DEPTH OF BOOLEAN ALGEBRAS

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ABSTRACT. Suppose D is an ultrafilter on κ and $\lambda^\kappa = \lambda$. We prove that if $\mathbf{B}_i$ is a Boolean algebra for every $i < \kappa$ and λ bounds the Depth of every $\mathbf{B}_i$, then the Depth of the ultraproduct of the $\mathbf{B}_i$'s mod D is bounded by λ^+. We also show that for singular cardinals with small cofinality, there is no gap at all. This gives a partial answer to problem No. 12 in [?].

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0. INTRODUCTION

Let $\mathbf{B}$ be a Boolean Algebra. We define the Depth of it as the supremum on the cardinalities of well-ordered subsets in $\mathbf{B}$. Now suppose that $\langle \mathbf{B_i} : i < \kappa \rangle$ is a sequence of Boolean algebras, and D is an ultrafilter on κ. Define the ultra-product algebra $\mathbf{B}$ as $\prod\limits_{i<\kappa} \mathbf{B}_i/D$. The question (raised also for other cardinal invariants, by Monk, in [?]) is about the relationship between $\text{Depth}(\mathbf{B})$ and $\prod_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D$.

Let us try to draw the picture:

$$\begin{tikzcd}
 & & \langle \mathbf{B_i} : i < \kappa \rangle, D \arrow[dll] \arrow[drr] & & \\
\mathbf{B} = \prod\limits_{i<\kappa} \mathbf{B}_i/D \arrow[dr] & & & & \langle \text{Depth}(\mathbf{B}_i) : i < \kappa \rangle \arrow[dl] \\
 & \text{Depth}(\mathbf{B}) & & \prod\limits_{i<\kappa} \text{Depth}\mathbf{B}_i/D &
\end{tikzcd}$$

As we can see from the picture, given a sequence of Boolean algebras (of length κ) and an ultrafilter on κ, we have two alternating ways to produce a cardinal value. The left course creates, first, a new Boolean algebra namely the ultraproduct algebra $\mathbf{B}$. Then we compute the Depth of it. In the second way, first of all we get rid of the algebraic structure, producing a sequence of cardinals (namely $\langle \text{Depth}(\mathbf{B}_i) : i < \kappa \rangle$). Then we compute the cardinality of its cartesian product divided by D.

Shelah proved in [?] §5, under the assumption $\mathbf{V} = \mathbf{L}$, that if $\kappa = \text{cf}(\kappa) < \lambda$ and $\lambda = \lambda^\kappa$ (so $\kappa < \text{cf}(\lambda)$), then you can build a sequence of Boolean algebras $\langle \mathbf{B_i} : i < \kappa \rangle$, such that $\text{Depth}(\mathbf{B_i}) \leq \lambda$ for every $i < \kappa$, and $\text{Depth}(\mathbf{B}) > \prod_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D$ for every uniform ultrafilter D. This result is based on the square principle, introduced and proved in $\mathbf{L}$ by Jensen.
A natural question is how far can this gap reach. We prove (in §2) that if $\mathbf{V} = \mathbf{L}$ then the gap is at most one cardinal. In other words, for every regular cardinal and for every singular cardinal with high cofinality we can create a gap (having the square for every infinite cardinal in $\mathbf{L}$), but it is limited to one cardinal.

The assumption $\mathbf{V} = \mathbf{L}$ is just to make sure that every ultrafilter is regular, so the results in §2 apply also outside $\mathbf{L}$. On the other hand, if $\mathbf{V}$ is far from $\mathbf{L}$ we get a different picture. By [?] (see conclusion 2.2 there, page 94), under some reasonable assumptions, there is no gap at all above a compact cardinal.
We can ask further what happens if $\text{cf}(\lambda) < \lambda$, and $\kappa \geq \text{cf}(\lambda)$. We prove here that if λ is singular with small cofinality, (i.e., the cases which are not covered in the previous paragraph), then $\prod_{i<\kappa} \text{Depth}(\mathbf{B}_i)/D \geq \text{Depth}(\mathbf{B})$. It is interesting to know that similar result holds above a compact cardinal

for singular cardinals with countable cofinality. We suspect that it holds (for such cardinals) in ZFC.
The proof of those results is based on an improvement to the main Theorem in [**?**]. It says that under some assumptions we can dominate the gap between $\mathrm{Depth}(\mathbf{B})$ and $\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$. In this paper we use weaker assumptions. We give here the full proof, so the paper is self-contained. We intend to shed light on the other side of the coin (i.e., under large cardinals assumptions) in a subsequent paper.

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1. The main theorem

Definition 1.1. Depth.
Let $\mathbf{B}$ be a Boolean Algebra.

$$\text{Depth}(\mathbf{B}) := \sup\{\theta : \exists \bar{b} = (b_\gamma : \gamma < \theta), \text{ increasing sequence in } \mathbf{B}\}$$

Remark 1.2. Clearly, we can use *decreasing* instead of *increasing* in the definition of Depth. We prefer the *increasing* version, since it is coherent with the terminology of [**?**].

Discussion 1.3. Depth($\mathbf{B}$) is always a cardinal, but it does not have to be a regular cardinal. It is achieved in the case of a successor cardinal (i.e., $\text{Depth}(\mathbf{B}) = \lambda^+$ for some infinite cardinal λ), and in the case of a singular cardinal with countable cofinality (i.e., $\text{Depth}(\mathbf{B}) = \lambda > \text{cf}(\lambda) = \aleph_0$). In all other cases, one can create an example of a Boolean Algebra $\mathbf{B}$, whose Depth is not attained. A detailed survey of these facts appears in [**?**].

We use also an important variant of the Depth:

Definition 1.4. Depth$^+$.
Let $\mathbf{B}$ be a Boolean Algebra.

$$\text{Depth}^+(\mathbf{B}) := \sup\{\theta^+ : \exists \bar{b} = (b_\gamma : \gamma < \theta), \text{ increasing sequence in } \mathbf{B}\}$$

Discussion 1.5. Assume λ is a limit cardinal. The question of achieving the Depth (for a Boolean Algebra $\mathbf{B}$ such that $\text{Depth}(\mathbf{B}) = \lambda$) demonstrates the difference between Depth and Depth$^+$. If $\text{cf}(\lambda)$ is uncountable, we can imagine two situations. In the first one the Depth is achieved, and in that case we have $\text{Depth}^+(\mathbf{B}) = \lambda^+$. In the second, the Depth is not achieved. Consequently, $\text{Depth}^+(\mathbf{B}) = \lambda$. Notice that $\text{Depth}(\mathbf{B}) = \lambda$ in both cases, so Depth$^+$ is more delicate and using it (as a scaffold) helps us to prove our results.

Throughout the paper, we use the following notation:

Notation 1.6.
(a) κ, λ are infinite cardinals
(b) D is a uniform ultrafilter on κ
(c) $\mathbf{B}_i$ is a Boolean Algebra, for any $i < \kappa$
(d) $\mathbf{B} = \prod\limits_{i<\kappa} \mathbf{B}_i / D$
(e) for $\kappa = \text{cf}(\kappa) < \lambda$, $S^\lambda_\kappa = \{\alpha < \lambda : \text{cf}(\alpha) = \kappa\}$.

We state our main result:

Theorem 1.7. *Assume*
(a) $\lambda \geq \text{cf}(\lambda) > \kappa$
(b) $\lambda = \lambda^\kappa$
(c) $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$.

Then $\text{Depth}^+(\mathbf{B}) \leq \lambda^+$.

Proof.
Assume towards a contradiction that $\langle a_\alpha : \alpha < \lambda^+\rangle$ is an increasing sequence in $\mathbf{B}$. Let us write a_α as $\langle a_i^\alpha : i < \kappa\rangle/D$ for every $\alpha < \lambda^+$. Let $\bar{M}$ be an approaching seauence of elementary submodels with nice properties (the detailed requirements are phrased in claim 1.8 below). We may assume that $\langle a_i^\alpha : \alpha < \lambda^+, i < \kappa\rangle \in M_0$. We also assume that $\mathbf{B}, \langle \mathbf{B}_i : i < \kappa\rangle, D \in M_0$.

We shall apply claim 1.8, so λ, κ, D are given and we define R_i for every $i < \kappa$ as the set $\{(\alpha, \beta) : \alpha < \beta < \lambda^+ \text{ and } a_i^\alpha < a_i^\beta\}$. As $\alpha < \beta \Rightarrow a_\alpha <_D a_\beta \Rightarrow \{i < \kappa : \mathbf{B}_i \models a_i^\alpha < a_i^\beta\} \in D$, all the assumptions of 1.8 hold, hence the conclusion also holds. So there are $i_* < \kappa$ and $Z \subseteq \lambda^+$ of order type λ as there.

Now, if $\alpha < \beta$ are from Z we have $\iota \in (\alpha, \beta)$ which satisfies $\alpha R_{i_*} \iota$ and $\iota R_{i_*} \beta$. It means that $a_{i_*}^\alpha <_{\mathbf{B}_{i_*}} a_{i_*}^\iota <_{\mathbf{B}_{i_*}} a_{i_*}^\beta$. By the transitivity of $<_{\mathbf{B}_{i_*}}$, we have $a_{i_*}^\alpha <_{\mathbf{B}_{i_*}} a_{i_*}^\beta$ for every $\alpha < \beta$ from Z.

Since $|Z| = \lambda$, we have an increasing sequence of length λ in $\mathbf{B}_{i_*}$, so $\text{Depth}^+(\mathbf{B}_{i_*}) \geq \lambda^+$, contradicting the assumptions of the Theorem.

$\square_{1.7}$

Claim 1.8. *Assume*

(a) $\lambda = \lambda^\kappa$
(b) D *is an ultrafilter on* κ
(c) $R_i \subseteq \{(\alpha, \beta) : \alpha < \beta < \lambda^+\}$ *is a two place relation on* λ^+ *for every* $i < \kappa$
(d) $\alpha < \beta \Rightarrow \{i < \kappa : (\alpha, \beta) \in R_i\} \in D$

Then *There exists* $i_* < \kappa$ *and* $Z \subseteq \lambda^+$ *of order type* λ, *such that for every* $\alpha < \beta$ *from* Z, *for some* $\iota \in (\alpha, \beta)$ *we have* $(\alpha, \iota), (\iota, \beta) \in R_i$.

Proof.
Let $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ be a continuous and increasing sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ for sufficiently large χ, with the following properties for every $\alpha < \lambda^+$:

(a) $\|M_\alpha\| = \lambda$
(b) $\lambda + 1 \subseteq M_\alpha$
(c) $\langle M_\beta : \beta \leq \alpha\rangle \in M_{\alpha+1}$
(d) $[M_{\alpha+1}]^\kappa \subseteq M_{\alpha+1}$.

For every $\alpha < \beta < \lambda^+$, define:

$$A_{\alpha,\beta} = \{i < \kappa : \alpha R_i \beta\}$$

By the assumption, $A_{\alpha,\beta} \in D$ for all $\alpha < \beta < \lambda^+$. Define $C := \{\gamma < \lambda^+ : \gamma = M_\gamma \cap \lambda^+\}$, and $S := C \cap S^{\lambda^+}_{\text{cf}(\lambda)}$. Since C is a club subset of λ^+, S is a stationary subset of λ^+. Choose δ^* as the λ-th member of S. For every $\alpha < \delta^*$, let A_α denote the set A_{α,δ^*}.

Let $u \subseteq \delta^*$, $|u| \leq \kappa$. Notice that $u \in M_{\delta^*}$, by the assumptions on $\bar{M}$.
Define:

$$S_u = \{\beta < \lambda^+ : \beta > \sup(u), \mathrm{cf}(\beta) = \mathrm{cf}(\lambda) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

Notice that $S_u \neq \emptyset$ as $\delta^* \in S_u$, hence if $u \subseteq \delta^*$ and $|u| \leq \kappa$ then $S_u \cap \delta^* \neq \emptyset$. We try to choose an increasing continuous sequence of ordinals from $C \cap \delta^*$, so that the non-limit points belong also to S. Choose $\delta_0 = 0$. Choose $\delta_{\epsilon+1}$ as the $(\epsilon+1)$-th member of $S \cap \delta^*$, and $\delta_\epsilon = \cup\{\delta_{\zeta+1} : \zeta < \epsilon\}$ for limit ϵ below λ. Since $\mathrm{otp}(S \cap \delta^*) = \lambda$, we have:

(a) $\langle \delta_\epsilon : \epsilon < \lambda \rangle$ is increasing and continuous
(b) $\sup\{\delta_\epsilon : \epsilon < \lambda\} = \delta^*$
(c) $\delta_{\epsilon+1} \in S$, for every $\epsilon < \lambda$

Define, for every $\epsilon < \lambda$, the following family:

$$\mathcal{A}_\epsilon = \{S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon : u \in [\delta_{\epsilon+1}]^{\leq\kappa}\}.$$

The crucial point is that $\mathcal{A}_\epsilon$ is not empty for each ϵ. We shall prove this in Lemma 1.9 below. So we have a family of non-empty sets, which is downward κ^+-directed. Hence, there is a κ^+-complete filter E_ϵ on $[\delta_\epsilon, \delta_{\epsilon+1})$, with $\mathcal{A}_\epsilon \subseteq E_\epsilon$, for every $\epsilon < \lambda$.
Define, for any $i < \kappa$ and $\epsilon < \lambda$, the sets $W_{\epsilon,i} \subseteq [\delta_\epsilon, \delta_{\epsilon+1})$ and $B_\epsilon \subseteq \kappa$, by:

$$W_{\epsilon,i} := \{\beta : \delta_\epsilon \leq \beta < \delta_{\epsilon+1} \text{ and } i \in A_{\beta,\delta_{\epsilon+1}}\}$$

$$B_\epsilon := \{i < \kappa : W_{\epsilon,i} \in E_\epsilon^+\}.$$

Finally, take a look at $W_\epsilon := \cap\{[\delta_\epsilon, \delta_{\epsilon+1}) \setminus W_{\epsilon,i} : i \in \kappa \setminus B_\epsilon\}$. For every $\epsilon < \lambda, W_\epsilon \in E_\epsilon$, since E_ϵ is κ^+-complete, so clearly $W_\epsilon \neq \emptyset$.

Choose $\beta = \beta_\epsilon \in W_\epsilon$. If $i \in A_{\beta,\delta_{\epsilon+1}}$, then $W_{\epsilon,i} \in E_\epsilon^+$, so $A_{\beta,\delta_{\epsilon+1}} \subseteq B_\epsilon$ (by the definition of B_ϵ). But, $A_{\beta,\delta_{\epsilon+1}} \in D$, so $B_\epsilon \in D$. For every $\epsilon < \lambda$, $A_{\delta_{\epsilon+1}}$ (which equals to $A_{\delta_{\epsilon+1},\delta^*}$) belongs to D, so $B_\epsilon \cap A_{\delta_{\epsilon+1}} \in D$.
Choose $i_\epsilon \in B_\epsilon \cap A_{\delta_{\epsilon+1}}$, for every $\epsilon < \lambda$. We choose, in this process, λ i_ϵ-s from κ, so as $\mathrm{cf}(\delta^*) = \mathrm{cf}(\lambda) > \kappa$, there is an ordinal $i_* \in \kappa$ such that the set $Y = \{\epsilon < \lambda : \epsilon \text{ is an even ordinal, and } i_\epsilon = i_*\}$ has cardinality λ.
The last step will be as follows:
Define $Z = \{\delta_{\epsilon+1} : \epsilon \in Y\}$. Clearly, $Z \in [\delta^*]^\lambda \subseteq [\lambda^+]^\lambda$. We will show that for $\alpha < \beta$ from Z we can find $\iota \in (\alpha, \beta)$ so that $(\alpha R_i \iota)$ and $(\iota R_i \beta)$. The idea is that if $\alpha < \beta$ and $\alpha, \beta \in Z$, then $i_* \in A_{\alpha,\beta}$.

Why? Recall that $\alpha = \delta_{\epsilon+1}$ and $\beta = \delta_{\zeta+1}$, for some $\epsilon < \zeta < \lambda$ (that's the form of the members of Z). Define:

$$U_1 := S_{\{\delta_{\epsilon+1}\}} \cap [\delta_\zeta, \delta_{\zeta+1}) \in \mathcal{A}_\zeta \subseteq E_\zeta.$$

$$U_2 := \{\gamma : \delta_\zeta \leq \gamma < \delta_{\zeta+1}, i_* \in A_{\gamma,\delta_{\zeta+1}}\} \in E_\zeta^+.$$

So, $U_1 \cap U_2 \neq \emptyset$, and we can choose $\iota \in U_1 \cap U_2$.
Now the following statements hold:

(a) $\alpha R_{i_*} \iota$
[Why? Well, $\iota \in U_1$, so $A_{\delta_{\epsilon+1,\iota}} = A_{\delta_{\epsilon+1}}$. But, $i_* \in B_\epsilon \cap A_{\delta_{\epsilon+1}} \subseteq A_{\delta_{\epsilon+1}}$, so $i_* \in A_{\delta_{\epsilon+1,\iota}}$, which means that $\delta_{\epsilon+1} R_{i_*} \iota$].

(b) $\iota R_{i_*} \beta$
[Why? Well, $\iota \in U_2$, so $i_* \in A_{\iota, \delta_{\zeta+1}}$, which means that $\iota R_{i_*} \delta_{\zeta+1}$].

(c) $\alpha R_{i_*} \beta$
[Why? By (a)+(b)].

So, we are done.

$\square_{1.8}$

Lemma 1.9. *Let* $\mathcal{A}_\epsilon = \{S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon : u \in [\delta_{\epsilon+1}]^{\leq\kappa}\}$.

(a) $\mathcal{A}_\epsilon$ *is not empty, for every* $\epsilon < \lambda$
(b) *Moreover,* $u \in [\delta_{\epsilon+1}]^{\leq\kappa} \Rightarrow S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon$ *is unbounded in* $\delta_{\epsilon+1}$.

Proof.
Clearly, (b) implies (a). Let us prove part (b).
First we observe that if $u \in [\delta_{\epsilon+1}]^{\leq\kappa}$ then $\sup(u) < \delta_{\epsilon+1}$ (since $\delta_{\epsilon+1} \in S \subseteq S^{\lambda^+}_{\mathrm{cf}(\lambda)}$, and $\kappa < \mathrm{cf}(\lambda)$). Second, $M_{\delta_{\epsilon+1}} = \bigcup\{M_\alpha : \alpha < \delta_{\epsilon+1}\}$ (since $\delta_{\epsilon+1}$ is a limit ordinal and $\bar{M}$ is continuous).
Consequently, there exists $\alpha < \delta_{\epsilon+1}$ so that $u \subseteq M_\alpha$. Choose such α, and observe that $u \in M_{\alpha+1}$ (again, this follows from the properties of $\bar{M}$). We derive $S_u \in M_{\alpha+1}$ as well (since it is definable from parameters in $M_{\alpha+1}$).
By the definition of S_u, $\delta^* \in S_u$. We conclude:

$$M_{\alpha+1} \cap \lambda^+ \subseteq M_{\delta_{\epsilon+1}} \cap \lambda^+ = \delta_{\epsilon+1} < \delta^* \in S_u$$

We can infer that $\sup(S_u) = \lambda^+$, so $M_{\delta_{\epsilon+1}} \models$ "$S_u \subseteq \lambda^+$, unbounded in λ^+". Since $M_{\delta_{\epsilon+1}} \cap \lambda^+ = \delta_{\epsilon+1}$ and by virtue of elementarity, $S_u \cap \delta_{\epsilon+1}$ is unbounded in $\delta_{\epsilon+1}$.
Recall that $\delta_\epsilon < \delta_{\epsilon+1}$, so $S_u \cap \delta_{\epsilon+1} \setminus \delta_\epsilon$ is also unbounded, and we are done.
$\square_{1.9}$

Corollary 1.10. *(GCH)*
Assume

(a) $\kappa < \mu$
(b) $\mathrm{Depth}(\mathbf{B}_i) \leq \mu$, *for every* $i < \kappa$.

Then $\mathrm{Depth}(\mathbf{B}) \leq \mu^+$.

Proof.
For every successor cardinal μ^+, and every $\kappa < \mu$, we have (under the GCH) $(\mu^+)^\kappa = \mu^+$. By assumption (b), we know that $\mathrm{Depth}^+(\mathbf{B}_i) \leq \mu^+$ for every $i < \kappa$. Now apply Theorem 1.7 (upon noticing that μ^+ here is standing for λ there), and conclude that $\mathrm{Depth}^+(\mathbf{B}) \leq \mu^{+2}$, so $\mathrm{Depth}(\mathbf{B}) \leq \mu^+$ as required.

$\square_{1.10}$

Remark 1.11. Notice that the corollary holds even if almost every $\mathbf{B}_i$ has μ as its Depth. So we may assume, without loss of generality, that $\mu = \lim_D(\langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa \rangle)$. This assumption becomes important if we try to phrase an equality (not just $\leq$), as in the Theorem of the next section.

2. Depth in L

We would like to draw some conclusions from the main Theorem in the previous section. We work in the constructible universe, for two reasons. The first one is that we can cover all the cases in **L**, with respect to the problem that we try to analyze. The second is that we can get, from the situation in **L**, a limitation *in ZFC* on one of the problems from [?].

We start with a short discussion on regular ultrafilters. A good source for the subject is [?], section 4.3. Recall:

Definition 2.1. Regular ultrafilters.
Let D be an ultrafilter on an infinite cardinal κ, and $\theta \leq \kappa$.

(a) D is θ-regular if there exits $E \subseteq D, |E| = \theta$, so that $\alpha < \kappa \Rightarrow |\{e \in E : \alpha \in e\}| < \aleph_0$
(b) D is called regular when D is κ-regular.

Remark 2.2. Measurability and $\aleph_0$-regular ultrafilters.
An ultrafilter D on κ is $\aleph_0$-regular iff D is $\aleph_1$-incomplete (The proof appears, for instance, in [?], proposition 4.3.4, page 249). If κ is below the first measurable cardinal, then every non-principal ultrafilter on κ is $\aleph_1$-incomplete, hence $\aleph_0$-regular.

The following is a fundamental result of Donder, from [?]:

Theorem 2.3. *Regular ultrafilters in the constructible universe.*
Assume $\mathbf{V} = \mathbf{L}$.
Let D be a non-principal ultrafilter on an infinite cardinal κ.
Then D is regular.

$\square_{2.3}$

It is proved (see [?], proposition 4.3.5, page 249) that for every infinite cardinal κ there exists a regular ultrafilter D over κ. Having a regular ultrafilter D, one can estimate the cardinality of an ultraproduct divided by D. A proof of the following claim can be found in [?], proposition 4.3.7 (page 250):

Claim 2.4. *Suppose D is a regular ultrafilter on κ.*
then $|\prod_{i<\kappa} \lambda/D| = \lambda^\kappa$.

$\square_{2.4}$

By [?], in §5, if λ is regular and $\kappa < \lambda$, or even $\lambda > \mathrm{cf}(\lambda) > \kappa$, we can build in **L** an example for $\mathrm{Depth}(\mathbf{B}) > \prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$, but the discrepancy is just one cardinal as shown in Corollary 1.10. We can ask what happens if λ is singular with small cofinality. The Theorem below says that equality holds.

The theorem answers problem No. 12 from [?], for the case of singular cardinals with countable cofinality (since then $\mathrm{cf}(\lambda) \leq \kappa$ for every infinite cardinal κ). Monk asks there whether an example with $\mathrm{Depth}(\prod_{i<\kappa} \mathbf{B}_i/D) > |\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D|$ is possible in ZFC. The equality in $\mathbf{L}$ below shows that such an example does not exist, in the case of countable cofinality.

Theorem 2.5. *Assume*

(a) $\lambda > \kappa \geq \mathrm{cf}(\lambda)$
(b) $\mathrm{Depth}(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$
(c) $\lambda = \lim_D(\langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa \rangle)$.

Then

($\aleph$) $\mathbf{V} = \mathbf{L}$ *implies* $\mathrm{Depth}(\mathbf{B}) = \prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$.
($\beth$) *Instead of* $\mathbf{V} = \mathbf{L}$ *it suffices that* D *is a* κ*-regular ultrafilter, and* $\lambda^\kappa = \lambda^+$.

Proof.

($\aleph$) First we claim that $\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D = \lambda^+$. It follows from the fact that in $\mathbf{L}$ we know that D is regular (by Theorem 2.3 of Donder, taken from [?]), so (using assumption (c), and Claim 2.4) $\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D = \lambda^\kappa = \lambda^+$ (recall that $\mathrm{cf}(\lambda) \leq \kappa$).
Now $\mathrm{Depth}(\mathbf{B}) \geq \prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D = \lambda^+$, by Theorem 4.14 from [?] (since $\mathbf{L} \models$ GCH). On the other hand, Corollary 1.10 makes sure that $\mathrm{Depth}(\mathbf{B}) \leq \lambda^+$ (by (b) of the present Theorem). So $\prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D = \lambda^+ = \mathrm{Depth}(\mathbf{B})$, and we are done.
($\beth$) Notice that in the proof of $\aleph$ we use just the regularity of D (and κ-regularity suffices), and the assumption that $\lambda^\kappa = \lambda$.

$\square_{2.5}$

We know that if κ is less than the first measurable cardinal, then every uniform ultrafilter on κ is $\aleph_0$-regular, as noted in Remark 2.2. It gives us the result of Theorem 2.5 for singular cardinals with countable cofinality, if the length of the sequence (i.e., κ) is below the first measurable.
We have good evidence that something similar holds for singular cardinals with countable cofinality above a compact cardinal. Moreover, if $\mathrm{cf}(\lambda) = \aleph_0$ then $\kappa \geq \mathrm{cf}(\lambda)$ for every infinite cardinal κ. It means that it is consistent with ZFC not to have a counterexample in this case. So the following conjecture does make sense:

Conjecture 2.6. (ZFC)
Assume

(a) $\aleph_0 = \mathrm{cf}(\lambda) < \lambda$
(b) $\kappa < \lambda$, and $2^\kappa < \lambda$
(c) $\mathrm{Depth}(\mathbf{B}_i) \leq \lambda$, for every $i < \kappa$
(d) $\lambda = \lim_D(\langle \mathrm{Depth}(\mathbf{B}_i) : i < \kappa \rangle)$
(e) λ is below the first measurable, or just D is not $\aleph_1$-complete.

Then $\mathrm{Depth}(\mathbf{B}) \leq \prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$.

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