# Absolutely Indecomposable Modules

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**Abstract**

A module is called *absolutely indecomposable* if it is directly indecomposable in every generic extension of the universe. We want to show the existence of large abelian groups that are absolutely indecomposable. This will follow from a more general result about $R$-modules over a large class of commutative rings $R$ with endomorphism ring $R$ which remains the same when passing to a generic extension of the universe. It turns out that 'large' in this context has a *precise meaning,* namely being smaller than the first $\omega$-Erdős cardinal defined below. We will first apply a result on large rigid valuated trees with a similar property established by Shelah [26] in 1982, and will prove the existence of related '$R_\omega$-modules' ($R$-modules with countably many distinguished submodules) and finally pass to $R$-modules. The passage through $R_\omega$-modules has the great advantage that the proofs become very transparent essentially using a few 'linear algebra' arguments accessible also for graduate students. The result closes a gap in [12, 11], provides a good starting point for [16] and gives a new construction of indecomposable modules in general using a counting argument.

## 1 Introduction

There is a whole industry transporting symmetry properties from one category to another: For example consider a tree or a graph (with extra properties if needed) together with its group of automorphisms. Then encode the tree or the graph into an object of your favored category in such a way that the branches (or vertices) of the tree (of the graph) are recognized in the new structure. If the new category are abelian group argue by (infinite) divisibility, in case of groups and fields you use of course infinite chains of roots (with legal primes) etc. Thus the automorphism group of the tree or the graph is respected in the new category and by density arguments (or killing unwanted automorphisms by prediction arguments 'on the way') it happens that the automorphism group we start with becomes (modulo inessential maps: inner automorphisms in case of groups and Frobenius automorphisms in case of fields) the automorphism group of an object of the new category. For a few illustrating details the reader may want to see papers by Heineken [22], Braun, Göbel [2] (in case of groups), Corner, Göbel [7] in case of modules (with group rings as the first category) or Fried and Kollar [14], Dugas, Göbel [9] in case of fields and [10] for automorphism groups of geometric lattices. In this paper we also argue with symmetry properties of trees, but they are of a different kind. Given a cardinal $\lambda$ which is

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not extraordinary large (we explain what we mean by 'extraordinary large' in the next section) then there is an absolute and rigid family of ($\omega$-)valuated trees based on this cardinal. This is a family of $\lambda$ subtrees $T$ of size $\lambda$ of the tree $T_\lambda = {}^{\omega>}\lambda$ of finite sequences of ordinals in $\lambda$ together with a valuation map $v : T \longrightarrow \omega$. Rigid means that there is no level preserving valuated homomorphisms between any two distinct members. (A tree homomorphism is valuated if the value of a branch is the same as the value of its image.) Moreover this property is preserved if we change the universe, passing to a generic extension of the given universe (of set theory) we live in. The existence of such trees was shown by Shelah [26]. These trees (used also in applied mathematics) were considered earlier in papers by Nash-Williams, see [24] for example. We will encode them into free $R_\omega$-modules over an arbitrary not extraordinary large commutative ring $R$ with $1 \neq 0$. To be definite we can assume that $R$ is the field $\mathbb{Q}$ of rationals or $\mathbb{Z}$. Recall that $R_\omega$-modules are $R$-modules with countably many ($\omega$) distinguished submodules and free means that the module and its distinguished submodules and factor modules are free as well. Such creatures are considered in Brenner, Butler, Corner (see [3, 4, 5, 1]) and Göbel, May [18] for arbitrary commutative rings and an account about the advanced theory in case of fields can be seen in [27] and in the references given there. We will show the existence of free $R_\omega$-modules with endomorphism algebra $R$ by transporting the absolute rigid trees into the category of $R_\omega$-modules. It turns out that the passage through $R_\omega$-modules makes the anticipated proofs very transparent. Moreover our main result on $R_\omega$-modules with distinguished submodules is only a few steps away from the desired result on $R$-modules if $R$ has enough primes (like $\mathbb{Z}$).

The corollary on the existence of large absolutely (fully) rigid abelian groups replaces the earlier unsuccessful approach in [12] and [11, Chapter XV]: Let $R \neq 0$ be any fixed countable ring. Then by Corollary 4.2 there exists an absolutely rigid $R_\omega$-module of size $\lambda$ (or an absolute family of size $\lambda$ of non-trivial $R$-modules with only the zero-homomorphism between distinct member) iff $\lambda < \kappa(\omega)$. The same holds if $R_\omega$-modules are replaced by abelian groups. Thus as a byproduct we present a new construction of large, absolutely indecomposable abelian groups, not using stationary sets as [25, 7]. So, if we restrict to the problem on the existence of large absolute indecomposable abelian groups addressed in [12, 11], then it follows from the above (realizing for example $\mathbb{Z}$ as the endomorphism ring in Corollary 4.2) that from $\lambda < \kappa(\omega)$ follows the existence of such abelian groups. The converse direction would need a strengthening of the Theorem 2.2 from [12] now showing the existence of non-trivial idempotents. (The second author believes that this guess might be true.)

It is also a different matter how to replace $R_\omega$-modules by $R_4$-modules or $R_5$-modules and the endomorphism algebra $R$ by a general not extraordinary large prescribed $R$-algebra $A$. This will follow from [16], a paper which had to wait for Theorem 4.1 in place of [12].

## 2 Rigid families of valuated trees and the first $\omega$–Erdős cardinal

We first describe the result on trees we want to apply by encoding them into modules with distinguished submodules.

Let $\kappa(\omega)$ denote the first *$\omega$-Erdős cardinal*. This is defined as the smallest cardinal $\kappa$ such that $\kappa \to (\omega)^{<\omega}$, i.e. for every function $f$ from the finite subsets of $\kappa$ to 2 there exist an infinite subset $X \subset \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ for all finite subsets $Y$ of $X$. The cardinal

$\kappa(\omega)$ is strongly inaccessible; see Jech [23, p. 392]. Thus $\kappa(\omega)$ is a large cardinal. We should also emphasize that $\kappa(\omega)$ may not exist in every universe.

If $\lambda < \kappa(\omega)$, then let

$$T_\lambda = {}^{\omega>}\lambda = \{f : n \longrightarrow \lambda : \text{ with } n < \omega \text{ and } n = \operatorname{Dom} f\}$$

be the tree of all finite sequences $f$ (of length or level $\lg(f) = n$) in $\lambda$. Since $n = \{0, \dots n-1\}$ as ordinal, we also write $f = f(0)^\wedge f(1)^\wedge \dots {}^\wedge f(n-1)$. By restriction $g = f \restriction m$ for any $m \leq n$ we obtain all *initial segments* of $f$. We will write $g \lhd f$. Thus

$$g \leq f \iff g \subseteq f \text{ as graphs} \iff g \lhd f.$$

A subtree $T$ of $T_\lambda$ is a subset which is closed under initial segments and a homomorphism between two subtrees of $T_\lambda$ is a map that preserves levels and initial segments. (Note that a homomorphism does not need to be injective or preserve $\not\leq$.) The tree $T$ is valuated if with the tree we have a valuation map $v : T \longrightarrow \omega$. In the following a tree will always come with a valuation and $\operatorname{Hom}(T_1, T_2)$ denotes the valuated homomorphisms between subtrees $T_1$ and $T_2$, i.e if $v_i$ is the valuation of $T_i$ $(i = 1, 2)$ and $\varphi$ is such a *valuated homomorphism*, then $v_2(\eta\varphi) = v_1(\eta)$ for all $\eta \in T_1$. Shelah [26] showed the existence of an *absolutely rigid* family of $2^\lambda$ valuated subtrees of $T_\lambda$.

**Theorem 2.1** *If $\lambda < \kappa(\omega)$ is infinite and $T_\lambda = {}^{\omega>}\lambda$, then there is a family $T_\alpha$ $(\alpha \in 2^\lambda)$ of valuated subtrees of $T_\lambda$ (of size $\lambda$) such that for $\alpha, \beta \in 2^\lambda$ and in any generic extension of the universe the following holds.*

$$\operatorname{Hom}(T_\alpha, T_\beta) \neq \emptyset \Longrightarrow \alpha = \beta.$$

*Proof.* The result is a consequence of the Main Theorem 5.3 in [26, p. 208]. The family of rigid trees is constructed in [26, p. 214, Theorem 5.7] and the proof, that the trees are rigid, follows from Theorem 5.8 using the Conclusion 2.14 in [26]. In Shelah's notation $\kappa(\omega)$ is the first beautiful cardinal $> \aleph_0$. ■

This property of rigid families of valuated trees in Theorem 2.1 fails, if we choose $\lambda \geq \kappa(\omega)$. In fact the following result from [12] on rigid families of $R$-modules reflects this.

**Theorem 2.2** (Eklof-Shelah [12]) Let $\lambda$ be a cardinal $\geq \kappa(\omega)$ and $R$ any ring with 1.

(i) If $\{M_\nu \mid \nu < \lambda\}$ is a family of non-zero left $R$-modules, then there are distinct ordinals $\mu, \nu < \lambda$, such that in some generic extension $V[G]$ of the universe $V$, there is an injective homomorphism $\phi : M_\mu \to M_\nu$.

(ii) If $M$ is an $R$-module of cardinality $\lambda$, then there exists a generic extension $V[G]$ of the universe $V$, such that $M$ has an endomorphism that is not multiplication by an element of $R$.

Thus $\kappa(\omega)$ is the precise border line for Theorem 3.1 and we can not expect absolute results on endomorphism rings and rigid families of abelian groups above $\kappa(\omega)$, see Corollary 4.2.

Combining Theorem 2.2 with our main result this also conversely shows that the implication of Theorem 2.1 fails whenever $\lambda \geq \kappa(\omega)$, i.e. there is a generic extension $V[G]$ of the universe $V$ and there are distinct ordinals $\alpha, \beta \in 2^\lambda$ with $\operatorname{Hom}(T_\alpha, T_\beta) \neq \emptyset$.

# 3 The main construction

Let $R \neq 0$ be a commutative ring. As we shall write endomorphisms on the right, it will be convenient to view $R$-modules as left $R$-modules. Next we define a free $R$-module $F$ of rank $\lambda$ over a suitable indexing set (obviously) used to encode trees $T_\alpha$ from Theorem 2.1 into the structure when turning the free $R$-module $F$ into an $R_\omega$-module module $\mathbf{F}$ with $\omega$ distinguished submodules.

We enumerate a subfamily of $\lambda$ valuated trees from Theorem 2.1 by the indexing set $I = {}^{\omega>}({}^{\omega>}\lambda)$. Thus

$$T_{\overline{\eta}} \text{ with valuation map } \ v_{\overline{\eta}} : T_{\overline{\eta}} \longrightarrow \omega \quad (\overline{\eta} \in I)$$

without repetition. Next define inductively subsets $S_n \subseteq {}^n({}^{\omega>}\lambda)$ such that the following holds.

(0) $S_0 = \{\perp\}$

(1) If $S_n$ is defined, then $S_{n+1} = \{\overline{\eta}^\wedge\langle\nu\rangle : \overline{\eta} \in S_n, \perp \neq \nu \in T_{\overline{\eta}}\}$.

Let $S = \bigcup\limits_{n\in\omega} S_n$ and also let $\overline{\eta}^\wedge\langle\perp\rangle = \overline{\eta}$ for $\perp \in T_{\overline{\eta}}$.

Put $S_{nk} = \{\overline{\eta}^\wedge\langle\nu\rangle \in S : \ \lg\overline{\eta} = n, \lg\nu = k\} \subseteq S_{n+1}$. Here $\nu = \nu_0{}^\wedge \dots {}^\wedge\nu_{k-1}$ with $\nu_i \in \ \lambda$ is a sequence of ordinals and $\overline{\eta} = \overline{\eta}_0{}^\wedge \dots {}^\wedge\overline{\eta}_{n-1}$ with $\overline{\eta}_i \in T_{\overline{\eta}_0{}^\wedge\dots{}^\wedge\overline{\eta}_{i-1}}$ a sequence of branches from special trees. Moreover write

$$T^k_{\overline{\eta}} = \{\nu \in T_{\overline{\eta}} : \ \lg\nu = k\} \subseteq T_{\overline{\eta}} \text{ and } T_{k\overline{\eta}} = \{\nu \in T_{\overline{\eta}} : \ v_{\overline{\eta}}(\nu) = k\} \quad (\overline{\eta} \in I).$$

Now we define the free $R$-modules:

(i) $F = \bigoplus\limits_{\overline{\eta}\in S} Re_{\overline{\eta}}$

(ii) $F_{nk} = \bigoplus\limits_{\overline{\eta}\in S_n} \bigoplus\limits_{\nu\in T^k_{\overline{\eta}}} R(e_{\overline{\eta}^\wedge\langle\nu\restriction k-1\rangle} - e_{\overline{\eta}^\wedge\langle\nu\rangle})$

(iii) $F^{nk} = \bigoplus\limits_{\overline{\eta}\in S_n} (\bigoplus\limits_{\nu\in T^k_{\overline{\eta}}} Re_{\overline{\eta}^\wedge\langle\nu\rangle})$

(iv) $F^k_n = \bigoplus\limits_{\overline{\eta}\in S_n} (\bigoplus\limits_{\nu\in T_{k\overline{\eta}}} Re_{\overline{\eta}^\wedge\langle\nu\rangle})$

(v) $F_0 = \langle R(e_{\overline{\eta}} \ - e_{\overline{\eta}'}) : \ \overline{\eta}, \overline{\eta}' \in S\rangle$ and $F_1 = Re_\perp$.

We note that $F_0 = \bigoplus\limits_{\perp\neq\overline{\eta}\in S} R(e_\perp - e_{\overline{\eta}})$ and $F = F_0 \oplus F_1$.

Next we define $R_\omega$-modules. These are $R$-modules with $\omega$ distinguished submodules. We enumerate the distinguished submodules by a particular well-ordered, countable indexing set

$$W = \langle 0, 1\rangle^\wedge L_1{}^\wedge L_2{}^\wedge L_3 \text{ with } L_i \text{ a copy of } \omega\times\omega \quad (i = 1, 2, 3).$$

We view $W$ as an ordinal. Then an *$R_\omega$-module* $\mathbf{X}$ is an $R$-module $X$ with a family of submodules $X_i$ $(i \in W)$. We will also say that $\mathbf{X}$ is *a free $R_\omega$-module* if $X, X_i, X/X_i$ $(i \in W)$ are free $R$-modules. In particular

$$\mathbf{F} = (F, F_0, F_1, F_{nk}, F^{pq}, F^r_s : (nk) \in L_1, (pq) \in L_2), (rs) \in L_3) \text{ is a free } R_\omega - \text{module.} \tag{3.1}$$

If $\mathbf{X}, \mathbf{Y}$ are $R_\omega$-modules, then $\varphi$ is an $R_\omega$-homomorphism ($\varphi \in \mathbf{Hom}_R(\mathbf{X}, \mathbf{Y})$) if $\varphi \in \mathrm{Hom}_R(X, Y)$ and $X_i\varphi \subseteq Y_i$ for all $i \in W$, where $\mathbf{Y} = (Y, Y_i : i \in W)$. We also write $\mathbf{Hom}_R(\mathbf{X}, \mathbf{X}) = \mathbf{End}_R\mathbf{X}$.

We want to show the following

**Theorem 3.1** *Let $R$ be a commutative ring with $1 \neq 0$ and $|R|, \lambda < \kappa(\omega)$. A free $R$-module $F$ of rank $\lambda$ can be made into a free $R_\omega$-module $\mathbf{F} = (F, F_i : i \in W)$ such that $\mathbf{End}_R\mathbf{F} = R$ holds in any generic extension of the given universe.*

Note that the size of $R$ and the rank $\lambda$ can be arbitrary $< \kappa(\omega)$; in particular $R = \mathbb{Z}/2\mathbb{Z}$. If $\lambda$ is finite, then we can choose directly a suitable finite family of $F_i$s with the required endomorphism ring. Otherwise $\lambda$ is infinite and we can apply Theorem 2.1. So we choose $\mathbf{F} = (F, F_i : i \in W)$ as in (3.1) depending on the valuated trees from Theorem 2.1. Then clearly it remains to show $\mathbf{End}_R\mathbf{F} = R$. We first show the following crucial

**Lemma 3.2** *Let $\varphi \in \mathbf{End}_R\mathbf{F}$ with $\mathbf{F}$ as in (3.1) and $F = \bigoplus_{\overline{\eta} \in S} Re_{\overline{\eta}}$. If $\overline{\eta} \in S$, then*

$$e_{\overline{\eta}}\varphi \in Re_{\overline{\eta}}.$$

*Proof.* Let $\overline{\eta} \in S$ be fixed and recall that $T^k_{\overline{\eta}} = T_{\overline{\eta}} \cap {}^k\lambda$. We consider its successors $\overline{\eta}^\wedge\langle\nu\rangle$ in $S$ with $\perp \neq \nu \in T_{\overline{\eta}}$ and let $\lg \overline{\eta} = n, \lg \nu = k$. Thus $\overline{\eta}^\wedge\langle\nu\rangle \in S_{nk}$ and $\nu \in T^k_{\overline{\eta}}$. If $\varphi \in \mathbf{End}_R\mathbf{F}$, then we claim

$$e_{\overline{\eta}^\wedge\langle\nu\rangle}\varphi = \sum_{l<l_\nu} r_{\nu l} e_{\overline{\rho}_{\nu l}{}^\wedge\langle\sigma_{\nu l}\rangle} \text{ with } \overline{\rho}_{\nu l} \in S_n,\ \sigma_{\nu l} \in T^k_{\overline{\rho}_{\nu l}} \text{ and } 0 \neq r_{\nu l} \in R. \tag{3.2}$$

If $e_{\overline{\eta}^\wedge\langle\nu\rangle}\varphi = 0$, we choose $l_\nu = 0$ and have the empty sum which is 0. By definition of $F^{nk}$ follows $e_{\overline{\eta}^\wedge\langle\nu\rangle} \in F^{nk}$, thus $e_{\overline{\eta}^\wedge\langle\nu\rangle}\varphi \in F^{nk}$ showing that $e_{\overline{\eta}^\wedge\langle\nu\rangle}\varphi$ is of the desired form (3.2).

We will now use $\mathbf{F}$ to derive further restrictions of the expressions in (3.2).

If $\nu_1 \in T^{k+1}_{\overline{\eta}}$, then $\nu_0 = \nu_1 \restriction k \in T^k_{\overline{\eta}}$ and $e_{\overline{\eta}^\wedge\langle\nu_0\rangle} - e_{\overline{\eta}^\wedge\langle\nu_1\rangle} \in F_{n\ k+1}$ hence $w := (e_{\overline{\eta}^\wedge\langle\nu_0\rangle} - e_{\overline{\eta}^\wedge\langle\nu_1\rangle})\varphi \in F_{n\ k+1}$ as well. Using (3.2) and the definition of $F_{n\ k+1}$ we get

$$w = \sum_{l<l_{\nu_0}} r_{\nu_0 l} e_{\overline{\rho}_{\nu_0 l}{}^\wedge\langle\sigma_{\nu_0 l}\rangle} - \sum_{l<l_{\nu_1}} r_{\nu_1 l} e_{\overline{\rho}_{\nu_1 l}{}^\wedge\langle\sigma_{\nu_1 l}\rangle} = \sum_{i<l_w} s_{wi} \left(e_{\overline{\rho}_{wi}{}^\wedge\langle\nu_{wi}\restriction\ k\rangle} - e_{\overline{\rho}_{wi}{}^\wedge\langle\nu_{wi}\rangle}\right)$$

with $\overline{\rho}_{wi} \in S_n,\ \nu_{wi} \in T^{k+1}_{\overline{\rho}_{wi}}$ and $0 \neq s_{wi} \in R$.

Now we collect terms of length $k$ and $k+1$ respectively, and it follows

$$\text{length k:} \quad \sum_{l<l_{\nu_0}} r_{\nu_0 l} e_{\overline{\rho}_{\nu_0 l}{}^\wedge\langle\sigma_{\nu_0 l}\rangle} = \sum_{i<l_w} s_{wi} e_{\overline{\rho}_{wi}{}^\wedge\langle\nu_{wi}\restriction\ k\rangle}$$

$$\text{length k+1:} \quad \sum_{l<l_{\nu_1}} r_{\nu_1 l} e_{\overline{\rho}_{\nu_1 l}{}^\wedge\langle\sigma_{\nu_1 l}\rangle} = \sum_{i<l_w} s_{wi} e_{\overline{\rho}_{wi}{}^\wedge\langle\nu_{wi}\rangle}.$$

We will apply the two displayed equations and suppose for contradiction that $e_{\overline{\eta}}\varphi \notin Re_{\overline{\eta}}$. Hence $e_{\overline{\eta}}\varphi = \sum\limits_{l<l_{\overline{\eta}}} r_l\, e_{\overline{\eta}_l}$ and there is $\overline{\eta}_0 \neq \overline{\eta}$ with $r_0 \neq 0$. We want to construct a (level preserving) valuated homomorphism

$$g : T_{\overline{\eta}} \longrightarrow T_{\overline{\eta}_0} \text{ with } v_{\overline{\eta}_0}(g(\nu)) = v_{\overline{\eta}}(\nu) \text{ for all } \nu \in T_{\overline{\eta}}.$$

Hence $T_{\overline{\eta}}, T_{\overline{\eta}_0}$ are not rigid and this would contradict the implication of Theorem 2.1. We will construct $g = \bigcup_{k\in\omega} g_k$ as the union of an ascending chain of valuated homomorphisms

$$g_k : T_{\overline{\eta}} \cap {}^{k\geq}\lambda \longrightarrow T_{\overline{\eta}_0} \cap {}^{k\geq}\lambda.$$

Let $g_0(\perp) = \perp$ and suppose that $g_k$ is defined subject to the following condition which we carry on by induction.

$$\text{If } \nu_1 \in T^k_{\overline{\eta}}, \text{ then } \overline{\eta}_0{}^\wedge\langle g_k(\nu_1)\rangle \in \{\overline{\rho}_{\nu_1 l}{}^\wedge\langle\sigma_{\nu_1 l}\rangle :\ l < l_{\nu_1}\} \tag{3.3}$$

thus $g_k(\nu_1) \in T_{\overline{\eta}_0}$ for $\overline{\eta}_0 = \overline{\rho}_{\nu_1 l}$. Note that (3.3) is satisfied for $k = 0$ by the assumption on $\varphi$. Thus we can proceed. If now $\nu_1 \in T^{k+1}_{\overline{\eta}}$ and $\nu_0 = \nu_1 \restriction k$, then $g_k(\nu_0) \in T^k_{\overline{\eta}_0}$ is given and we want to determine $g_{k+1}(\nu_1)$. By induction hypothesis we have some $l_* < l_{\nu_0}$ with $\overline{\rho}_{\nu_0 l_*} = \overline{\eta}_0$ and $g_k(\nu_0) = \sigma_{\nu_0 l_*} \in T_{\overline{\eta}_0}$.

We must find $l' < l_{\nu_1}$ (see (3.2)) such that $\overline{\rho}_{\nu_1 l'} = \overline{\eta}_0$ and $\sigma_{\nu_0 l_*} = \sigma_{\nu_1 l'} \restriction k$. The second condition ensures that $g$ will be the union of an ascending chain of $g_k's$ and also level preserving. The first assertion is our induction-bag which we must carry along. It is also the link to the undesired map $\varphi$.

By the displayed equation for length $k$, there is some $i$ (perhaps more than one) such that $s_{wi} \neq 0$ and $\overline{\rho}_{w_i} = \overline{\eta}_0$ and $\nu_{wi} \restriction k = \sigma_{\nu_0 l^*}$. Then the other displayed equation of length $k+1$, by picking one of the preceding $i$, yields the desired $l'$.

We now have $l' < l_{\nu_1}$ with $\overline{\rho}_{\nu_1 l'} = \overline{\eta}_0$ and $\sigma_{\nu_1 l'} \in T_{\overline{\eta}_0}$ of length $k+1$ with $\sigma_{\nu_1 l'} \restriction k = \sigma_{\nu_0 l_*}$. So we can map $g_{k+1}(\nu_1) \in T_{\overline{\eta}_0}$. If $v_{\overline{\eta}}(\nu_1) = k$, then (using $\lg(\overline{\eta}) = n$) $e_{\overline{\eta}^\wedge\langle\nu_1\rangle} \in F^k_n$ and by (iv) also $e_{\overline{\eta}^\wedge\langle\nu_1\rangle}\varphi \in F^k_n$ and $v\overline{\eta}_0(g_{k+1}(\nu_1)) = k = v\overline{\eta}(\nu_1)$ follows. Thus valuation is preserved.

We argue like this for all $\nu_1 \in T_{\overline{\eta}}$ of length $k+1$. This completes the definition of $g_{k+1}$. Thus $g : T_{\overline{\eta}} \longrightarrow T_{\overline{\eta}_0}$ exists, a contradiction. ∎

*Proof. (of Theorem 3.1)* From Lemma 3.2 follows $e_\perp\varphi = re_\perp$, $e_{\overline{\eta}}\varphi = r_{\overline{\eta}}e_{\overline{\eta}}$ for some $r, r_{\overline{\eta}} \in R$ and all $\perp \neq \overline{\eta} \in S$. Moreover $(e_\perp - e_{\overline{\eta}}) \in F_0$, and therefore $(e_\perp - e_{\overline{\eta}})\varphi \in F_0$ and $(e_\perp - e_{\overline{\eta}})\varphi = re_\perp - r_{\overline{\eta}}e_{\overline{\eta}} \in R(e_\perp - e_{\overline{\eta}})$ by support (in the direct sum). Hence $re_\perp - r_{\overline{\eta}}e_{\overline{\eta}} = r'(e_\perp - e_{\overline{\eta}})$ for some $r' \in R$ and $r = r', r_{\overline{\eta}} = r'$ implies $r_{\overline{\eta}} = r$ for all $\overline{\eta} \in S$. Thus $\varphi = r \in R$. ∎

# 4 Extension to fully rigid systems

We want to strengthen Theorem 3.1 showing the existence of fully rigid systems of $R_\omega$-modules on $\lambda$. This is a family $\mathbf{F}_U$ $(U \subseteq \lambda)$ of $R_\omega$-modules such that the following holds.

$$\mathbf{Hom}_R(\mathbf{F}_U, \mathbf{F}_V) = \begin{cases} R \text{ if} & U \subseteq V \\ 0 \text{ if} & U \not\subseteq V \end{cases}$$

This result will be the starting point for realizing $R$-algebras $A$ as endomorphism algebras $\mathbf{End}_R\mathbf{F} = A$ which are also absolute, see Fuchs, Göbel [16]. We first extend the well-ordered indexing set $W$ for $\mathbf{F}$ by one more element and let

$$W' := \langle 0,1,2\rangle^\wedge L_1{}^\wedge L_2{}^\wedge L_3 \text{ with } L_i \cong \omega\times\omega.$$

Hence $W'$ and $W$ are both order-isomorphic to $\omega\times\omega$ but $W'$ has virtually one more element than $W$ added at place 2 to the definition of $\mathbf{F}$. This allows us to replace $\mathbf{F}$ from Theorem 3.1 by $\mathbf{F}_U$ where the new place is

$$F_2 := F_U := \bigoplus_{e\in U} eR \text{ for any } U\subseteq S.$$

From Theorem 3.1 follows

$$\mathbf{Hom}_R(\mathbf{F}_U.\mathbf{F}_V)\subseteq R \text{ for any } U,V\subseteq S.$$

Clearly $\mathbf{Hom}_R(\mathbf{F}_U.\mathbf{F}_V) = R$ if $U\subseteq V$. On the other hand, if $u\in U\setminus V$, then $e_u\varphi = re_u$ by the displayed formula. But $re_u\in F_V$ only if $r=0$. Hence $\mathbf{Hom}_R(\mathbf{F}_U.\mathbf{F}_V) = 0$ whenever $U\not\subseteq V$. Finally note that $|S|=\lambda$. We established the existence of fully rigid systems.

**Theorem 4.1** *If $R$ is any commutative ring with $1\neq 0$ and $\lambda, |R| < \kappa(\omega)$, then there is a fully rigid system $\mathbf{F}_U$ $(U\subseteq\lambda)$ of free $R_\omega$-modules with the following properties.*

*(i) $F$ is free of rank $\lambda$ and $\mathbf{F}_U = (F, F_0, F_1, F_U, F_i :\ i\in L_1{}^\wedge L_2{}^\wedge L_3)$, thus only $F_2 = F_U$ depends on $U$.*

*(ii) The family $\mathbf{F}_U$ $(U\subseteq\lambda)$ is absolute, i.e. if the given universe is replaced by a generic extension, then the family is still fully rigid.*

The last theorem and a result from [12] (see Theorem 2.2) immediately characterize the first $\omega$-Erdős cardinal. For clarity we restrict ourself to countable rings $R$.

**Corollary 4.2** *Let $R$ by any countable commutative ring. Then the following conditions for a cardinal $\lambda$ are equivalent.*

*(i) There is an absolute $R_\omega$-module $\mathbf{X}$ of size $\lambda$ with $\mathrm{End}_R M = R$.*

*(ii) There is a fully rigid family $\mathbf{F}_U$ $(U\subseteq\lambda)$ of free $R_\omega$-modules.*

*(iii) There is a family of $R_\omega$-modules of size $\lambda$ with only the zero-homomorphism between two distinct members.*

*(iv) $\lambda<\kappa(\omega)$ with $\kappa(\omega)$ the first $\omega$- Erdős cardinal.*

We note, that the last theorem can also be applied to vector spaces (and $\omega$ in (i), (ii) and (iii) can be replaced by 4 or 5 as demonstrated in [16])

# 5 Passing to $R$-modules

We will restrict ourself to only one application of Theorem 4.1. A forthcoming paper by Fuchs, Göbel [16] will exploit Theorem 4.1 and new results will be obtained in two directions. Firstly the number of primes needed in Corollary 5.1 will be reduced to four (which is minimal), moreover $R$-algebras $A$ will be realized as $\mathrm{End}_R\, M = A$ in order to give more absolute results. These applications were obtained earlier but had to wait for publication until it became possible to replace certain results in [12] by Theorem 4.1.

**Corollary 5.1** *Let $R$ be a domain with infinitely many comaximal primes. If $\lambda, |R| < \kappa(\omega)$, then there is an absolute fully rigid family $M_U$ $(U \subseteq \lambda)$ of torsion-free $R$-modules $M_U$ of size $\lambda$. Thus the following holds in any generic extension of the given universe of set theory.*

$$\mathrm{Hom}_R(M_U, M_V) = \begin{cases} R \textit{ if } & U \subseteq V \\ 0 \textit{ if } & U \not\subseteq V \end{cases}$$

*Proof.* Let $p_i$ $(i \in W')$ be a countable family of comaximal primes of $R$ and choose $\mathbf{F}_U = (F, F_0, F_1, F_U, F_i : i \in L_1{}^\wedge L_2{}^\wedge L_3)$ from Theorem 4.1. We will now construct $R$-modules $M_U$ with

$$F \subseteq M_U \subseteq Q \otimes F$$

where $Q$ denotes the quotient field of $R$. Also, if $X \subseteq F$, then we denote by

$$p^{-\infty} X := \bigcup_{n \in \omega} p^{-n} X \subseteq Q \otimes F.$$

Now let

$$M_U := \langle p_i^{-\infty} F_i, p_2^{-\infty} F_U : i \in W \rangle.$$

Thus $F \subseteq M_U \subseteq Q \otimes F$ because $F_0 + F_1 = F$ and $Q \otimes \mathbf{F}_U := (Q \otimes F, Q \otimes F_0, Q \otimes F_1, Q \otimes F_U, Q \otimes F_i : i \in L_1{}^\wedge L_2{}^\wedge L_3)$ satisfies $\mathrm{End}_Q(Q \otimes \mathbf{F}_U) = Q$ by Theorem 4.1. Consider now any $\varphi \in \mathrm{End}_R\, M_U$. The primes ensure that $p_i^{-\infty} F_i \varphi \subseteq p_i^{-\infty} F_i$ for all $i \in W'$ and $\varphi$ extends uniquely to an endomorphism (also called) $\varphi \in \mathrm{End}_Q(Q \otimes \mathbf{F}_U)$. It follows that $\varphi = q \in Q$, thus $\varphi$ is scalar multiplication by $q$ on the right. It remains to show that $(\varphi =) q \in R$ and possibly $\varphi = 0$.

Now we recall that the family of primes, in particular $p_0$ and $p_1$ are comaximal, thus $p_0^{-\infty} R \cap p_1^{-\infty} R = R$. Choose any $e_{\overline{\eta}} \in F_1$. Then $e_{\overline{\eta}} \varphi \in p_1^{-\infty} R e_{\overline{\eta}}$, hence $q \in p_1^{-\infty} R$. Similarly, $e_\perp \varphi \in p_0^{-\infty} R e_\perp$, thus also $q \in p_0^{-\infty} R$ and $q \in p_0^{-\infty} R \cap p_1^{-\infty} R = R$ as required. If $U \not\subseteq V$, then $\mathrm{Hom}_Q(Q \otimes F_U, Q \otimes F_V) = 0$ by Theorem 4.1 and the unique extension of $\varphi$ to the corresponding $Q$-vector space must by zero. Hence $\varphi = 0$ and the corollary follows. ∎

We would like to mention that the infinite set of primes in the corollary can be replaced by 4 primes, see [16]; and primes can also be replaces by comaximal multiplicatively closed subsets. The latter is a natural straight extension suggested by Tony Corner (unpublished); this can be looked up in [13].

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