DIAMONDS
SH922

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Abstract. If $\lambda = \chi^+ = 2^\chi > \aleph_1$ then diamond on λ holds. Moreover, if $\lambda = \chi^+ = 2^\chi$ and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \neq \mathrm{cf}(\chi)\}$ is stationary then $\diamondsuit_S$ holds. Earlier this was known only under additional assumptions on χ and/or S.

Date: Oct.7, 2009.

Research supported by the United-States-Israel Binational Science Foundation (Grant No. 2002323), Publication No. 922. The author thanks Alice Leonhardt for the beautiful typing. First Typed - 07/Sept/19 Previous Version - 09/Oct/5.

1. Introduction

We prove in this paper several results about diamonds. Let us recall the basic definitions and sketch the (pretty long) history of the related questions.

The diamond principle was formulated by Jensen, who proved that it holds in $\mathbf{L}$ for every regular uncountable cardinal κ and stationary $S \subseteq \kappa$. This is a prediction principle, which asserts the following:

Definition 1.1. $\diamondsuit_S$ (the set version).

Assume $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $S \subseteq \kappa$ is stationary, $\diamondsuit_S$ holds <u>when</u> there is a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that $A_\alpha \subseteq \alpha$ for every $\alpha \in S$ and the set $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is a stationary subset of κ for every $A \subseteq \kappa$.

The diamond sequence $\langle A_\alpha : \alpha \in S\rangle$ guesses enough (i.e., stationarily many) initial segments of every $A \subseteq \kappa$. Several variants of this principle were formulated, for example:

Definition 1.2. $\diamondsuit^*_S$.

Assume $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and S is a stationary subset of κ. Now $\diamondsuit^*_S$ holds <u>when</u> there is a sequence $\langle \mathcal{A}_\alpha : \alpha \in S\rangle$ such that each $\mathcal{A}_\alpha$ is a subfamily of $\mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| \leq |\alpha|$ and for every $A \subseteq \kappa$ there exists a club $C \subseteq \kappa$ such that $A \cap \alpha \in \mathcal{A}_\alpha$ for every $\alpha \in C \cap S$.

We know that $\diamondsuit^*_S$ holds in $\mathbf{L}$ for every regular uncountable κ and stationary $S \subseteq \kappa$. Kunen proved that $\diamondsuit^*_S \Rightarrow \diamondsuit_S$. Moreover, if $S_1 \subseteq S_2$ are stationary subsets of κ then $\diamondsuit^*_{S_2} \Rightarrow \diamondsuit^*_{S_1}$ (hence $\diamondsuit_{S_1}$). But the assumption $\mathbf{V} = \mathbf{L}$ is heavy. Trying to avoid it, we can walk in several directions. On weaker relatives see [**?**] and references there. We can also use other methods, aiming to prove the diamond without assuming $\mathbf{V} = \mathbf{L}$.

There is another formulation of the diamond principle, phrased via functions (instead of sets). Since we use this version in our proof, we introduce the following:

Definition 1.3. $\diamondsuit_S$ (the functional version).

Assume $\lambda = \mathrm{cf}(\lambda) > \aleph_0, S \subseteq \lambda, S$ is stationary. $\diamondsuit_S$ holds if there exists a diamond sequence $\langle g_\delta : \delta \in S\rangle$ which means that $g_\delta \in {}^\delta\delta$ for every $\delta \in S$, and for every $g \in {}^\lambda\lambda$ the set $\{\delta \in S : g{\restriction}\delta = g_\delta\}$ is a stationary subset of λ.

By Gregory [**?**] and Shelah [**?**] we know that assuming $\lambda = \chi^+ = 2^\chi$ and $\kappa = \mathrm{cf}(\kappa) \neq \mathrm{cf}(\chi), \kappa < \lambda$, and GCH holds (or actually just $\chi^\kappa = \chi$ or $(\forall \alpha < \chi)(|\alpha|^\kappa < \chi) \wedge \mathrm{cf}(\chi) < \kappa$), then $\diamondsuit^*_{S^\lambda_\kappa}$ holds (recall that $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$).

We have got also results which show that the failures of the diamond above a strong limit cardinal are limited. For instance, if $\lambda = \chi^+ = 2^\chi > \mu$ and $\mu > \aleph_0$ is strong limit, then (by [**?**]) the set $\{\kappa < \mu : \diamondsuit^*_{S^\lambda_\kappa}$ fails$\}$ is bounded in μ (recall that κ is regular). Note that the result here does not completely subsume the earlier results when $\lambda = 2^\chi = \chi^+$ as we get "diamond on every stationary set $S \subseteq \lambda \backslash S^\lambda_{\mathrm{cf}(\chi)}$", but not $\diamondsuit^*_S$; this is inherent as noted in 3.4. In [**?**], a similar, stronger result is proved for $\diamondsuit_{S^\lambda_\kappa}$: for every $\lambda = \chi^+ = 2^\chi > \mu, \mu$ strong limit for some <u>finite</u> $\mathfrak{d} \subseteq$ Reg $\cap\, \mu$, for every regular $\kappa < \mu$ not from $\mathfrak{d}$ we have $\diamondsuit_{S^\lambda_\kappa}$, and even $\diamondsuit_S$ for "most" stationary $S \subseteq S^\lambda_\kappa$. In fact, for the relevant good stationary sets $S \subseteq S^\lambda_\kappa$ we get $\diamondsuit^*_S$. Also weaker related results are proved there for other regular λ (mainly $\lambda = \mathrm{cf}(2^\chi)$).

The present work does not resolve:

Problem 1.4. Assume χ is singular, $\lambda = \chi^+ = 2^\chi$, do we have $\diamondsuit_{S^\lambda_{\mathrm{cf}(\chi)}}$? (You may even assume G.C.H.).

However, the full analog result for 1.4 consistently fails, see [?] or [?]; that is: if G.C.H., $\chi > \mathrm{cf}(\chi) = \kappa$ then we can force a non-reflecting stationary $S \subseteq S^{\chi^+}_\kappa$ such that the diamond on S fails and cardinalities and cofinalities are preserved; also G.C.H. continue to hold. But if χ is strong limit, $\lambda = \chi^+ = 2^\chi$, still we know something on guessing equalities for every stationary $S \subseteq S^\lambda_\kappa$; see [?].

Note that this S (by [?], [?]) in some circumstances has to be "small"

(∗) if (χ is singular, $2^\chi = \chi^+ = \lambda, \kappa = \mathrm{cf}(\chi)$ and) we have the square $\square_\chi$ (i.e. there exists a sequence $\langle C_\delta : \delta < \lambda, \delta \text{ is a limit ordinal}\rangle$, so that C_δ is closed and unbounded in δ, $\mathrm{cf}(\delta) < \chi \Rightarrow |C_\delta| < \chi$ and if γ is a limit point of C_δ then $C_\gamma = C_\delta \cap \gamma$) <u>then</u> $\diamondsuit_{S^\lambda_\kappa}$ holds, moreover, if $S \subseteq S^\lambda_\kappa$ reflects in a stationary set of $\delta < \lambda$ then $\diamondsuit_S$ holds, see [?, §3].

Also note that our results are of the form "$\diamondsuit_S$ for every stationary $S \subseteq S^*$" for suitable $S^* \subseteq \lambda$. Now usually this was deduced from the stronger statement $\diamondsuit^*_S$. However, the results on $\diamondsuit^*_S$ cannot be improved, see 3.4.

Also if χ is regular we cannot improve the result to $\diamondsuit_{S^\lambda_\chi}$, see [?] or [?], even assuming G.C.H. Furthermore the question on $\diamondsuit_{\aleph_2}$ when $2^{\aleph_1} = \aleph_2 = 2^{\aleph_0}$ was raised. Concerning this we show in 3.2 that $\diamondsuit_{S^{\aleph_2}_{\aleph_1}}$ may fail (this works in other cases, too).

Question 1.5. Can we deduce any ZFC result on λ strongly inaccessible?

By Džamonja-Shelah [?] we know that failure of SNR helps (SNR stands for strong non-reflection), a parallel here is 2.3(2).

For $\lambda = \lambda^{<\lambda} = 2^\mu$ weakly inaccessible we know less, still see [?], [?] getting a weaker relative of diamond, see Definition 3.5(2). Again failure of SNR helps.

On consistency results on SNR see Cummings-Džamonja-Shelah [?], Džamonja-Shelah [?].

We thank the audience in the Jerusalem Logic Seminar for their comments in the lecture on this work in Fall 2006. We thank the referee for many helpful remarks. In particular, as he urged for details, the proof of 2.5(1),(2) just said "like 2.3", instead we make the old proof of 2.3(1) prove 2.5(1),(2) by minor changes and make explicit 2.5(3) which earlier was proved by "repeat of the second half of the proof of 2.3(1)". We thank Shimoni Garti for his help in the proofreading.

Notation 1.6. 1) If $\kappa = \mathrm{cf}(\kappa) < \lambda = \mathrm{cf}(\lambda)$ then we let $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.
2) $\mathcal{D}_\lambda$ is the club filter on λ for λ a regular uncountable cardinal.

2. Diamond on successor cardinals

Recall (needed only for part (2) of the Theorem 2.3):

Definition 2.1. 1) We say λ on S has κ-SNR or $\mathrm{SNR}(\lambda,S,\kappa)$ or λ has strong non-reflection for S in κ <u>when</u> $S \subseteq S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ so $\lambda = \mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa)$ and there are $h : \lambda \to \kappa$ and a club E of λ such that for every $\delta \in S \cap E$ for some club C of δ, the function $h \restriction C$ is one-to-one and even increasing; (note that without loss of generality $\alpha \in \mathrm{nacc}(E) \Rightarrow \alpha$ successor and without loss of generality $E = \lambda$, so $\mu \in \mathrm{Reg} \cap \lambda \backslash \kappa^+ \Rightarrow \mathrm{SNR}(\mu, S \cap \mu, \kappa))$. If $S = S^\lambda_\kappa$ we may omit it.

Remark 2.2. Note that by Fodor's lemma if $\mathrm{cf}(\delta) = \kappa > \aleph_0$ and h is a function from some set $\supseteq \delta$ and the range of h is $\subseteq \kappa$ <u>then</u> the following conditions are equivalent:

- (a) h is one-to-one on some club of δ
- (b) h is increasing on some club of δ
- (c) $\mathrm{Rang}(h \restriction S)$ is unbounded in κ for every stationary subset S of δ.

Our main theorem is:

Claim 2.3. *Assume $\lambda = 2^\chi = \chi^+$.*
1) If $S \subseteq \lambda$ is stationary and $\delta \in S \Rightarrow \mathrm{cf}(\delta) \neq \mathrm{cf}(\chi)$ <u>then</u> $\diamondsuit_S$ holds.
2) If $\aleph_0 < \kappa = \mathrm{cf}(\chi) < \chi$ and $\diamondsuit_S$ fails where $S = S^\lambda_\kappa$ (or just $S \subseteq S^\lambda_\kappa$ is a stationary subset of λ) <u>then</u> we have $\mathrm{SNR}(\lambda,\kappa)$ or just λ has strong non-reflection for $S \subseteq S^\lambda_\kappa$ in κ.

Definition 2.4. 1) For a filter D on a set I let $\mathrm{Dom}(D) := I$ and S is called D-positive when $S \subseteq I \wedge (I \backslash S) \notin D$ and $D^+ = \{S \subseteq \mathrm{Dom}(D) : S \text{ is } D\text{-positive}\}$ and we let $D + A = \{B \subseteq I : B \cup (I \backslash A) \in D\}$ (so if $D = \mathcal{D}_\lambda$, the club filter on the regular uncountable λ then D^+ is the family of stationary subsets of X).
2) For D a filter on a regular uncountable cardinal λ which extends the club filter, let $\diamondsuit_D$ means: there is $\bar f = \langle f_\alpha : \alpha \in S\rangle$ which is a diamond sequence for D (or a D-diamond sequence) which means that $S \in D^+$ and for every $g \in {}^\lambda\lambda$ the set $\{\alpha < \lambda : g \restriction \alpha = f_\alpha\}$ belongs to D^+; so $\bar f$ is also a diamond sequence for the filter $D + S$, (clearly $\diamondsuit_S$ means $\diamondsuit_{\mathcal{D}_\lambda + S}$ for S a stationary subset of the regular uncountable λ).

A somewhat more general version of the theorem is

Claim 2.5. *1) Assume $\lambda = \chi^+ = 2^\chi$ and D is a λ-complete filter on λ which extends the club filter. If $S \in D^+$ and $\delta \in S \Rightarrow \mathrm{cf}(\delta) \neq \mathrm{cf}(\chi)$ <u>then</u> we have $\diamondsuit_{D+S}$.*
2) We have $\diamondsuit_D$ <u>when</u>:

- (a) $\lambda = \lambda^{<\lambda}$
- (b) $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ *lists* $\cup\{{}^\alpha\lambda : \alpha < \lambda\}$
- (c) $S \in D^+$
- (d) $\bar u = \langle u_\alpha : \alpha \in S\rangle$ *and* $u_\alpha \subseteq \alpha$ *for every* $\alpha \in S$
- (e) $\chi = \sup\{|u_\alpha|^+ : \alpha < \lambda\} < \lambda$
- (f) D *is a* χ^+*-complete filter on* λ *extending the club filter*
- (g) $(\forall g \in {}^\lambda\lambda)(\exists^{D^+} \delta \in S)[\delta = \sup\{\alpha \in u_\delta : g \restriction \alpha \in \{f_\beta : \beta \in u_\delta\}\}]$.

3) Assume $\lambda = \chi^+ = 2^\chi$ and $\aleph_0 < \kappa = \mathrm{cf}(\chi) < \chi$, $S \subseteq S^\lambda_\kappa$ is stationary, and D is a λ-complete filter extending the club filter on λ to which S belongs. If $\diamondsuit_D$ fails <u>*then*</u> $\mathrm{SNR}(\lambda, S, \kappa)$.

Proof. <u>Proof of 2.3</u> Part (1) follows from 2.5(1) for D the filter $\mathcal{D}_\lambda + S$. Part (2) follows from 2.5(3) for D the filter $\mathcal{D}_\lambda + S$. $\square_{2.3}$

Proof. <u>Proof of 2.5</u>
<u>Proof of part (1)</u>
Clearly we can assume

$\circledast_0$ $\chi > \aleph_0$ as for $\chi = \aleph_0$ the statement is empty.

Let

$\circledast_1$ $\langle f_\alpha : \alpha < \lambda \rangle$ list the set $\{f : f$ is a function from β to λ for some $\beta < \lambda\}$.

For each $\alpha < \lambda$ clearly $|\alpha| \leq \chi$ so let

$\circledast_{2,\alpha}$ $\langle u_{\alpha,\varepsilon} : \varepsilon < \chi \rangle$ be $\subseteq$-increasing continuous with union α such that $\varepsilon < \chi \Rightarrow |u_{\alpha,\varepsilon}| \leq \aleph_0 + |\varepsilon| < \chi$.

For $g \in {}^\lambda\lambda$ let $h_g \in {}^\lambda\lambda$ be defined by

$\circledast_{3,g}$ $h_g(\alpha) = \mathrm{Min}\{\beta < \lambda : g \restriction \alpha = f_\beta\}$.

Let cd, $\langle \mathrm{cd}_\varepsilon : \varepsilon < \chi \rangle$ be such that

$\circledast_4$ (a) cd is a one-to-one function from ${}^\chi\lambda$ onto λ such that $\mathrm{cd}(\bar\alpha) \geq \sup\{\alpha_\varepsilon : \varepsilon < \chi\}$ (when $\bar\alpha = \langle \alpha_\varepsilon : \varepsilon < \chi \rangle$)
(b) for $\varepsilon < \chi$, cd_ε is a function from λ to λ such that $\bar\alpha = \langle \alpha_\varepsilon : \varepsilon < \chi \rangle \in {}^\chi\lambda \Rightarrow \mathrm{cd}_\varepsilon(\mathrm{cd}(\bar\alpha)) = \alpha_\varepsilon$

(they exist as $\lambda = \lambda^\chi$, in the present case this holds as $2^\chi = \chi^+ = \lambda$).
Now we let (for $\beta < \lambda, \varepsilon < \chi$):

$\circledast_5$ $f^1_{\beta,\varepsilon}$ be the function from $\mathrm{Dom}(f_\beta)$ into λ defined by $f^1_{\beta,\varepsilon}(\alpha) = \mathrm{cd}_\varepsilon(f_\beta(\alpha))$ so $\mathrm{Dom}(f^1_{\beta,\varepsilon}) = \mathrm{Dom}(f_\beta)$.

Without loss of generality

$\circledast_6$ $\alpha \in S \Rightarrow \alpha$ is a limit ordinal.

For $g \in {}^\lambda\lambda$ and $\varepsilon < \chi$ we let

$$S^\varepsilon_g = \Big\{\delta \in S : \quad \delta = \sup\{\alpha \in u_{\delta,\varepsilon} : \text{ for some } \beta \in u_{\delta,\varepsilon} \text{ we have } g \restriction \alpha = f^1_{\beta,\varepsilon}\}\Big\}.$$

Next we shall show

$\circledast_7$ for some $\varepsilon(*) < \chi$ for every $g \in {}^\lambda\lambda$ the set $S^{\varepsilon(*)}_g$ is a D-positive subset of λ.

Proof of $\circledast_7$ Assume this fails, so for every $\varepsilon < \chi$ there is $g_\varepsilon \in {}^\lambda\lambda$ such that $S^\varepsilon_{g_\varepsilon}$ is not D-positive and let E_ε be a member of D disjoint to $S^\varepsilon_{g_\varepsilon}$. Define $g \in {}^\lambda\lambda$ by $g(\alpha) := \mathrm{cd}(\langle g_\varepsilon(\alpha) : \varepsilon < \chi\rangle)$ and let $h_g \in {}^\lambda\lambda$ be as in $\circledast_{3,g}$, i.e. $h_g(\alpha) = \mathrm{Min}\{\beta : g \restriction \alpha = f_\beta\}$.

Let $E_* = \{\delta < \lambda : \delta$ is a limit ordinal such that $\alpha < \delta \Rightarrow h_g(\alpha) < \delta\}$, clearly it is a club of λ hence it belongs to D and so $E = \cap\{E_\varepsilon : \varepsilon < \chi\} \cap E_*$ belongs to D as D is λ-complete and $\chi + 1 < \lambda$.

As S is a D-positive subset of λ there is $\delta_* \in E \cap S$. For each $\alpha < \delta_*$ as $\delta_* \in E \subseteq E_*$ clearly $h_g(\alpha) < \delta_*$ and α as well as $h_g(\alpha)$ belong to $\cup\{u_{\delta_*,\varepsilon} : \varepsilon < \chi\} = \delta_*$, but $\langle u_{\delta_*,\varepsilon} : \varepsilon < \chi\rangle$ is $\subseteq$-increasing hence $\varepsilon_{\delta_*,\alpha} = \min\{\varepsilon : \alpha \in u_{\delta_*,\varepsilon}$ and $h_g(\alpha) \in u_{\delta_*,\varepsilon}\}$ is not just well defined but also $\varepsilon \in [\varepsilon_{\delta_*,\alpha}, \chi) \Rightarrow \{\alpha, h_g(\alpha)\} \subseteq u_{\delta_*,\varepsilon}$. As $\mathrm{cf}(\delta_*) \neq \mathrm{cf}(\chi)$, by an assumption on S, it follows that for some $\varepsilon(*) < \chi$ the set $B := \{\alpha < \delta_* : \varepsilon_{\delta_*,\alpha} < \varepsilon(*)\}$ is unbounded below δ_*.

So

(a) $\alpha \in B \Rightarrow \{\alpha, h_g(\alpha)\} \subseteq u_{\delta_*,\varepsilon(*)}$ and
(b) $\alpha \in B \Rightarrow g{\restriction}\alpha = f_{h_g(\alpha)} \Rightarrow \bigwedge\limits_{\varepsilon<\chi} [g_\varepsilon{\restriction}\alpha = f^1_{h_g(\alpha),\varepsilon}] \Rightarrow g_{\varepsilon(*)}{\restriction}\alpha = f^1_{h_g(\alpha),\varepsilon(*)}$.

But $\delta_* \in E \subseteq E_{\varepsilon(*)}$ hence $\delta_* \notin S^{\varepsilon(*)}_{g_{\varepsilon(*)}}$ by the choice of $E_{\varepsilon(*)}$, but by (a) + (b) and the definition of $S^{\varepsilon(*)}_{g_{\varepsilon(*)}}$ recalling $\delta_* \in S$ we have $\sup(B) = \delta_* \Rightarrow \delta_* \in S^{\varepsilon(*)}_{g_{\varepsilon(*)}}$, (where $h_g(\alpha)$ plays the role of β in the definition of S^ε_g above), contradiction. So the proof of $\circledast_7$ is finished.

Let $\chi_* = (|\varepsilon(*)| + \aleph_0)$ hence $\delta \in S \Rightarrow |u_{\delta,\varepsilon(*)}| \le \chi_*$ and $\chi^+_* < \lambda$ as $\chi_* < \chi < \lambda$ because $\aleph_0, \varepsilon(*) < \chi < \lambda$. Now we apply 2.5(2) which is proved below with $\lambda, S, D, \chi^+_*, \langle f^1_{\beta,\varepsilon(*)} : \beta < \lambda\rangle, \langle u_{\delta,\varepsilon(*)} : \delta \in S\rangle$ here standing for $\lambda, S, D, \chi, \bar f, \bar u$ there. The conditions there are satisfied hence also the conclusion which says that $\diamondsuit_D$ holds. $\square_{2.5(1)}$

Proof. Proof of 2.5(2)

Let

$\boxtimes_1$ $\langle \mathrm{cd}_\varepsilon : \varepsilon < \chi\rangle$ and cd be as in $\circledast_4$ in the proof of part (1), possible as we are assuming $\chi < \lambda = \lambda^{<\lambda}$

$\boxtimes_2$ for $\beta < \lambda$ and $\zeta < \chi$ let $f^2_{\beta,\zeta}$ be the function with domain $\mathrm{Dom}(f_\beta)$ such that $f^2_{\beta,\zeta}(\alpha) = \mathrm{cd}_\zeta(f_\beta(\alpha))$

$\boxtimes_3$ for $g \in {}^\lambda\lambda$ define $h_g \in {}^\lambda\lambda$ as in $\circledast_3$ in the proof of part (1), i.e. $h_g(\alpha) = \mathrm{Min}\{\beta : g{\restriction}\alpha = f_\beta\}$.

If $2^{<\chi} < \lambda$ our life is easier but we do not assume this. For $\delta \in S$ let ξ^*_δ be a cardinal, and let $\langle(\alpha^1_{\delta,\xi}, \alpha^2_{\delta,\xi}) : \xi < \xi^*_\delta\rangle$ list the set $\{(\alpha_1, \alpha_2) \in u_\delta \times u_\delta : \mathrm{Dom}(f_{\alpha_2}) = \alpha_1\}$, note that $\xi^*_\delta < \chi$, recalling $|u_\delta| < \chi$ by clause (e) of the assumption. We now try to choose $(\bar v_\varepsilon, g_\varepsilon, E_\varepsilon)$ by induction on $\varepsilon < \chi$, (note that $\bar v_\varepsilon$ is defined from $\langle g_\zeta : \zeta < \varepsilon\rangle$ (see clause (e) of $\boxtimes_4$ below) so we choose just $(g_\varepsilon, E_\varepsilon)$), such that:

$\boxtimes_4$ (a) E_ε is a member of D and $\langle E_\zeta : \zeta \le \varepsilon\rangle$ is $\subseteq$-decreasing with ζ
(b) $\bar v_\varepsilon = \langle v^\varepsilon_\delta : \delta \in S \cap E'_\varepsilon\rangle$ when $E'_\varepsilon = \cap\{E_\zeta : \zeta < \varepsilon\} \cap \lambda$ so is λ if $\varepsilon = 0$
(c) $\langle v^\zeta_\delta : \zeta \le \varepsilon\rangle$ is $\subseteq$-decreasing with ζ for each $\delta \in S \cap E'_\varepsilon$
(d) $g_\varepsilon \in {}^\lambda\lambda$

(e) $v^\varepsilon_\delta = \{\xi < \xi^*_\delta$: if $\zeta < \varepsilon$ then $g_\zeta \restriction \alpha^1_{\delta,\xi} = f^2_{\alpha^2_{\delta,\xi},\zeta}\}$
(so if ε is a limit ordinal then $v^\varepsilon_\delta = \bigcap\limits_{\zeta<\varepsilon} v^\zeta_\delta$ and $\varepsilon = 0 \Rightarrow v^\varepsilon_\delta = \xi^*_\delta$)

(f) if $\delta \in E'_\varepsilon \cap S$ then $v^{\varepsilon+1}_\delta \subsetneqq v^\varepsilon_\delta$ or $\delta > \sup\{\alpha^1_{\delta,\xi} : \xi \in v^{\varepsilon+1}_\delta\}$.

Next

$\oplus_1$ we cannot carry the induction, that is for all $\varepsilon < \chi$.

Why? Assume toward contradiction that $\langle(\bar{v}_\varepsilon, g_\varepsilon, E_\varepsilon) : \varepsilon < \chi\rangle$ is well defined. Let $E := \cap\{E_\varepsilon : \varepsilon < \chi\}$, it is a member of D as D is χ^+-complete. Define $g \in {}^\lambda\lambda$ by $g(\alpha) :=$ cd$(\langle g_\varepsilon(\alpha) : \varepsilon < \chi\rangle)$. Let $E_* = \{\delta < \lambda : \delta$ a limit ordinal such that $h_g(\alpha) < \delta$ and $\delta > \sup(\mathrm{Dom}(f_\alpha) \cup \mathrm{Rang}(f_\alpha))$ for every $\alpha < \delta\}$, so E_* is a club of λ hence it belongs to D. By assumption (g) of the claim the set

$$S_g := \{\delta \in S : \delta = \sup\{\alpha \in u_\delta : (\exists \beta \in u_\delta)(f_\beta = g \restriction \alpha)\}\}$$

is D-positive, so we can choose $\delta \in E \cap E_* \cap S_g$. Hence $B := \{\alpha \in u_\delta : (\exists \beta \in u_\delta)(f_\beta = g{\restriction}\alpha)\}$ is an unbounded subset of u_δ and let $h : B \to u_\delta$ be $h(\alpha) = \min\{\beta \in u_\delta : f_\beta = g{\restriction}\alpha\}$, clearly h is a function from B into u_δ. Now $\alpha \in B \wedge \zeta < \chi \Rightarrow f_{h(\alpha)} = g{\restriction}\alpha \wedge \zeta < \chi \Rightarrow f^2_{h(\alpha),\zeta} = g_\zeta{\restriction}\alpha$, so for $\alpha \in B$ the pair $(\alpha, h(\alpha))$ belongs to $\{(\alpha^1_{\delta,\xi}, \alpha^2_{\delta,\xi}) : \xi \in v^\varepsilon_\delta\}$ for every $\varepsilon < \chi$. Hence for any $\varepsilon < \chi$ we have $B \subseteq \{\alpha^1_{\delta,\xi} : \xi \in v^\varepsilon_\delta\}$ so $\delta = \sup\{\alpha^1_{\delta,\xi} : \xi \in v^\varepsilon_\delta\}$.

So for the present δ, in clause (f) of $\boxtimes_4$ the second possibility never occurs.

So clearly $\langle v^\varepsilon_{\delta_*} : \varepsilon < \chi\rangle$ is strictly $\subseteq$-decreasing, i.e. is $\subset$-decreasing which is impossible as $|v^0_{\delta_*}| = \xi^*_{\delta_*} < \chi$. So we have proved $\oplus_1$ hence we can assume

$\oplus_2$ there is $\varepsilon < \chi$ such that we have defined our triple for every $\zeta < \varepsilon$ but we cannot define for ε. So we have $\langle(\bar{v}_\zeta, g_\zeta, E_\zeta) : \zeta < \varepsilon\rangle$.

As in $\boxplus_4(e)$, let

$\odot_1$ E'_ε be λ if $\varepsilon = 0$ and $\cap\{E_\zeta : \zeta < \varepsilon\}$ if $\varepsilon > 0$ and let $S_* := S \cap E'_\varepsilon$.

Clearly $\bar{v}_\varepsilon$ is well defined, see clauses (b),(e) of $\boxtimes_4$, and for $\delta \in S_*$ let $\mathcal{F}_\delta = \{f^2_{\alpha^2_{\delta,\xi},\varepsilon} : \xi \in v^\varepsilon_\delta\}$, so each member is a function from some $\alpha \in u_\delta \subseteq \delta$ into some ordinal $< \delta$.

Let

$\odot_2$ $S^*_1 := \{\delta \in S_* :$ there are $f', f'' \in \mathcal{F}_\delta$ which are incompatible as functions$\}$

$\odot_3$ $S^*_2 := \{\delta \in S_* : \delta \notin S^*_1$ but the function $\cup\{f : f \in \mathcal{F}_\delta\}$ has domain $\neq \delta\}$

$\odot_4$ $S^*_3 = S_* \backslash (S^*_1 \cup S^*_2)$.

For $\delta \in S^*_3$ let $g^*_\delta = \cup\{f : f \in \mathcal{F}_\delta\}$, so by the definition of $\langle S^*_\ell : \ell = 1, 2, 3\rangle$ clearly $g^*_\delta \in {}^\delta\delta$. Now if $\langle g^*_\delta : \delta \in S^*_3\rangle$ is a diamond sequence for D then we are done.

So assume that this fails, so for some $g \in {}^\lambda\lambda$ and member E of D we have $\delta \in S^*_3 \cap E \Rightarrow g^*_\delta \neq g \restriction \delta$. Without loss of generality E is included in E'_ε. But then we could have chosen (g, E) as $(g_\varepsilon, E_\varepsilon)$, recalling $\bar{v}_\varepsilon$ was already chosen. Easily the triple $(g_\varepsilon, E_\varepsilon, \bar{v}_\varepsilon)$ is as required in $\oplus_1$, contradicting the choice of ε in $\oplus_2$ so we are done proving part (2) of Theorem 2.3 hence also part (1). $\square_{2.5(2)}$

Proof. Proof of part (3)

We use cd,cd_ε (for $\varepsilon < \chi$), $\langle\langle u_{\alpha,\varepsilon} : \varepsilon < \chi\rangle : \alpha < \lambda\rangle, \langle f_\alpha : \alpha < \lambda\rangle, \langle f^1_{\alpha,\varepsilon} : \alpha < \lambda, \varepsilon < \chi\rangle$ and S^ε_g for $\varepsilon < \kappa$ as in the proof of part (1).

Recall κ, a regular uncountable cardinal, is the cofinality of the singular cardinal χ and let $\langle\chi_\gamma : \gamma < \kappa\rangle$ be increasing with limit χ. For every $\gamma < \kappa$ we ask: The γ-Question: Do we have: for every $g \in {}^\lambda\lambda$, the following is a D-positive subset of λ:

$\{\delta \in S : S_\gamma[g] \cap \delta$ is a stationary subset of $\delta\}$ where $S_\gamma[g] := \{\zeta < \lambda$: $\mathrm{cf}(\zeta) \in [\aleph_0, \kappa)$, $\sup(u_{\zeta,\chi_\gamma}) = \zeta$ and for arbitrarily large $\alpha \in u_{\zeta,\chi_\gamma}$ for some $\beta \in u_{\zeta,\chi_\gamma}$ and $\varepsilon < \chi_\gamma$, we have $\mathrm{Dom}(f_\beta) = \alpha$ and $g \restriction \alpha = f^1_{\beta,\varepsilon}\}$.

Case 1: For some $\gamma < \kappa$, the answer is yes.

Choose $\langle C_\delta : \delta \in S\rangle$ such that C_δ is a club of δ of order type $\mathrm{cf}(\delta) = \kappa$.

For $\delta \in S \subseteq S^\lambda_\kappa$ let $u_\delta := \cup\{u_{\alpha,\chi_\gamma} : \alpha \in C_\delta\}$.

Clearly

$\boxplus_2$ $|u_\delta| \le \kappa + \chi_\gamma < \chi$

$\boxplus_3$ for every $g \in {}^\lambda\lambda$ for D-positively many $\delta \in S$, we have $\delta = \sup\{\alpha \in u_\delta : g{\restriction}\alpha \in \{f^1_{\beta,\varepsilon} : \varepsilon < \chi_\gamma$ and $\beta \in u_\delta\}\}$.

Why $\boxplus_3$ holds? Given $g \in {}^\lambda\lambda$, let $h_g \in {}^\lambda\lambda$ be defined by $h_g(\alpha) = \min\{\beta < \lambda : g{\restriction}\alpha = f_\beta\}$, so $h_g(\alpha) \ge \alpha$ (but is less than λ). Let $E_g = \{\delta < \lambda : \delta$ is a limit ordinal such that $(\forall\alpha < \delta)h_g(\alpha) < \delta\}$, so E_g is a club of λ and let E'_g be the set of accumulation points of E_g, so E'_g, too, is a club of λ. By the assumption of this case, the set $S' := \{\delta \in S : \delta \cap S_\gamma[g]$ is a stationary subset of $\lambda\}$ is D-positive, hence $S'' := S' \cap E'_g$ is a D-positive subset of λ. Let $\delta \in S''$, by E'_g's definition, we can find $B^0_\delta \subseteq E_g \cap \delta$ unbounded in δ, so without loss of generality B^0_δ is closed. But $S_\gamma[g] \cap \delta$ is a stationary subset of δ, recalling $\delta \in S''$, so $B^1_\delta = B^0_\delta \cap S_\gamma[g] \cap C_\delta$ is a stationary subset of δ as B^0_δ, C_δ are closed unbounded subsets of δ.

Clearly $\zeta \in B^1_\delta \Rightarrow \zeta \in C_\delta \Rightarrow u_{\zeta,\chi_\gamma} \subseteq u_\delta$ by the definitions of B^1_δ and u_δ. Also $\zeta \in B^1_\delta \Rightarrow \zeta \in S_\gamma[g] \Rightarrow (\zeta$ is a limit ordinal$) \wedge \zeta = \sup(u_{\zeta,\chi_\gamma}) = \sup\{\alpha \in u_{\zeta,\chi_\gamma} : (\exists\beta \in u_{\zeta,\chi_\gamma})(\exists\varepsilon < \chi_\gamma)(g{\restriction}\alpha = f^1_{\beta,\varepsilon})\} \Rightarrow ((\zeta$ is a limit ordinal$) \wedge \zeta = \sup\{\alpha \in u_\delta \cap \zeta : (\exists\beta \in u_\delta\backslash\alpha)(\exists\varepsilon < \chi_\gamma)(g{\restriction}\alpha = f^1_{\beta,\varepsilon})\})$.

As B^1_δ is unbounded in δ being stationary we are done proving $\boxplus_3$.

Now without loss of generality every $\delta \in S$ is divisible by χ hence $\delta = \chi_\gamma\delta$ and let $u'_\delta = u_\delta \cup \{\chi_\gamma\alpha + \varepsilon : \alpha \in u_\delta, \varepsilon < \chi_\gamma\}$, so u_δ is an unbounded subset of δ, and let $f'_\beta = f^1_{\alpha,\varepsilon}$ when $\beta = \chi_\gamma\alpha + \varepsilon, \varepsilon < \chi_\gamma$. So translating what we have is:

$\boxplus_4$ (a) $\langle f'_\alpha : \alpha < \lambda\rangle$ is a sequence of members of $\cup\{{}^\beta\lambda : \beta < \lambda\}$

(b) for $\delta \in S, u'_\delta$ is an unbounded subset of δ of cardinality $\le \chi_\gamma \times \chi_\gamma = \chi_\gamma (< \chi)$

(c) for every $g \in {}^\lambda\lambda$ for D-positively many $\delta \in S$ we have $\delta = \sup\{\alpha \in u'_\delta : (\exists\beta \in u'_\delta)(g{\restriction}\alpha = f'_\beta)\}$.

Now we can apply part (2) with $\langle f'_\alpha : \alpha < \lambda\rangle, \langle u'_\delta : \delta \in S\rangle$ replacing $\bar f, \langle u_\delta : \delta \in S\rangle$.

So as there we can prove $\diamondsuit_S$, hence we are done.

Case 2: For every $\gamma < \kappa$ the answer is no.

Let (g_γ, E_γ) exemplify that the answer for γ is no; so $g_\gamma \in {}^\lambda\lambda$ and $E_\gamma \in D$. Let $E = \bigcap\limits_{\gamma<\kappa} E_\gamma$, so E is a member of D. Let $g \in {}^\lambda\lambda$ be defined by $g(\alpha) = \mathrm{cd}(\langle g_\gamma(\alpha) : \gamma < \kappa\rangle\hat{}\,(0)_\chi)$, i.e. $\mathrm{cd}_\varepsilon(g(\alpha))$ is $g_\gamma(\alpha)$ if $\gamma < \kappa$ and is 0 if $\varepsilon \in [\kappa, \chi)$.

Let

$$E_g := \{\delta < \lambda : \ \delta \text{ a limit ordinal such that if } \alpha < \lambda \text{ then } h_g(\alpha) < \delta \text{ and } \delta > \sup(\mathrm{Dom}(f_\alpha) \cup \mathrm{Rang}(f_\alpha))\}.$$

We now define $h : \lambda \to \kappa$ as follows

$\boxplus_5$ for $\beta < \lambda$

(a) if $\mathrm{cf}(\beta) \notin [\aleph_0, \kappa)$ or $\beta \notin E_g$ then $h(\beta) = 0$

(b) otherwise

$$h(\beta) = \min\{\gamma < \kappa : \ \beta = \sup\{\alpha_1 \in u_{\beta,\chi_\gamma} : \text{ for some } \alpha_2 \in u_{\beta,\chi_\gamma} \text{ and } \varepsilon < \chi_\gamma \text{ we have } g \restriction \alpha_1 = f^1_{\alpha_2,\varepsilon}\}\}.$$

Now

$\boxplus_6$ $h : \lambda \to \kappa$ is well defined.

Why $\boxplus_6$ holds? Let $\beta < \lambda$. If $\mathrm{cf}(\beta) \notin [\aleph_0, \kappa)$ or $\beta \notin E_g$ then $h(\alpha) = 0 < \kappa$ by clause (a) of $\boxplus_5$. So assume $\mathrm{cf}(\beta) \in [\aleph_0, \kappa)$ and $\beta \in E_g$. Let $\langle \gamma^1_{\beta,\varepsilon} : \varepsilon < \mathrm{cf}(\beta)\rangle$ be increasing with limit β and let $\gamma^2_{\beta,\varepsilon} = \min\{\gamma : g{\restriction}\gamma^1_{\beta,\varepsilon} = f_\gamma\}$, so $\varepsilon < \mathrm{cf}(\beta) \Rightarrow \gamma^2_{\beta,\varepsilon} < \beta$ as $\beta \in E_g$. But $\langle u_{\beta,\chi_\zeta} : \zeta < \mathrm{cf}(\chi)\rangle$ is $\subseteq$-increasing with union β so for each $\varepsilon < \mathrm{cf}(\beta)$ there is $\zeta = \zeta_{\beta,\varepsilon} < \mathrm{cf}(\chi)$ such that $\{\gamma^1_{\beta,\varepsilon}, \gamma^2_{\beta,\varepsilon}\} \subseteq u_{\beta,\chi_\zeta}$. As $\mathrm{cf}(\beta) < \kappa = \mathrm{cf}(\chi)$ for some $\zeta < \kappa$ the set $\{\varepsilon < \mathrm{cf}(\beta) : \zeta_{\beta,\varepsilon} < \zeta\}$ is unbounded in $\mathrm{cf}(\beta)$. So ζ can serve as γ in clause (b) of $\boxplus_5$ so $h(\beta)$ is well defined, in particular is less than κ so we have proved $\boxplus_6$.

$\boxplus_7$ if $\delta \in S \cap E_\gamma$ then for some club C of δ the function $h \restriction C$ is increasing.

Why $\boxplus_7$ holds? If not, then by Fodor's lemma for some $\gamma < \kappa$ the set $\{\delta' \in \delta \cap S : h(\delta') \leq \gamma\}$ is a stationary subset of δ, and we get contradiction to the choice of E_γ so $\boxplus_7$ holds indeed.

So h is as promised in the claim. $\square$

Note

Observation 2.6. If $\kappa_* < \lambda$ are regular, $S^\lambda_{\kappa_*}$ strongly does not reflect in λ for every $\kappa \in \mathrm{Reg} \cap \kappa_*$ and $\Pi(\mathrm{Reg} \cap \kappa_*) < \lambda$, <u>then</u> :

(a) $S^\lambda_{<\kappa_*}$ can be divided to $\leq \Pi(\mathrm{Reg} \cap \kappa_*)$ sets, each not reflecting in any $\delta \in S^\lambda_{<\kappa_*}$
in particular

(b) $S^\lambda_{\aleph_0}$ can be divided to $\leq \Pi(\mathrm{Reg} \cap \kappa_*)$ sets each not reflecting in any $\delta \in S^\lambda_{<\kappa_*}$.

Remark 2.7. 1) Of course if λ has κ-SNR then this holds for every regular $\lambda' \in (\kappa, \lambda)$.
2) We may state the results, using λ^*_κ (see below).

Definition 2.8. For each regular κ let $\lambda^*_\kappa = \mathrm{Min}\{\lambda : \lambda$ regular fails to have $\kappa-$SNR$\}$, and let λ^*_κ be ∞ (or not defined) if there is no such λ.

3. Consistent failure on S^2_1

A known question was:

Question 3.1. For $\theta \in \{\aleph_0, \aleph_1\}$ do we have $(2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 \Rightarrow \diamondsuit_{S^{\aleph_2}_\theta})$?

So for $\theta = \aleph_0$ the answer is yes (by 2.3(1)), but what about $\theta = \aleph_1$? We noted some years ago that easily:

Claim 3.2. *Assume* $\mathbf{V} \models$ GCH *or even just* $2^{\aleph_\ell} = \aleph_{\ell+1}$ *for* $\ell = 0, 1, 2$. *Then some forcing notion* $\mathbb{P}$ *satisfies*

(a) $\mathbb{P}$ *is of cardinality* $\aleph_3$
(b) *forcing with* $\mathbb{P}$ *preserves cardinals and cofinalities*
(c) *in* $\mathbf{V}^{\mathbb{P}}, 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2, 2^{\aleph_2} = \aleph_3$
(d) *in* $\mathbf{V}^{\mathbb{P}}, \diamondsuit_S$ *fails where* $S = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$, *moreover*
  $(*)$ *there is a sequence* $\bar{A} = \langle A_\delta : \delta \in S\rangle$ *where* A_δ *an unbounded subset of* δ *of order type* ω_1 *satisfying*
  $(**)$ *if* $\bar{f} = \langle f_\delta : \delta \in S\rangle, f_\delta \in {}^{(A_\delta)}(\omega_1)$, *then there is* $f \in {}^{(\omega_2)}(\omega_1)$ *such that* $\delta \in S \Rightarrow \delta > \sup(\{\alpha \in A_\delta : f(\alpha) \le f_\delta(\alpha)\})$.

Remark 3.3. Similarly for other cardinals.

Proof. There is an $\aleph_1$-complete $\aleph_3$-c.c. forcing notion $\mathbb{P}$ not collapsing cardinals, not changing cofinalities, preserving $2^{\aleph_\ell} = \aleph_\ell$ for $\ell = 0, 1, 2$ and $|\mathbb{P}| = \aleph_3$ such that in $\mathbf{V}^{\mathbb{P}}$, we have $(*)$, in fact more[1] than $(*)$ holds - see [**?**]. Let $\mathbb{Q}$ be the forcing of adding $\aleph_2$ Cohen or just any c.c.c. forcing notion of cardinality $\aleph_2$ adding $\aleph_2$ reals (can be $\underset{\sim}{\mathbb{Q}}$, a $\mathbb{P}$-name). Now we shall show that $\mathbb{P} * \mathbb{Q}$, equivalently $\mathbb{P} \times \mathbb{Q}$ is as required:

Clause (a):
$|\mathbb{P} * \mathbb{Q}| = \aleph_3$; trivial.

Clause (b):
Preserving cardinals and cofinalities; obvious as both $\mathbb{P}$ and $\mathbb{Q}$ do this.

Clause (c): Easy.

Clause (d): In $\mathbf{V}^{\mathbb{P}}$ we have $(*)$ as exemplified by say $\bar{A} = \langle A_\delta : \delta \in S\rangle$. We shall show that $\mathbf{V}^{\mathbb{P}*\mathbb{Q}} \models$ "$\bar{A}$ satisfies $(**)$". Otherwise in $\mathbf{V}^{\mathbb{P}*\mathbb{Q}}$ we have $\bar{f} = \langle f_\delta : \delta \in S\rangle$ say in $\mathbf{V}[G_{\mathbb{P}}, G_{\mathbb{Q}}]$ a counterexample then in $\mathbf{V}[G_{\mathbb{P}}]$ for some $q \in \mathbb{Q}$ and $\underset{\sim}{\bar{f}}$ we have

$$\begin{aligned}\mathbf{V}[G_{\mathbb{P}}] \models (q \Vdash_{\mathbb{Q}} \text{``}\underset{\sim}{\bar{f}} = \ & \langle \underset{\sim}{f}_\delta : \delta \in S\rangle \text{ where } \underset{\sim}{f}_\delta : A_\delta \to \omega_1 \text{ for each } \delta \in S \\ & \text{form a counterexample to } (*)\text{''}).\end{aligned}$$

Now in $\mathbf{V}[G_{\mathbb{P}}]$ we can define $\bar{g} = \langle g^1_\delta : \delta \in S\rangle \in \mathbf{V}[G_{\mathbb{P}}]$ where g^1_δ a function with domain A_δ, by

$$g^1_\delta(\alpha) = \{i : q \nVdash \underset{\sim}{f}_\delta(\alpha) \neq i\}.$$

[1] I.e. there is $\bar{A} = \langle A_\delta : \delta \in S\rangle$ where A_δ is an unbounded subset of δ of order type ω_1 satisfying:

$\oplus$ if $\bar{f} = \langle f_\delta : \delta \in S\rangle, f_\delta \in {}^{(A_\delta)}\omega_1$ then there is $f \in {}^{(\omega_2)}\omega_1$ such that for every $\delta \in S^{\aleph_2}_{\aleph_1}$ for every $\alpha \in A_\delta$ large enough we have $f(\alpha) = f_\delta(\alpha)$.

So in $\mathbf{V}[G_{\mathbb{P}}]$ we have $q \Vdash_{\mathbb{Q}}$ " $\bigwedge\limits_{\delta\in S}(\forall\alpha\in A_\delta)\underset{\sim}{f}_\delta(\alpha)\in g^1_\delta(\alpha)\}$". Also $g^1_\delta(\alpha)$ is a countable subset of ω_1 as $\mathbb{Q}$ satisfies the c.c.c.

For $\delta\in S$ we define a function $g_\delta : A_\delta\to\omega_1$ by letting $g_\delta(\alpha) = (\sup(g^1_\delta(\alpha))+1$ hence $g_\delta(\alpha)<\omega_1$ so $\langle g_\delta : \delta\in S\rangle$ is as required on $\bar f$ in $(**)$ in $\mathbf{V}[G_{\mathbb{P}}]$, of course. Apply clause $(**)$ in $\mathbf{V}[G_{\mathbb{P}}]$ to $\langle g_\delta : \delta\in S\rangle$ so we can find $g:\omega_2\to\omega_1$ such that $\bigwedge\limits_{\delta\in S}\delta > \sup\{\alpha\in A_\delta, g_\delta(\alpha)>g(\alpha)\}$. Now g is as required also in $\mathbf{V}[G_{\mathbb{P}}][G_{\mathbb{Q}}]$. $\square_{3.2}$

We may wonder can we strengthen the conclusion of 2.3 to $\diamondsuit^*_S$ (of course the demand in clause (e) and (f) in claim 3.4 below are necessary, i.e. otherwise $\diamondsuit^*_S$ holds). The answer is <u>not</u> as: (the restriction in (e) and in (f) are best possible).

Observation 3.4. Assume $\lambda=\lambda^{<\lambda}, S\subseteq S^\lambda_\kappa$.

Then for some $\mathbb{P}$

- (a) $\mathbb{P}$ is a forcing notion
- (b) $\mathbb{P}$ is of cardinality λ^+ satisfying the λ^+-c.c.
- (c) forcing with $\mathbb{P}$ does not collapse cardinals and does not change cofinality
- (d) forcing with $\mathbb{P}$ adds no new $\eta\in {}^{\lambda>}\mathrm{Ord}$
- (e) $\diamondsuit^*_S$ fails for every stationary subset S of λ such that
  - (α) $S\subseteq S^\lambda_\kappa$ when $(\exists\mu<\lambda)[\mu^{<\kappa>_{\rm tr}}=\lambda]$
    or just
  - (β) $\alpha\in S\Rightarrow |\alpha|^{<\mathrm{cf}(\alpha)>_{\rm tr}}>|\alpha|$
- (f) $(D\ell)_S$, see below, fails for every $S\subseteq S^\lambda_\kappa$ when $\alpha\in S\Rightarrow|\alpha|^{<\mathrm{cf}(\alpha)>_{\rm tr}}=\lambda$.

Recalling

Definition 3.5. 1) For $\mu\geq\kappa=\mathrm{cf}(\kappa)$ let $\mu^{<\kappa>_{\rm tr}}=\{|\mathcal{T}| : \mathcal{T}\subseteq {}^{\kappa\geq}\mu$ is closed under initial segments (i.e. a subtree) such that $|\mathcal{T}\cap {}^{\kappa>}\mu|\leq\mu\}$.
2) For λ regular uncountable and stationary $S\subseteq\lambda$ let $(D\ell)_S$ mean that there is a sequence $\bar{\mathcal{P}}=\langle\mathcal{P}_\delta:\delta\in S\rangle$ witnessing it which means:

$(*)_{\bar{\mathcal{P}}}$ (a) $\mathcal{P}_\delta\subseteq {}^\delta\delta$ has cardinality $<\lambda$
(b) for every $f\in{}^\lambda\lambda$ the set $\{\delta\in S : f{\restriction}\delta\in\mathcal{P}_\delta\}$ is stationary

(for λ successor it is equivalent to $\diamondsuit_S$; for λ strong inaccessible it is trivial).

Proof. <u>Proof of 3.4</u> Use $\mathbb{P}$ = adding λ^+,λ-Cohen subsets.
The proof is straight. $\square_{3.4}$

Remark 3.6. The consistency results in 3.4 are best possible, see [**?**].

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