ) For $\bar{x} \in \mathbf{S}_{u,i}$ and $h \in \mathrm{pos}^u_i$ we let $\mathrm{suc}_{\bar{x}}(h)$ be $\bar{x}^\frown\langle y\rangle$ where $y \in \mathrm{OB}^u_i$ is defined by:

- $(f_y(\alpha), g_y(\alpha)) = h(\alpha, g_{x_{i-1}}(\alpha))$ for $\alpha \in u$,
- $e_y(\alpha)(\pi) = h(\alpha, \pi)$ for $\alpha \in u$ and $\pi \in \mathrm{Per}({}^{n_*(i-1)}2)$.

(J) (α) $\underline{\mathrm{CR}}^u_i$ is the set of all pairs $\mathfrak{c} = (\mathcal{F}, m) = (\mathcal{F}_\mathfrak{c}, m_\mathfrak{c})$ such that m is a non-negative real and $\mathcal{F} \in \mathrm{wpos}^u_i$ and $\mathrm{nor}^0_i(\mathcal{F}) \geq m$. We also let $\mathrm{CR}^u_i = \{\mathfrak{c} \in \underline{\mathrm{CR}}^u_i : \mathcal{F}_\mathfrak{c} \in \mathrm{vpos}^u_i\}$.

(β) For $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, we let $\mathrm{nor}^1_i(\mathfrak{c}) = (\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) - m_\mathfrak{c})$ and $\mathrm{nor}^2_i(\mathfrak{c}) = \log_{\ell^*_i}(\mathrm{nor}^1_i(\mathfrak{c}))$ if non-negative and well defined, and it is zero otherwise. (Remember, $\ell^*_i = \beth(30i+10)$.) We will write $\mathrm{nor}_i(\mathfrak{c}) = \mathrm{nor}^2_i(\mathfrak{c})$.

(γ) For $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$ let $\underline{\Sigma}(\mathfrak{c})$ be the set of all $\mathfrak{d} \in \mathrm{CR}^u_i$ such that $\mathcal{F}_\mathfrak{d} \leq \mathcal{F}_\mathfrak{c}$ and $m_\mathfrak{d} \geq m_\mathfrak{c}$. For $\mathfrak{c} \in \mathrm{CR}^u_i$ we let $\Sigma(\mathfrak{c}) = \underline{\Sigma}(\mathfrak{c}) \cap \mathrm{CR}^u_i$.

(K) $\mathbb{Q}_u = (\mathbb{Q}_u, \leq_{\mathbb{Q}_u})$ is defined by

(α) conditions in $\mathbb{Q}_u$ are pairs $p = (\bar{x}, \bar{\mathfrak{c}}) = (\bar{x}_p, \bar{\mathfrak{c}}_p)$ such that

(a) $\bar{x} \in \mathbf{S}_{u,i}$ for some $i = \mathbf{i}(p) < \omega$, so $\bar{x}_p = \langle x_{p,j} : j < \mathbf{i}(p)\rangle$,

(b) $\bar{\mathfrak{c}} = \langle \mathfrak{c}_j : j \in [\mathbf{i}(p), \omega)\rangle$, so $\mathfrak{c}_j = \mathfrak{c}^p_j$, and $\mathfrak{c}_j \in \mathrm{CR}^u_j$,

(c) the sequence $\langle \mathrm{nor}_j(\mathfrak{c}_j) : j \in [\mathbf{i}(p), \omega)\rangle$ diverges to ∞;

(β) $p \leq_{\mathbb{Q}_u} q$ if and only if (both are from $\mathbb{Q}_u$ and)

(a) $\bar{x}_p \trianglelefteq \bar{x}_q$, and

(b) if $\mathbf{i}(p) \leq j < \mathbf{i}(q)$, then for some $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{c}^p_j})$ we have $\bar{x}_q{\restriction}(j+1) = \mathrm{suc}_{\bar{x}_q{\restriction}j}(h)$ (see clause (I)(ζ) above),

(c) if $i \in [\mathbf{i}(q), \omega)$, then $\mathfrak{c}^q_i \in \Sigma(\mathfrak{c}^p_i)$.

$\underline{\mathbb{Q}}_u = (\underline{\mathbb{Q}}_u, \leq_{\underline{\mathbb{Q}}_u})$ is defined similarly, replacing CR^u_j, Σ by $\underline{\mathrm{CR}}^u_j$, $\underline{\Sigma}$, respectively.

(L) If $u_1, u_2 \subseteq \mathrm{Ord}$ are finite, $|u_1| = |u_2|$ and $h : u_1 \longrightarrow u_2$ is the order preserving bijection, then $\hat{h}$ is the isomorphism from $\mathbb{Q}_{u_1}$ onto $\mathbb{Q}_{u_2}$ induced by h in a natural way.

Proposition 2.4. *Let $u \subseteq \mathrm{Ord}$ be a finite non-empty set, $i \in (1, \omega)$ and $|u| \leq n_*(i-1)$. Then*

(a) $|\mathrm{pos}^u_{i-1}| < \beth(30i+3)$, $|\mathrm{vpos}^u_{i-1}| < \beth(30i+4)$, $\mathrm{nor}^0_i(\mathrm{pos}^u_i) = k^*_i$ *and* $\mathrm{nor}_i(\mathfrak{c}^{\max}_{u,i}) = \beth(30i+19)/\beth(30i+9)$ *and* $\mathrm{CR}^u_i = \Sigma(\mathfrak{c}^{\max}_{u,i})$, *where* $\mathfrak{c}^{\max}_{u,i} = (\mathrm{pos}^u_i, 0)$.

(b) $|\mathbf{S}_{u,i}| < \ell^*_i$ *and if* $\bar{x} \in \mathbf{S}_{u,i}$ *and* $h \in \mathrm{pos}^u_i$, *then* $\mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$.

(c) *If* $\mathcal{F}_1 \leq \mathcal{F}_2$ *are from* wpos^u_i, *then* $0 \leq \mathrm{nor}^0_i(\mathcal{F}_1) \leq \mathrm{nor}^0_i(\mathcal{F}_2)$.

(d) *If* $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$ *and* $\mathrm{nor}^1_i(\mathfrak{c}) \geq 1$, *then* $\mathfrak{c}$ *has* k^*_i*–bigness with respect to* nor^1_i, *which means that:*
if $\mathcal{F}_\mathfrak{c} = \sum\{\mathcal{Y}_k : k < k^*_i\}$ *then* $\mathrm{nor}^1_i(\mathfrak{c}) \leq \max\{\mathrm{nor}^1_i(\mathcal{Y}_m, m_\mathfrak{c}) + 1 : k < k^*_i\}$;
moreover, if $\mathcal{F}' \leq \mathcal{F}_\mathfrak{c}$, $\|\mathcal{F}'\| \geq \|\mathcal{F}_\mathfrak{c}\|/k^*_i$ *then* $\mathrm{nor}^0_i(\mathcal{F}') \geq \mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) - 1$.

(e) *Both* CR^u_i *and* $\underline{\mathrm{CR}}^u_i$ *have halving with respect to* nor^1_i, *that is*

(α) *if* $\mathfrak{c} = (\mathcal{F}_\mathfrak{c}, m_\mathfrak{c})$, $m_1 = (\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) + m_\mathfrak{c})/2$, $\mathfrak{d} = (\mathcal{F}_\mathfrak{c}, m_1)$, *then* $\mathrm{nor}^1_i(\mathfrak{d}) \geq \mathrm{nor}^1_i(\mathfrak{c})/2$, *and*

(β) *if* $\mathfrak{d}' \in \Sigma(\mathfrak{d})$ *is such that* $\mathrm{nor}^1_i(\mathfrak{d}') \geq 1$, *then* $\mathfrak{d}'' := (\mathcal{F}_{\mathfrak{d}'}, m_\mathfrak{c})$ *satisfies*

$$\mathfrak{d}'' \in \Sigma(\mathfrak{c}), \qquad \mathrm{nor}^1_i(\mathfrak{d}'') \geq \mathrm{nor}^1_i(\mathfrak{c})/2 \qquad \textit{and} \qquad \mathcal{F}_{\mathfrak{d}''} = \mathcal{F}_{\mathfrak{d}'}.$$

Proof. Clause (a): Clearly by the definition $\mathfrak{c}^{\max}_{u,i} = (\mathrm{pos}^u_i, 0) \in \mathrm{CR}^u_i = \Sigma(\mathfrak{c}^{\max}_{u,i})$ and

$$\mathrm{nor}^0_i(\mathrm{pos}^u_i) = k^*_i - \log_3\left(\log_{k^*_i}(k^*_i)\right) = k^*_i,$$

so $\mathrm{nor}^1_i(\mathfrak{c}^{\max}_{u,i}) = k^*_i - 0 = k^*_i$ and $\mathrm{nor}_i(\mathfrak{c}^{\max}_{u,i}) = \log_{\ell^*_i}(k^*_i) = \log_{\beth(30i+10)}\big(\beth(30i+20)\big) = \log_2\big(\beth(30i+20)\big)/\log_2\big(\beth(30i+10)\big) = \beth(30i+19)/\beth(30i+9)$. Now, for every $j>0$, letting $A_j = \mathrm{Per}({}^{n_*(j)}2)\times\mathrm{Per}({}^{n_*(j)}2)$ and recalling 2.3(I)(α), we have

$$|D^u_j| \le (2^{n_*(j-1)}!)\times|u| \le 2^{(2^{n_*(j-1)})^2}\times|u| \quad\text{and}\quad |A_j|\le(2^{n_*(j)}!)^2\le 2^{2^{2n_*(j)+1}}\le 2^{2^{3n_*(j)}}.$$

Since $|u|\le n_*(i-1)$, we get $|D^u_j|\le 2^{2^{2n_*(j-1)}}\times n_*(i-1)$. Since $2^{2^{2n_*(i-2)}}\le n_*(i-1)$, $n_*(i-1)^2\le 2^{n_*(i-1)}$ and $4n_*(i-1)+1\le 2^{n_*(i-1)}$, we conclude now that

$$|\mathrm{pos}^u_{i-1}|\le|A_{i-1}|^{|D^u_{i-1}|}\le(2^{2^{3n_*(i-1)}})^{|D^u_{i-1}|}\le 2^{2^{3n_*(i-1)}\times 2^{2^{2n_*(i-2)}}\times n_*(i-1)}\le 2^{2^{4n_*(i-1)}}<\beth(30i+3)$$

and

$$|\mathrm{vpos}^u_{i-1}| = (2^{n_*(i-1)}+1)^{|\mathrm{pos}^u_{i-1}|}<2^{(n_*(i-1)+1)\times 2^{2^{4n_*(i-1)}}}<2^{2^{2^{4n_*(i-1)+1}}}<\beth(30i+4).$$

Clause (b): Let B_j be the set of all functions from $\mathrm{Per}({}^{n_*(j-1)}2)$ to $\mathrm{Per}({}^{n_*(j)}2)\times\mathrm{Per}({}^{n_*(j)}2)$. Then we have

$$|B_j| = \Big(2^{n_*(j)}!\Big)^{2\cdot(2^{n_*(j-1)}!)}\le 2^{2^{2n_*(j)}\cdot 2\cdot(2^{n_*(j-1)}!)}\le 2^{2^{4n_*(j)}}$$

and hence for $j<i$:

$$|\mathrm{OB}^u_j|\le|{}^u\mathrm{Per}({}^{n_*(j)}2)|\cdot|{}^u\mathrm{Per}({}^{n_*(j)}2)|\cdot|{}^uB_j|\le(2^{n_*(j)}!)^{2|u|}\cdot 2^{2^{4n_*(j)}\cdot|u|}\le$$

$$2^{2^{2n_*(j)+1}\cdot|u|+2^{4n_*(j)}\cdot|u|}\le 2^{2^{7n_*(j)}\cdot n_*(i-1)}\le 2^{2^{8n_*(i-1)}}.$$

Therefore,

$$|\mathbf{S}_{u,i}|\le\prod_{j<i}|\mathrm{OB}^u_j|\le(2^{2^{8n_*(i-1)}})^i<2^{2^{9n_*(i-1)}}<\ell^*_i.$$

Clause (d): Assume $\mathfrak{c}\in\underline{\mathrm{CR}}^u_i$ and $\mathcal{F}_\mathfrak{c}=\sum\{\mathcal{Y}_k : k<k^*_i\}$, hence $\|\mathcal{F}_\mathfrak{c}\|=\sum\{\|\mathcal{Y}_k\| : k<k^*_i\}$. Let $k(*)<k^*_i$ be such that $\|\mathcal{Y}_{k(*)}\|$ is maximal. Plainly $\|\mathcal{F}_\mathfrak{c}\|\le k^*_i\times\|\mathcal{Y}_{k(*)}\|$ and therefore it suffices to prove the "moreover" part. So assume $\mathcal{Y}\le\mathcal{F}_\mathfrak{c}$, $\|\mathcal{F}_\mathfrak{c}\|\le k^*_i\times\|\mathcal{Y}\|$. Then

$$\mathrm{nor}^0_i(\mathcal{Y}) = k^*_i-\log_3\Big(\log_{k^*_i}\Big(\frac{k^*_i\cdot|\mathrm{pos}^u_i|}{\|\mathcal{Y}\|}\Big)\Big)\ge k^*_i-\log_3\Big(\log_{k^*_i}\Big(\frac{k^*_i\cdot|\mathrm{pos}^u_i|}{\|\mathcal{F}_\mathfrak{c}\|}\cdot k^*_i\Big)\Big)\ge$$

$$k^*_i-\log_3\Big(3\log_{k^*_i}\Big(\frac{k^*_i\cdot|\mathrm{pos}^u_i|}{\|\mathcal{F}_\mathfrak{c}\|}\Big)\Big)=\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c})-1,$$

so we are done.

Clauses (c) and (e): Obvious. $\square$

Observation 2.5. (1) *$\mathbb{Q}_u$, $\underline{\mathbb{Q}}_u$ are non-trivial partial orders.*
(2) *$\mathbb{Q}_u$ is a dense subset of $\underline{\mathbb{Q}}_u$.*

Proof. (1) Should be clear.
(2) For $\mathfrak{c}\in\underline{\mathrm{CR}}^u_i$ such that $\mathrm{nor}^1_i(\mathfrak{c})>1$ we set $[\mathfrak{c}]=([\mathcal{F}_\mathfrak{c}],m_\mathfrak{c})$ (see 2.3(I)(ε)). Note that $\frac{\|[\mathcal{F}_\mathfrak{c}]\|}{|\mathrm{pos}^u_i|}\ge\frac{\|\mathcal{F}_\mathfrak{c}\|}{|\mathrm{pos}^u_i|}-\frac{1}{2^{n_*(i)}}$ and hence (as $(k^*_i)^{3^{k^*_i}}<2^{n_*(i)}$ and $\frac{\|\mathcal{F}_\mathfrak{c}\|}{|\mathrm{pos}^u_i|}>(k^*_i)^{1-3^{k^*_i}}$) we have $\frac{\|[\mathcal{F}_\mathfrak{c}]\|}{|\mathrm{pos}^u_i|}\ge\Big(\frac{\|\mathcal{F}_\mathfrak{c}\|}{|\mathrm{pos}^u_i|}\Big)^3\cdot\frac{1}{k^2}$ and hence easily $\mathrm{nor}^0_i([\mathcal{F}_\mathfrak{c}])\ge\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c})-1$. Consequently, $[\mathfrak{c}]\in\mathrm{CR}^u_i$ and $\mathrm{nor}^1_i([\mathfrak{c}])\ge\mathrm{nor}^1_i(\mathfrak{c})-1$.

Now suppose that $p \in \mathbb{Q}_u$. We may assume that $\mathrm{nor}_i(\mathfrak{c}^p_i) > 1$ for all $i \geq \mathbf{i}(p)$. Put $\mathbf{i}(q) = \mathbf{i}(p)$, $\mathfrak{c}^q_i = [\mathfrak{c}^p_i]$ for $i \geq \mathbf{i}(q)$ and $\bar{x}_q = \bar{x}_p$. Then $q = (\bar{x}_q, \langle \mathfrak{c}^q_i : i \geq \mathbf{i}(q)\rangle) \in \mathbb{Q}_u$ is a condition stronger than p. $\square$

Definition 2.6. Let $u \subseteq$ Ord be a finite non-empty set.

(1) Let $\underset{\sim}{\bar{x}}$ and $\underset{\sim}{\varkappa}_\alpha, \underset{\sim}{t}_\alpha$ for $\alpha \in u$ be the following $\mathbb{Q}_u$-names:
  (a) $\underset{\sim}{\bar{x}} = \underset{\sim}{\bar{x}}_u = \bigcup\{\bar{x}_p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}\}$ and $\underset{\sim}{\varkappa}_\alpha = \langle \underset{\sim}{\pi}_{\alpha,i} : i < \omega\rangle$, where

$$\underset{\sim}{\pi}_{\alpha,i}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}] = \pi \quad \text{if and only if} \quad \text{for some } p \in \underset{\sim}{\mathbf{G}} \text{ we have } \ell g(\bar{x}_p) > i \text{ and } f_{x_{p,i}}(\alpha) = \pi.$$

  (b) $\underset{\sim}{t}_\alpha = t^*_{\underset{\sim}{\varkappa}_\alpha}$, i.e., it is a tree (see 1.2(4)).

(2) For $p \in \mathbb{Q}_u$ let $\mathrm{pos}(p) = \{\bar{x}_q : p \leq_{\mathbb{Q}_u} q\}$ and for $\bar{x} \in \mathrm{pos}(p)$ let $p^{[\bar{x}]} = (\bar{x}, \langle \mathfrak{c}^p_i : i \in [\ell g(\bar{x}), \omega)\rangle)$.

Observation 2.7. *Let $u \subseteq$ Ord be a finite non-empty set, $\alpha \in u$. Then:*

(1) $\Vdash_{\mathbb{Q}_u}$ "$\underset{\sim}{\bar{x}} \in \mathbf{S}_{u,\omega}$".
(2) *We can reconstruct $\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}$ from $\underset{\sim}{\bar{x}}$. As a matter of fact, $\langle e_{\underset{\sim}{\bar{x}}_i} : i < \omega\rangle$ determines $\langle f_{\underset{\sim}{\bar{x}}_i}, g_{\underset{\sim}{\bar{x}}_i} : i < \omega\rangle$ (and also $\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}$).*
(3) $\underset{\sim}{\varkappa}_\alpha = \bigcup\{\varkappa^\alpha_{\bar{x}} : \bar{x} = \bar{x}_p$ *and* $p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}\}$.
(4) $\Vdash_{\mathbb{Q}_u}$ "$\underset{\sim}{\varkappa}_\alpha \in \mathbf{T}_\omega$".
(5) *If $h : u \longrightarrow$ Ord is one-to-one, then $\hat{h}$ (see 2.3(L)) maps $\underset{\sim}{\bar{x}}_u$ to $\underset{\sim}{\bar{x}}_{h[u]}$, $(\underset{\sim}{\bar{x}}_u)_i$ to $(\underset{\sim}{\bar{x}}_{h[u]})_i$, etc.*

Observation 2.8. (1) *$p^{[\bar{x}]} \in \mathbb{Q}_u$ and $p \leq_{\mathbb{Q}_u} p^{[\bar{x}]}$ for every $\bar{x} \in \mathrm{pos}(p)$.*
(2) *If $p \in \mathbb{Q}_u$ and $i \in [\ell g(\bar{x}_p), \omega)$, then the set $\mathcal{I}_{p,i} := \{p^{[\bar{x}]} : \bar{x} \in \mathrm{pos}(p) \cap \mathbf{S}_{u,i}\}$ is predense above p in $\mathbb{Q}_u$.*

Proposition 2.9. *$\mathbb{Q}_u$ is a proper ${}^\omega\omega$–bounding forcing notion with rapid continuous reading of names, i.e., if $p \in \mathbb{Q}_u$ and $p \Vdash$ "$\underset{\sim}{h}$ is a function from ω to $\mathbf{V}$", then for some $q \in \mathbb{Q}_u$ we have:*

(a) *$p \leq q$ and $\mathbf{i}(p) = \mathbf{i}(q)$,*
(b) *for every $i < \omega$ the set $\{y : q \nVdash_{\mathbb{Q}_u}$ "$\underset{\sim}{h}(i) \neq y$"$\}$ is finite, moreover, for some $j \in [\ell g(\bar{x}_q), \omega)$, for each $\bar{x} \in \mathrm{pos}(q) \cap \mathbf{S}_{u,j}$ the condition $q^{[\bar{x}]}$ forces a value to $\underset{\sim}{h}(i)$,*
(c) *if $p \Vdash_{\mathbb{Q}_u}$ "$(\forall i < \omega)(\underset{\sim}{h}(i) < k^*_i)$", then:*
  (⊛) *if $\bar{x} \in \mathrm{pos}(q)$ has length $i > \mathbf{i}(q)$, then $q^{[\bar{x}]}$ forces a value to $\underset{\sim}{h}(i)$.*

Proof. It is a consequence of [?], so in the proof below we will follow definitions and notation as there. First note that we may assume $|u| < \mathbf{i}(p)$ (as otherwise we fix $i > |u|$ and we carry out the construction successively for all $\bar{x} \in \mathrm{pos}(p)$ of length i).

For $i < \mathbf{i}(p)$ let $\mathbf{H}(i) = \{x_{p,i}\}$ and for $i \geq \mathbf{i}(p)$ let $\mathbf{H}(i) = \mathrm{pos}^u_i$. Let K^* consists of all creatures $t = (\mathrm{nor}[t], \mathrm{val}[t], \mathrm{dis}[t])$ such that

- for some $i \geq \mathbf{i}(p)$ and $\mathfrak{c} \in \mathrm{CR}^u_i$ we have $\mathrm{dis}[t] = (\mathfrak{c}, i)$ and $\mathrm{nor}[t] = \mathrm{nor}^1_i(\mathfrak{c})$, and
- $\mathrm{val}[t] = \{(\bar{w}, \bar{w}^\frown\langle h\rangle) : \bar{w} \in \prod\limits_{j<i} \mathbf{H}(j)\ \&\ h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})\}$.

(Note the use of nor^1_i and not nor^2_i above.) For $t \in K^*$ with $\mathrm{dis}[t] = (\mathfrak{c}, i)$ we let

$$\Sigma^*(t) = \{s \in K : \mathrm{dis}[s] = (\mathfrak{d}, i)\ \&\ \mathfrak{d} \in \Sigma(\mathfrak{c})\}.$$

Then (K^*, Σ^*) is a local finitary big creating pair (for $\mathbf{H}$) with the Halving Property (remember 2.4(d,e)). Now define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by $f(j,i) = (\ell^*_i)^{j+1}$. Let

$p^*\in\mathbb{Q}^*_f(K^*,\Sigma^*)$ be a condition such that $w^{p^*}=\bar{x}_p$ and $\mathrm{dis}[t^{p^*}_i]=(\mathfrak{c}^p_{i+\mathbf{i}(p)},i+\mathbf{i}(p))$ for $i<\omega$. Note that $\mathbb{Q}_u$ above p is essentially the same as $\mathbb{Q}^*_f(K^*,\Sigma^*)$ above p^* (compare 2.7(2)). It should be clear that it is enough to find a condition $q^*\geq p^*$ with the properties (a)–(c) restated for $\mathbb{Q}^*_f(K,\Sigma)$.

Let $\varphi_{\mathbf{H}}(i)=|\prod\limits_{j<i}\mathbf{H}(j)|$. It follows from 2.4(a) that $\varphi_{\mathbf{H}}(i)\leq|\mathrm{pos}^u_{i-1}|^i<(\beth(30i+3))^i<\beth(30i+4)$ and $2^{\varphi_{\mathbf{H}}(i)}<\beth(30i+5)$. Therefore,

$$2^{\varphi_{\mathbf{H}}(i)}\cdot\big(f(j,i)+\varphi_{\mathbf{H}}(i)+2\big)\leq\beth(30i+5)\cdot\Big(\big(\beth(30i+10)\big)^{j+1}+\beth(30i+4)+2\Big)<\beth(30i+7)\cdot\big(\beth(30i+10)\big)^{j+1}<\big(\beth(30i+10)\big)^{j+2}=f(j+1,i).$$

Since plainly $f(j,i)\leq f(j,i+1)$, we conclude that the function f is $\mathbf{H}$–fast. Therefore [**?**, Theorem 2.2.11] gives us a condition q^* satisfying (a)+(b) (restated for $\mathbb{Q}^*_f(K^*,\Sigma^*)$). Proceeding as in [**?**, Theorem 5.1.12] but using the large amount of bigness here (see 2.4(d)) we may find a stronger condition saisfying also demand (c).

Note that to claim just properness of $\mathbb{Q}_u$ one could use the quite strong halving of nor_i and [**?**]. $\square$

Observation 2.10. *(1) $D^{u_1\cup u_2}_i=D^{u_1}_i\cup D^{u_2}_i$.*

(2) $h\in\mathrm{pos}^{u_1\cup u_2}_i$ if and only if h is a function with domain $D^{u_1\cup u_2}_i$ and $h{\restriction}D^{u_\ell}_i\in\mathrm{pos}^{u_\ell}_i$ for $\ell=1,2$.

Definition 2.11. Assume that $\emptyset\neq w\subseteq u\subseteq\mathrm{Ord}$ are finite, $v=u\setminus w\neq\emptyset$. Let $\mathcal{F}\in\mathrm{wpos}^u_i$. We define $\mathcal{F}\upharpoonright w:\mathrm{pos}^w_i\longrightarrow[0,1]$ by

$$(\mathcal{F}\upharpoonright w)(h)=\frac{\sum\{\mathcal{F}(e):h\subseteq e\in\mathrm{pos}^u_i\}}{|\mathrm{pos}^v_i|}\qquad\text{for } h\in\mathrm{pos}^w_i.$$

We will also keep the convention that if $u\subseteq\mathrm{Ord}$ and $\mathcal{F}\in\mathrm{pos}^u_i$, then $\mathcal{F}\upharpoonright u=\mathcal{F}$.

Proposition 2.12. *Assume that $\emptyset\neq u_0\subseteq u_1\subseteq\mathrm{Ord}$ are finite, $u_0\neq u_1$ and $\mathcal{F}_1\in\mathrm{wpos}^{u_1}_i$. Let $\mathcal{F}_0:=\mathcal{F}_1\upharpoonright u_0$. Then*

1. *$\mathcal{F}_0\in\mathrm{wpos}^{u_0}_i$ and $\frac{\|\mathcal{F}_0\|}{|\mathrm{pos}^{u_0}_i|}=\frac{\|\mathcal{F}_1\|}{|\mathrm{pos}^{u_1}_i|}$.*
2. *If $\mathcal{F}_2\in\mathrm{wpos}^{u_0}_i$, $\mathcal{F}_2\leq\mathcal{F}_0$, then there is $\mathcal{F}_3\in\mathrm{wpos}^{u_1}_i$ such that $\mathcal{F}_3\leq\mathcal{F}_1$ and $\mathcal{F}_3\upharpoonright u_0=\mathcal{F}_2$.*

Proof. Let $v=u_1\setminus u_0$.

(1) Plainly, $\mathcal{F}_0\in\mathrm{wpos}^{u_0}_i$. Also

$$\|\mathcal{F}_0\|=\frac{1}{|\mathrm{pos}^v_i|}\sum\Big\{\sum\{\mathcal{F}_1(e):h\subseteq e\in\mathrm{pos}^{u_1}_i\}:h\in\mathrm{pos}^{u_0}_i\Big\}=\frac{\|\mathcal{F}_1\|}{|\mathrm{pos}^v_i|}=\frac{|\mathrm{pos}^{u_0}_i|}{|\mathrm{pos}^{u_1}_i|}\cdot\|\mathcal{F}_1\|.$$

(2) Suppose $\mathcal{F}_2\in\mathrm{wpos}^{u_0}_i$, $\mathcal{F}_2\leq\mathcal{F}_0$. For $e\in\mathrm{pos}^{u_1}_i$ such that $\mathcal{F}_0(e\upharpoonright u_0)>0$ we put

$$\mathcal{F}_3(e)=\mathcal{F}_1(e)\cdot\frac{\mathcal{F}_2(e\upharpoonright u_0)}{\mathcal{F}_0(e\upharpoonright u_0)},$$

and for $e\in\mathrm{pos}^{u_1}_i$ such that $\mathcal{F}_0(e\upharpoonright u_0)=0$ we let $\mathcal{F}_3(e)=0$. Then clearly $\mathcal{F}_3\in\mathrm{wpos}^{u_1}_i$, $\mathcal{F}_3\leq\mathcal{F}_1$ and for $h\in\mathrm{pos}^{u_0}_i$ we have:

$$(\mathcal{F}_3\upharpoonright u_0)(h)=\frac{\sum\{\mathcal{F}_3(e):h\subseteq e\in\mathrm{pos}^{u_1}_i\}}{|\mathrm{pos}^v_i|}=\frac{\mathcal{F}_2(h)}{\mathcal{F}_0(h)}\cdot\frac{\sum\{\mathcal{F}_1(e):h\subseteq e\in\mathrm{pos}^{u_1}_i\}}{|\mathrm{pos}^v_i|}=\mathcal{F}_2(h).$$

$\square$

Definition 2.13. (1) We say that a pair $(\mathcal{F}_1, \mathcal{F}_2)$ is *balanced* when for some $i < \omega$ and finite non-empty sets $u_1, u_2 \subseteq \mathrm{Ord}$ we have $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$ and $\|\mathcal{F}_1\|/|\mathrm{pos}_i^{u_1}| = \|\mathcal{F}_2\|/|\mathrm{pos}_i^{u_2}|$ and, moreover, if $u_1 \cap u_2 \neq \emptyset$ then also $\mathcal{F}_1 \restriction (u_1 \cap u_2) = \mathcal{F}_2 \restriction (u_1 \cap u_2)$.

(2) A pair $(\mathcal{F}_1, \mathcal{F}_2)$ is *strongly balanced* if it is balanced and $0 \neq |u_1 \setminus u_2| = |u_2 \setminus u_1|$ (where $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$).

(3) Assume $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ (for $\ell = 1, 2$). Let $u = u_1 \cup u_2$. We define $\mathcal{F} = \mathcal{F}_1 * \mathcal{F}_2 \in \mathrm{ypos}_i^{u_1 \cup u_2}$ (see 2.3(I)(γ)) by putting for $h \in \mathrm{pos}_i^{u_1 \cup u_2}$

$$\mathcal{F}(h) = \mathcal{F}_1(h \restriction u_1) \cdot \mathcal{F}_2(h \restriction u_2).$$

Remark 2.14. (1) Note that $\mathcal{F}_1 * \mathcal{F}_2$ can be constantly zero, so it does not have to be a member of wpos. However, below we will apply to it our notation and definitions formulated for wpos.

(2) If $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ ($\ell = 1, 2$), $u_0 = u_1 \cap u_2 \neq \emptyset$, and $\mathcal{F}_3 = \mathcal{F}_1 * \mathcal{F}_2$, then $\mathcal{F}_3 \restriction u_0 = (\mathcal{F}_1 \restriction u_0) \cdot (\mathcal{F}_2 \restriction u_0)$.

(3) If $u_1 \cap u_2 = \emptyset$, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$, then $\|\mathcal{F}_1 * \mathcal{F}_2\| = \|\mathcal{F}_1\| \cdot \|\mathcal{F}_2\|$.

(4) Suppose $(\mathcal{F}_1, \mathcal{F}_2)$ is balanced, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ (for $\ell = 1, 2$). Choose finite $u_1', u_2' \subseteq \mathrm{Ord}$ such that $u_1 \subseteq u_1'$, $u_2 \subseteq u_2'$, $u_1 \cap u_2 = u_1' \cap u_2'$ and $|u_1' \setminus u_2'| = |u_2' \setminus u_1'| \neq 0$. For $\ell = 1, 2$ and $h \in \mathrm{pos}_i^{u_\ell'}$ put $\mathcal{F}_\ell'(h) = \mathcal{F}_\ell(h \restriction u_\ell)$. Then $(\mathcal{F}_1', \mathcal{F}_2')$ is strongly balanced and $\mathcal{F}_\ell' \restriction u_\ell = \mathcal{F}_\ell$.

Proposition 2.15. (1) *If (u_1, u_2) is a Δ–system pair, $u_1 \neq u_2 \neq \emptyset$, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$, and $\mathcal{F}_2 = \mathrm{OP}_{u_2, u_1}(\mathcal{F}_1)$, then the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is strongly balanced.*

(2) *If $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$ and $\|\mathcal{F}_\ell\|/|\mathrm{pos}_i^{u_\ell}| \geq a > 0$, the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is balanced, $u_3 = u_1 \cup u_2$ and $\mathcal{F} =: \mathcal{F}_1 * \mathcal{F}_2$, then $\|\mathcal{F}\|/|\mathrm{pos}_i^{u_3}| \geq \frac{a^3}{8}$.*

Proof. (1) Straightforward.
(2) Let $u_0 = u_1 \cap u_2$. We may assume $u_0 \neq \emptyset$ (see 2.14(3)). Let $\mathcal{F}_3 := \mathcal{F}$ and $\mathcal{F}_0 = \mathcal{F}_1 \restriction u_0 = \mathcal{F}_2 \restriction u_0$. For $h \in \mathrm{pos}_i^{u_0}$ and $\ell \leq 3$ let $\mathcal{F}_\ell^{[h]} : \mathrm{pos}_i^{u_\ell} \longrightarrow [0, 1]$ be defined by

$$\mathcal{F}_\ell^{[h]}(e) = \begin{cases} \mathcal{F}_\ell(e) & \text{if } h \subseteq e, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$(*)_0$ $k_\ell = |\{e \in \mathrm{pos}_i^{u_\ell} : h \subseteq e\}|$ for $h \in \mathrm{pos}_i^{u_0}$, $\ell = 1, 2$, i.e., this number does not depend on h.

[Why? By the definition of $\mathrm{pos}_i^{u_\ell}$ and 2.10.]

$(*)_1$ $\mathcal{F}_\ell$ is the disjoint sum of $\langle \mathcal{F}_\ell^{[h]} : h \in \mathrm{pos}_i^{u_0} \rangle$ for $\ell = 1, 2, 3$; the "disjoint" means that $\langle \mathrm{set}(\mathcal{F}_\ell^{[h]}) : h \in \mathrm{pos}_i^{u_0} \rangle$ are pairwise disjoint. Hence $\|\mathcal{F}_\ell\| = \sum \{ \|\mathcal{F}_\ell^{[h]}\| : h \in \mathrm{pos}_i^{u_0} \}$.

[Why? By the definition of $\mathrm{pos}_i^{u_\ell}$ and $\mathcal{F}_\ell^{[h]}$.]

$(*)_2$ $k_\ell \geq \|\mathcal{F}_\ell^{[h]}\| = \mathcal{F}_0(h) \cdot k_\ell$ for $\ell = 1, 2$.

[Why? By Defintion 2.11.]

$(*)_3$ $\|\mathcal{F}_3^{[h]}\| = \|\mathcal{F}_2^{[h]}\| \times \|\mathcal{F}_1^{[h]}\|$.

[Why? By the choice of $\mathcal{F}_3^{[h]}$.]

Let (noting that $0 < a \leq 1$)

$(*)_4$ $A_0 = \{h \in \mathrm{pos}_i^{u_0} : \mathcal{F}_0(h) \geq \frac{a}{2}\}$.

Now

$(*)_5$ $|A_0| \geq \frac{a}{2-a} \times |\mathrm{pos}_i^{u_0}|$.

[Why? Letting $d = |A_0|/|\mathrm{pos}_i^{u_0}|$ and $b = \frac{a}{2}$ (so $0 < b \leq \frac{1}{2}$) we have

$$h \in \mathrm{pos}_i^{u_0} \setminus A_0 \quad \Rightarrow \quad \|\mathcal{F}_1^{[h]}\| \leq \frac{a}{2} k_1 = bk_1$$

(remember $(*)_2$). Also $\|\mathcal{F}_1^{[h]}\| \leq k_1$ for all $h \in \mathrm{pos}_i^{u_0}$ and $k_1 \cdot |\mathrm{pos}_i^{u_0}| = |\mathrm{pos}_i^{u_1}|$. Hence

$$\begin{aligned}&a \times |\mathrm{pos}_i^{u_1}| \leq \|\mathcal{F}_1\| = \sum\{\|\mathcal{F}_1^{[h]}\| : h \in \mathrm{pos}_i^{u_0}\} = \\ &\sum\{\|\mathcal{F}_1^{[h]}\| : h \in \mathrm{pos}_i^{u_0} \setminus A_0\} + \sum\{\|\mathcal{F}_1^{[h]}\| : h \in A_0\} \leq bk_1 \cdot (|\mathrm{pos}_i^{u_0}| - |A_0|) + k_1|A_0| = \\ &bk_1(1-d)|\mathrm{pos}_i^{u_0}| + k_1 d|\mathrm{pos}_i^{u_0}| = k_1 \cdot |\mathrm{pos}_i^{u_0}| \cdot (b(1-d)+d) = |\mathrm{pos}_i^{u_1}|(b + (1-b)d).\end{aligned}$$

Hence $a \leq b + (1-b)d$ and $\frac{a-b}{1-b} \leq d$. So, as $b = a/2$, we have $d \geq \frac{a/2}{1-a/2} = \frac{a}{2-a}$. By the choice of d we conclude $|A_0| = d \times |\mathrm{pos}_i^{u_0}| \geq \frac{a}{2-a} \times |\mathrm{pos}_i^{u_0}|$, i.e., $(*)_5$ holds.]

Now

$(*)_6$ $\|\mathcal{F}_3\| \geq \frac{a^2}{4} \times k_1 \times k_2 \times |A_0|$.

[Why? By $(*)_3$, $\|\mathcal{F}_3^{[h]}\| = \|\mathcal{F}_1^{[h]}\| \times \|\mathcal{F}_2^{[h]}\|$ for all $h \in \mathrm{pos}_i^{u_0}$ and hence

$$\begin{aligned}&\|\mathcal{F}_3\| = \sum\{\|\mathcal{F}_3^{[h]}\| : h \in \mathrm{pos}_i^{u_0}\} = \sum\{\|\mathcal{F}_1^{[h]}\| \times \|\mathcal{F}_2^{[h]}\| : h \in \mathrm{pos}_i^{u_0}\} \geq \\ &\sum\{\|\mathcal{F}_1^{[h]}\| \times \|\mathcal{F}_2^{[h]}\| : h \in A_0\} \geq \sum\{\tfrac{a^2}{4} \cdot k_1 \cdot k_2 : h \in A_0\} = \tfrac{a^2}{4} \cdot k_1 \cdot k_2 \cdot |A_0|.\end{aligned}$$

So $(*)_6$ holds.]

Lastly,

$(*)_7$ $\|\mathcal{F}_3\| \geq \frac{a^3}{8}|\mathrm{pos}_i^{u_3}|$.

Why? Note that $k_1 \cdot k_2 \cdot |\mathrm{pos}_i^{u_0}| = |\mathrm{pos}_i^{u_3}|$ and hence

$$\begin{aligned}&\|\mathcal{F}_3\| \geq \tfrac{a^2}{4} \times k_1 \times k_2 \times |A_0| = \tfrac{a^2}{4}(|A_0|/|\mathrm{pos}_i^{u_0}|)(k_1 \times k_2 \times |\mathrm{pos}_i^{u_0}|) = \\ &\tfrac{a^2}{4} \times (|A_0|/|\mathrm{pos}_i^{u_0}|) \times |\mathrm{pos}_i^{u_3}| \geq \tfrac{a^2}{4} \times \tfrac{a}{2-a} \times |\mathrm{pos}_i^{u_3}| \geq \tfrac{a^3}{8}|\mathrm{pos}_i^{u_3}|.\end{aligned}$$

So $(*)_7$ holds and we are done. $\square$

Remark 2.16. In 2.15(2) we can get a better bound, the proof gives $\frac{a^4}{4(2-a)^2}$ and we can point out the minimal value, gotten when all are equal.

Definition 2.17. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions.

(1) A mapping $\mathbf{j} : \mathbb{P} \longrightarrow \mathbb{Q}$ is called *a projection of* $\mathbb{P}$ *onto* $\mathbb{Q}$ when:
  (a) $\mathbf{j}$ is "onto" $\mathbb{Q}$ and
  (b) $p_1 \leq_{\mathbb{P}} p_2 \quad \Rightarrow \quad \mathbf{j}(p_1) \leq_{\mathbb{Q}} \mathbf{j}(p_2)$.
(2) A projection $\mathbf{j} : \mathbb{P} \longrightarrow \mathbb{Q}$ is $\lessdot$*–complete* if (in addition to (a), (b) above):
  (c) if $\mathbb{Q} \models$" $\mathbf{j}(p) \leq q$ ", then some p_1 satisfies $p \leq_{\mathbb{P}} p_1$ and $q \leq_{\mathbb{Q}} \mathbf{j}(p_1)$.

Definition 2.18. If $\emptyset \neq u \subseteq v \subset \mathrm{Ord}$ are finite, then $\mathbf{j}_{u,v}$ is a function from $\underset{\sim}{\mathbb{Q}}_v$ onto $\underset{\sim}{\mathbb{Q}}_u$ defined by:

for $q \in \underset{\sim}{\mathbb{Q}}_v$ we have $\mathbf{j}_{u,v}(q) = p \in \underset{\sim}{\mathbb{Q}}_u$ if and only if

(α) $\mathbf{i}(p) = \mathbf{i}(q)$ and $\bar{x}_p = \bar{x}_q \restriction u$, and
(β) for $i \in [\mathbf{i}(p), \omega)$ we have $\mathfrak{c}_i^p := \mathrm{proj}_u(\mathfrak{c}_i^q)$ which means $\mathfrak{c}_i^p = (\mathcal{F}_{\mathfrak{c}_i^q} \restriction u, m_{\mathfrak{c}_i^p})$.

Proposition 2.19. *If* $u \subseteq v \in \mathrm{Ord}^{<\aleph_0}$*, then* $\mathbf{j}_{u,v}$ *is a (well defined)* $\lessdot$*–complete projection from* $\underset{\sim}{\mathbb{Q}}_v$ *onto* $\underset{\sim}{\mathbb{Q}}_u$.

Proof. It follows from 2.12 that

$(*)_1$ if $\mathfrak{c} \in \underline{\mathrm{CR}}^v_i$, then $\mathrm{proj}_u(\mathfrak{c}) \in \underline{\mathrm{CR}}^u_i$ and $\mathrm{nor}_i(\mathrm{proj}_u(\mathfrak{c})) = \mathrm{nor}_i(\mathfrak{c})$.

Also, by the definition of proj_u and 2.11, easily

$(*)_2$ if $\mathfrak{c} \in \underline{\mathrm{CR}}^v_i$, $\mathfrak{d} \in \underline{\Sigma}(\mathfrak{c})$, then $\mathrm{proj}_u(\mathfrak{d}) \in \underline{\Sigma}(\mathrm{proj}_u(\mathfrak{c}))$, and

$(*)_3$ if $\mathfrak{d} \in \underline{\mathrm{CR}}^u_i$, $\mathcal{F} : \mathrm{pos}^v_i \longrightarrow [0,1]$ is defined by $\mathcal{F}(h) = \mathcal{F}_{\mathfrak{d}}(h \restriction u)$, then $(\mathcal{F}, m_{\mathfrak{d}}) \in \underline{\mathrm{CR}}^v_i$, $\mathrm{nor}_i\big((\mathcal{F}, m_{\mathfrak{d}})\big) = \mathrm{nor}_i(\mathfrak{d})$ and $\mathrm{proj}_u\big((\mathcal{F}, m_{\mathfrak{d}})\big) = \mathfrak{d}$.

Therefore $\mathbf{j}_{u,v}$ is a projection from $\underline{\mathbb{Q}}_v$ onto $\underline{\mathbb{Q}}_u$. To show that it is $\lessdot$–complete we note that, by 2.12(2),

$(*)_4$ if $\mathfrak{c}_1 \in \underline{\mathrm{CR}}^v_i$, $\mathfrak{c}_0 = \mathrm{proj}_u(\mathfrak{c}_1)$ and $\mathfrak{c}_2 \in \underline{\Sigma}(\mathfrak{c}_0)$, then some $\mathfrak{c}_3 \in \underline{\mathrm{CR}}^v_i$ satisfies $\mathfrak{c}_3 \in \underline{\Sigma}(\mathfrak{c}_1)$ and $\mathrm{proj}_u(\mathfrak{c}_3) = \mathfrak{c}_2$.

The rest should be clear. □

Proposition 2.20. *Assume (u_1, u_2) is a Δ–system pair, i.e., $u_1, u_2 \subseteq \mathrm{Ord}$, $|u_1| = |u_2| < \aleph_0$ and so OP_{u_2,u_1} (the order isomorphism from u_1 onto u_2, see 0.5(10)) is the identity on $u_1 \cap u_2$. Let $u = u_1 \cup u_2$. Further assume that $p_\ell \in \underline{\mathbb{Q}}_{u_\ell}$ for $\ell = 1, 2$, $\mathrm{nor}^1_i(\mathfrak{c}^{p_\ell}_i) \geq 1$ for all $i \geq \mathbf{i}(p_\ell)$ and OP_{u_1,u_2} maps p_1 to p_2. Then there is a condition $q \in \underline{\mathbb{Q}}_u$ such that:*

(a) *$\mathbf{i}(q) = \mathbf{i}(p_1)$ and $p_\ell \leq_{\underline{\mathbb{Q}}_{u_\ell}} \mathbf{j}_{u_\ell,u}(q)$ for $\ell = 1, 2$, and*

(b) *$\mathrm{nor}^1_i(\mathfrak{c}^q_i) \geq \mathrm{nor}^1_i(\mathfrak{c}^{p_1}_i) - 1$ for $i \in [\mathbf{i}(q), \omega)$.*

Proof. We shall mainly use clause (2) of 2.15.

First, we set $\mathbf{i}(q) = \mathbf{i}(p_1)$ and we let $\bar{x} = \langle x_i : i < \mathbf{i}(q)\rangle$, where $x_i = (f_{x_i}, g_{x_i}, e_{x_i})$ is defined by

$(\bullet_1)$ $f_{x_i} = f_{x^{p_1}_i} \cup f_{x^{p_2}_i}$, it is well defined function because $f_{x^{p_\ell}_i} \in {}^{u_\ell}(\mathrm{Per}({}^{n_*(i)}2))$ for $\ell = 1, 2$ are well defined functions, with the same restriction to $u_0 = u_1 \cap u_2$;

$(\bullet_2)$ $g_{x_i} = g_{x^{p_1}_i} \cup g_{x^{p_2}_i}$ (similarly well defined);

$(\bullet_3)$ $e_{x_i} = e_{x^{p_1}_i} \cup e_{x^{p_2}_i}$ (again, it is well defined).

Easily,

$(\bullet_4)$ $\bar{x} \in \mathbf{S}_{u,\mathbf{i}(q)}$.

Second, we let $\bar{\mathfrak{c}} = \langle \mathfrak{c}_i : i \in [\mathbf{i}(q), \omega)\rangle$ where for $i \in [\mathbf{i}(q), \omega)$ we let $\mathfrak{c}_i = (\mathcal{F}_i, m_i)$, where

$(\bullet_5)$ $\mathcal{F}_i = \mathcal{F}_{\mathfrak{c}^{p_1}_i} * \mathcal{F}_{\mathfrak{c}^{p_2}_i}$,

$(\bullet_6)$ $m_i = m_{\mathfrak{c}^{p_\ell}_i}$ for $\ell = 1, 2$.

Let $i \in [\mathbf{i}(q), \omega)$. By Proposition 2.15(1) we know that the pair $(\mathcal{F}_{\mathfrak{c}^{p_1}_i}, \mathcal{F}_{\mathfrak{c}^{p_2}_i})$ is (strongly) balanced. Let $a = \frac{\|\mathcal{F}_{\mathfrak{c}^{p_1}_i}\|}{|\mathrm{pos}^{u_1}_i|} = \frac{\|\mathcal{F}_{\mathfrak{c}^{p_2}_i}\|}{|\mathrm{pos}^{u_2}_i|}$. Then, by 2.15(2) we have $\|\mathcal{F}_i\| \geq \frac{a^3}{8} \times |\mathrm{pos}^u_i|$. Hence, recalling $k^*_i \geq 3$,

$$\mathrm{nor}^0_i(\mathcal{F}_i) = k^*_i - \log_3\big(\log_{k^*_i}\big(\tfrac{k^*_i \cdot |\mathrm{pos}^u_i|}{\|\mathcal{F}_i\|}\big)\big) \geq k^*_i - \log_3\big(\log_{k^*_i}\big(\tfrac{8k^*_i}{a^3}\big)\big) \geq k^*_i - \log_3\big(3\log_{k^*_i}\big(\tfrac{k^*_i}{a}\big)\big) = k^*_i - \log_3\big(\log_{k^*_i}\big(\tfrac{k^*_i \cdot |\mathrm{pos}^{u_1}_i|}{\|\mathcal{F}_{\mathfrak{c}^{p_1}_i}\|}\big)\big) - 1 = \mathrm{nor}^0_i(\mathcal{F}_{\mathfrak{c}^{p_1}_i}) - 1 = \mathrm{nor}^0_i(\mathcal{F}_{\mathfrak{c}^{p_2}_i}) - 1.$$

Now clearly $q := (\bar{x}, \bar{\mathfrak{c}})$ is as required. □

3. Definable branches and disjoint cones

Now we come to the claim on creatures specifically to deal with the bounded intersection of branches. We think below of H_ℓ as part of a name of a branch of the α-th tree.

Lemma 3.1. *Assume that $u = u_1 \cup u_2$ are finite non-empty sets of ordinals, $|u_2 \setminus u_1| = |u_1 \setminus u_2| \neq 0$, $w = u_1 \cap u_2$. Suppose also that $i = j+1 < \omega$, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ (for $\ell = 1,2$) and the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is balanced. Let S be a finite set (e.g., ${}^{n_*(i)}2$) and $H_\ell : \mathrm{pos}_i^{u_\ell} \longrightarrow S$. Then there are $\mathcal{F}'_1, \mathcal{F}'_2, \mathcal{F}$ such that:*

(a) $\mathcal{F} \in \mathrm{wpos}_i^u$,
(b) $\mathcal{F}'_\ell \le \mathcal{F}_\ell$ *for* $\ell = 1,2$ *and* $\mathcal{F} = \mathcal{F}'_1 * \mathcal{F}'_2$,
(c) *the pair* $(\mathcal{F}'_1, \mathcal{F}'_2)$ *is balanced,*
(d) $\|\mathcal{F}'_\ell\| \ge \frac{1}{8}\|\mathcal{F}_\ell\|$ *for* $\ell = 1,2$,
(e) *one of the following occurs:*
  (α) *if* $h \in \mathrm{set}(\mathcal{F})$ *then* $H_1(h \restriction u_1) \neq H_2(h \restriction u_2)$,
  (β) (Case 1) $u_1 \cap u_2 = \emptyset$: *for some* $s \in S$ *we have* $h \in \mathrm{set}(\mathcal{F}) \Rightarrow H_1(h \restriction u_1) = s = H_2(h \restriction u_2)$;
  (Case 2) *general: for some function* H' *from* pos_i^w *to* S *we have:*

$$h \in \mathrm{set}(\mathcal{F}) \quad \Rightarrow \quad H_1(h \restriction u_1) = H'(h \restriction (u_1 \cap u_2)) = H_2(h \restriction u_2).$$

Proof. Let $\langle s_m : m < m_* \rangle$ list of all members of S. Let $g \in \mathcal{G} := \mathrm{pos}_i^w$. Now for every $m \le m_*$ we define

($\oplus_1$) (a) $\mathcal{F}_{\ell,g} : \mathrm{pos}_i^{u_\ell} \longrightarrow [0,1]$ is given by $\mathcal{F}_{\ell,g}(h) = \mathcal{F}_\ell(h)$ if $g \subseteq h$ and $\mathcal{F}_{\ell,g}(h) = 0$ otherwise,
(b) $k_{\ell,g} := \|\mathcal{F}_{\ell,g}\|$,
(c) $k^<_{\ell,m,g} := \sum\big\{\mathcal{F}_{\ell,g}(h) : g \subseteq h \in \mathrm{pos}_i^{u_\ell} \ \&\ H_\ell(h) \in \{s_{m_1} : m_1 < m\}\big\}$,
(d) $k^=_{\ell,m,g} := \sum\big\{\mathcal{F}_{\ell,g}(h) : g \subseteq h \in \mathrm{pos}_i^{u_\ell} \ \&\ H_\ell(h) = s_m\big\}$,
(e) $k^\ge_{\ell,m,g} := \sum\big\{\mathcal{F}_{\ell,g}(h) : g \subseteq h \in \mathrm{pos}_i^{u_\ell} \ \&\ H_\ell(h) \in \{s_{m_1} : m \le m_1 < m_*\}\big\}$.

Since we are assuming that $(\mathcal{F}_1, \mathcal{F}_2)$ is strongly balanced, we have

($\oplus_2$) $k_{1,g} = k_{2,g}$, call it k_g.

Plainly, $k^<_{\ell,m,g}, k^=_{\ell,m,g}, k^\ge_{\ell,m,g}, k_g$ are non-negative reals and

$(*)_1$ $k^<_{\ell,m,g} + k^\ge_{\ell,m,g} = k_g$.

Hence

$(*)_2$ $\max\{k^<_{\ell,m,g}, k^\ge_{\ell,m,g}\} \ge k_g/2$.

Also,

$(*)_3$ $k^<_{\ell,m,g} \le k^<_{\ell,m+1,g}$ and $k^\ge_{\ell,m,g} \ge k^\ge_{\ell,m+1,g}$, in fact $k^<_{\ell,m,g} + k^=_{\ell,m,g} = k^<_{\ell,m+1,g}$ and $k^\ge_{\ell,m+1,g} + k^=_{\ell,m,g} = k^\ge_{\ell,m,g}$, and

$(*)_4$ $k^<_{\ell,0,g} = 0 = k^\ge_{\ell,m_*,g}$.

Hence for some $m_{\ell,g}$ we have

$(*)_5$ $k^<_{\ell,m_{\ell,g}+1,g} \ge k_g/2$ and $k^\ge_{\ell,m_{\ell,g},g} \ge k_g/2$.

Therefore:

$(*)_6$ one of the following possibilities holds:
  (a) both $k^<_{\ell,m_{\ell,g},g}$ and $k^\ge_{\ell,m_{\ell,g}+1,g}$ are greater than or equal to $k_g/4$, or

(b) $k^{=}_{\ell,m_{\ell,g},g} \geq k_g/4$.

[Why? If clause (b) fails then by $(*)_5$ we get clause (a).]

Choose $(\iota_g, \mathcal{F}^*_{1,g}, \mathcal{F}^*_{2,g})$ as follows.

$(*)_7$ *Case 1:* $k^{=}_{1,m_{1,g},g} \geq k_g/4$ and $k^{=}_{2,m_{2,g},g} \geq k_g/4$.

Let $\iota_g = 1$, and $\mathcal{F}^*_{\ell,g} : \mathrm{pos}^{u_\ell}_i \longrightarrow [0,1]$ be such that $\mathcal{F}^*_{\ell,g}(h) = \mathcal{F}_{\ell,g}(h)$ if $g \subseteq h$ and $H_\ell(h) = s_{m_{\ell,g}}$, and $\mathcal{F}^*_{\ell,g}(h) = 0$ otherwise (for $\ell = 1,2$).

Case 2: $k^{=}_{1,m_{1,g},g} \geq k_g/4$ and $k^{=}_{2,m_{2,g},g} < k_g/4$.

Let $\iota_g = 2$ and $\mathcal{F}^*_{\ell,g} : \mathrm{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1,2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) = s_{m_{1,g}}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) \neq s_{m_{1,g}}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.

Case 3: $k^{=}_{1,m_{1,g},g} < k_g/4$ and $k^{=}_{2,m_{2,g},g} \geq k_g/4$.

Let $\iota_g = 3$ and $\mathcal{F}^*_{\ell,g} : \mathrm{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1,2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) \neq s_{m_{2,g}}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) = s_{m_{2,g}}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.

Case 4: $k^{=}_{1,m_{1,g},g} < k_g/4$, $k^{=}_{2,m_{2,g},g} < k_g/4$ and $m_{1,g} \leq m_{2,g}$.

Let $\iota_g = 4$ and $\mathcal{F}^*_{\ell,g} : \mathrm{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1,2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) \in \{s_0, \dots, s_{m_{1,g}-1}\}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) \in \{s_{m_{1,g}}, \dots, s_{m_*-1}\}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.

Case 5: $k^{=}_{1,m_{1,g},g} < k_g/4$, $k^{=}_{2,m_{2,g},g} < k_g/4$ and $m_{1,g} > m_{2,g}$.

Let $\iota_g = 5$ and $\mathcal{F}^*_{\ell,g} : \mathrm{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1,2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) \in \{s_{m_{2,g}}, \dots, s_{m_*-1}\}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) \in \{s_0, \dots, s_{m_{2,g}-1}\}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.

Now:

$(*)_8$ $\|\mathcal{F}^*_{\ell,g}\| \geq \frac{1}{4}\|\mathcal{F}_{\ell,g}\| = \frac{1}{4}k_g$ for $\ell = 1,2$.

[Why? By $(\oplus_2)$ and $(*)_7$ - check each case.]

Finally choose $\mathcal{F}^{**}_{\ell,g}$ (for $\ell = 1,2$ and $g \in \mathcal{G}$) such that:

$(*)_9$ (a) $\mathcal{F}^{**}_{\ell,g} \leq \mathcal{F}^*_{\ell,g}$, $\|\mathcal{F}^{**}_{\ell,g}\| \geq \frac{1}{4}k_g$, and $\|\mathcal{F}^{**}_{1,g}\| = \|\mathcal{F}^{**}_{2,g}\|$,

(b) *if* $(\iota_g = 1 \wedge m_{1,g} = m_{2,g})$ *then* for some $s = s(g) \in S$

$$h_1 \in \mathrm{set}(\mathcal{F}^{**}_{1,g}) \wedge h_2 \in \mathrm{set}(\mathcal{F}^{**}_{2,g}) \quad \Rightarrow \quad H_1(h_1) = H_2(h_2) = s,$$

(c) *if* $(\iota_g \neq 1 \vee m_{1,g} \neq m_{2,g})$ *then*

$$h_1 \in \mathrm{set}(\mathcal{F}^{**}_{1,g}) \wedge h_2 \in \mathrm{set}(\mathcal{F}^{**}_{2,g}) \quad \Rightarrow \quad H_1(h_1) \neq H_2(h_2).$$

[Why possible? We can choose them to satisfy clause (a) by $(*)_8$ and clauses (b),(c) follow - look at the choices inside $(*)_7$.]

Now we stop fixing $g \in \mathcal{G}$. Put

$$\mathcal{G}^1 = \{g \in \mathcal{G} : \iota_g = 1 \text{ and } m_{1,g} = m_{2,g}\} \quad \text{and} \quad \mathcal{G}^2 = \{g \in \mathcal{G} : \iota_g \neq 1 \text{ or } m_{1,g} \neq m_{2,g}\}.$$

When we vary $g \in \mathcal{G}$, obviously

$(\circledast_1)$ $\mathcal{F}_\ell$ is the disjoint sum of $\langle \mathcal{F}_{\ell,g} : g \in \mathcal{G} \rangle$,

and hence

$(\circledast_2)$ $\|\mathcal{F}_\ell\| = \sum\{k_g : g \in \mathcal{G}\}$.

As $\mathcal{G} = \mathrm{pos}_i^w$ is the disjoint union of $\mathcal{G}^1, \mathcal{G}^2$, plainly

($\circledast_3$) for some $\mathcal{G}' \in \{\mathcal{G}^1, \mathcal{G}^2\}$ the following occurs:

$$\sum\{k_g : g \in \mathcal{G}'\} \geq \|\mathcal{F}_1\|/2 = \|\mathcal{F}_2\|/2.$$

Lastly, we put $\mathcal{F}'_\ell = \sum\{\mathcal{F}^{**}_{\ell,g} : g \in \mathcal{G}'\}$ (for $\ell = 1, 2$). We note that

$$\|\mathcal{F}'_\ell\| = \sum\{\|\mathcal{F}^{**}_{\ell,g}\| : g \in \mathcal{G}'\} \geq \sum\{\frac{1}{4}k_g : g \in \mathcal{G}'\} \geq \frac{1}{4}(\|\mathcal{F}_\ell\|/2) = \frac{1}{8}\|\mathcal{F}_\ell\|.$$

Now it should be clear that $\mathcal{F}'_1, \mathcal{F}'_2$ and $\mathcal{F} = \mathcal{F}'_1 * \mathcal{F}'_2$ are as required. $\square$

Crucial Lemma 3.2. *Assume that*

(a) u_1, u_2 *are finite subsets of* Ord, $|u_1 \setminus u_2| = |u_2 \setminus u_1| \neq 0$,
(b) $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$, $i < \omega$ *and* $\|\mathcal{F}_\ell\| \geq a \times |\mathrm{pos}_i^{u_\ell}| > 0$,
(c) H_ℓ *is a function from* $\mathbf{S}_{u_\ell,i+1}$ *to* ${}^{n_*(i)}2$,
(d) *the pair* $(\mathcal{F}_1, \mathcal{F}_2)$ *is balanced.*

Let $u = u_1 \cup u_2$ *and* $w = u_1 \cap u_2$ *and* $|u| < n_*(i-1)$. *Then we can find* $\mathcal{F}'_\ell \in \mathrm{wpos}_i^{u_\ell}$ *and partial functions* $\mathbf{h}_\ell$ *from* $\mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1}$ *into* ${}^{n_*(i)}2$ *for* $\ell = 1, 2$ *and* $\mathcal{F} \in \mathrm{wpos}_i^u$ *such that:*

(α) $\mathcal{F}'_\ell \leq \mathcal{F}_\ell$, $\|\mathcal{F}'_\ell\| \geq 8^{-k_*}\|\mathcal{F}_\ell\|$, *where* $k_* = |\mathbf{S}_{u,i}| < \ell^*_i$, *and the pair* $(\mathcal{F}'_1, \mathcal{F}'_2)$ *is balanced,*
(β) $\mathcal{F} = \mathcal{F}'_1 * \mathcal{F}'_2$ *and so* $\mathcal{F} \restriction u_\ell \leq \mathcal{F}_\ell$ *for* $\ell = 1, 2$ *and* $\|\mathcal{F}\|/|\mathrm{pos}_i^u| \geq \frac{a^3}{2^{9k_*+3}}$,
(γ) *if* $h \in \mathrm{set}(\mathcal{F})$, $\bar{x} \in \mathbf{S}_{u,i}$ *(so* $\ell g(\bar{x}) = i$*) and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, *then*

$$H_1(\bar{y} \restriction u_1) = H_2(\bar{y} \restriction u_2) \quad \Rightarrow \quad \mathbf{h}_1(\bar{x} \restriction u_1, \bar{y} \restriction w) = \mathbf{h}_2(\bar{x} \restriction u_2, \bar{y} \restriction w) = H_1(\bar{y} \restriction u_1) = H_2(\bar{y} \restriction u_2).$$

(δ) *moreover, for each* $\bar{x} \in \mathbf{S}_{u,i}$ *the truth value of the equality* $H_1(\bar{y} \restriction u_1) = H_2(\bar{y} \restriction u_2)$ *in clause* (γ) *is the same for all* $h \in \mathrm{set}(\mathcal{F})$.

Proof. Let $\langle \bar{x}_k : k < k_* \rangle$ list $\mathbf{S}_{u,i}$ (without repetitions). We choose $(\mathcal{F}_k, \mathcal{F}_{1,k}, \mathcal{F}_{2,k})$ by induction on $k \leq k_*$ such that:

(i) $\mathcal{F}_{\ell,k} \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$,
(ii) if $k = 0$, then $\mathcal{F}_{\ell,k} = \mathcal{F}_\ell$,
(iii) $\mathcal{F}_{\ell,k}$ is $\leq$–decreasing with k, i.e., $\mathcal{F}_{\ell,k+1} \leq \mathcal{F}_{\ell,k}$,
(iv) $\|\mathcal{F}_{\ell,k}\| \geq \frac{1}{8^k}\|\mathcal{F}_\ell\|$,
(v) $(\mathcal{F}_{1,k}, \mathcal{F}_{2,k})$ is balanced,
(vi) $\mathcal{F}_k = \mathcal{F}_{1,k} * \mathcal{F}_{2,k}$, so also $\leq$–decreasing with k,
(vii) for each k one of the following occurs:
 (α) *if* $h \in \mathrm{set}(\mathcal{F}_{k+1})$ and $\bar{y} = \mathrm{suc}_{\bar{x}_k}(h) \in \mathbf{S}_{u,i+1}$, *then* $H_1(\bar{y} \restriction u_1) \neq H_2(\bar{y} \restriction u_2)$;
 (β) *if* $h', h'' \in \mathrm{set}(\mathcal{F}_{k+1})$ and $h' \restriction w = h'' \restriction w$, $\bar{y}' = \mathrm{suc}_{\bar{x}_k}(h')$, $\bar{y}'' = \mathrm{suc}_{\bar{x}_k}(h'')$, *then*

$$H_1(\bar{y}' \restriction u_1) = H_1(\bar{y}'' \restriction u_1) = H_2(\bar{y}' \restriction u_2) = H_2(\bar{y}'' \restriction u_2).$$

If we carry out the definition then $\mathcal{F} = \mathcal{F}_{k_*}$ is as required. Note that $\|\mathcal{F}_{\ell,k_*}\| \geq \frac{\|\mathcal{F}_\ell\|}{8^{k_*}}$, hence the bound on $\|\mathcal{F}\|$, i.e. clause (β) of 3.2 holds by 2.15; that is we choose $8^{-k_*}a$ here for a there and $\frac{a^3}{8}$ there means $\frac{(8^{-k_*}a)^3}{8} = \frac{a^3}{2^{9k_*+3}}$ here.

The initial step of $k = 0$ is obvious. For the inductive step, for $k+1$ we define $H_{\ell,k}$ as follows: for $h \in \mathrm{pos}_i^{u_\ell}$ we put $H_{\ell,k}(h) = H_\ell(\mathrm{suc}_{\bar{x}_k \restriction u_\ell}(h))$ and we apply Lemma 3.1 to $\mathcal{F}_{1,k}, \mathcal{F}_{2,k}, H_{1,k}, H_{2,k}$ here standing for $\mathcal{F}_1, \mathcal{F}_2, H_1, H_2$ there. This way we obtain $\mathcal{F}_{1,k+1}, \mathcal{F}_{2,k+1}$ and we set $\mathcal{F}_{k+1} = \mathcal{F}_{1,k+1} * \mathcal{F}_{2,k+1}$. If in clause 3.1(e) subclause (α) holds, then the demand in (vii)(α) is satisfied. Otherwise, we get a function H' such that for each $h \in \mathrm{set}(\mathcal{F}_{k+1})$ we have

$$H_{1,k}(h \restriction u_1) = H'(h \restriction w) = H_{2,k}(h \restriction u_2).$$

Consequently, the demand in (vii)(β) is fulfilled. Moreover this choice is O.K. for any $\mathcal{F}' \subseteq \mathcal{F}_{k+1}$, so we are done. $\square$

Lemma 3.3. (1) *Assume that* $u \subseteq \mathrm{Ord}$ *is finite,* $\alpha \in u$ *and* $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, $i > 0$. *Suppose also that there are* $\bar{x} \in \mathbf{S}_{u,i}$ *and functions* $\mathbf{h}_1, \mathbf{h}_2$ *such that*
if $h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})$ *and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$ *(see 2.3(I)(ζ)),*
then $\eta_\ell := \mathbf{h}_\ell(h \restriction (u \setminus \{\alpha\})) \in {}^{n_*(i)}2$ *is well defined for* $\ell = 1, 2$ *and* $(g_y(\alpha)^{-1} \circ f_y(\alpha))(\eta_1) = \eta_2$.
Then $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$.

(2) *Assume that* $w \subseteq u \subseteq \mathrm{Ord}$ *are finite,* $\alpha_1, \alpha_2 \in u \setminus w$, $\alpha_1 \neq \alpha_2$ *and* $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, $i > 0$. *Suppose also that* $\bar{x} \in \mathbf{S}_{u,i}$ *and there are functions* $\mathbf{h}_1, \mathbf{h}_2$ *such that*
if $h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})$ *and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$,
then $\eta_\ell := \mathbf{h}_\ell(\bar{x}, \bar{y} \restriction w) \in {}^{n_*(i)}2$ *is well defined for* $\ell = 1, 2$ *and*

$$(g_y(\alpha_1)^{-1} \circ f_y(\alpha_1))(\eta_1) = (g_y(\alpha_2)^{-1} \circ f_y(\alpha_2))(\eta_2).$$

Then $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$.

Proof. (1) First we try to give an upper bound to $|\mathrm{set}(\mathcal{F}_\mathfrak{c})|/|\mathrm{pos}^u_i|$. Thinking of "randomly drawing" $h_0 \in \mathrm{pos}^{u\setminus\{\alpha\}}_i$ with equal probability, we get an upper bound to the fraction of $h \in \mathrm{pos}^u_i$, $h \restriction (u \setminus \{\alpha\}) = h_0$ such that if $\mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$, then $\eta_\ell := \mathbf{h}_\ell(h \restriction (u \setminus \{\alpha\})) \in {}^{n_*(i)}2$ is well defined for $\ell = 1, 2$ and $(g_y^{-1}(\alpha) \circ f_y(\alpha))(\eta_1) = \eta_2$.
Since

$$g_y(\alpha)(\nu){\restriction}n_*(i-1) = g_{x_{i-1}}(\alpha)(\nu{\restriction}n_*(i-1)) = f_y(\alpha)(\nu){\restriction}n_*(i-1)) \quad \text{for all } \nu \in {}^{n_*(i)}2,$$

clearly it is $\leq 1/2^{n_*(i)-n_*(i-1)}$. So $\|\mathcal{F}_\mathfrak{c}\|/|\mathrm{pos}^u_i| \leq |\mathrm{set}(\mathcal{F}_\mathfrak{c})|/|\mathrm{pos}^u_i| \leq 1/2^{n_*(i)-n_*(i-1)} < (k^*_i)^{1-3^{k^*_i}}$ and consequently $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$.

(2) For $e \in \mathrm{pos}^{u\setminus\{\alpha_1\}}_i$ let $\bar{y}_e = \mathrm{suc}_{\bar{x}\restriction(u\setminus\{\alpha_1\})}(e) = (\bar{x} \restriction (u \setminus \{\alpha_1\}))^\frown\langle y_e\rangle$, $\mathbf{h}'_1(e) = \mathbf{h}_1(\bar{x}, \bar{y}_e \restriction w)$ and $\mathbf{h}'_2(e) = \big(g_{y_e}(\alpha_2)^{-1} \circ f_{y_e}(\alpha_2)\big)\big(\mathbf{h}_2(\bar{x}, \bar{y}_e \restriction w)\big)$. Since $\alpha_1, \alpha_2 \notin w$ and $\alpha_2 \in u \setminus \{\alpha_1\}$, for each $h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})$ the values $\mathbf{h}'_1\big(h \restriction (u \setminus \{\alpha_1\})\big), \mathbf{h}'_2\big(h \restriction (u \setminus \{\alpha_1\})\big)$ are well defined and, letting $\bar{y} = \mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$,

$$\big(g_y(\alpha_1)^{-1} \circ f_y(\alpha_1)\big)\big(\mathbf{h}'_1(h \restriction (u \setminus \{\alpha_1\}))\big) = \mathbf{h}'_2(h \restriction (u \setminus \{\alpha_1\})).$$

Therefore clause (1) applies and $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$. $\square$

Before we state the main corollary to Crucial Lemma 3.2, let us recall that if $\emptyset \neq w \subseteq u$, $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, then $\mathrm{proj}_w(\mathfrak{c}) = (\mathcal{F}_\mathfrak{c} \restriction w, m_\mathfrak{c}) \in \underline{\mathrm{CR}}^w_i$ (see Definition 2.18(β)). Also, if $\emptyset = w = u_1 \cap u_2$ and $\mathfrak{c}_\ell \in \underline{\mathrm{CR}}^{u_\ell}_i$, then $\mathrm{proj}_w(\mathfrak{c}_1) = \mathrm{proj}_w(\mathfrak{c}_2)$ will mean that $\mathrm{nor}_i(\mathfrak{c}_1) = \mathrm{nor}_i(\mathfrak{c}_2)$ and $m_{\mathfrak{c}_1} = m_{\mathfrak{c}_2}$.

Crucial Corollary 3.4. *Assume that*

(a) u_1, u_2 *are finite subsets of* Ord, $|u_1 \setminus u_2| = |u_2 \setminus u_1|$, $u = u_1 \cup u_2$, $w = u_1 \cap u_2$, $\alpha_1 \in u_1 \setminus u_2$ *and* $\alpha_2 \in u_2 \setminus u_1$, $1 < i < \omega$, $|u| < n_*(i-1)$,
(b) $\mathfrak{c}_\ell \in \underline{\mathrm{CR}}^{u_\ell}_i$ *and* $\mathrm{nor}_i(\mathfrak{c}_\ell) > 2$ *(for* $\ell = 1, 2$*), and* $\mathrm{proj}_w(\mathfrak{c}_1) = \mathrm{proj}_w(\mathfrak{c}_2)$,
(c) $H_\ell : \mathbf{S}_{u_\ell,i+1} \longrightarrow {}^{n_*(i)}2$.

Then we can find $\mathfrak{d}_\ell \in \underline{\Sigma}(\mathfrak{c}_\ell)$, $\ell = 1, 2$, *such that:*

(α) $\mathrm{proj}_w(\mathfrak{d}_1) = \mathrm{proj}_w(\mathfrak{d}_2)$,
(β) $\mathrm{nor}_i(\mathfrak{d}_\ell) \geq \mathrm{nor}_i(\mathfrak{c}_\ell) - 1$,
(γ) *if* $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{d}_1} * \mathcal{F}_{\mathfrak{d}_2})$, $\bar{x} \in \mathbf{S}_{u,i}$ *and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, *and* $\eta_\ell = H_\ell(\bar{y} \upharpoonright u_\ell) \in {}^{n_*(i)}2$ *(for* $\ell = 1, 2$*), then*

$$\eta_1 = \eta_2 \quad \Rightarrow \quad \big(g_{y_i}(\alpha_1)^{-1} \circ f_{y_i}(\alpha_1)\big)(\eta_1) \neq \big(g_{y_i}(\alpha_2)^{-1} \circ f_{y_i}(\alpha_2)\big)(\eta_2).$$

Proof. Let $\mathcal{F}_\ell = \mathcal{F}_{\mathfrak{c}_\ell}$. By assumptions (a,b), the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is strongly balanced and $\mathrm{nor}^0_i(\mathcal{F}_\ell) > (\ell^*_i)^2$. Apply Crucial Lemma 3.2 to choose $\mathcal{F}'_1, \mathcal{F}'_2, \mathbf{h}_1, \mathbf{h}_2$ such that

$(*)_1$ $\mathcal{F}'_\ell \in \mathrm{wpos}^{u_\ell}_i$, $\mathcal{F}'_\ell \leq \mathcal{F}_\ell$, $\|\mathcal{F}'_\ell\| \geq 8^{-k_*} \cdot \|\mathcal{F}_\ell\|$ (where $k_* = |\mathbf{S}_{u,i}|$), and the pair $(\mathcal{F}'_1, \mathcal{F}'_2)$ is balanced,
$(*)_2$ $\mathbf{h}_\ell : \mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1} \longrightarrow {}^{n_*(i)}2$,
$(*)_3$ if $h \in \mathrm{set}(\mathcal{F}'_1 * \mathcal{F}'_2)$, $\bar{x} \in \mathbf{S}_{u,i}$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, then

$$H_1(\bar{y} \upharpoonright u_1) = H_2(\bar{y} \upharpoonright u_2) \quad \Rightarrow \quad \mathbf{h}_1(\bar{x} \upharpoonright u_1, \bar{y} \upharpoonright w) = \mathbf{h}_2(\bar{x} \upharpoonright u_2, \bar{y} \upharpoonright w) = H_1(\bar{y} \upharpoonright u_1) = H_2(\bar{y} \upharpoonright u_2).$$

Next, for $\bar{y} \in \mathbf{S}_{u_\ell,i+1}$, $\ell = 1, 2$, put

$$H'_\ell(\bar{y}) = \big(g_{y_i}(\alpha_\ell)^{-1} \circ f_{y_i}(\alpha_\ell)\big)\big(\mathbf{h}_\ell(\bar{y}{\restriction}i, \bar{y} \upharpoonright w)\big) \in {}^{n_*(i)}2.$$

Apply 3.2 again (this time using clause (δ) there too) to choose $\mathcal{F}''_1, \mathcal{F}''_2, \mathbf{h}''_1, \mathbf{h}''_2$ such that

$(*)_4$ $\mathcal{F}''_\ell \in \mathrm{wpos}^{u_\ell}_i$, $\mathcal{F}''_\ell \leq \mathcal{F}'_\ell$, $\|\mathcal{F}''_\ell\| \geq 8^{-k_*} \cdot \|\mathcal{F}'_\ell\|$, and the pair $(\mathcal{F}'_1, \mathcal{F}'_2)$ is balanced,
$(*)_5$ $\mathbf{h}''_\ell : \mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1} \longrightarrow {}^{n_*(i)}2$,
$(*)_6$ for each $\bar{x} \in \mathbf{S}_{u,i}$ one of the following occurs:
  $(\alpha)_{\bar{x}}$ if $h \in \mathrm{set}(\mathcal{F}''_1 * \mathcal{F}''_2)$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, then $H'_1(\bar{y} \upharpoonright u_1) \neq H'_2(\bar{y} \upharpoonright u_2)$, or
  $(\beta)_{\bar{x}}$ if $h \in \mathrm{set}(\mathcal{F}''_1 * \mathcal{F}''_2)$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, then

$$\mathbf{h}''_1(\bar{x} \upharpoonright u_1, \bar{y} \upharpoonright w) = \mathbf{h}''_2(\bar{x} \upharpoonright u_2, \bar{y} \upharpoonright w) = H'_1(\bar{y} \upharpoonright u_1) = H'_2(\bar{y} \upharpoonright u_2).$$

It follows from $(*)_1 + (*)_4$ that $\frac{|\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}''_\ell\|} \leq 64^{k_*} \cdot \frac{|\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}_\ell\|} < 64^{\ell^*_i} \cdot \frac{|\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}_\ell\|}$ and hence (remembering that $\mathrm{nor}^0_i(\mathcal{F}_\ell) > (\ell^*_i)^2$) we have

$$\mathrm{nor}^0_i(\mathcal{F}''_\ell) \geq k^*_i - \log_3\Big(\log_{k^*_i}\big(\frac{k^*_i \cdot |\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}_\ell\|} \cdot 64^{\ell^*_i}\big)\Big) \geq k^*_i - \log_3\Big(\log_{k^*_i}\big(\frac{k^*_i \cdot |\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}_\ell\|} \cdot k^*_i\big)\Big) \geq$$

$$k^*_i - \log_3\Big(\log_{k^*_i}\big((\frac{k^*_i \cdot |\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}_\ell\|})^3\big)\Big) = k^*_i - \log_3\Big(3\log_{k^*_i}\big(\frac{k^*_i \cdot |\mathrm{pos}^{u_\ell}_i|}{\|\mathcal{F}_\ell\|}\big)\Big) = \mathrm{nor}^0_i(\mathcal{F}_\ell) - 1 > \ell^*_i.$$

In particular, $\|\mathcal{F}''_\ell\| / |\mathrm{pos}^{u_\ell}_i| > (k^*_i)^{1 - 3^{k^*_i - \ell^*_i}}$ and by 2.15(2) we get

$$\frac{\|\mathcal{F}''_1 * \mathcal{F}''_2\|}{|\mathrm{pos}^u_i|} \geq \Big(\frac{1}{2}(k^*_i)^{1 - 3^{k^*_i - \ell^*_i}}\Big)^3,$$

so

$(*)_7$ $\mathrm{nor}^0_i(\mathcal{F}''_1 * \mathcal{F}''_2) \geq k^*_i - \log_3\big(\log_{k^*_i}\big(k^*_i \cdot (2(k^*_i)^{3^{k^*_i - \ell^*_i} - 1})^3\big) > \ell^*_i - 2 > 0$.

Now we claim that

$(*)_8$ in clause $(*)_6$ before, the possibility $(\beta)_{\bar{x}}$ cannot occur.

Suppose towards contradiction that for some $\bar{x} \in \mathbf{S}_{u,i}$ the statement in $(\beta)_{\bar{x}}$ holds true. Then, remembering $\mathbf{h}_\ell : \mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1} \longrightarrow {}^{n_*(i)}2$, we have

($\circledast$) if $h \in \mathrm{set}(\mathcal{F}''_1 * \mathcal{F}''_2)$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h)$ and $\eta_\ell = \mathbf{h}_\ell(\bar{x} \upharpoonright u_\ell, \bar{y} \upharpoonright w)$ (for $\ell = 1, 2$), then $\big(g_{y_i}(\alpha_1)^{-1} \circ f_{y_i}(\alpha_1)\big)(\eta_1) = \big(g_{y_i}(\alpha_2)^{-1} \circ f_{y_i}(\alpha_2)\big)(\eta_2)$.

Since $\alpha_1 \neq \alpha_2$ are in $u \backslash w$ we may apply Lemma 3.3(2) to get that $\mathrm{nor}^0_i(\mathcal{F}''_1 * \mathcal{F}''_2) = 0$, contradicting $(*)_7$.

Thus, putting together $(*)_3$ and $(*)_6 + (*)_8$ we conclude that

$(*)_9$ if $h \in \mathrm{set}(\mathcal{F}''_1 * \mathcal{F}''_2)$, $\bar{x} \in \mathbf{S}_{u,i}$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h)$, $\eta_\ell = H_\ell(\bar{y} \restriction u_\ell)$ (for $\ell = 1,2$), then

$$\eta_1 = \eta_2 \quad \Rightarrow \quad \big(g_{y_i}(\alpha_1)^{-1} \circ f_{y_i}(\alpha_1)\big)(\eta_1) \neq \big(g_{y_i}(\alpha_2)^{-1} \circ f_{y_i}(\alpha_2)\big)(\eta_2).$$

Now we set $\mathfrak{d}_\ell = (\mathcal{F}''_\ell, m_{\mathfrak{c}_\ell})$ (for $\ell = 1,2$). Since $\mathcal{F}''_\ell \leq \mathcal{F}'_\ell \leq \mathcal{F}_\ell$ and $\mathrm{nor}^0_i(\mathcal{F}''_\ell) \geq \mathrm{nor}^0_i(\mathcal{F}_\ell) - 1 > m_{\mathfrak{c}_\ell}$, we know that $\mathfrak{d}_\ell \in \underset{\sim}{\Sigma}(\mathfrak{c}_\ell)$, and since $(\mathcal{F}''_1, \mathcal{F}''_2)$ is balanced we conclude $\mathrm{proj}_w(\mathfrak{d}_1) = \mathrm{proj}_w(\mathfrak{d}_2)$. Also $\mathrm{nor}_i(\mathfrak{d}_\ell) \geq \mathrm{nor}_i(\mathfrak{c}_\ell) - 1$ and thus $\mathfrak{d}_1, \mathfrak{d}_2$ are as required in $(\alpha), (\beta)$. Finally, the demand (γ) is given by $(*)_9$. $\square$

Lemma 3.5. *Assume that*

(a) $u_1, u_2 \subseteq \mathrm{Ord}$ *are finite non-empty sets of the same size,* $|u_1 \setminus u_2| = |u_2 \setminus u_1|$,
(b) $w = u_1 \cap u_2$, $u = u_1 \cup u_2$, *and for* $\ell = 1,2$*:*
(c) $p_\ell \in \underset{\sim}{\mathbb{Q}}_{u_\ell}$ *and* $\alpha_{\ell,k} \in u_\ell \setminus w$ *and* $\underset{\sim}{\rho}_{\ell,k}$ *is a* $\underset{\sim}{\mathbb{Q}}_{u_\ell}$*–name for a branch of* $\underset{\sim}{t}_{\alpha_{\ell,k}}$ *(i.e., this is forced) for* $k < \omega$*, and*
(d) $\mathbf{j}_{w,u_1}(p_1), \mathbf{j}_{w,u_2}(p_2)$ *are compatible in* $\underset{\sim}{\mathbb{Q}}_w$ *(see 2.18, 2.19).*

Then there is $q \in \underset{\sim}{\mathbb{Q}}_u$ *such that* $p_\ell \leq_{\underset{\sim}{\mathbb{Q}}_{u_\ell}} \mathbf{j}_{u_\ell,u}(q)$ *for* $\ell = 1,2$ *and*

$$q \Vdash_{\underset{\sim}{\mathbb{Q}}_u} \text{“ } \underset{\sim}{\rho}_{1,k}, \underset{\sim}{\rho}_{2,k} \text{ have bounded intersection ”}.$$

Proof. Without loss of generality

($\circledast$) for $\underset{\sim}{\mathbb{Q}}_{u_\ell}$, for each $j < \omega$ the sequence $\underset{\sim}{\rho}_{\ell,j}$ can be read continuously above p_ℓ; moreover for every large enough i, say $i \geq i_\ell(j)$ the sequence $\underset{\sim}{\rho}_{\ell,j} \restriction i$ can be read from $\underset{\sim}{\bar{x}}_{u_\ell} \restriction i$.

[Why? First by Proposition 2.19 there is q_1 such that $p_1 \leq_{\underset{\sim}{\mathbb{Q}}_{u_1}} q_1$ and

$$(\forall q)[q_1 \leq_{\underset{\sim}{\mathbb{Q}}_{u_1}} q \ \Rightarrow \ \mathbf{j}_{w,u_1}(q), \mathbf{j}_{w,u_2}(p_2) \text{ are compatible in } \underset{\sim}{\mathbb{Q}}_w].$$

Second, by 2.5+2.9, there is $p'_1 \in \underset{\sim}{\mathbb{Q}}_{u_1}$ satisfying ($\circledast$) and such that $q_1 \leq_{\underset{\sim}{\mathbb{Q}}_{u_1}} p'_1$. Third, we may choose $q_2 \geq_{\underset{\sim}{\mathbb{Q}}_{u_2}} p_2$ such that

$$(\forall q)[q_2 \leq_{\underset{\sim}{\mathbb{Q}}_{u_2}} q \Rightarrow \mathbf{j}_{w,u_1}(p'_1), \mathbf{j}_{w,u_2}(q) \text{ are compatible in } \underset{\sim}{\mathbb{Q}}_w].$$

Fourth, by 2.9, there is $p'_2 \in \underset{\sim}{\mathbb{Q}}_{u_2}$ satisfying ($\circledast$) and such that $q_2 \leq_{\underset{\sim}{\mathbb{Q}}_{u_2}} p'_2$. Clearly (p'_1, p'_2) are as required.]

Passing to stronger conditions if needed we may also require that $\mathbf{i}(p_1) = \mathbf{i}(p_2) = \mathbf{i}$, $\mathbf{j}_{w,u_1}(p_1) = \mathbf{j}_{w,u_2}(p_2)$ (note $(*)_4$ from the proof of 2.19), $|u| < n_*(\mathbf{i} - 1)$ and $\mathrm{nor}_i(\mathfrak{c}^{p_\ell}_i) > 100$ for $i \geq \mathbf{i}$. Without loss of generality, letting $i(j) = \max\{i_1(j), i_2(j)\}$, it satisfies $i(0) = \mathbf{i}$, $i(j+1) > i(j) + 10$ and

$$\mathrm{nor}_i(\mathfrak{c}^{p_1}_i) = \mathrm{nor}_i(\mathfrak{c}^{p_2}_i) > 2j + 2 \qquad \text{for } i \geq i(j).$$

Fix $i \geq \mathbf{i}$ for a moment. Let k be such that $i(k) \leq i < i(k+1)$. We shall shrink $\mathfrak{c}^{p_1}_i, \mathfrak{c}^{p_2}_i$ in order to take care of $(\alpha_{1,m}, \underset{\sim}{\rho}_{1,m}, \alpha_{2,m}, \underset{\sim}{\rho}_{2,m})$ for $m \leq k$. By ($\circledast$) from the beginning of the proof we know that

(i) if $\bar{y} \in \mathbf{S}_{u_\ell, i+1} \cap \mathrm{pos}(p_\ell)$, then the condition $(p_\ell)^{[\bar{y}]} \in \underset{\sim}{\mathbb{Q}}_{u_\ell}$ decides $\underset{\sim}{\rho}_{\ell,m}(i)$ for $m \leq k$, say $(p_\ell)^{[\bar{y}]} \Vdash_{\underset{\sim}{\mathbb{Q}}_{u_\ell}}$ “ $\underset{\sim}{\rho}_{\ell,m}(i) = H_{\ell,m}(\bar{y})$ ”, where $H_{\ell,m} : \mathbf{S}_{u_\ell,i+1} \longrightarrow {}^{n_*(i)}2$.

Use Crucial Corollary 3.4 $(k+1)$ times to choose $\mathfrak{d}^1_i \in \underline{\Sigma}(\mathfrak{c}^{p_1}_i)$ and $\mathfrak{d}^2_i \in \underline{\Sigma}(\mathfrak{c}^{p_2}_i)$ such that:

(ii) $\mathrm{proj}_w(\mathfrak{d}^1_i) = \mathrm{proj}_w(\mathfrak{d}^2_i)$,
(iii) $\mathrm{nor}_i(\mathfrak{d}^\ell_i) \geq \mathrm{nor}_i(\mathfrak{c}^{p_\ell}_i) - (k+1)$ (for $\ell = 1, 2$),
(iv) *if* $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{d}^1_i} * \mathcal{F}_{\mathfrak{d}^2_i})$, $\bar{x} \in \mathbf{S}_{u,i}$, $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, $m \leq k$, $\ell = 1, 2$ and $\eta_\ell = H_{\ell,m}(\bar{y} \restriction u_\ell) \in {}^{n_*(i)}2$, *then*

$$\eta_{1,m} = \eta_{2,m} \quad \Rightarrow \quad \big(g_{y_i}(\alpha_{1,m})^{-1} \circ f_{y_i}(\alpha_{1,m})\big)(\eta_{1,m}) \neq \big(g_{y_i}(\alpha_{2,m})^{-1} \circ f_{y_i}(\alpha_{2,m})\big)(\eta_{2,m}).$$

After this construction is carried out for every $i \geq \mathbf{i}$ we define

- $q_\ell = (\bar{x}_{p_\ell}, \bar{\mathfrak{d}}^\ell)$, where $\bar{\mathfrak{d}}^\ell = \langle \mathfrak{d}^\ell_i : i \in [\mathbf{i}, \omega) \rangle$, $\ell = 1, 2$,
- $q = (\bar{x}_{p_1} \cup \bar{x}_{p_2}, \bar{\mathfrak{d}})$, where $\bar{\mathfrak{d}} = \langle \mathfrak{d}_i : i \in [\mathbf{i}, \omega) \rangle$, $\mathcal{F}_{\mathfrak{d}_i} = \mathcal{F}_{\mathfrak{d}^1_i} * \mathcal{F}_{\mathfrak{d}^2_i}$, $m_{\mathfrak{d}_i} = m_{\mathfrak{d}^1_i} = m_{\mathfrak{d}^2_i}$.

It follows from (iii) (and the choice of $i(j)$) that $q_\ell \in \underline{\mathbb{Q}}_{u_\ell}$ and, by 2.15(2), $q \in \underline{\mathbb{Q}}_u$. Plainly $p_\ell \leq_{\underline{\mathbb{Q}}_{u_\ell}} q_\ell \leq_{\underline{\mathbb{Q}}_{u_\ell}} \mathbf{j}_{u_\ell,u}(q)$.

Now, let $k < \omega$ and consider $i \geq i(k)$. It follows from (iv)+2.2(5) that for each $\bar{x} \in \mathbf{S}_{u,i} \cap \mathrm{pos}(q)$ and $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{d}_i})$, if $\bar{y} = \mathrm{suc}_{\bar{x}}(h)$ and $\eta_{\ell,k} = H_{\ell,k}(\bar{y} \restriction u_\ell)$, then

$$\eta_{1,k} = \eta_{2,k} \quad \Rightarrow \quad q^{[\bar{y}]} \Vdash_{\underline{\mathbb{Q}}_u} \text{“ } \{\rho : \eta_{1,k} <_{t_{\varkappa \alpha_{1,k}}} \rho\} \cap \{\rho : \eta_{2,k} <_{t_{\varkappa \alpha_{2,k}}} \rho\} = \emptyset \text{ ”}.$$

Since $q^{[\bar{y}]} \Vdash_{\underline{\mathbb{Q}}_u}$ “ $\underset{\sim}{\rho}_{\ell,k}(i) = \eta_{\ell,k}$ ” (for $\ell = 1, 2$) we may conclude that

$$q^{[\bar{y}]} \Vdash_{\underline{\mathbb{Q}}_u} \text{“ either } \underset{\sim}{\rho}_{1,k}(i) \neq \underset{\sim}{\rho}_{2,k}(i) \text{ or } (\forall j > i)(\underset{\sim}{\rho}_{1,k}(j) \neq \underset{\sim}{\rho}_{2,k}(j)) \text{ ”}.$$

Hence immediately we see that q is as required in the assertion of the lemma. $\square$

Remark 3.6. (1) If we can deal only with one case (i.e., one k in clause (c) of 3.5), we have to use $\mathcal{A} = \mathbf{T}^*_\omega$, not “any uncountable” $\mathcal{A} \subseteq \mathbf{T}^*_\omega$. But actually it is enough in 3.5 to deal with finitely many pairs.
(2) We can prove in 3.5 that there is a pair (p'_1, p'_2) such that:
  (a) $p_\ell \leq_{\mathbb{Q}_{u_\ell}} p'_\ell$ for $\ell = 1, 2$,
  (b) $\mathbf{j}_{w,u_1}(p'_1), \mathbf{j}_{w,u_2}(p'_2)$ are compatible,
  (c) if $p \in \mathbb{Q}_u$ satisfies $p'_\ell \leq_{\mathbb{Q}_{u_\ell}} \mathbf{j}_{u,u_\ell}(p)$, then p is as required.

If $u = \{\alpha\}$ is a singleton, then considering $\mathrm{OB}^u_i, \mathbf{S}_{u,i}, \mathbf{S}_u, \mathrm{pos}^u_i, \mathrm{wpos}^u_i, \underline{\mathbb{Q}}_u$ we may ignore u (and α) in a natural way arriving to the definitions of $\mathrm{OB}_i, \mathbf{S}_i, \mathbf{S}, \mathrm{pos}_i, \mathrm{wpos}_i, \underline{\mathbb{Q}}$, respectively. Let $\varkappa : \mathbf{S}_\omega \longrightarrow \mathbf{T}_\omega$ be the mapping given by $\varkappa(\bar{x}) = \langle f_{x_i} : i < \omega \rangle$ (on $\mathbf{T}$ see Definition 1.2(2), concerning $\varkappa$ compare Definition 2.1(G)).

The following proposition finishes the proof of Theorem 1.1.

Proposition 3.7. *Let $N_* \prec (\mathcal{H}(\beth^+_7), \theta)$ be countable.*

(1) *There is a perfect subtree $\mathbf{S}^* \subseteq \mathbf{S}$ (so $\mathbf{S}^*_\omega = \lim_\omega(\mathbf{S}^*) \subseteq \mathbf{S}_\omega$) such that:*
 if $n < \omega$, $\bar{x}_\ell \in \mathbf{S}^*_\omega$ for $\ell < n$ are pairwise distinct then $(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is a generic for $\underline{\mathbb{Q}}_n$ over N_*.
(2) *Moreover, $\varkappa[\mathbf{S}^*_\omega] \subseteq \mathbf{T}_\omega$ is strongly pbd (see Definition 1.4(3)) and* $\mathrm{ar\text{-}cl}\{A_{\varkappa(\bar{x})} : \bar{x} \in \mathbf{S}^*_\omega\}$ *is Borel.*

Proof. By 2.9 and 2.20 and (for part (2)) by 3.5. In details, let $\mathcal{T}$ be a perfect subtree of ${}^{\omega>}2$ such that in each level only in one node we have splitting and let $\mathcal{T}_i = \{\eta \in \mathcal{T} : \eta \text{ of the } i\text{-th level}\}$.

Let $h_i : |\mathcal{T}_i| \longrightarrow \mathcal{T}_i$ be a bijection such that

$$m' < m'' < n_i \;\Leftrightarrow\; h_i(m') <_{\text{lex}} h_i(m''),$$

where $n_i = |\mathcal{T}_i|$. Let $\langle (m_j, k_j, \underset{\sim}{\rho}_j) : j < \omega \rangle$ list all the triples $(m, k, \underset{\sim}{\rho})$ satisfying: $m < \omega$, $k < m$ and $\underset{\sim}{\rho}$ is a $\mathbb{Q}_{m \setminus \{k\}}$–name of a branch of $\underset{\sim}{t}_k$ such that $\underset{\sim}{\rho}$ belongs to N_*.

Let η_i be the unique member of $\mathcal{T}_i$ such that $\{\eta_i{}^\frown\langle 0\rangle, \eta_i{}^\frown\langle 1\rangle\} \in \mathcal{T}_{i+1}$. For $\ell = 0, 1$ let $f_{i,\ell} : \mathcal{T}_i \longrightarrow \mathcal{T}_{i+1}$ be such that

$$[\eta \in \mathcal{T}_i \setminus \{\eta_i\} \quad\Rightarrow\quad f_{i,\ell}(\eta){\restriction} i = \eta] \qquad \text{and} \qquad f_{i,\ell}(\eta_i) = \eta_i{}^\frown\langle \ell\rangle.$$

Let $u_{i,\ell} =\ \text{Rang}(g_{i,\ell})$ where $g_{i,\ell} = h_{i+1}^{-1} \circ f_{i,\ell} \circ h_i$. For an order preserving function g from the finite $u \subset$ Ord into Ord let $\hat{g}$ be the isomorphism from $\mathbb{Q}_u$ onto $\mathbb{Q}_{g[u]}$ induced by g.

Let $\langle \mathcal{I}_{n,i} : i < \omega \rangle$ list all the dense open subsets of $\mathbb{Q}_n$ which belong to N_*. By induction on $i < \omega$ choose p_i such that if $\ell \in \{1, 2\}$ then (recalling $\mathbf{j}_{u_{i,\ell}, n_j}$ is a complete projection from $\mathbb{Q}_{n_j}$ onto $\mathbb{Q}_{u_{i,\ell}}$) we have

- (i) $p_i \in \mathbb{Q}_{n_i}$, $\hat{g}_{i,\ell}(p_i) \leq_{\mathbb{Q}_{u_{i,\ell}}} \mathbf{j}_{u_{i,\ell}, n_{i+1},}(p_{i+1})$ for $\ell = 0, 1$.
- (ii) If $u \subseteq n_i$ and h_u^* is $\text{OP}_{u,|u|}$, i.e., the order preserving function from $\{0, \ldots, |u|-1\}$ onto u, and $\hat{h}_u^*$ is defined as above and $k < i$, then $\mathbf{j}_{u,n_i}(p_i) \in \mathbb{Q}_u$ belongs to $\hat{h}_u^*(\mathcal{I}_{|u|,k})$.
- (iii) Assume that for $\ell = 0, 1$ the objects $j_\ell < \omega$, $u_\ell \subseteq \mathcal{T}_i$ satisfy

  $$\eta_i \in u_\ell,\ |u_\ell| = m_{j_\ell},\ h_{u_\ell}^*(k_{j_\ell}) = h_i^{-1}(\eta_i)$$

  and let $\underset{\sim}{\rho}_\ell = \hat{g}_{i,\ell}(\hat{h}_{u_\ell}^*(\underset{\sim}{\rho}_{j_\ell}))$ (so it is a $\mathbb{Q}_{n_{i+1}}$–name for a branch of $\underset{\sim}{t}_{g_{i,\ell}(h_u^*(\eta_i))}$). *Then* $\Vdash_{\mathbb{Q}_{n_{i+1}}}$ " the branches $\underset{\sim}{\rho}_0$ of $\underset{\sim}{t}_{f_{i,0}(\eta_i)}$ and $\underset{\sim}{\rho}_1$ of $\underset{\sim}{t}_{f_{i,1}(k_{\eta_i})}$ have bounded intersection ".

This is straightforward. $\square$

Theorem 3.8. (1) *There is a Borel arithmetically closed set $\mathbf{B} \subseteq \mathcal{P}(\omega)$ such that there is no arithmetically closed 2-Ramsey ultrafilter on it.*
(2) *Moreover, there is a Borel*[1] *$\mathcal{A}_* \subseteq \mathcal{B}$ such that for every uncountable $\mathcal{A}' \subseteq A$, there is no definably closed minimal ultrafilter on the arithmetic closure of* ar-cl$(\mathcal{A}')$ *of $\mathcal{A}'$.*
(3) *We can demand that above each* ar-cl$(\mathcal{A}')$ *is a standard system.*

Proof. (1) and (2) Let $\mathcal{A} = \mathbf{T}_\omega^*$ be as in the proof of 3.7 and let $\mathcal{B}$ be the arithmetic closure ar-cl$(\mathcal{A})$ of $\mathcal{A}$. For every $A_t \in \mathcal{A}$ there towards contradiction assume D is a $\mathbf{B}$–minimal ultrafilter where $\underset{\sim}{B} =$ ar-cl$(\mathcal{A}')$, $\mathcal{A}' \subseteq \mathcal{A}$ is uncountable.

Now for every $A_t \in \mathcal{A}'$, $(\mathbb{N}, <_t)$ is a tree with finite levels (hence finite splittings), a root and the set of levels is $\mathbb{N}$. For every $i < \omega$ the set $\{n < \omega$: in $<_t^*$ the level of n is $< i\}$ is finite and hence its compliment belongs to D. The rest is divided to $\langle \{m : b \leq_t^* m\} : b$ is of level exactly i for $<_t^* \rangle$. This is a finite division hence for some unique $b = b_i^t$ of level i such that $\{m : b \leq_t^* m\} \in D$. As D is a 2-Ramsey ultrafilter

- (i) $\langle b_i^t : i < \omega \rangle$ is definable in $\mathbb{N}_{\mathcal{A}'}$.

We define a function g_t on $\mathbb{N}$ by $g_t(c) = \max\{i : b_i^t \leq_t c\}$. Again

- (ii) g_t is definable in $\mathbb{N}_{\mathcal{A}'}$.

[1] to eliminate it we have to force over $\mathbb{N}$

As D is minimal there is $C_t \subseteq \mathbb{N}$ definable in $\mathbb{N}_{\mathcal{A}'}$ and such that

(iii) $g_t \restriction C_t$ is one-to-one.

Let C_t be the first order definable in $\mathbb{N}_{\mathcal{A}_t}$ where $\mathcal{A}_t \subseteq \mathcal{A}'$ is finite, $t \in \mathcal{A}_t$ for simplicity and so is the set $\{b^t_i : i < \omega\}$. As each $\mathbb{Q}_u$ is ${}^\omega\omega$–bounding and we can further shred c_t below there is $h_* \in N_*$ such that [recall we are forcing over the countable $N_* \prec (H(\chi), \in)$, so our $\mathcal{B}$ is $\bigcup\{\mathcal{P}(\omega) \cap N[t_0, \ldots, t_{n-1}] : t_\ell \in T^*_\omega\}$] such that

(iv) $h_* \in {}^\omega\omega$ is increasing, $h_*(0) = 0$, and

(v) if $c \in C_t$ and $g_t(c) <_t h_*(i)$ then $c <_{\mathbb{N}} h_*(i+1)$.

Without loss of generality now by the infinite Δ–system for finite sets for some $t_1 \neq t_2$ we have $\{t_1, t_2\} \cap (\mathcal{A}_{t_1} \cap \mathcal{A}_{t_2}) = \emptyset$, etc.

Moreover, replacing $\mathcal{A}_{t_1} \cup A_2$, $\mathcal{A}_1, \mathcal{A}_2, t_1, t_2$ by $u = u_1 \cup u_2$, u_1, u_0, $\alpha_1 \in u_1 \setminus u_2$, $\alpha_2 \in u_2 \setminus u_1$ we have the situation in §2 by similar proof. We get $C_{t_2} \cap C_{t_2}$ is finite, but both are in an ultrafilter, so we are done.

(3) We let $\mathbb{Q}$ be as in [?] for $\lambda \geq \beth_{\omega_1}$, use what is proved there. $\square$

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