

\aleph_n -Free Modules With Trivial Duals

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Abstract. In the first part of this paper we introduce a simplified version of a new Black Box from Shelah [11] which can be used to construct complicated \aleph_n -free abelian groups for any natural number $n \in \mathbb{N}$. In the second part we apply this prediction principle to derive for many commutative rings R the existence of \aleph_n -free R -modules M with trivial dual $M^* = 0$, where $M^* = \text{Hom}(M, R)$. The minimal size of the \aleph_n -free abelian groups constructed below is \beth_n , and this lower bound is also necessary as can be seen immediately if we apply GCH.

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1. Introduction

The existence of almost free R -modules M over countable principal ideal domains (but not fields) with trivial dual $M^* := \text{Hom}(M, R) = 0$ is a well-known fact by results using strong prediction principles like diamonds in $V = L$. Only note that any (non-trivial) R -module with endomorphism ring $\text{End } M = R$ is such an example. For every regular cardinal $\kappa > |R|$ (which is not weakly compact) we can find (strongly) κ -free R -modules M of size κ with $\text{End } M = R$. But from the singular compactness theorem follows, that such modules M do not exist for singular cardinals, see e.g. [4] or [8]. Thus we want to get rid of additional set theoretic restrictions and work exclusively with ZFC:

If we restrict to \aleph_1 -free R -modules (meaning that all countable submodules are free) and do not care about the size of M , then we have an abundance of such modules and those of minimal cardinal 2^{\aleph_0} can be constructed by applications of

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the (ordinary) Black Box, see for example [8]. However, it is much harder to find examples like this of size \aleph_1 (recall that \aleph_1 may be much smaller than 2^{\aleph_0} and hence we do not have that many possible extension of a module to eliminate unwanted homomorphisms. A first example of an \aleph_1 -free R -module M of size \aleph_1 with trivial dual was given in Eda [3]. Some years later we improved this result showing the existence of such modules with endomorphism ring R , see [7, 2] or [8]. If we want to replace \aleph_1 by \aleph_2 or any higher cardinal, then we necessarily encounter additional set theoretic restriction, see [6]. If we require only the existence of indecomposable abelian groups, then their κ -freeness is restricted to small cardinals; see [10]. Thus, in order to construct \aleph_n -free R -modules M with trivial dual we must relax the restriction on the size of M . Clearly (note that GCH is not excluded, in which case $\aleph_n = \beth_n$), the size of M must be at least \beth_n . These cardinals are defined inductively as $\beth_0 = 2^{\aleph_0}$ and $\beth_{n+1} = 2^{\beth_n}$; see Jech [9]. Using a recent refined Black Box from Shelah [11] which takes care of additional freeness of the module, we can give a reasonably short proof of the existence of \aleph_n -free R -modules M with trivial dual $M^* = 0$ of cardinality $|M| = \beth_n$ for any natural number n ; see Theorem 4.3 and Corollary 4.4. It remains an open question if we can go any further and pass \aleph_ω or possibly replace $M^* = 0$ by $\text{End } M = R$. In this context it is also worthwhile to recall (from [10]) that there are models of ZFC in which \aleph_{ω^2+1} -free implies \aleph_{ω^2+2} -free.

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2. The Combinatorial Black Box for $\bar{\lambda}$

The new Black Box depends on a finite sequence of cardinals satisfying some cardinal conditions. Thus let $k_* < \omega$ and $\bar{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ be a sequence of cardinals such that for $\chi_l := \lambda_l^{\aleph_0}$ ($l \leq k_*$) the following \blacksquare -conditions holds.

$$\chi_{l+1}^{\chi_l} = \chi_{l+1} \quad (l < k_*). \quad (2.1)$$

We will also say that $\bar{\lambda}$ is a \blacksquare -sequence and note that $\chi_l = \chi_l^{\aleph_0} < \chi_{l+1}$.

Condition (2.1) is used to enumerate all maps which we want to predict before constructing the modules. If λ is any cardinal, then we can define inductively a $\bar{\lambda}$ -sequence: Let $\chi_1 = \lambda^{\aleph_0}$ and if λ_l is defined for $l < k_*$, then choose a suitable $\lambda_{l+1} > \lambda_l$ with (2.1), e.g. put $\lambda_{l+1} = \chi_l^{\chi_l}$. The sequence $\langle \beth_1, \dots, \beth_{k_*} \rangle$ is an example of such a \blacksquare -sequence.

If λ is a cardinal, then $\omega^\uparrow \lambda$ will denote all *order preserving* maps $\eta : \omega \rightarrow \lambda$ (which we also call *infinite branches*) on λ , while $\omega^{\uparrow >} \lambda$ denotes the family of all order preserving *finite branches* $\eta : n \rightarrow \lambda$ on λ , where the natural number n , λ and ω (the first infinite ordinal) are considered as sets, e.g. $n = \{0, \dots, n-1\}$, thus the finite branch η has length n .

For the sake of generality we first consider any sequence $\bar{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ of cardinals such that $\lambda_l^{\aleph_0} = \chi_l$ ($l \leq k_*$) (which later will be strengthened to a \blacksquare -sequence). Moreover, we associate with $\bar{\lambda}$ two sets Λ and Λ_* . Thus let

$$\Lambda = \omega^\uparrow \lambda_1 \times \dots \times \omega^\uparrow \lambda_{k_*}.$$

For the second set we replace the m -th (and only the m -th) coordinate $\omega^\uparrow \lambda_m$ by the finite branches $\omega^{\uparrow >} \lambda_m$, thus

$$\Lambda_m = \omega^\uparrow \lambda_1 \times \dots \times \omega^{\uparrow >} \lambda_m \times \dots \times \omega^\uparrow \lambda_{k_*} \text{ for } m \leq k_* \text{ and let } \Lambda_* = \bigcup_{m \leq k_*} \Lambda_m. \quad (2.2)$$

The elements of Λ, Λ_* will be written as sequences $\bar{\eta} = (\eta_1, \dots, \eta_{k_*})$ with $\eta_l \in \omega^\uparrow \lambda$ or $\eta_l \in \omega^{\uparrow >} \lambda$, respectively. Using these $\bar{\eta}$ s as support of elements of the module will make enough room for linear independence which will then give \aleph_n -freeness.

With each member of Λ we can associate a subset of Λ_* :

Definition 2.1. *If $\bar{\eta} = (\eta_1, \dots, \eta_{k_*}) \in \Lambda$ and $m \leq k_*, n < \omega$, then let $\bar{\eta} \upharpoonright \langle m, n \rangle$ be the following element in Λ_m (thus in Λ_*)*

$$(\bar{\eta} \upharpoonright \langle m, n \rangle)_l = \begin{cases} \eta_l & \text{if } m \neq l \leq k_* \\ \eta_m \upharpoonright n & \text{if } l = m. \end{cases}$$

We associate with $\bar{\eta}$ its support $[\bar{\eta}] = \{\bar{\eta} \upharpoonright \langle m, n \rangle \mid m \leq k_*, n < \omega\}$ which is a countable subset of Λ_* . Similarly, for $m \leq k_*$ also let $[\bar{\eta} \upharpoonright m] = \{\bar{\eta} \upharpoonright \langle m, n \rangle \mid n < \omega\} \subseteq [\bar{\eta}]$.

Definition 2.2. *Let $\bar{C} = \langle C_1, \dots, C_{k_*} \rangle$ be a sequence of sets C_m satisfying $|C_m| \leq \chi_m$ for all $m \leq k_*$. We let $C = \bigcup_{m \leq k_*} C_m$ and define a set-trap (for Λ, \bar{C}) as a map $\varphi_{\bar{\eta}} : [\bar{\eta}] \rightarrow C$ with a label $\bar{\eta} \in \Lambda$.*

The following lemma will be used for the inductive proof of our next theorem.

Lemma 2.3. *Let λ be an infinite cardinal, $\chi = \lambda^{\aleph_0}$ and \mathfrak{P} a set of size $|\mathfrak{P}| = \chi$. Then there is a sequence $\langle \Phi_\eta \mid \eta \in \omega^\uparrow \lambda \rangle$ such that*

- (a) $\Phi_\eta = \langle \Phi_{\eta n} \mid n < \omega \rangle$, with $\Phi_{\eta n} \in \mathfrak{P}$,
- (b) If $\bar{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in \omega^{\uparrow >} \lambda\}$, $\alpha \in \lambda$ and $\rho \in \omega^{\uparrow >} \lambda$, then there is $\eta \in \omega^\uparrow \lambda$ such that $0\eta = \alpha$, $\rho \subset \eta$ and $\Phi_{\eta n} = f_{\eta \upharpoonright n}$ for all $n < \omega$.

Proof. Since $|\mathfrak{P}| = \chi = \lambda^{\aleph_0} = |\omega^\uparrow \lambda|$, we can fix an embedding

$$\pi : \mathfrak{P} \hookrightarrow \omega^\uparrow \lambda.$$

And since $|\omega^{\uparrow >} \lambda| = \lambda$ there is also a list $\omega^{\uparrow >} \lambda = \langle \mu_\alpha \mid \alpha < \lambda \rangle$ with enough repetitions for each $\eta \in \omega^{\uparrow >} \lambda$:

$$\{\alpha < \lambda \mid \mu_\alpha = \eta\} \subseteq \lambda \text{ is unbounded.}$$

Moreover we define for each $n < \omega$ a coding map

$$\pi_n : {}^n \mathfrak{P} \longrightarrow {}^{n^2} \lambda \subseteq \omega^{\uparrow >} \lambda$$

$$\bar{\varphi} = \langle \varphi_0, \dots, \varphi_{n-1} \rangle \mapsto \bar{\varphi}\pi_n = (\varphi_0\pi \upharpoonright n)^\wedge \dots^\wedge (\varphi_{n-1}\pi \upharpoonright n).$$

Finally let $X \subseteq {}^\omega\lambda$ be the collection of all order preserving maps $\eta : \omega \rightarrow \lambda$ such that the following holds.

$$\exists \bar{\varphi} = \langle \varphi_i \mid i < \omega \rangle \in {}^\omega\mathfrak{P} \text{ and } \exists \kappa < \omega \text{ with } (\bar{\varphi} \upharpoonright n)\pi_n = \mu_{n\eta} \text{ for all } n > \kappa. \quad (2.3)$$

By definition of π_n it follows that $\bar{\varphi}$ is uniquely determined by (2.3). (Just note that, $\mu_{n\eta}$ determines $\varphi_m\pi \upharpoonright n$ for all $n > \kappa, m$.)

We now prove the two statements of the lemma. For (a) we consider any $\eta \in {}^\omega\lambda$. If $\eta \notin X$, then we can choose arbitrary members $\Phi_{\eta m} \in \mathfrak{P}$, and if $\eta \in X$, then choose the uniquely determined sequence $\bar{\varphi}$ from (2.3) and let $\Phi_{\eta m} = \varphi_n$, so $\Phi_\eta = \bar{\varphi}$.

For (b) we consider some $\bar{f} = \{f_\nu \mid f_\nu \in \mathfrak{P}, \nu \in {}^\omega\lambda\}$ and $\rho \in {}^\omega\lambda$. In this case we must define an extension $\eta = \langle \alpha_n \mid n < \omega \rangle \in {}^\omega\lambda$ of ρ . Thus put $0\eta = \alpha$, $\alpha_n = n\rho$ for $n < \text{lg}(\rho)$. And if $n \geq \text{lg}(\rho)$, then using the above and that the list of μ_α s is unbounded, we can choose inductively $\alpha_n > \alpha_{n-1}$ with $\langle f_\eta \upharpoonright m \mid m < n \rangle \pi_n = \mu_{\alpha_n}$.

Finally we check statement (b). Using (2.3) it will follow that the sequence η belongs to X :

If $\bar{\varphi} = \langle f_{\eta \upharpoonright i} \mid i < \omega \rangle \in {}^\omega\mathfrak{P}$ and $k = \text{lg}(\rho)$, then we have

$$(\bar{\varphi} \upharpoonright n)\pi_n = \langle f_{\eta \upharpoonright m} \mid m < n \rangle \pi_n = \mu_{\alpha_n} = \mu_{n\eta} \text{ for all } n > k$$

and $\Phi_{n\eta} = \varphi_n = f_{\eta \upharpoonright n}$ for all $n < \omega$ is immediate. \square

The $\bar{\lambda}$ -Black Box 2.4. Let $\langle \lambda_1, \dots, \lambda_{k_*} \rangle$ be a \blacksquare -sequence satisfying (2.1), Λ, Λ_* as above and C as in Definition 2.2. Then there is a family of set-traps $\langle \varphi_{\bar{\eta}} \mid \bar{\eta} \in \Lambda \rangle$ satisfying the following

PREDICTION PRINCIPLE: If $\varphi : \Lambda_* \rightarrow C$ is any map with the trap-condition $\Lambda_m\varphi \subseteq C_m$ ($m \leq k_*$) and $\alpha \in \lambda_{k_*}$, then for some $\bar{\eta} \in \Lambda$ there is a set-trap $\varphi_{\bar{\eta}}$ with $\varphi_{\bar{\eta}} \subseteq \varphi$ and $0\eta_{k_*} = \alpha$.

Proof. The proof of Theorem 2.4 will follow by induction on k_* . Temporarily we will attach parameters k_* to the above symbols like $\Lambda^{k_*}, \bar{C}^{k_*}, \varphi_{\bar{\eta}}^{k_*}, \bar{\eta}^{k_*}, \dots$

The first step is $k_* = 1$. In this case the claim is a special case of Lemma 2.3. Indeed, we have $\Lambda^{k_*} = {}^\omega\lambda_{k_*}$ and $\Lambda_*^{k_*} = {}^\omega\lambda_{k_*}$ and $\bar{\eta} \upharpoonright \langle m, n \rangle = \eta_{k_*} \upharpoonright n$ holds. We put $\mathfrak{P} = C^{k_*} = C_{k_*}^{k_*}$ and note that $|\mathfrak{P}| \leq \chi_{k_*} = \chi_{k_*}^{\aleph_0}$. The trap functions $\varphi_{\bar{\eta}}$ are defined by

$$(\bar{\eta} \upharpoonright \langle m, n \rangle)\varphi_{\bar{\eta}} = \Phi_{\eta_{k_*} n}$$

and with $f_\nu = \nu\varphi$ condition (b) of Lemma 2.3 reads as

$$(\bar{\eta} \upharpoonright \langle m, n \rangle)\varphi_{\bar{\eta}} = \Phi_{\eta_{k_*} n} = f_{\eta_{k_*} \upharpoonright n} = (\bar{\eta} \upharpoonright \langle m, n \rangle)\varphi$$

and the prediction principle in Theorem 2.4 is clear.

The induction step $k_* = k + 1$:

Suppose that Theorem 2.4 is shown for k . We must find a family of traps $\{\varphi_{\bar{\eta}}^{k_*} \mid \bar{\eta} \in$

Λ^{k_*} for $\Lambda^{k_*}, \overline{C}^{k_*}$ and verify the prediction principle in Theorem 2.4. By induction hypothesis there is such a family $\{\varphi_{\overline{\eta}}^k \mid \overline{\eta} \in \Lambda^k\}$ for $\Lambda^k, \overline{C}^k$.

Let $\chi_{k_*} = \chi$, $\lambda_{k_*} = \lambda$ and recall that $\chi = \lambda^{\aleph_0}$. Moreover, by assumption $|C^{k_*}| \leq \chi$. We now consider $\mathfrak{P} = \text{map}(\Lambda^k, C_{k_*}^{k_*})$ which has size $|\mathfrak{P}| = |C_{k_*}^{k_*}|^{|\Lambda^k|} \leq \chi^{\chi^k} = \chi$ by condition (2.1) of the \blacksquare -sequence.

If $\overline{\eta} \in \Lambda^{k_*}$, then let $\varphi_{\overline{\eta}}^{k_*} : [\overline{\eta}] \rightarrow C^{k_*}$ be the following map in (2.4). Recall that $\overline{\eta} = \langle \eta_1, \dots, \eta_{k_*} \rangle$ and thus $\eta_{k_*} \in \omega^\uparrow \lambda$ and $\Phi_{\eta_{k_*} n} \in \text{map}(\Lambda^k, C_{k_*}^{k_*})$ by Lemma 2.3. If $\overline{\eta}' = \langle \eta_1, \dots, \eta_k \rangle \in \Lambda^k$, then $\overline{\eta}' \Phi_{\eta_{k_*} n}$ is a well-defined element of $C_{k_*}^{k_*}$ and $\varphi_{\overline{\eta}'}^k$ is given by induction hypothesis. We can now define

$$(\overline{\eta} \upharpoonright \langle m, n \rangle) \varphi_{\overline{\eta}}^{k_*} = \begin{cases} \overline{\eta}' \Phi_{\eta_{k_*} n} & \text{if } m = k_* \\ (\overline{\eta}' \upharpoonright \langle m, n \rangle) \varphi_{\overline{\eta}'}^k & \text{if } m < k_*. \end{cases} \quad (2.4)$$

In order to show the prediction principle we consider an arbitrary map $\varphi : \Lambda^{k_*} \rightarrow C^{k_*}$ satisfying the trap-condition $\Lambda_m \varphi \subseteq C_m$ for all $m \leq k_*$. We want to find $\overline{\eta} \in \Lambda^k$ and $\mu \in \omega^\uparrow \lambda$ such that $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle \in \Lambda^{k_*}$ satisfies $\varphi \upharpoonright [\overline{\eta}^*] = \varphi_{\overline{\eta}^*}^{k_*}$ and $0\eta_{k_*}^* = \alpha$ (which is the claim of Theorem 2.4).

First we search for μ and define for each $\nu \in \omega^\uparrow \lambda$ a map $f_\nu : \Lambda^k \rightarrow C_{k_*}^{k_*}$ from \mathfrak{P} depending on φ . If $\overline{\eta} \in \Lambda^k$, then $\overline{\eta}^\wedge \langle \nu \rangle \in \Lambda^{k_*}$, thus

$$\overline{\eta} f_\nu := (\overline{\eta}^\wedge \langle \nu \rangle) \varphi \quad (2.5)$$

is well-defined. By the Lemma 2.3 we find $\mu \in \omega^\uparrow \lambda$ such that

$$0\mu = \alpha \text{ and } f_{\mu \upharpoonright n} = \Phi_{\mu n} : \Lambda^k \rightarrow C_{k_*}^{k_*} \text{ for all } n \in \omega.$$

By Lemma 2.3(b), (2.4) and (2.5) we have for any $\overline{\eta}$ and $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle$ that

$$(\overline{\eta}^* \upharpoonright \langle k_*, n \rangle) \varphi_{\overline{\eta}^*}^{k_*} = \overline{\eta} \Phi_{\mu n} = \overline{\eta} f_{\mu \upharpoonright n} = (\overline{\eta}^\wedge \langle \mu \upharpoonright n \rangle) \varphi = (\overline{\eta}^* \upharpoonright \langle k_*, n \rangle) \varphi$$

and $0\mu = \alpha$ which is the prediction as required for $m = k_*$.

Now we consider the case when $m < k_*$ and define a map $\varphi' : \Lambda^k \rightarrow C^k$ depending on φ and μ . If $\overline{\eta}' \in \Lambda^k$, then $\overline{\eta}'^\wedge \langle \mu \rangle \in \Lambda^{k_*}$ (because $\mu \in \omega^\uparrow \lambda$), thus

$$\overline{\eta}' \varphi' := (\overline{\eta}'^\wedge \langle \mu \rangle) \varphi$$

is well-defined and by induction hypothesis on the traps $\varphi_{\overline{\eta}'}^k$ there is some $\overline{\eta} \in \Lambda^k$ such that

$$(\overline{\eta} \upharpoonright \langle m, n \rangle) \varphi_{\overline{\eta}}^k = (\overline{\eta} \upharpoonright \langle m, n \rangle) \varphi' \text{ for } m \leq k \text{ and } n < \omega.$$

Now let $\overline{\eta}^* = \overline{\eta}^\wedge \langle \mu \rangle \in \Lambda^{k_*}$. By the last displayed equation and (2.4) we have for $m < k_*$ that

$$(\overline{\eta}^* \upharpoonright \langle m, n \rangle) \varphi_{\overline{\eta}^*}^{k_*} = (\overline{\eta} \upharpoonright \langle m, n \rangle) \varphi_{\overline{\eta}}^k = (\overline{\eta} \upharpoonright \langle m, n \rangle) \varphi' = (\overline{\eta}^* \upharpoonright \langle m, n \rangle) \varphi.$$

Thus $\varphi_{\overline{\eta}^*}^{k_*}$ predicts φ with $0\eta_{k_*}^* = 0\mu = \alpha$ as suggested above. \square

Definition 2.5. Let $F : \Lambda \rightarrow \Lambda_*$ be a given map. A subset $\Omega \subseteq \Lambda$ is free (with respect to F) if there is an enumeration $\langle \bar{\eta}^\alpha \mid \alpha < \alpha_* \rangle$ of Ω (we write $\Omega_\alpha = \{\bar{\eta}^\beta \mid \beta < \alpha\}$) and there are $\ell_\alpha \leq k_*, n_\alpha < \omega$ ($\alpha < \alpha_*$) such that for $\alpha < \alpha_*$ and $n_\alpha \leq n$

$$\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \Omega_\alpha F.$$

Moreover, Ω is κ -free (with respect to F) for some cardinal κ if the above holds for all subsets of Ω of cardinality $< \kappa$.

This is to say, that every newly chosen element $\bar{\eta}^\alpha$ picks up some unused element from Λ_* in its support. Note that the enumeration of Ω in Definition 2.5 does not permit repetitions. We want to show the following

Freeness-Proposition 2.6. With the notions from Theorem 2.4 and Definition 2.5 the set Λ is \aleph_{k_*} -free with respect to any function $F : \Lambda \rightarrow \Lambda_*$. For any $k < k_*$, $\Omega \subseteq \Lambda$ of cardinality $|\Omega| \leq \aleph_k$ and $\langle u_{\bar{\eta}} \subseteq \{1, \dots, k_*\} \mid |u_{\bar{\eta}}| > k, \bar{\eta} \in \Omega \rangle$ we can find an enumeration $\langle \bar{\eta}^\alpha \mid \alpha < \aleph_k \rangle$ of Ω , $\ell_\alpha \in u_{\bar{\eta}^\alpha}$ and $n_\alpha < \omega$ ($\alpha < \aleph_k$) such that

$$\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \Omega_\alpha F \text{ for all } n \geq n_\alpha.$$

Proof. The proof follows by induction on k . We begin with $k = 0$, hence we may assume that $|\Omega| = \aleph_0$. Let $\Omega = \{\bar{\eta}^\alpha \mid \alpha < \omega\}$ be an enumeration without repetitions. From $0 = k < |u_{\bar{\eta}}|$ follows $u_{\bar{\eta}} \neq \emptyset$ and we can choose any $\ell_\alpha \in u_{\bar{\eta}^\alpha}$ for all $\alpha < \omega$. To be definite we may choose $\ell_\alpha = \min u_{\bar{\eta}^\alpha}$. If $\alpha \neq \beta < \omega$, then $\bar{\eta}^\alpha \neq \bar{\eta}^\beta$ and there is $n_{\alpha,\beta} \in \omega$ such that $\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \neq \bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle$ for all $n \geq n_{\alpha,\beta}$. Since $\Omega_\alpha F$ is finite, we may enlarge $n_{\alpha,\beta}$, if necessary, such that $\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \Omega_\alpha F$ for all $n \geq n_{\alpha,\beta}$. If $n_\alpha = \max_{\beta < \alpha} n_{\alpha,\beta}$, then $\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \Omega_\alpha F$ for all $n \geq n_\alpha$. Hence case $k = 0$ is settled and we let $k' = k + 1$ and assume that the proposition holds for k .

Let $|\Omega| = \aleph_{k'}$ and choose an $\aleph_{k'}$ -filtration $\Omega = \bigcup_{\delta < \aleph_{k'}} \Omega_\delta$ with $\Omega_0 = \emptyset$ and $|\Omega_1| = \aleph_k$. The crucial idea comes from [11]: We can also assume that this chain is closed, meaning that for any $\delta < \aleph_{k'}$, $\bar{v}, \bar{v}' \in \Omega_\delta$ and $\bar{\eta} \in \Omega$ with

$$\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\bar{v}F)_m, (\bar{v}'F)_m \mid m \leq k_*\}$$

follows $\bar{\eta} \in \Omega_\delta$. Thus, if $\bar{\eta} \in \Omega_{\delta+1} \setminus \Omega_\delta$, then the set

$$u_{\bar{\eta}}^* = \{\ell \leq k_* \mid \exists n < \omega, \bar{v} \in \Omega_\delta \text{ such that } \bar{\eta} \upharpoonright \langle \ell, n \rangle = \bar{v} \upharpoonright \langle \ell, n \rangle \text{ or } \bar{\eta} \upharpoonright \langle \ell, n \rangle = \bar{v}F\}$$

is empty or a singleton. Otherwise there are $n, n' < \omega$ and distinct $\ell, \ell' \leq k_*$ with $\bar{\eta} \upharpoonright \langle \ell, n \rangle \in \{\bar{v} \upharpoonright \langle \ell, n \rangle, \bar{v}F\}$ and $\bar{\eta} \upharpoonright \langle \ell', n' \rangle \in \{\bar{v}' \upharpoonright \langle \ell', n' \rangle, \bar{v}'F\}$ for certain $\bar{v}, \bar{v}' \in \Omega_\delta$. Hence $\{\eta_m \mid m \leq k_*\} \subseteq \{\nu_m, \nu'_m, (\bar{v}F)_m, (\bar{v}'F)_m \mid m \leq k_*\}$, and the closure property implies the contradiction $\bar{\eta} \in \Omega_\delta$.

If $\delta < \aleph_{k'}$, then let $D_\delta = \Omega_{\delta+1} \setminus \Omega_\delta$ and $u_{\bar{\eta}}' := u_{\bar{\eta}} \setminus u_{\bar{\eta}}^*$ must have size $> k' - 1 = k$. Thus the induction hypothesis applies and we find an enumeration $\bar{\eta}^{\delta\alpha}$ ($\alpha < \aleph_k$) of D_δ as in the proposition. Finally we put these chains for each $\delta < \aleph_{k'}$ together with the induced ordering to get an enumeration $\langle \bar{\eta}^\alpha \mid \alpha < \aleph_{k'} \rangle$ of Ω satisfying the proposition. \square

3. The Black Box for \aleph_n -free modules

Let R be a commutative ring with \mathbb{S} a countable multiplicatively closed subset such that the following holds.

- (i) The elements of \mathbb{S} are not zero-divisors, i.e. if $s \in \mathbb{S}, r \in R$ and $sr = 0$, then $r = 0$.
- (ii) $\bigcap_{s \in \mathbb{S}} sR = 0$.

We also say that R is an \mathbb{S} -ring. If (i) holds, then R is \mathbb{S} -torsion-free and if (ii) holds, then R is \mathbb{S} -reduced, see [8]. To ease notations we use the letter \mathbb{S} only if we want to emphasize that the argument depends on it. If M is an R -module, then these definitions naturally carry over to M . Finally we enumerate $\mathbb{S} = \{s_n \mid n < \omega\}$, let $s_0 = 1$ and put $q_n = \prod_{i \leq n} s_i$, thus $q_{n+1} = q_n s_{n+1}$.

Similar to the Black Box in [1], we first define the basic R -module B , which is

$$B = \bigoplus_{\bar{\eta} \in \Lambda_*} Re_{\bar{\eta}}.$$

Definition 3.1. *If $U \subset \Lambda_*$, then we get a canonical summand $B_U = \bigoplus_{\bar{\eta} \in U} Re_{\bar{\eta}}$ of B , and in particular, let $B_{\bar{\eta}} = B_{\{\bar{\eta}\}}$ and $B_{\bar{\eta} \upharpoonright m} = B_{\{\bar{\eta} \upharpoonright m\}}$ be the canonical summand of $\bar{\eta}$ and $\bar{\eta} \upharpoonright m$ ($\bar{\eta} \in \Lambda$), respectively.*

We have several R -free summands

$$B_{\bar{\eta} \upharpoonright m} = \bigoplus_{n < \omega} Re_{\bar{\eta} \upharpoonright \langle m, n \rangle} \text{ and } B_{\bar{\eta}} = \bigoplus_{m \leq k_*} B_{\bar{\eta} \upharpoonright m}.$$

The \mathbb{S} -topology (generated by the basis sB ($s \in \mathbb{S}$)) of neighbourhoods of 0 is Hausdorff on B and (as usual) we can consider the \mathbb{S} -completion \widehat{B} of B ; see [8] for elementary facts on the elements of \widehat{B} . Let $\widetilde{B} = \bigoplus_{\bar{\eta} \in \Lambda_*} \widehat{R}e_{\bar{\eta}}$. Every element $b \in \widehat{B}$ has a natural Λ_* -support $[b]_{\Lambda_*} \subseteq \Lambda_*$ which are those $\bar{\eta} \in \Lambda_*$ which contribute to the sum-representation $b = \sum_{\bar{\eta} \in \Lambda_*} b_{\bar{\eta}}e_{\bar{\eta}}$ with coefficients $0 \neq b_{\bar{\eta}} \in \widehat{R}$. Thus let $[b]_{\Lambda_*} = \{\bar{\eta} \in \Lambda_* \mid b_{\bar{\eta}} \neq 0\}$. Note that $[b]_{\Lambda_*}$ is at most countable and $b \in \widetilde{B}$ iff $[b]_{\Lambda_*}$ is finite. As in the earlier Black Boxes (see [8]) we use conditions on the support (given by the prediction) to select (carefully) elements from \widehat{B} added to B to get the final structure M , such that

$$B \subseteq M \subseteq_* \widehat{B}$$

which is an \mathbb{S} -pure submodule of \widehat{B} , thus satisfying

$$M \cap s\widehat{B} \subseteq sM$$

for all $s \in \mathbb{S}$. Thus \mathbb{S} -topological arguments can be used carelessly switching between these three modules.

We will now use B, Λ_*, Λ to define the Black Box for \aleph_n -free R -modules. As in [1] we will also use the notion of a trap.

Definition 3.2. Let G be any R -module. A trap (for B, G) is a partial R -homomorphism of B into G with a label $\bar{\eta} \in \Lambda$, say $\varphi_{\bar{\eta}} : \text{Dom}(\varphi_{\bar{\eta}}) \rightarrow G$, such that $B_{\bar{\eta}} \subseteq \text{Dom}(\varphi_{\bar{\eta}}) \subseteq B$.

The $\bar{\lambda}$ -Black Box 3.3. Given a \blacksquare -sequence $\bar{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ with (2.1) and an R -module G of size $|G| \leq \chi_1$, let Λ, Λ_* be as above. Then there is a family of traps $\varphi_{\bar{\eta}}$ ($\bar{\eta} \in \Lambda$) with the following property:

THE PREDICTION: If $\varphi : B \rightarrow G$ is an R -homomorphism and $\alpha \in \lambda_{k_*}$, then there is $\bar{\eta} \in \Lambda$ with $0\eta_{k_*} = \alpha$ and $\varphi_{\bar{\eta}} \subseteq \varphi$.

Proof. The theorem is an immediate consequence of Theorem 2.4. We view the set maps in Theorem 2.4 as the restrictions of the R -homomorphisms in Theorem 3.3 to the canonical R -basis $\{e_{\bar{\nu}} \mid \bar{\nu} \in \Lambda_*\}$ of B . There is a one-to-one correspondence between these maps and thus Theorem 3.3 follows. \square

4. The R -modules

Let R be an \mathbb{S} -torsion-free and \mathbb{S} -reduced commutative ring of size $|R| < 2^{\aleph_0}$, $\chi_{k_*} = \lambda_{k_*}^{\aleph_0} = \lambda_{k_*}$ be as before, $B = \bigoplus_{\bar{\nu} \in \Lambda_*} R e_{\bar{\nu}}$ the R -module freely generated by $\{e_{\bar{\nu}} \mid \bar{\nu} \in \Lambda_*\}$ and

$$\Lambda_* = \bigcup_{\bar{\eta} \in \Lambda} [\bar{\eta}] \text{ with } [\bar{\eta}] = \{\bar{\eta} \upharpoonright \langle m, n \rangle \mid m \leq k_*, n < \omega\}.$$

We also choose any bijection

$$\delta : \lambda_{k_*} \longrightarrow \Lambda_*.$$

Thus we can write the basis elements of B in the form $e_{\delta(\alpha)}$ for any $\alpha \in \lambda_{k_*}$.

From [5] follows that the \mathbb{S} -adic completion \widehat{R} of R has 2^{\aleph_0} algebraically independent elements over R , and in particular $|\widehat{R}| = 2^{\aleph_0}$.

Next we define particular elements in \widehat{B} . If $\bar{\eta} \in \Lambda$, then let

$$y_{\bar{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} \left(\sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, n \rangle} + b_{\bar{\eta}n} e_{\delta(0\eta_{k_*})} \right)$$

where $b_{\bar{\eta}n} \in R$. Moreover let $y_{\bar{\eta}} = y_{\bar{\eta}0}$. We will choose $\pi_{\bar{\eta}} \in \widehat{R}$ and write $\pi_{\bar{\eta}} = \sum_{n < \omega} q_n b_{\bar{\eta}n}$ and let $\pi_{\bar{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} b_{\bar{\eta}n}$. Thus

$$y_{\bar{\eta}k} = \sum_{n \geq k} \frac{q_n}{q_k} \left(\sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, n \rangle} \right) + \pi_{\bar{\eta}k} e_{\delta(0\eta_{k_*})}$$

and from

$$s_{k+1} y_{\bar{\eta}k+1} = \sum_{n \geq k+1} \frac{q_n}{q_k} \left(\sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, n \rangle} + b_{\bar{\eta}n} e_{\delta(0\eta_{k_*})} \right)$$

and $y_{\bar{\eta}k} - s_{k+1}y_{\bar{\eta}k+1} = \sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, k \rangle} + b_{\bar{\eta}k}e_{\delta(0\eta_{k_*})}$, follows

$$s_{k+1}y_{\bar{\eta}k+1} = y_{\bar{\eta}k} - \sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, k \rangle} - b_{\bar{\eta}k}e_{\delta(0\eta_{k_*})}. \quad (4.1)$$

We want to define an R -module M with $B \subseteq M \subseteq_* \widehat{B}$ which is \mathbb{S} -pure in \widehat{B} . Thus M/B is \mathbb{S} -torsion-free and \mathbb{S} -divisible. It follows that for any non-trivial homomorphism $\sigma : M \rightarrow R$ there is $\bar{v} \in \Lambda_*$ with $e_{\bar{v}}\sigma \neq 0$. If $\bar{\eta} \in \Lambda$, then we will adjoin to B for some suitable $\pi_{\bar{\eta}} \in \widehat{R}$ the element $y_{\bar{\eta}} = \sum_{n < \omega} q_n(\sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, n \rangle}) + \pi_{\bar{\eta}}e_{\delta(0\eta_{k_*})}$. This will follow with the help of the next

Proposition 4.1. *Let R be an \mathbb{S} -torsion-free and \mathbb{S} -reduced commutative ring of size $< 2^{\aleph_0}$. Then for any $\bar{\eta} \in \Lambda$ there are $\pi_{\bar{\eta}} \in \widehat{R}$ and*

$$y_{\bar{\eta}} = \sum_{n < \omega} q_n \left(\sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, n \rangle} \right) + \pi_{\bar{\eta}}e_{\delta(0\eta_{k_*})} \quad (4.2)$$

with no homomorphism $\varphi : \langle B, y_{\bar{\eta}} \rangle_* \rightarrow R$ such that $\varphi \upharpoonright B_{[\bar{\eta}]} = \varphi_{\bar{\eta}}$ and $e_{\delta(0\eta_{k_*})}\varphi \neq 0$.

Proof. Let $e = e_{\delta(0\eta_{k_*})}$ and choose pairwise distinct elements $\pi_\alpha \in \widehat{R}$ ($\alpha < 2^{\aleph_0}$). Moreover let $y = \sum_{n < \omega} q_n(\sum_{m=1}^{k_*} e_{\bar{\eta} \upharpoonright \langle m, n \rangle})$ and put $y_\alpha = y + \pi_\alpha e$. Suppose that for each $\alpha < 2^{\aleph_0}$ there is a homomorphism $\varphi_\alpha : \langle B, y_\alpha \rangle_* \rightarrow R$ with $\varphi_\alpha \upharpoonright B_{[\bar{\eta}]} = \varphi_{\bar{\eta}}$ and $e\varphi_\alpha \neq 0$. By a pigeon hole argument there are distinct $\alpha, \beta < 2^{\aleph_0}$ with the same images $y_\alpha\varphi_\alpha = y_\beta\varphi_\beta$ and also $e\varphi_\alpha = e\varphi_\beta =: c \neq 0$. But this implies

$$\begin{aligned} 0 &= y_\alpha\varphi_\alpha - y_\beta\varphi_\beta = (y + \pi_\alpha e)\varphi_\alpha - (y + \pi_\beta e)\varphi_\beta \\ &= y\varphi_{\bar{\eta}} + \pi_\alpha e\varphi_\alpha - y\varphi_{\bar{\eta}} - \pi_\beta e\varphi_\beta = (\pi_\alpha - \pi_\beta)c. \end{aligned}$$

And from $\pi_\alpha - \pi_\beta \neq 0$ follows $c = 0$, a contradiction. \square

Finally we define the R -module

$$M = \langle B, y_{\bar{\eta}} \mid \bar{\eta} \in \Lambda \rangle_* \subseteq \widehat{B}. \quad (4.3)$$

Here we let $y_{\bar{\eta}}$ be as in (4.2) and apply Proposition 4.1.

First we will take care of the freeness of M by applying the set-theoretic version of freeness, i.e. Proposition 2.6. In order to apply our results to rings which are not necessarily PIDs, we more generally say that an R -module M is κ -free if any subset of size $< \kappa$ is contained in a free R -submodule of M .

Freeness-Proposition 4.2. *The module M as defined in (4.3) is \aleph_{k_*} -free.*

Proof. Besides the Λ_* -support $[g]_{\Lambda_*}$ (discussed at the beginning of the last section) any element g of the module $M = \langle B, y_{\bar{\eta}} \mid \bar{\eta} \in \Lambda \rangle_*$ has a refined natural finite support $[g]$ arriving from the definition (4.3). It consists of all those elements of Λ and Λ_* contributing to g . We observe that g is generated by elements $y_{\bar{\eta}}$ and $e_{\bar{\eta} \upharpoonright \langle m, n \rangle}$ and simply collect the $\bar{\eta}$ s and $\bar{\eta} \upharpoonright \langle m, n \rangle$ needed. Clearly $[g]$ is a finite subset of $\Lambda \cup \Lambda_*$. Hence any submodule H of M has a natural support $[H]$ taking the union of supports of its elements and if $|H| < \kappa$ for any cardinal $\kappa > |R|$, then there is a subset $\Omega \subseteq \Lambda$ of size $|\Omega| < \kappa$ such that H is a submodule of the pure R -submodule

$$M_\Omega = \langle e_{\bar{\eta} \upharpoonright \langle m, n \rangle}, e_{\delta(0\eta_{k_*})}, y_{\bar{\eta}} \mid \bar{\eta} \in \Omega, m \leq k_*, n < \omega \rangle_* \subseteq \widehat{B},$$

which also has size $< \kappa$. Thus, in order to show \aleph_{k_*} -freeness of M , we only must consider any $\Omega \subseteq \Lambda$ of size $|\Omega| < \aleph_{k_*}$ and show the freeness of the module M_Ω . We may assume that $|\Omega| = \aleph_{k_*-1}$. Let $F : \Lambda \rightarrow \Lambda_*$ be the map which assigns to $\bar{\eta} \in \Lambda$ the element $\bar{\eta}F = \delta(0\eta_{k_*}) \in \Lambda_*$.

By Proposition 2.6 we can express the generators of M_Ω of the form

$$M_\Omega = \langle e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle}, e_{\bar{\eta}^\alpha F}, y_{\bar{\eta}^\alpha n} \mid \alpha < \aleph_{k_*-1}, m \leq k_*, n < \omega \rangle$$

and find a sequence of pairs $(\ell_\alpha, n_\alpha) \in (k_* + 1) \times \omega$ such that for $n \geq n_\alpha$

$$\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{ \bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha \} \cup \{ \bar{\eta}^\beta F \mid \beta < \alpha \}. \quad (4.4)$$

Let $M_\alpha = \langle e_{\bar{\eta}^\gamma \upharpoonright \langle m, n \rangle}, e_{\bar{\eta}^\gamma F}, y_{\bar{\eta}^\gamma n} \mid \gamma < \alpha, m \leq k_*, n < \omega \rangle$ for any $\alpha < \aleph_{k_*-1}$; thus

$$\begin{aligned} M_{\alpha+1} &= M_\alpha + \langle e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle}, e_{\bar{\eta}^\alpha F}, y_{\bar{\eta}^\alpha n} \mid m \leq k_*, n < \omega \rangle \\ &= M_\alpha + \langle e_{\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle} \mid n < n_\alpha \rangle + \langle y_{\bar{\eta}^\alpha n} \mid n \geq n_\alpha \rangle \\ &\quad + \langle e_{\bar{\eta}^\alpha F}, e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle} \mid \ell_\alpha \neq m \leq k_*, n < \omega \rangle. \end{aligned}$$

Hence any element in $M_{\alpha+1}/M_\alpha$ can be represented in $M_{\alpha+1}$ modulo M_α of the form

$$\sum_{n \geq n_\alpha} r_n y_{\bar{\eta}^\alpha n} + \sum_{n < n_\alpha} r'_n e_{\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle} + r e_{\bar{\eta}^\alpha F} + \sum_{n < \omega} \sum_{\ell_\alpha \neq m \leq k_*} r''_{mn} e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle}.$$

Moreover, the summands involving the $e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle}$ s have disjoint supports. Now condition (4.4) applies recursively. And by the disjointness (identifying $e_{\bar{\eta}^\alpha F}$ with one of the $e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle}$ s if possible) it also follows that all coefficients r, r'_n, r''_{mn} must be zero, showing that the set

$$\{ e_{\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, k \rangle}, e_{\bar{\eta}^\alpha F}, e_{\bar{\eta}^\alpha \upharpoonright \langle m, n \rangle} \mid k < n_\alpha, \ell_\alpha \neq m \leq k_*, n < \omega \} \setminus M_\alpha$$

freely generates $M_{\alpha+1}/M_\alpha$. Thus M_Ω has an ascending chain with only free factors; it follows that M_Ω is free. \square

We can finally show the

Theorem 4.3. *Let R be an \mathbb{S} -torsion-free and \mathbb{S} -reduced commutative ring of size $< 2^{\aleph_0}$. Then for any \blacksquare -sequence $\bar{\lambda} = \langle \lambda_1, \dots, \lambda_{k_*} \rangle$ with (2.1) there exists an \aleph_{k_*} -free R -module M of size χ_{k_*} with trivial dual $\text{Hom}(M, R) = 0$. In particular, if R is a principal ideal domain but not a field of size $< 2^{\aleph_0}$, then there is an \aleph_{k_*} -free R -module M of size χ_{k_*} with trivial dual.*

Proof. If M is the R -module above, then M is \aleph_{k_*} -free by Proposition 4.2. Obviously M has size χ_{k_*} . If $\varphi : M \rightarrow R$ is a non-trivial homomorphism, then there is $\bar{\nu} \in \Lambda_*$ such that for some basis element $e_{\bar{\nu}}\varphi \neq 0$. By the Black Box 3.3 there is $\bar{\eta} \in \Lambda$ with $\delta(0\eta_{k_*}) = \bar{\nu}$ and $\varphi \upharpoonright B_{[\bar{\eta}]} = \varphi_{\bar{\eta}}$. We apply Proposition 4.1 to see that this is a contradiction. Hence $\text{Hom}(M, R) = 0$ follows. \square

Corollary 4.4. *If n is a natural number, then we find \aleph_n -free abelian groups of size \beth_n with trivial dual.*

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