ON KERNELS OF CELLULAR COVERS

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Dedicated to Avinoam Mann on the occasion of his retirement, 2006.

Abstract. In the present paper we continue to examine cellular covers of groups, focusing on the cardinality and the structure of the kernel K of the cellular map $G \to M$. We show that in general a torsion free reduced abelian group M may have a proper class of non-isomorphic cellular covers. In other words, the cardinality of the kernels is unbounded. In the opposite direction we show that if the kernel of a cellular cover of any group M has certain "freeness" properties, then its cardinality must be bounded by $|M|$.

Introduction and main results

In this paper we continue the discussion of cellular covers in the category of groups begun in [FGS1, FGS2], where this notion is also motivated. Given a map of groups $c\colon G \to M$, we say that (G, c) is *a cellular cover of* M or that $c\colon G \to M$ is a cellular cover, if every group map $\varphi\colon G \to M$ factors uniquely through c, or, equivalently, the natural map $\mathrm{Hom}(G, G) \to \mathrm{Hom}(G, M)$, induced by c, is an isomorphism of sets. Explicitly this means that there exists a unique *lift* $\widetilde{\varphi} \in \mathrm{End}(G)$ such that $\widetilde{\varphi} \circ c = \varphi$ (maps are composed from left to right).

It has been shown before [FGS1, FlR] that cellular covers are values of general augmented ($FM \to M$) and idempotent ($F \circ F = F$) functors on the category of groups. More concretely, such functors are of the form $cell_A(-)$, namely A-cellular approximation with respect to some group A.

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The functors $cell_A(-)$ had been used fruitfully in the category of groups, topological spaces and chain complexes over rings or DGAs (= differential graded algebras); compare, for example, Dwyer *et al* [DGrI], [RSc, FlR], Shoham (see [Sho]). The present results shade some light on the possible values of the functor $cell_A(-)$ when A is abelian. (We note that very different groups A can give rise to the same functor.) It is possible that the values of all such functors (i.e. $\{cell_A M \mid A \text{ a group}\}$) on a fixed group M yields only a set of results, up to isomorphism. In some topological analogous situations it has been shown that indeed only a set of values occurs (cf. [DP]). We have seen in [FGS1, FGS2] that this is the case when M is a finite group, a finitely generated nilpotent group or a divisible abelian group. One aim of the present paper is to show that *there are abelian groups M for which $\{cell_A M \mid A \text{ an abelian group}\}$ is a proper class of isomorphism types.* This is a consequence of the following.

Theorem 1. *For any infinite cardinality λ, there exists an abelian group M of cardinality λ with $\operatorname{End} M \cong \mathbb{Z}$, such that for any infinite cardinality κ there exists an abelian group K of cardinality κ and with $\operatorname{Hom}(K, M) = 0$ such that K is the kernel of some cellular cover $G \twoheadrightarrow M$.*

Theorem 1 is Theorem 2.11 of §2; its proof relies on Theorem 2.5 which may be of independent interest.

Let $c\colon G \to M$ be a cellular cover. In previous papers we have noticed that G inherits several important properties from M: First the kernel $K = \ker c$ is central in G, that is, G is a central extension of M, and further, if M is nilpotent, then G is nilpotent of the same class; if M is finite then so is G. In addition, we have classified all possible covers of divisible abelian groups ([FGS2, §4]) and showed that when M is abelian the kernel K is reduced and torsion-free ([FGS1, Thm. 4.7]. The case when M is abelian was independently investigated in [BD] and [D]. Amongst other results it was shown there that when M is (abelian and) reduced, K is cotorsion free.

In [FGS1] we have already observed that if M is perfect, and G is the so-called universal central extension of M (so that K is the Schur-multiplier), then $G \twoheadrightarrow M$ is a cellular cover, and, since *any* abelian group is a Schur-multiplier, in general, there is no restriction on the *structure* of K (other than being in the center of G and hence K is abelian).

Note that the covers in Theorem 1 are very special covers in which the only map $K \to M$ from the kernel to M is the zero map. This class of maps are both cellular cover and localization maps. Namely $c\colon G \to M$ is both a cellular cover and a localization. Recall that "c is a localization" means that for any $\varphi \in \operatorname{Hom}(G, M)$ there is a unique *corresponding* $\widetilde{\varphi} \in \operatorname{End}(M)$ such that $c \circ \widetilde{\varphi} = \varphi$. Therefore, this class of localization-cellular maps $V \to W$ have the property that they induce isomorphisms on endomorphism sets: $\operatorname{End} V \cong \operatorname{Hom}(V, W) \cong \operatorname{End} W$.

The kernel K in Theorem 1 cannot be of an arbitrary nature:

Theorem 2. *For any cellular cover $c : G \to M$ (where M is an arbitrary, not necessarily abelian, group), if the kernel K of c is a free abelian group then $|K| \le |M|$.*

In fact, the results in §1 (see Proposition 1.4) are somewhat more general than Theorem 2. We note that [FuG] continues the investigation of cellular covers of abelian groups begun in [FGS2] and in Theorem 1 of this paper, and in particular, further results on "large" cellular covers of "small" abelian groups are obtained there.

1. Free kernels are small

In this section we consider the kernel K of a cellular cover $c\colon G \to M$. We impose some additional "freeness" assumptions on K. We show that under these restrictions the cardinality of G is bounded in terms of the cardinality of M.

Definition 1.1 (Compare with [EMe], p. 90, [Fu], p. 184)**.** Let K be an abelian group and α, β be cardinal numbers such that $\alpha \leq \beta$. We say that K is *weakly-*(α,β)*-separable* iff any subgroup $K_1 \leq K$ of size $\leq \alpha$ is contained in a direct summand $K_2 \leq K$ of size $\leq \beta$. Notice that when $\alpha = \beta$, then our notion coincides with the notion of (weakly) α^+-separable group as in [EMe], p. 90. In this case we will say that K is weakly-α-separable (and not weakly α^+-separable as in [EMe]).

We recall the following well-known fact.

Lemma 1.2. *Let K be a free abelian group. Then K is weakly-α-separable, for every infinite cardinal number α.*

Proof. Let K_1 be a subgroup of K. Of course we may assume that $K_1 \neq 0$. Let $\mathcal{B}$ be a basis of K and for each $x \in K_1$ let $\mathcal{B}_x \subseteq \mathcal{B}$ be a finite subset such that $x \in \langle \mathcal{B}_x \rangle$. Let $K_2 := \langle \mathcal{B}_x \mid x \in K_1 \rangle$. Then $K_1 \leq K_2$, $\mid K_1 \mid = \mid K_2 \mid$, and $K = K_2 \oplus F$, where $F = \langle \mathcal{B} \smallsetminus \bigcup_{x\in K_1} \mathcal{B}_x \rangle$. □

Lemma 1.3. *If G, M are groups and $c \in \mathrm{Hom}(G,M)$ is surjective, then there exists $G_1 \leq G$ such that $\mid G_1 \mid \leq \mid M \mid + \aleph_0$ with $c(G_1) = M$.*

Proof. For each $m \in M$ choose a preimage $g_m \in G$ (i.e. $c(g_m) = m$) and let $G_1 = \langle g_m \mid m \in M \rangle$. □

Proposition 1.4. *Let $c\colon G \to M$ be a cellular cover of the infinite group M and set $K := \ker c$. Let β be a cardinal number such that $\beta \geq \mid M \mid$. Then*

(1) *if K is weakly $(\mid M \mid, \beta)$-separable, then $\mid G \mid \leq \beta$; in particular,*
(2) *if K is a free abelian group, then $\mid G \mid \leq \mid M \mid$.*

Proof. Notice that (2) is an immediate consequence of (1) and Lemma 1.2. It remains to prove (1). Notice that if we restrict the image and consider the map $c\colon G \to c(G)$ we still get a cellular cover. It follows that if $c(G)$ is finite, then G is finite (see [FGS1, Theorem 5.4]). We may thus assume without loss that c is surjective. Let $G_1 \leq G$ be a subgroup such that $c(G_1) = M$ and such that

$$\mid G_1 \mid = \mid M \mid,$$

whose existence is guaranteed by Lemma 1.3 (note that since M is infinite, $|M|+\aleph_0 = |M|$). Since $c(G_1) = M$, we have that

$$G = KG_1.$$

Let $K_1 := G_1 \cap K$; then $| K_1 |\leq| M |$, so by hypothesis there exists a subgroup $K_2 \leq K$ such that $K_1 \leq K_2$, $| K_2 |\leq \beta$ and such that $K = K_2 \times F$, for some $F \leq K$. It is easy to check that it follows that

$$G = (G_1K_2) \times F.$$

In particular, if $F \neq 1$, then, since $F \leq K$, $\mathrm{Hom}(G, K) \neq 0$, a contradiction. Thus $F = 1$, so $G = G_1K_2$ and hence $| G |\leq \beta$. □

2. Cellular covers with large kernels

A. Preliminaries.

Before describing the main construction we introduce some definitions, prove a few lemmas about them and recall an existence result about "large" rigid abelian groups to be used below.

Definitions 2.1. Let A be an abelian group, q a prime and π a set of primes. Then

(1) A is *q-reduced* if $\bigcap_{i=1}^{\infty} q^i A = 0$.
(2) A is π-reduced if A is p-reduced, for all $p \in \pi$.
(3) An element $a \in A$ is q-pure (in A) if a is not divisible by q in A.
(4) A is q-divisible if each element $a \in A$ is divisible by q in L.
(5) An integer n is a π-number, if n is divisible only by primes from π (1 and -1 are always π-numbers).
(6) A torsion element $a \in A$ is a π-element if the order of a is a π-number (or $a = 0$).
(7) A is a π-group, if each element of A is a π-element.
(8) $\mathbb{Z}[1/\pi] := \mathbb{Z}[1/p \mid p \in \pi]$ (and if $\pi = \emptyset$, then $\mathbb{Z}[1/\pi] = \mathbb{Z}$).

Remarks 2.2 (Tensor products, see [Fu])**.** (1) Let A be a torsion free abelian group. Then $V := \mathbb{Q} \otimes A$ is a vector space over $\mathbb{Q}$ which contains a copy of A. Thus we always think of A as being contained in a vector space V over $\mathbb{Q}$ such that V/A is a torsion abelian group. Hence it makes sense to talk about the group $\langle A \cup \{\frac{a_i}{m_i} \mid i \in I\}\rangle$ where I is an index set, $\{a_i \mid i \in I\} \subseteq A$ and $\{m_i \mid i \in I\} \subseteq \mathbb{Z} \setminus \{0\}$. This is the subgroup of V generated by $A \cup \{\frac{a_i}{m_i} \mid i \in I\}$.

(2) Note that if $S \subseteq V$ and π is a set of primes such that for each $s \in S$ there exists a π-number n with $ns \in A$, then $\langle A \cup S\rangle/A$ is a π-group. In particular, for a subring $R \subseteq Q$ we view $R \otimes A$ as a subgroup of V and if $R = \mathbb{Z}[1/\pi]$, then $(R \otimes A)/A$ is a π-group.

(3) Note further that if π_1 and π_2 are disjoint sets of primes and $B \subseteq V$ is a subgroup containing A such that A is π_1-reduced and B/A is a π_2-group, then B is π_1 reduced.

Notation 2.3. Let L be a torsion free abelian group and let q be a prime. Let $0 \neq x \in L$ we denote, using Remark 2.2(1),

$$L \oplus_x \mathbb{Z}[1/q] := \langle L \cup \{\tfrac{x}{q^i} \mid 1 \leq i \in \mathbb{Z}\}\rangle.$$

We write $H = x\mathbb{Z}[1/q]$ for the subgroup of $L \oplus_x \mathbb{Z}[1/q]$ consisting of the elements

$$H := \{\tfrac{m}{q^i}x \mid m \in \mathbb{Z} \text{ and } 1 \leq i \in \mathbb{Z}\}.$$

Remark 2.4. Assume L is a torsion free abelian group, q is a prime and $0 \neq x \in L$ is a q-pure element. Then

$$L \oplus_x \mathbb{Z}[1/q] \cong (L \oplus \mathbb{Z}[1/q])/\langle(-x, 1)\rangle.$$

Furthermore, let $\widehat{M}$ be a group such that $\widehat{M} = L \oplus H$ where L, H are subgroups of $\widehat{M}$, L is torsion free and H is isomorphic to $\mathbb{Z}[1/q]$ under an isomorphism taking some $0 \neq h \in H$ to 1. Let $0 \neq y \in L$ be a q-pure element and let $M := \widehat{M}/\langle y - h\rangle$. Then M is isomorphic to the group $L \oplus_y \mathbb{Z}[1/q]$ constructed in Notation 2.3.

B. Existence of large rigid groups.

The following is our main stepping stone for proving the existence of covers with arbitrarily large kernels.

Theorem 2.5. *Let P be a set of at least four primes, Q its complementary set of primes and λ any infinite cardinal. Then there is a torsion-free abelian group H of cardinality λ with the following three properties.*

1. *H is Q-reduced;*
2. *if $Q_0 \subseteq Q$ is a set of primes and A is a torsion free abelian group containing H such that A/H is a Q_0-group, then* $\mathrm{End}(A) \subseteq \mathbb{Z}[1/Q_0]$*;*
3. *H contains a free abelian group F of cardinality λ such that H/F is a P-group.*

Proof. Let $R := \mathbb{Z}[1/Q]$. By [Sh, Thm. 2.1] (see also [GT, Corollary 14.5.3(b), p. 577]), there exists an R-module M of cardinality λ such that $\mathrm{End}(M) = R$. Let $\mathcal{B}$ be a maximal ($\mathbb{Z}$)-independent subset of M. We let

$$F := \langle \mathcal{B}\rangle \text{ and } H := \{x \in M \mid \text{there exists a } P\text{-number } n \in \mathbb{Z} \text{ with } nx \in F\}.$$

We claim that H satisfies all the required properties. By construction (3) holds. Also, since F is a free abelian group and since H/F is a P-group, H is Q-reduced (see Remark 2.2(3)), so (1) holds.

We now show (2). By construction, M/H is a Q-group, so $R \otimes H = M$. Thus for any group $H \subseteq A \subseteq M$, $R \otimes A = M$. Let A be as in (2). Then $H \subseteq A \subseteq R \otimes A = M$, and since $R \otimes A = M$, it follows that any endomorphism of A extends to an endomorphism of M, thus $\mathrm{End}(A) \subseteq R$. Let now $Q_0 \subseteq Q$ and suppose that A/H is a Q_0-group. Let $f \in \mathrm{End}(A)$ so that f is multiplication by $\frac{m}{n}$, where $\gcd(m, n) = 1$ and n is a Q-number. Assume there exists a prime $q \in Q \setminus Q_0$ such that $q \mid n$. Then, after multiplying by an appropriate integer, we may assume that $n = q$. Writing $1 = \alpha q + \beta m$, with $\alpha, \beta \in \mathbb{Z}$, we see that $\frac{1}{q} = \alpha + \beta \frac{m}{q}$, so multiplication by $\frac{1}{q}$ is an endomorphism of A. However, $q \notin Q_0$, H is Q-reduced and A/H is a Q_0-group, so Remark 2.2(3) implies that A is q-reduced. This is a contradiction. Thus n is a Q_0-number, so $\mathrm{End}(A) \subseteq \mathbb{Z}[1/Q_0]$ and (2) holds.

$\square$

Remark. The set of primes P in Theorem 2.5 is the set of primes that are used to construct the $\mathbb{Z}[1/Q]$-module M as in the begining of the proof of the theorem. Thus we only work with the complimentary set of primes Q when using the theorem to construct groups L that have some desirable properties. Below we will fix the set Q of primes which will be used for our constructions (in fact we only need 3 primes in Q, see Corollary 2.6 below). The set P will be the complimentary set of primes.

The variant of Theorem 2.5 which we actually use in subsection C below is the following Corollary.

Corollary 2.6. *Let λ be any infinite cardinal and let $Q := \{q_L, q_K, q\}$ be a set consisting of three primes. Then there exists an abelian group L whose cardinality is λ such that*

(1) *L is torsion free and q_L-divisible;*
(2) *L is $Q \setminus \{q_L\}$-reduced;*
(3) *if $M \supseteq L$ is a torsion free abelian group such that M/L is a q-group, then $\mathrm{End}(M) \subseteq \mathbb{Z}[1/\{q_L, q\}]$.*
(4) *there exists a q-pure element $x_L \in L$ such that for $M := L \oplus_{x_L} \mathbb{Z}[1/q]$ we have $\bigcap_{i=1}^{\infty} q^i M = x\mathbb{Z}[1/q]$.*

Proof. We use Theorem 2.5 with Q playing the role of Q in that theorem. Let H be as in Theorem 2.5, let $R = \mathbb{Z}[1/q_L]$ and let $L := R \otimes H$. Notice that by Remark 2.2(3), L is $Q \setminus \{q_L\}$ reduced. Of course L is q_L-divisible.

Next if $M \supseteq L$ is a torsion free abelian group such that M/L is a q-group, then, by construction, M/H is a $\{q_L, q\}$-group, so (3) follows from Theorem 2.5(2).

To prove (4) let F be as in part (3) of Theorem 2.5. Let $\mathcal{B} \subseteq F$ be a free generating set of F, pick $x_L \in \mathcal{B}$ and set $x := X_L$. Clearly x is q-pure. Assume (4) is false and write $U := \bigcap_{i=1}^{\infty} q^i (L \oplus_x \mathbb{Z}[1/q])$. Since $x\mathbb{Z}[1/q] \subseteq U$, there exists $\ell \in L \setminus \langle x \rangle$ such that $\ell \in U$. But then writing $\ell = \sum_{i=1}^{t} \alpha_i x_i$, with $\alpha_i \in \mathbb{Z}[1/(P \cup \{q_L\})]$, $x_i \in \mathcal{B}$ and $x_1 \neq x$, we see that there exists $0 < j \in \mathbb{Z}$ such that q^j does not divide $\alpha_1 x_1$, and hence q^j does not divide $\ell + sx$, for any $s \in \mathbb{Z}$, and this contradicts the fact that $\ell \in U$. $\square$

C. Constructing covers with arbitrarily large kernels.

In this section we use Corollary 2.6 above to construct an abelian group M and, for arbitrarily large cardinal κ, a cellular cover $G \to M$ whose kernel K has cardinality κ. The group M will be as in Corollary 2.6(4). Lemma 2.8 below describes the nice properties of such a group M.

We start with a very simple lemma that allows us to conclude that the canonical homomorphism $G \to G/K$ from the abelian group G to the factor group G/K is a cellular cover. The rest of the section is devoted to building arbitrarily large groups K satisfying the conditions of the lemma (while G/K remains fixed).

Lemma 2.7. *Let G be an abelian group and $K \leq G$ be a subgroup. Set $M := G/K$ and let $c\colon G \to G/K$ be the canonical homomorphism. Assume that*

(i) $\mathrm{End}(M) \cong \mathbb{Z}$;
(ii) *K is a fully invariant subgroup of G;*
(iii) $\mathrm{Hom}(K, M) = 0 = \mathrm{Hom}(G, K)$.

Then $\mathrm{End}(G) = \mathbb{Z}$ *and c is a cellular cover.*

Proof. Let $\mu \in \mathrm{End}(G)$. By (ii), $\mu(K) \leq K$ so μ induces $\hat{\mu} \in \mathrm{End}(M)$ defined by $\hat{\mu}(g + K) = \mu(g) + K$. By (i), there exists $n \in \mathbb{Z}$ such that $\hat{\mu}$ is multiplication by n. Thus the map $g \to (\mu(g) - ng)$ is in $\mathrm{Hom}(G, K)$, so by (iii) it is the zero map and it follows that μ is multiplication by n. This shows that $\mathrm{End}(G) \cong \mathbb{Z}$.

Let now $\varphi \in \mathrm{Hom}(G, M)$. Then by (iii), $\varphi(K) = 0$, so φ induces $\hat{\varphi} \in \mathrm{End}(M)$ defined by $\hat{\varphi}(g + K) = \varphi(g)$. Thus by (i) there is $n \in \mathbb{Z}$ such that $\varphi(g) = ng + K$, for all $g \in G$. Consequently, the map $\check{\varphi} \in \mathrm{End}(G)$ defined by $\check{\varphi}(g) = ng$ lifts φ, so any $\varphi \in \mathrm{Hom}(G, M)$ lifts. Since $\mathrm{Hom}(G, K) = 0$, [FGS1, Lemma 3.6] shows that c is a cellular cover. □

Lemma 2.8. *Let $Q := \{q_L, q_K, q\}$ be a set consisting of three primes, and let L be an abelian group satisfying (1)–(4) of Corollary 2.6. Let $x_L \in L$ be a q-pure element as in (4) of Corollary 2.6, and set $M = L \oplus_{x_L} \mathbb{Z}[1/q]$. Then M is torsion free, it is q_K-reduced and* $\mathrm{End}(M) \cong \mathbb{Z}$.

Proof. That M is torsion free is by construction. By Remark 2.2(3), M is q_K-reduced.

Recall that by (4) of Corollary 2.6,

$$(*) \qquad H = \bigcap_{i=1}^{\infty} q^i M,$$

where $H = x\mathbb{Z}[1/q]$ is as in Notation 2.3.

Let $\varphi \in \mathrm{End}(M)$. Since M/L is a q-group part (3) of Corollary 2.6 implies that there exists $\frac{m}{n} \in \mathbb{Q}$, with $\gcd(m, n) = 1$ such that $n \geq 1$ is a $\{q_L, q\}$-number and such that $\varphi(x) = \frac{m}{n}x$, for all $x \in M$. Suppose $n \neq 1$ and let $p \in \{q_L, q\}$ such that $p \mid n$. Since $\frac{m}{n}x \in M$, for all $x \in M$ also $\frac{m}{p}x \in M$, for all $x \in M$ and then writing $1 = \alpha m + \beta p$, $\alpha, \beta \in \mathbb{Z}$ we see that $\frac{1}{p}x = \frac{\alpha m}{p}x + \beta x \in M$. Thus M is p-divisible. Now if $p = q$, then (*) implies that L is not q-divisible, a contradiction. If $p = q_L$, then, since by (*) H is a fully invariant subgroup of M, it follows that H is q_L-divisible (because multiplication by $1/q_L$ is an endomorphism of M). But of course H is not q_L divisible. Thus $n = 1$ and this completes the proof of the lemma. □

Lemma 2.9. *Let G be an abelian group containing subgroups K and $\widehat{M}$ such that $G = K + \widehat{M}$. Set $M := G/K$ and let $c\colon G \to M$ be the canonical homomorphism. Assume that*

(i) *K is a torsion free fully invariant subgroup of G;*
(ii) *K is an R-module for some subring $R \subset \mathbb{Q}$ and $\mathrm{End}(K) = R$;*
(iii) *M is torsion free and $\mathrm{End}(M) \cong \mathbb{Z}$;*
(iv) *$\mathrm{Hom}(\widehat{M}, K) = 0 = \mathrm{Hom}(K, M)$;*
(v) *$K \cap \widehat{M} \neq 0$.*

Then $\mathrm{End}(G) \cong \mathbb{Z}$ and c is a cellular cover.

Proof. We use Lemma 2.7. It only remains to show that $\mathrm{Hom}(G, K) = 0$. Let $\mu \in \mathrm{Hom}(G, K)$. By hypothesis (iv), $\mu(\widehat{M}) = 0$. By hypothesis (i), $\mu(K) \leq K$, so by hypothesis (ii) there exists $r \in R$ such that $\mu(v) = rv$, for all $v \in K$. Let $0 \neq v \in \widehat{M} \cap K$. Then $rv = \mu(v) = 0$, so since K is torsion free, $r = 0$, and it follows that $\mu(K) = 0$ and then $\mu = 0$. $\square$

Proposition 2.10. *Let $Q := \{q_L, q_K, q\}$ be a set consisting of three primes. Let K and L be abelian groups and assume that*

(i) *K is torsion free, it is q_K-divisible and $Q \setminus \{q_K\}$-reduced.*
(ii) *L and the element $x_L \in L$ satisfy (1)–(4) of Corollary 2.6*

Let $0 \neq x_K \in K$ be an arbitrary element, and let

$$G = (K \oplus L) \oplus_{(x_K - x_L)} \mathbb{Z}[1/q]$$

be the group constructed in Notation 2.3, with $K \oplus L$ in place of L and $x_K - x_L$ in place of x. Set

$$H := (x_K - x_L)\mathbb{Z}[1/q], \qquad \textit{and} \qquad \widehat{M} = L + H.$$

Then G, K and $\widehat{M}$ satisfy all the hypotheses of Lemma 2.9. In particular, the canonical homomorphism $c\colon G \to G/K$ is a cellular cover.

Proof. Clearly $G = K + \widehat{M}$. Now since $(K + L) \cap H = \langle x_K - x_L \rangle$, it is easy to check that

$$\text{(I)} \qquad K \cap \widehat{M} = \langle x_K \rangle.$$

Note that $L \cap H = 0$, because if $g := n(x_K - x_L)/q^i \in L$, then $n(x_K - x_L) \in L$, which implies that $nx_K \in L$. But K is torsion free and $K \cap L = 0$, so $n = 0$ and then $g = 0$. Thus $\widehat{M} = L \oplus H$, also $x_K = x_L + (x_K - x_L)$ and $H \cong \mathbb{Z}[1/q]$ by an isomorphism sending $(x_K - x_L) \to 1$, so by (I) and Remark 2.4, $M \cong \widehat{M}/\langle x_K \rangle \cong L \oplus_{x_L} \mathbb{Z}[1/q]$. From (ii) and Lemma 2.8 it follows that

$$\text{(II)} \qquad M \text{ is torsion free, } M \text{ is } q_K\text{-reduced and } \mathrm{End}(M) = \mathbb{Z}.$$

Since K is q_K-divisible, we conclude that

$$\text{(III)} \qquad \mathrm{Hom}(K, M) = 0,$$

and also, since M is q_K-reduced, we have: $\bigcap_{i=0}^{\infty} q_K^i G = K$, so

(IV) $$K \text{ is a fully invariant subgroup of } G.$$

Next, since L is q_L-divisible and K is q_L-reduced, $\mathrm{Hom}(L, K) = 0$. Similarly, since H is q-divisible, $\mathrm{Hom}(H, K) = 0$. Hence

(V) $$\mathrm{Hom}(\widehat{M}, K) = 0.$$

Thus all hypotheses of Lemma 2.9 have been verified. □

As a Corollary to Proposition 2.10 we get Theorem 1 of the introduction.

Theorem 2.11. *Let λ be any infinite cardinal. There exists an abelian group M of cardinality λ such that for any infinite cardinal $\kappa \geq \lambda$ there exists a cellular cover $c\colon G \to M$ with $|\ker c| = \kappa$.*

Proof. Corollary 2.6 guarantees the existence of groups L and K of cardinality λ and κ respectively, and primes q_K, q_L and q satisfying all hypotheses of Proposition 2.10. Let K and G be as in Proposition 2.10 and set $M := G/K$. By Proposition 2.10, $c\colon G \to M$ is a cellular cover and of course $|K| = \kappa$ and $|M| = \lambda$. Notice that we saw in the proof of Proposition 2.10 that $M \cong L \oplus_{x_L} \mathbb{Z}[1/q]$, so the structure of M is independent of the choice of K. □

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