MEASURED CREATURES

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Abstract. We prove that two basic questions on outer measure are undecidable. First we show that consistently

- every sup-measurable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable.

The interest in sup-measurable functions comes from differential equations and the question for which functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ the Cauchy problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has a unique almost-everywhere solution in the class $AC_l(\mathbb{R})$ of locally absolutely continuous functions on $\mathbb{R}$.

Next we prove that consistently

- every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on some set of positive outer Lebesgue measure.

This says that in a strong sense the family of continuous functions (from the reals to the reals) is dense in the space of arbitrary such functions.

For the proofs we discover and investigate a new family of nicely definable forcing notions (so indirectly we deal with nice ideals of subsets of the reals – the two classical ones being the ideal of null sets and the ideal of meagre ones).

Concerning the method, i.e., the development of a family of forcing notions, the point is that whereas there are many such objects close to the Cohen forcing (corresponding to the ideal of meagre sets), little has been known on the existence of relatives of the random real forcing (corresponding to the ideal of null sets), and we look exactly at such forcing notions.

0. Introduction

The present paper deals with two, as it occurs closely related, problems concerning real functions. The first one is the question if it is possible that all superposition–measurable functions are measurable.

Definition 0.1. A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is *superposition–measurable* (in short: *sup–measurable*) if for every Lebesgue measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}$ the superposition

$$f_g : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto f(x, g(x))$$

is Lebesgue measurable.

The interest in sup-measurable functions comes from differential equations and the question for which functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ the Cauchy problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

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has a unique almost-everywhere solution in the class $AC_l(\mathbb{R})$ of locally absolutely continuous functions on $\mathbb{R}$. For the detailed discussion of this area we refer the reader to Balcerzak [?], Balcerzak and Ciesielski [?] and Kharazishvili [?]. Grande and Lipiński [?] proved that, under CH, there is a non-measurable function which is sup-measurable. Later the assumption of CH was weakened (see Balcerzak [?, Thm 2.1]), however the question if one can build a non-measurable sup-measurable function in ZFC remained open (it was formulated in Balcerzak [?, Problem 1.10] and Ciesielski [?, Problem 5], and implicitly in Kharazishvili [?, Remark 4]). In the third section we will answer this question by showing that, consistently, every sup-measurable function is Lebesgue measurable.

Next we deal with von Weizsäcker's problem. It has enjoyed considerable popularity, and it has origins in measure theory and topology. In [?], von Weizsäcker noted that if

$(*)$ $\mathrm{non}(\mathcal{N}) \stackrel{\rm def}{=} \min\{|X| : X \subseteq \mathbb{R}$ has positive outer Lebesgue measure $\} = \mathfrak{c}$,

then

$(\otimes)$ there is a function $f : [0,1] \longrightarrow [0,1]$ such that the graph of f is of (two dimensional) outer measure 1 but for every Borel function $g : [0,1] \longrightarrow [0,1]$ the set $\{x \in [0,1] : f(x) = g(x)\}$ is of measure zero.

Then he showed that $(\otimes)$ implies

$(\boxtimes)$ there is a countably generated σ–algebra $\mathcal{A}$ containing Borel$([0,1])$ such that the Lebesgue measure can be extended to $\mathcal{A}$, but there is no extremal extension to $\mathcal{A}$.

So it was natural to ask if the statement in $(\otimes)$ can be proved in ZFC (i.e., without assuming $(*)$). A way to formulate this question was to ask

$(\circledast)_{\rm vW}$ Is it consistent to suppose that for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ there is a Borel measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) = g(x)\}$ is not Lebesgue negligible ?

One can arrive to question $(\circledast)_{\rm vW}$ also from the topological side. In [?], Blumberg proved that if X is a separable complete metric space and $f : X \longrightarrow \mathbb{R}$, then there exists a dense (but possibly countable) subset D of X such that the restriction $f \restriction D$ is continuous. This result has been generalized in many ways: by considering functions on other topological spaces, or by aiming at getting "a large set" on which the function is continuous. For example, in the second direction, we may restrict ourselves to $X = \mathbb{R}$ and ask if above we may request that the set D is uncountable. That was answered by Abraham, Rubin and Shelah who showed in [?] that, consistently, every real function is continuous on an uncountable set. The next natural step is to ask if we can demand that the set D is of positive outer measure, and this is von Weizsäcker's question $(\circledast)_{\rm vW}$. It appears in Fremlin's list of problems as [?, Problem AR(a)] and in Ciesielski [?, Problem 1].

We will answer question $(\circledast)_{\rm vW}$ in affirmative in the fourth section. The respective model is built by a small modification of the iteration used to deal with the sup-measurability problem (and, as a matter of fact, it may serve for both purposes). We do not know if $(\boxtimes)$ fails in our model (and the question if $\neg(\boxtimes)$ is consistent remains open).

Let us note that the close relation of the two problems solved here is not very surprising. Some connections were noticed already in Balcerzak and Ciesielski [?].

Also, among others, these connections motivated the following strengthening of $(\circledast)_{\rm vW}$:

$(\circledast)^+_{\rm vW}$ Is it consistent that for every subset Y of $\mathbb{R}$ of positive outer measure and every function $f : Y \longrightarrow \mathbb{R}$, there exists a set $X \subseteq Y$ of positive outer measure such that $f \restriction X$ is continuous?

However, as Fremlin points out, the answer to $(\circledast)^+_{\rm vW}$ is NO:

Proposition 0.2 (Fremlin [?])**.** *There are a set $Y \subseteq \mathbb{R}$ of positive outer measure and a function $f : Y \longrightarrow \mathbb{R}$ such that $f \restriction X$ is not continuous for any $X \subseteq Y$ of positive outer measure.*

Proof. Recall that a Hausdorff space Z is *universally negligible* if there is no Borel probability measure on Z that vanishes at singletons. By Grzegorek [?], there is a universally negligible set $Z \subseteq \mathbb{R}$ of cardinality ${\rm non}(\mathcal{N})$ (see also [?, Volume IV, 439E(c)]). Pick a non-null set $Y \subseteq \mathbb{R}$ of size ${\rm non}(\mathcal{N})$ and fix a bijection $f : Y \longrightarrow Z$.

If $X \subseteq Y$ is such that $f \restriction X$ is continuous, then we may transport Borel measures on X to Z, and therefore X is universally negligible and thus Lebesgue negligible. (See also [?, Volume IV, 439C(f)].) $\square$

The notion of sup-measurability has its category version (defined naturally by replacing "Lebesgue measurability" by "Baire property"). It was investigated in E.Grande and Z.Grande [?], Balcerzak [?], and Ciesielski and Shelah [?]. The latter paper presents a model in which every Baire-sup-measurable function has the Baire property. Also von Weizsäcker problem has its category counterpart which was answered in Shelah [?]. What is somewhat surprising, is that the models of [?] and [?] seem to be totally unrelated (while for the measure case presented here the connection is striking). Moreover, neither the forcing used in [?] (based on the oracle-cc method of Shelah [?, Chapter IV]), nor the one applied in Shelah [?], are parallel to the method presented here.

The present paper is a part of the authors' program to investigate the family of forcing notions with *norms on possibilities*, and we here further develop the theory of that forcing notions introducing *measured creatures*. This enrichment of the method of norms on possibilities creates a bridge between the forcings of [?] and the random real forcing (including the latter in our framework), and we present here ω^ω–bounding friends of the random forcing. Though they are not ccc, they do make random not so lonely. [One of the points is that we know many forcing notions in the neighbourhood of the Cohen forcing notion (see, e.g., Rosłanowski and Shelah [?], [?]), but this is the first time that we find many relatives of the random real forcing].

Our presentation is self-contained, and though we use the notation of [?], the two basic definitions we need from there are stated in somewhat restricted form below (in 0.3, 0.4). The general construction of forcing notions using measured (tree) creatures is presented in the first section, and only in the following section we define the particular example that works for us. The forcing notion $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$ (defined in section 2) is the basic ingredient of our construction. The required models are obtained by CS iterations of $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$; in the fourth section we also add in the iteration random reals (on a stationary set of coordinates).

Let us point out that "measured creatures" presented here have their ccc relative which appeared in [?, §2.1].

Notation: Most of our notation is standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [**?**]). However in forcing we keep the convention that *the stronger condition is the larger one* (i.e., $p \leq q$ means that q is stronger than p).

(1) $\mathbb{R}^{\geq 0}$ stands for the set of non-negative reals. For a real number r and a set A, the function with domain A and the constant value r will be denoted r_A.
(2) For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.
(3) A *tree* is a family T of finite sequences such that for some $\mathrm{root}(T) \in T$ we have
$$(\forall \nu \in T)(\mathrm{root}(T) \trianglelefteq \nu) \quad \text{and} \quad \mathrm{root}(T) \trianglelefteq \nu \trianglelefteq \eta \in T \ \Rightarrow \ \nu \in T.$$
(4) For a tree T, the family of all ω–branches through T is denoted by $[T]$, and we let
$$\max(T) \stackrel{\mathrm{def}}{=} \{\nu \in T : \text{ there is no } \rho \in T \text{ such that } \nu \vartriangleleft \rho\}.$$
If η is a node in the tree T then
$$\begin{array}{lcl} \mathrm{succ}_T(\eta) & = & \{\nu \in T : \eta \vartriangleleft \nu \ \& \ \mathrm{lh}(\nu) = \mathrm{lh}(\eta) + 1\} \ \text{ and} \\ T^{[\eta]} & = & \{\nu \in T : \eta \trianglelefteq \nu\}. \end{array}$$
A set $F \subseteq T$ is *a front of* T if
$$(\forall \eta \in [T])(\exists k \in \omega)(\eta \restriction k \in F).$$
(5) The Cantor space 2^ω (the spaces of all functions from ω to 2) and the space $\prod\limits_{i<\omega} N_i$ (where N_i are positive integers thought of as non-empty finite sets) are equipped with natural (Polish) topologies, as well with as with standard product measure structures.
(6) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a dot above (e.g. $\dot{\tau}$, $\dot{X}$), but we do not notationally distinguish between objects in the ground model and their names in the forcing language.
(7) For a relation R (a set of ordered pairs), $\mathrm{rng}(R)$ and $\mathrm{dom}(R)$ stand for the range and the domain of R, respectively.
(8) We will keep the convention that $\sup(\emptyset)$ is 0. Similarly, the sum over an empty set of reals is assumed to be 0.

Let us recall the definition of tree creating pairs. Since we are going to use local tree creating pairs only, we restrict ourselves to this case. For more information and properties of tree creating pairs and related forcing notions we refer the reader to [**?**, §1.3, 2.3].

Definition 0.3. Let $\mathbf{H}$ be a function with domain ω.

(1) *A local tree–creature for* $\mathbf{H}$ is a triple
$$t = (\mathbf{nor}, \mathbf{val}, \mathbf{dis}) = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t])$$

such that $\mathbf{nor} \in \mathbb{R}^{\geq 0}$, $\mathbf{dis} \in \mathcal{H}(\aleph_1)$ (i.e., $\mathbf{dis}$ is hereditarily countable), and for some sequence $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$, $n < \omega$, we have

$$\emptyset \neq \mathbf{val} \subseteq \{\langle \eta, \nu \rangle : \eta \lhd \nu \in \prod_{i \leq \mathrm{lh}(\eta)} \mathbf{H}(i)\}.$$

(Thus for $\langle \eta, \nu \rangle \in \mathbf{val}$ we have $\mathrm{lh}(\nu) = \mathrm{lh}(\eta) + 1$.) For a tree–creature t we let $\mathrm{pos}(t) \stackrel{\mathrm{def}}{=} \mathrm{rng}(\mathbf{val}[t])$.

The set of all local tree–creatures for $\mathbf{H}$ will be denoted by $\mathrm{LTCR}[\mathbf{H}]$, and for $\eta \in \bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}(i)$ we let $\mathrm{LTCR}_\eta[\mathbf{H}] = \{t \in \mathrm{LTCR}[\mathbf{H}] : \mathrm{dom}(\mathbf{val}[t]) = \{\eta\}\}$.

(2) Let $K \subseteq \mathrm{LTCR}[\mathbf{H}]$. We say that a function $\Sigma : K \longrightarrow \mathcal{P}(K)$ is a *local tree composition on* K whenever the following conditions are satisfied.
   (a) If $t \in K \cap \mathrm{LTCR}_\eta[\mathbf{H}]$, $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$, $n < \omega$, then $\Sigma(t) \subseteq \mathrm{LTCR}_\eta[\mathbf{H}]$ and $t \in \Sigma(t)$.
   (b) If $s \in \Sigma(t)$, then $\mathbf{val}[s] \subseteq \mathbf{val}[t]$.
   (c) [*transitivity*] If $s \in \Sigma(t)$, then $\Sigma(s) \subseteq \Sigma(t)$.

(3) If $K \subseteq \mathrm{LTCR}[\mathbf{H}]$ and Σ is a local tree composition operation on K, then (K, Σ) is called *a local tree–creating pair for* $\mathbf{H}$.

(4) We say that (K, Σ) is *strongly finitary* if $\mathbf{H}(m)$ is finite (for $m < \omega$) and $\mathrm{LTCR}_\eta[\mathbf{H}] \cap K$ is finite (for each η).

Definition 0.4 (See [**?**, Definition 1.3.5])**.** Let (K, Σ) be a local tree–creating pair for $\mathbf{H}$. The forcing notion $\mathbb{Q}_4^{\mathrm{tree}}(K, \Sigma)$ is defined as follows.

A condition is a system $p = \langle t_\eta : \eta \in T \rangle$ such that

(a) $T \subseteq \bigcup\limits_{n \in \omega} \prod\limits_{i<n} \mathbf{H}(i)$ is a non-empty tree with $\max(T) = \emptyset$,
(b) for all $\eta \in T$, $t_\eta \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K$ and $\mathrm{pos}(t_\eta) = \mathrm{succ}_T(\eta)$,
(c)$_4$ for every $n < \omega$, the set

$$\{\nu \in T : (\forall \rho \in T)(\nu \lhd \rho \;\Rightarrow\; \mathbf{nor}[t_\rho] \geq n)\}$$

contains a front of the tree T.

The order is given by:
$\langle t^1_\eta : \eta \in T^1 \rangle \leq \langle t^2_\eta : \eta \in T^2 \rangle$ (remember, this means that $\langle t^2_\eta : \eta \in T^2 \rangle$ is stronger than $\langle t^1_\eta : \eta \in T^1 \rangle$) if and only if
$T^2 \subseteq T^1$ and $t^2_\eta \in \Sigma(t^1_\eta)$ for each $\eta \in T^2$.

If $p = \langle t_\eta : \eta \in T \rangle$, then we write $\mathrm{root}(p) = \mathrm{root}(T)$, $T^p = T$, $t^p_\eta = t_\eta$ etc.

The forcing notion $\mathbb{Q}_\emptyset^{\mathrm{tree}}(K, \Sigma)$ is defined similarly, but we omit the norm requirement (c)$_4$. (So $\mathbb{Q}_\emptyset^{\mathrm{tree}}(K, \Sigma)$ is trivial in a sense; we will use it for notational convenience only.)

1. Measured Creatures

Below we introduce a relative of the *mixtures with random* presented in [**?**, §2.1]. Here, however, the interplay between the norm of a tree creature t, the set of possibilities $\mathrm{pos}(t)$ and the averaging function F_t assigned to t is different.

Basic Notation: In this section, $\mathbf{H}$ stands for a function with domain ω and such that $(\forall m\in\omega)(|\mathbf{H}(m)|\geq 2)$. Moreover we demand $\mathbf{H}\in\mathcal{H}(\aleph_1)$ (i.e., $\mathbf{H}$ is hereditarily countable).

Definition 1.1. (1) *A measured (tree) creature for* $\mathbf{H}$ is a pair (t,F) such that $t\in \mathrm{LTCR}[\mathbf{H}]$ and

$$F:[0,1]^{\mathrm{pos}(t)}\longrightarrow [0,1].$$

(2) We say that $(K,\Sigma,\mathbf{F})$ is a *measured tree creating triple for* $\mathbf{H}$ if
  (a) (K,Σ) is a local tree–creating pair for $\mathbf{H}$,
  (b) $\mathbf{F}$ is a function with domain K, $\mathbf{F}:t\mapsto F_t$, such that (t,F_t) is a measured (tree) creature (for each $t\in K$).

(3) If $(K,\Sigma,\mathbf{F})$ is as above, $t\in K$, $X\subseteq \mathrm{pos}(t)$, and $\langle r_\nu:\nu\in X\rangle\in[0,1]^X$, then we define $F_t(r_\nu:\nu\in X)$ as $F_t(r^*_\nu:\nu\in\mathrm{pos}(t))$, where

$$r^*_\nu=\begin{cases} r_\nu & \text{if } \nu\in X,\\ 0 & \text{if } \nu\in\mathrm{pos}(t)\setminus X.\end{cases}$$

We think of F_t as a kind of averaging function. At the first reading the reader may think that $\mathrm{pos}(t)$ is finite and

$$F_t(r_\nu:\nu\in\mathrm{pos}(t))=\frac{\sum\{r_\nu:\nu\in\mathrm{pos}(t)\}}{|\mathrm{pos}(t)|}.$$

For this particular function, our construction results in the random real forcing. However in general our averaging function does not have to be additive (as long as it has the properties stated in 1.2 below), and the result is not the random forcing (and this is one of the points of our construction). Also having F_t depend on t allows us to "cheat": if we do not like the results of our averaging we may pass to a tree creature $s\in\Sigma(t)$ (dropping the norm a little) with an averaging function F_s that is better for us.

Regarding the requirements of 1.2, note that they are meant to provide us with some features of the Lebesgue measure, without imposing additivity on the averaging functions F_t (specifically see 1.2(β)).

Definition 1.2. A measured tree creating triple $(K,\Sigma,\mathbf{F})$ is *nice* if for every $t\in K$:

(α) if $\langle r_\nu:\nu\in\mathrm{pos}(t)\rangle,\langle r'_\nu:\nu\in\mathrm{pos}(t)\rangle\subseteq[0,1]$, $r_\nu\leq r'_\nu$ for all $\nu\in\mathrm{pos}(t)$, then

$$F_t(r_\nu:\nu\in\mathrm{pos}(t))\leq F_t(r'_\nu:\nu\in\mathrm{pos}(t)),$$

(β) if $\mathbf{nor}[t]>1$, $\{\eta\}=\mathrm{dom}(\mathbf{val}[t])$, $r_\nu,r^0_\nu,r^1_\nu\in[0,1]$ (for $\nu\in\mathrm{pos}(t)$) are such that $r^0_\nu+r^1_\nu\geq r_\nu$ and $F_t(r_\nu:\nu\in\mathrm{pos}(t))\geq 2^{-2^{\mathrm{lh}(\eta)}}$, then there are real numbers c_0,c_1 and tree creatures $s_0,s_1\in\Sigma(t)$ such that

$$c_0+c_1=(1-2^{-2^{\mathrm{lh}(\eta)}})F_t(r_\nu:\nu\in\mathrm{pos}(t))$$

and

($\otimes$) if $\ell<2$, $c_\ell>0$, then $\mathbf{nor}[s_\ell]\geq\mathbf{nor}[t]-1$, $\mathrm{pos}(s_\ell)\subseteq\{\nu\in\mathrm{pos}(t): r^\ell_\nu>0\}$, and

$$F_{s_\ell}(r^\ell_\nu:\nu\in\mathrm{pos}(s_\ell))\geq c_\ell,$$

(γ) if $b\in[0,1]$ and $r_\nu\in[0,1]$ (for $\nu\in\mathrm{pos}(t)$), then

$$F_t(b\cdot r_\nu:\nu\in\mathrm{pos}(t))=b\cdot F_t(r_\nu:\nu\in\mathrm{pos}(t)),$$

(δ) if $\langle r_\nu : \nu \in \mathrm{pos}(t)\rangle \subseteq [0,1]$, $\varepsilon > 0$, then there are $r'_\nu > r_\nu$ (for $\nu \in \mathrm{pos}(t)$) such that for each $\langle r''_\nu : \nu \in \mathrm{pos}(t)\rangle \subseteq [0,1]$ satisfying $r_\nu \le r''_\nu < r'_\nu$ (for $\nu \in \mathrm{pos}(t)$) we have

$$F_t(r''_\nu : \nu \in \mathrm{pos}(t)) < F_t(r_\nu : \nu \in \mathrm{pos}(t)) + \varepsilon.$$

(Why do we have r''_ν's above? Only to avoid notational difficulties when $r_\nu = 1$ for some ν. Otherwise one may think that we demand just $F_t(r'_\nu : \nu \in \mathrm{pos}(t)) < F_t(r_\nu : \nu \in \mathrm{pos}(t)) + \varepsilon$.)

From now on (till the end of this section), let $(K, \Sigma, \mathbf{F})$ be a fixed strongly finitary and nice measured tree creating triple for $\mathbf{H}$. Note that then the condition $(c)_4$ of Definition 0.4 is equivalent to

$(c)_5$ $(\forall k \in \omega)(\exists n \in \omega)(\forall \eta \in T^p)(\mathrm{lh}(\eta) \ge n \;\Rightarrow\; \mathbf{nor}[t_\eta] \ge k)$.

Proposition 1.3. *Let $t \in K$. Then:*

(ε) *If $r_\nu = 0$ for $\nu \in \mathrm{pos}(t)$, then $F_t(r_\nu : \nu \in \mathrm{pos}(t)) = 0$.*

(ζ) *If $\langle r_\nu : \nu \in \mathrm{pos}(t)\rangle \subseteq [0,1]$, $\varepsilon > 0$, then there are $r'_\nu < r_\nu$ (for $\nu \in \mathrm{pos}(t)$) such that for each $\langle r''_\nu : \nu \in \mathrm{pos}(t)\rangle \subseteq [0,1]$ satisfying $r'_\nu < r''_\nu \le r_\nu$ (for $\nu \in \mathrm{pos}(t)$) we have*

$$F_t(r_\nu : \nu \in \mathrm{pos}(t)) - \varepsilon < F_t(r''_\nu : \nu \in \mathrm{pos}(t)).$$

Proof. (ε) Follows from 1.2(γ) (take $b = 0$).
(ζ) If $F_t(r_\nu : \nu \in \mathrm{pos}(t)) < \varepsilon$, then any $r'_\nu < r_\nu$ (for $\nu \in \mathrm{pos}(t)$) work. So assume $F_t(r_\nu : \nu \in \mathrm{pos}(t)) \ge \varepsilon$ and let $b = \frac{F_t(r_\nu : \nu \in \mathrm{pos}(t)) - \varepsilon/2}{F_t(r_\nu : \nu \in \mathrm{pos}(t))}$. Then $0 < b < 1$. For $\nu \in \mathrm{pos}(t)$ put

$$r'_\nu = \begin{cases} -1 & \text{if } r_\nu = 0, \\ b \cdot r_\nu & \text{otherwise.} \end{cases}$$

We are going to show that these r'_ν's are as required. To this end suppose that $\langle r''_\nu : \nu \in \mathrm{pos}(t)\rangle \subseteq [0,1]$ is such that $r'_\nu < r''_\nu \le r_\nu$ (for all $\nu \in \mathrm{pos}(t)$). Then also $b \cdot r_\nu \le r''_\nu$ (for $\nu \in \mathrm{pos}(t)$) and by 1.2(α, γ) we get

$$\begin{aligned} &F_t(r''_\nu : \nu \in \mathrm{pos}(t)) \ge F_t(b \cdot r_\nu : \nu \in \mathrm{pos}(t)) = b \cdot F_t(r_\nu : \nu \in \mathrm{pos}(t)) = \\ &F_t(r_\nu : \nu \in \mathrm{pos}(t)) - \varepsilon/2 > F_t(r_\nu : \nu \in \mathrm{pos}(t)) - \varepsilon. \end{aligned}$$

$\square$

Definition 1.4. Let $p = \langle t^p_\eta : \eta \in T^p\rangle \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$.

(1) For a front $A \subseteq T^p$ of T^p, we let $T[p, A] = \{\eta \in T^p : (\exists \rho \in A)(\eta \trianglelefteq \rho)\}$.
(2) Let A be a front of T^p and let $f : A \longrightarrow [0,1]$. By downward induction on $\eta \in T[p, A]$ we define a mapping $\mu^f_{p,A} : T[p, A] \longrightarrow [0,1]$ as follows:
- if $\eta \in A$ then $\mu^f_{p,A}(\eta) = f(\eta)$,
- if $\mu^f_{p,A}(\nu)$ has been defined for all $\nu \in \mathrm{pos}(t^p_\eta)$, $\eta \in T[p, A] \setminus A$, then we put $\mu^f_{p,A}(\eta) = F_{t^p_\eta}(\mu^f_{p,A}(\nu) : \nu \in \mathrm{pos}(t^p_\eta))$.

(3) For $\eta \in T^p$ we define

$$\mu^{\mathbf{F}}_p(\eta) = \inf\{\mu^f_{p^{[\eta]},A}(\eta) : A \text{ is a front of } (T^p)^{[\eta]} \text{ and } f = 1_A\},$$

and we let $\mu^{\mathbf{F}}(p) = \mu^{\mathbf{F}}_p(\mathrm{root}(p))$.

(4) For $e \in \{\emptyset, 4\}$ we let[1]
$$\mathbb{Q}^{\rm mt}_e(K,\Sigma,\mathbf{F}) = \{p \in \mathbb{Q}^{\rm tree}_e(K,\Sigma) : \mu^{\mathbf{F}}(p) > 0\}.$$
It is equipped with the partial order inherited from $\mathbb{Q}^{\rm tree}_e(K,\Sigma)$.

Proposition 1.5. *Assume $p \in \mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma)$ and A is a front of T^p.*

(1) *If $f_0, f_1 : A \longrightarrow [0,1]$ are such that $f_0(\nu) \leq f_1(\nu)$ for all $\nu \in A$, then*
$$(\forall \eta \in T[p,A])(\mu^{f_0}_{p,A}(\eta) \leq \mu^{f_1}_{p,A}(\eta)).$$
(2) *If $f_0 : A \longrightarrow [0,1]$, $b \in [0,1]$, and $f_1(\nu) = b \cdot f_0(\nu)$ (for $\nu \in A$), then*
$$(\forall \eta \in T[p,A])(\mu^{f_1}_{p,A}(\eta) = b \cdot \mu^{f_0}_{p,A}(\eta)).$$
(3) *If A' is a front of T^p above A (that is, $(\forall \nu' \in A')(\exists \nu \in A)(\nu \vartriangleleft \nu')$) and $\eta \in T[p,A]$, then $\mu^{1_{A'}}_{p,A'}(\eta) \leq \mu^{1_A}_{p,A}(\eta)$.*

Definition 1.6. Let $p \in \mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma,\mathbf{F})$.

(1) A function $\mu : T^p \longrightarrow [0,1]$ is *a semi–$\mathbf{F}$–measure on p* if
$$(\forall \eta \in T^p)\big(\mu(\eta) \leq F_{t^p_\eta}(\mu(\nu) : \nu \in {\rm pos}(t^p_\eta))\big).$$
(2) If above the equality holds for each $\eta \in T^p$, then μ is called *an $\mathbf{F}$–measure.*

Proposition 1.7. *Let $p \in \mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma)$.*

(1) *If $\mu : T^p \longrightarrow [0,1]$ is a semi–$\mathbf{F}$–measure on p, then for each $\eta \in T^p$ we have $\mu(\eta) \leq \mu^{\mathbf{F}}_p(\eta)$.*
(2) *If there is a semi–$\mathbf{F}$–measure μ on p such that $\mu({\rm root}(p)) > 0$, then $p \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$.*
(3) *If $p \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$, then the mapping $\eta \mapsto \mu^{\mathbf{F}}_p(\eta) : T^p \longrightarrow [0,1]$ is an $\mathbf{F}$–measure on p.*

Lemma 1.8. *Assume $p \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$ and $0 < \varepsilon < 1$. Then there is $\eta \in T^p$ such that $\mu^{\mathbf{F}}_p(\eta) \geq 1 - \varepsilon$.*

Proof. Assume towards a contradiction that $\mu^{\mathbf{F}}_p(\eta) < 1-\varepsilon$ for all $\eta \in T^p$. Choose inductively fronts A_k of T^p such that

- $A_0 = \{{\rm root}(p)\}$,
- $(\forall \eta \in A_{k+1})(\exists \nu \in A_k)(\nu \vartriangleleft \eta)$,
- $\mu^{1_{A_{k+1}}}_{p,A_{k+1}}(\nu) < 1-\varepsilon$ for all $\nu \in A_k$.

Note that then (by 1.5(1,2)) for each $k < \omega$ we have
$$\mu(p) \leq \mu^{1_{A_{k+1}}}_{p,A_{k+1}}({\rm root}(p)) \leq (1-\varepsilon)^{k+1}.$$
Since the right hand side of the inequality above approaches 0 (as $k \to \infty$), we get an immediate contradiction with the demand $\mu^{\mathbf{F}}(p) > 0$. $\square$

Definition 1.9. A condition $p \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$ is called *normal* if for every $\eta \in T^p$ we have $\mu^{\mathbf{F}}_p(\eta) > 0$. We say that p is *special* if for every $\eta \in T^p$ we have $\mu^{\mathbf{F}}_p(\eta) \geq 2^{-2^{{\rm lh}(\eta)+1}}$.

Proposition 1.10. (1) *Special conditions are dense in $\mathbb{Q}^{\rm mt}_4(K,\Sigma,\mathbf{F})$. (So also normal conditions are dense.)*

[1] "mt" stands for **m**easured **t**ree

(2) *If p is normal, and A is a front of T^p, then $\mu^{\mathbf{F}}(p) = \mu^f_{p,A}(\mathrm{root}(p))$, where $f(\nu) = \mu^{\mathbf{F}}_p(\nu)$ (for $\nu \in A$).*

Proof. 1) Let $p \in \mathbb{Q}^{\mathrm{mt}}_4(K, \Sigma, \mathbf{F})$; clearly we may assume that $\mathbf{nor}[t^p_\eta] > 1$ for all $\eta \in T^p$. Also we may assume that $\mu^{\mathbf{F}}(p) > 3/4$ (remember 1.8) and $\mathrm{lh}(\mathrm{root}(p)) > 4$.

Fix $\eta \in T^p$ such that $\mu^{\mathbf{F}}_p(\eta) \geq 2^{-2^{\mathrm{lh}(\eta)}}$ for a moment. Let $1 < a < 2$. For each $\nu \in \mathrm{pos}(t^p_\eta)$ pick a front A_ν of $(T^p)^{[\nu]}$ such that

- if $\mu^{\mathbf{F}}_p(\nu) < 2^{-2^{\mathrm{lh}(\eta)+1}}$, then $\mu^{1_{A_\nu}}_{p^{[\nu]},A_\nu}(\nu) < 2^{-2^{\mathrm{lh}(\eta)+1}}$,
- if $\mu^{\mathbf{F}}_p(\nu) \geq 2^{-2^{\mathrm{lh}(\eta)+1}}$, then $\mu^{1_{A_\nu}}_{p^{[\nu]},A_\nu}(\nu) < a \cdot \mu^{\mathbf{F}}_p(\nu)$.

Let $X_0 = \{\nu \in \mathrm{pos}(t^p_\eta) : \mu^{\mathbf{F}}_p(\nu) < 2^{-2^{\mathrm{lh}(\eta)+1}}\}$, $X_1 = \mathrm{pos}(t^p_\eta) \setminus X_0$, $r_\nu = \mu^{1_{A_\nu}}_{p^{[\nu]},A_\nu}(\nu)$, and

$$r^\ell_\nu = \begin{cases} r_\nu & \text{if } \nu \in X_\ell, \\ 0 & \text{if } \nu \in X_{1-\ell}. \end{cases}$$

Apply 1.2(β) for $t^p_\eta, r^0_\nu, r^1_\nu, r_\nu$ (note that $F_{t^p_\eta}(r_\nu : r_\nu \in \mathrm{pos}(t^p_\eta)) \geq \mu^{\mathbf{F}}_p(\eta) \geq 2^{-2^{\mathrm{lh}(\eta)}}$) to pick $s^a_0, s^a_1 \in \Sigma(t^p_\eta)$ and c^a_0, c^a_1 such that

$$c^a_0 + c^a_1 = (1 - 2^{-2^{\mathrm{lh}(\eta)}}) F_{t^p_\eta}(r_\nu : \nu \in \mathrm{pos}(t^p_\eta)),$$

and

$(\otimes)^a$ if $\ell < 2$, $c^a_\ell > 0$, then $\mathbf{nor}[s^a_\ell] \geq \mathbf{nor}[t^p_\eta] - 1$, $\mathrm{pos}(s^a_\ell) \subseteq X_\ell$, and

$$F_{s^a_\ell}(r_\nu : \nu \in \mathrm{pos}(s^a_\ell)) \geq c^a_\ell.$$

Note that, if $c^a_0 > 0$, then $c^a_0 \leq F_{s^a_0}(r_\nu : \nu \in \mathrm{pos}(s^a_\ell)) \leq 2^{-2^{\mathrm{lh}(\eta)+1}}$, and thus

$$c^a_1 \geq (1 - 2^{-2^{\mathrm{lh}(\eta)}}) F_{t^p_\eta}(r_\nu : \nu \in \mathrm{pos}(t^p_\eta)) - 2^{-2^{\mathrm{lh}(\eta)+1}} > 0.$$

Also, letting $r^*_\nu = \min\{a \cdot \mu^{\mathbf{F}}_p(\nu), 1\}$,

$$F_{s^a_1}(r_\nu : \nu \in \mathrm{pos}(s^a_1)) \leq F_{s^a_1}(r^*_\nu : \nu \in \mathrm{pos}(s^a_1)) \leq a \cdot F_{s^a_1}(\mu^{\mathbf{F}}_p(\nu) : \nu \in \mathrm{pos}(s^a_1)).$$

Together

$(*)_a$ $\quad (1 - 2^{-2^{\mathrm{lh}(\eta)}}) F_{t^p_\eta}(r_\nu : \nu \in \mathrm{pos}(t^p_\eta)) - 2^{-2^{\mathrm{lh}(\eta)+1}} \leq a \cdot F_{s^a_1}(\mu^{\mathbf{F}}_p(\nu) : \nu \in \mathrm{pos}(s^a_1)).$

Since (K, Σ) is strongly finitary, considering $a \to 1$ (and using 1.2(δ)), we find $s_\eta \in \Sigma(t^p_\eta)$ such that $\mathbf{nor}[s_\eta] \geq \mathbf{nor}[t^p_\eta] - 1$ and $\mu^{\mathbf{F}}_p(\nu) \geq 2^{-2^{\mathrm{lh}(\eta)+1}}$ for all $\nu \in \mathrm{pos}(s_\eta)$, and

$$\mu^{\mathbf{F}}_p(\eta) = F_{t^p_\eta}(\mu^{\mathbf{F}}_p(\nu) : \nu \in \mathrm{pos}(t^p_\eta)) \leq \frac{F_{s_\eta}(\mu^{\mathbf{F}}_p(\nu) : \nu \in \mathrm{pos}(s_\eta)) + 2^{-2^{\mathrm{lh}(\eta)+1}}}{1 - 2^{-2^{\mathrm{lh}(\eta)}}}.$$

Note that also, as $2^{-2^{\mathrm{lh}(\eta)}} \leq \mu^{\mathbf{F}}_p(\eta)$,

$$\mu^{\mathbf{F}}_p(\eta)(1 - 2^{-2^{\mathrm{lh}(\eta)}}) - 2^{-2^{\mathrm{lh}(\eta)+1}} \geq \mu^{\mathbf{F}}_p(\eta)(1 - 2^{1-2^{\mathrm{lh}(\eta)}}),$$

so

$(**)$ $\quad \mu^{\mathbf{F}}_p(\eta) \cdot (1 - 2^{1-2^{\mathrm{lh}(\eta)}}) \leq F_{s_\eta}(\mu^{\mathbf{F}}_p(\nu) : \nu \in \mathrm{pos}(s_\eta)).$

Now, starting with $\mathrm{root}(p)$, build a tree S and a system $q = \langle s_\eta : \eta \in S \rangle$ such that $\mathrm{succ}_S(\eta) = \mathrm{pos}(s_\eta)$. It should be clear that in this way we will get a condition in $\mathbb{Q}^{\mathrm{tree}}_4(K, \Sigma)$ (stronger than p). Why is q in $\mathbb{Q}^{\mathrm{mt}}_4(K, \Sigma, \mathbf{F})$? Let $k^* > \mathrm{lh}(\mathrm{root}(q))$,

$A=\{\nu\in S:\mathrm{lh}(\nu)=k^*\}$ and $f=1_A$. Using $(**)$, we may show by downward induction that for every $\eta\in T[q,A]$ we have

$$\mu^f_{q,A}(\eta)\geq\mu^{\mathbf{F}}_p(\eta)\cdot\prod_{k=\mathrm{lh}(\eta)}^{k^*-1}(1-2^{1-2^k})\geq\mu^{\mathbf{F}}_p(\eta)\cdot(1-2^{2-2^{\mathrm{lh}(\eta)}})\geq$$
$$2^{-2^{\mathrm{lh}(\eta)}}(1-2^{2-2^{\mathrm{lh}(\eta)}})\geq 2^{-2^{\mathrm{lh}(\eta)+1}}.$$

Now we may easily conclude that $q\in\mathbb{Q}^{\mathrm{mt}}_4(K,\Sigma,\mathbf{F})$ is special.

2) Let A be a front of T^p, p normal (so, in particular, $\mu^{\mathbf{F}}_p(\nu)>0$ for $\nu\in A$). Fix $a>1$ for a moment.

For each $\nu\in A$ pick a front A_ν of $(T^p)^{[\nu]}$ such that $\mu^{1_{A_\nu}}_{p,A_\nu}(\nu)<a\cdot\mu^{\mathbf{F}}_p(\nu)$. Let $B=\bigcup\limits_{\nu\in A}A_\nu$ and $f(\nu)=\mu^{\mathbf{F}}_p(\nu)$ for $\nu\in A$. By downward induction one can show that for all $\rho\in T[p,A]$ we have $\mu^{1_B}_{p,B}(\rho)\leq a\cdot\mu^f_{p,A}(\rho)$. Then, in particular, we have

$$\mu^{\mathbf{F}}(p)\leq\mu^{1_B}_{p,B}(\mathrm{root}(p))\leq a\cdot\mu^f_{p,A}(\mathrm{root}(p)),$$

and hence (letting $a\to 1$) $\mu^{\mathbf{F}}(p)\leq\mu^f_{p,A}(\mathrm{root}(p))$. The reverse inequality is even easier (remember 1.5(1)). $\square$

Lemma 1.11. *Let $p\in\mathbb{Q}^{\mathrm{mt}}_4(K,\Sigma)$ be a normal condition such that $\mu^{\mathbf{F}}(p)>\frac{1}{2}$, $\mathbf{nor}[t^p_\eta]>2$ for all $\eta\in T^p$, and let $k_0=\mathrm{lh}(\mathrm{root}(p))>4$, $0<\varepsilon\leq 2^{-(1+k_0)}$. Suppose that B is an antichain of T^p, and that for each $\nu\in B$ we are given a normal condition $q_\nu\geq p^{[\nu]}$ such that*

$$\mathrm{root}(q_\nu)=\nu\quad\textit{and}\quad\mu^{\mathbf{F}}(q_\nu)\geq 1-\varepsilon.$$

Then at least one of the following conditions holds.

(i) *There is a normal condition $q\in\mathbb{Q}^{\mathrm{mt}}_4(K,\Sigma,\mathbf{F})$ such that*

$$q\geq p,\quad \mathrm{root}(q)=\mathrm{root}(p),\quad\textit{and}\quad T^q\cap B=\emptyset.$$

(ii) *There is a normal condition $q\in\mathbb{Q}^{\mathrm{mt}}_4(K,\Sigma,\mathbf{F})$ such that*
- *$q\geq p$, $\mathrm{root}(q)=\mathrm{root}(p)$, $\mu^{\mathbf{F}}(q)\geq(1-2^{-k_0})\mu^{\mathbf{F}}(p)$, and*
- *$T^q\cap B$ is a front of T^q, and $q^{[\nu]}=q_\nu$ for $\nu\in T^q\cap B$, and*
- *if $\eta\in T^q$, $\eta\vartriangleleft\nu\in B$, then $\mathbf{nor}[t^q_\eta]\geq\mathbf{nor}[t^p_\eta]-2$.*

Proof. Let $e_\ell=2^{1-2^\ell}$ (for $\ell<\omega$); note that $(e_\ell)^2=2e_{\ell+1}$.

Fix $k>\mathrm{lh}(\mathrm{root}(p))$ for a while. Let A be a front of T^p such that

$$\{\nu\in B:\mathrm{lh}(\nu)\leq k\}\subseteq A\quad\text{and}\quad(\forall\nu\in A)(\nu\notin B\ \Rightarrow\ \mathrm{lh}(\nu)=k).$$

By downward induction, for each $\nu\in T[p,A]$, we define $r^0_\nu,r^1_\nu\in[0,1]$ and $s^0_\nu,s^1_\nu\in\Sigma(t^p_\nu)$ such that

(α) If $\nu\in A\cap B$, then $r^0_\nu=0$, $r^1_\nu=\mu^{\mathbf{F}}(q_\nu)$.
(β) If $\nu\in A\setminus B$, then $r^0_\nu=\mu^{\mathbf{F}}_p(\nu)$, $r^1_\nu=0$.
(γ) If $\nu\in T[p,A]\setminus A$, $\mathrm{lh}(\nu)=m$, then:

if $\mu^{\mathbf{F}}_p(\nu)\cdot(1-\varepsilon)\cdot\prod\limits_{\ell=m}^{k-1}(1-3e_\ell)<e_m$, then $r^0_\nu=r^1_\nu=0$,

else $r^0_\nu+r^1_\nu\geq\mu^{\mathbf{F}}_p(\nu)\cdot(1-\varepsilon)\cdot\prod\limits_{\ell=m}^{k-1}(1-3e_\ell)$.

Clauses $(\alpha), (\beta)$ define r^0_ν, r^1_ν for $\nu \in A$; s^0_ν, s^1_ν are not defined then (or are arbitrary).

Suppose $\eta \in T[p,A] \setminus A$, $\mathrm{lh}(\eta) = k-1$. If $\mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot (1-3e_{k-1}) < e_{k-1}$, then we let $r^0_\eta = r^1_\eta = 0$ (and s^0_η, s^1_η are not defined). So assume now that

$$\mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot (1-3e_{k-1}) \geq e_{k-1}.$$

Then also (as $r^0_\nu + r^1_\nu \geq \mu^{\mathbf{F}}_p(\nu) \cdot (1-\varepsilon)$ for $\nu \in \mathrm{pos}(t^p_\eta)$)

$$\begin{aligned} F_{t^p_\eta}(r^0_\nu + r^1_\nu : \nu \in \mathrm{pos}(t^p_\eta)) \geq \mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \geq \\ \mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot (1-3e_{k-1}) \geq \quad e_{k-1} > 2^{-2^{k-1}}, \end{aligned}$$

and we may apply 1.2(β) to pick r^0_η, r^1_η and $s^0_\eta, s^1_\eta \in \Sigma(t^p_\eta)$ such that

(i) $r^0_\eta + r^1_\eta \geq (1-e_{k-1}) \cdot F_{t^p_\eta}(r^0_\nu + r^1_\nu : \nu \in \mathrm{pos}(t^p_\eta)) \geq \mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot (1-3e_{k-1})$,

(ii) if $r^\ell_\eta > 0$, $\ell < 2$, then $\mathbf{nor}[s^\ell_\eta] \geq \mathbf{nor}[t^p_\eta] - 1$, $\mathrm{pos}(s^\ell_\eta) \subseteq \{\nu \in \mathrm{pos}(t^p_\eta) : r^\ell_\nu > 0\}$, and $F_{s^\ell_\eta}(r^\ell_\nu : \nu \in \mathrm{pos}(s^\ell_\eta)) \geq r^\ell_\eta$.

Suppose now that $\eta \in T[p,A] \setminus A$, $\mathrm{lh}(\eta) = m-1 < k-1$, and r^0_ν, r^1_ν have been defined for all $\nu \in \mathrm{pos}(t^p_\eta)$ (and they satisfy clause (γ)). If

$$\mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot \prod_{\ell=m-1}^{k-1} (1-3e_\ell) < e_{m-1},$$

then we let $r^0_\eta = r^1_\eta = 0$ (and s^0_η, s^1_η are not defined). So assume

$$\mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot \prod_{\ell=m-1}^{k-1} (1-3e_\ell) \geq e_{m-1}.$$

Then for $\nu \in \mathrm{pos}(t^p_\eta)$ we let

$$r^*_\nu = \begin{cases} r^0_\nu + r^1_\nu & \text{if } r^0_\nu + r^1_\nu > 0, \\ e_m & \text{otherwise,} \end{cases}$$

and we note that

$$F_{t^p_\eta}(r^*_\nu : \nu \in \mathrm{pos}(t^p_\eta)) \geq \mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot \prod_{\ell=m}^{k-1} (1-3e_\ell) \geq e_{m-1} > 2^{-2^{m-1}}.$$

Applying 1.2(β) choose $t^0, t^1 \in \Sigma(t^p_\eta)$ and c_0, c_1 such that $c_0 + c_1 \geq (1-e_{m-1}) F_{t^p_\eta}(r^*_\nu : \nu \in \mathrm{pos}(t^p_\eta))$ and

- if $c_0 > 0$, then $\mathrm{pos}(t^0) \subseteq \{\nu \in \mathrm{pos}(t^p_\eta) : r^0_\nu + r^1_\nu = 0\}$, $\mathbf{nor}[t^0] \geq \mathbf{nor}[t^p_\eta] - 1$ and $F_{t^0}(r^*_\nu : \nu \in \mathrm{pos}(t^0)) \geq c_0$,
- if $c_1 > 0$, then $\mathrm{pos}(t^1) \subseteq \{\nu \in \mathrm{pos}(t^p_\eta) : r^0_\nu + r^1_\nu > 0\}$, $\mathbf{nor}[t^1] \geq \mathbf{nor}[t^p_\eta] - 1$ and $F_{t^1}(r^*_\nu : \nu \in \mathrm{pos}(t^1)) \geq c_1$.

Now look at the definition of r^*_ν. If $c_0 > 0$, then $F_{t^0}(r^*_\nu : \nu \in \mathrm{pos}(t^0)) \leq e_m$, so $c_0 \leq e_m \leq (e_{m-1})^2$. Therefore

$$\begin{aligned} & F_{t^1}(r^*_\nu : \nu \in \mathrm{pos}(t^1)) \geq c_1 \geq (1-e_{m-1}) \cdot \mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot \prod_{\ell=m}^{k-1} (1-3e_\ell) - e_m \geq \\ & (1-e_{m-1}) \mu^{\mathbf{F}}_p(\eta)(1-\varepsilon) \cdot \prod_{\ell=m}^{k-1} (1-3e_\ell) - e_{m-1} \mu^{\mathbf{F}}_p(\eta)(1-\varepsilon) \cdot \prod_{\ell=m}^{k-1} (1-3e_\ell) = \\ & \mu^{\mathbf{F}}_p(\eta)(1-\varepsilon) \cdot \prod_{\ell=m}^{k-1} (1-3e_\ell) \cdot (1-2e_{m-1}) \geq e_{m-1} \cdot (1-2e_{m-1}) > 2^{-2^{m-1}}. \end{aligned}$$

Hence we may apply 1.2(β) again and get r^0_η, r^1_η and $s^0_\eta, s^1_\eta \in \Sigma(t^1) \subseteq \Sigma(t^p_\eta)$ such that

$$\begin{aligned} &r^0_\eta + r^1_\eta \geq (1-e_{m-1}) \cdot F_{t^1}(r^*_\nu : \nu \in \mathrm{pos}(t^1)) \geq \\ &\mu^{\mathbf{F}}_p(\eta)(1-\varepsilon) \cdot \prod_{\ell=m}^{k-1} (1-3e_\ell) \cdot (1-2e_{m-1}) \cdot (1-e_{m-1}) \geq \\ &\mu^{\mathbf{F}}_p(\eta) \cdot (1-\varepsilon) \cdot \prod_{\ell=m-1}^{k-1} (1-3e_\ell), \end{aligned}$$

and if $r^\ell_\eta > 0$, $\ell < 2$, then $\mathrm{pos}(s^\ell_\eta) \subseteq \{\nu \in \mathrm{pos}(t^p_\eta) : r^\ell_\nu > 0\}$, $\mathbf{nor}[s^\ell_\eta] \geq \mathbf{nor}[t^p_\eta] - 2$ and $F_{s^\ell_\eta}(r^\ell_\nu : \nu \in \mathrm{pos}(s^\ell_\eta)) \geq r^\ell_\eta$. This finishes the definition of $r^0_\nu, r^1_\nu, s^0_\nu$ and s^1_ν for $\nu \in T[p, A]$.

Note that (as $k_0 > 4$)

$$\varepsilon + \sum_{\ell=k_0}^{k-1} 3e_\ell \leq \frac{1}{2^{k_0+1}} + 6 \cdot \sum_{\ell=k_0}^{\infty} \frac{1}{2^{2^\ell}} \leq \frac{3}{2^{k_0+2}}.$$

Therefore,

$$\begin{aligned} \mu^{\mathbf{F}}_p(\mathrm{root}(p)) \cdot \prod_{\ell=k_0}^{k-1} (1-3e_\ell) \cdot (1-\varepsilon) \geq \mu^{\mathbf{F}}_p(\mathrm{root}(p)) \cdot (1 - (\varepsilon + \sum_{\ell=k_0}^{k-1} 3e_\ell)) \geq \\ \mu^{\mathbf{F}}_p(\mathrm{root}(p)) \cdot (1 - \tfrac{3}{2^{k_0+2}}) > \tfrac{1}{2} \cdot \tfrac{29}{32} > e_{k_0}. \end{aligned}$$

Hence also (by (γ))

$$\mu^{\mathbf{F}}_p(\mathrm{root}(p))(1 - \frac{3}{2^{k_0+2}}) \leq \mu^{\mathbf{F}}_p(\mathrm{root}(p)) \cdot \prod_{\ell=k_0}^{k-1} (1-3e_\ell) \cdot (1-\varepsilon) \leq r^0_{\mathrm{root}(p)} + r^1_{\mathrm{root}(p)}.$$

Now, if $r^\ell_{\mathrm{root}(p)} > 0$, $\ell < 2$, then we build inductively a finite tree $S^k_\ell \subseteq T[p, A]$ as follows. We declare that $\mathrm{root}(S^k_\ell) = \mathrm{root}(p)$, $s^{\ell,k}_{\mathrm{root}(p)} = s^\ell_{\mathrm{root}(p)}$, and $\mathrm{succ}_{S^k_\ell}(\mathrm{root}(p)) = \mathrm{pos}(s^{\ell,k}_{\mathrm{root}(p)})$. If we have decided that $\eta \in S^k_\ell$, $\eta \notin A$ (and $r^\ell_\eta > 0$), then we also declare $s^{\ell,k}_\eta = s^\ell_\eta$, $\mathrm{succ}_{S^k_\ell}(\eta) = \mathrm{pos}(s^{\ell,k}_\eta)$ (note $r^\ell_\nu > 0$ for $\nu \in \mathrm{pos}(s^{\ell,k}_\eta)$).

Then, if S^k_0 is defined, $S^k_0 \cap B = \emptyset$, and, if S^k_1 is defined, $S^k_1 \cap A \subseteq B$. Also, if we "extend" S^k_0 using $p^{[\nu]}$ (for $\nu \in S^k_0 \cap A$), then we get a condition $q^k_0 \geq p$ such that $\mu^{\mathbf{F}}(q^k_0) \geq r^0_{\mathrm{root}(p)} \stackrel{\mathrm{def}}{=} r^{0,k}$. Likewise, if we "extend" S^k_1 using q_ν (for $\nu \in S^k_1 \cap A$), then we get a condition $q^k_1 \geq p$ such that $\mu^{\mathbf{F}}(q^k_1) \geq r^1_{\mathrm{root}(p)} \stackrel{\mathrm{def}}{=} r^{1,k}$.

If for some $k > \mathrm{lh}(\mathrm{root}(p))$ we have $r^{1,k} \geq (1 - 2^{-k_0})\mu^{\mathbf{F}}(p)$, then we use the respective condition q^k_1 to witness the demand (ii) of the lemma. So assume that for each $k > \mathrm{lh}(\mathrm{root}(p))$ we have $r^{1,k} < (1 - 2^{-k_0})\mu^{\mathbf{F}}(p)$, and thus

$$r^{0,k} > (1 - \frac{3}{2^{k_0+2}})\mu^{\mathbf{F}}(p) - (1 - \frac{1}{2^{k_0}})\mu^{\mathbf{F}}(p) = \frac{1}{2^{k_0+2}}\mu^{\mathbf{F}}(p) > 0.$$

Apply the König Lemma to find an infinite set $I \subseteq \omega \setminus (k_0 + 1)$ such that for all $k, k', k'' \in I$, $k < k' < k''$, we have

$$(\forall \eta \in S^{k'}_0)(\mathrm{lh}(\eta) \leq k \ \Rightarrow\ \eta \in S^{k''}_0 \ \&\ s^{0,k'}_\eta = s^{0,k''}_\eta).$$

Then $S^q = \{\eta : (\forall^\infty k \in I)(\eta \in S^k_0)\}$, $s^q_\eta = s^{0,k}_\eta$ (for sufficiently large $k \in I$) determine a condition q witnessing the first assertion of the lemma. $\square$

Lemma 1.12. *Assume that $\dot\tau$ is a $\mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$–name for an ordinal, $n \le m < \omega$ and $p \in \mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$ is a normal condition such that $\mu^{\mathbf{F}}(p) > \frac{1}{2}$, and $\mathbf{nor}[t^p_\eta] > n+2$ for $\eta \in T^p$. Let $k_0 = {\rm lh}({\rm root}(p)) > 4$. Then there is a normal condition $q \in \mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$ such that*

(a) *$q \ge p$, ${\rm root}(q) = {\rm root}(p)$, $\mu^{\mathbf{F}}(q) \ge (1-2^{-k_0})\mu^{\mathbf{F}}(p)$, and*
(b) *$(\forall \eta \in T^q)(\mathbf{nor}[t^q_\eta] \ge n)$, and*
(c) *there is a front A of T^q such that for every $\nu \in A$:*
- *the condition $q^{[\nu]}$ forces a value to $\dot\tau$,*
- *$\mu^{\mathbf{F}}_q(\nu) > \frac{7}{8}$, ${\rm lh}(\nu) > k_0$,*
- *if $\nu \trianglelefteq \eta \in T^q$, then $\mathbf{nor}[t^q_\eta] \ge m$.*

Proof. Let B consist of all $\nu \in T^p$ such that

(α) ${\rm lh}(\nu) > k_0$ and there is a normal condition $q \in \mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$ stronger than $p^{[\nu]}$ and such that ${\rm root}(q) = \nu$, $\mu^{\mathbf{F}}(q) \ge (1-2^{-(2+k_0)})$, $(\forall \eta \in T^q)(\mathbf{nor}[t^q_\eta] \ge m)$, and for some front A of T^q, for every $\eta \in A$:
($\otimes$) $\mu^{\mathbf{F}}_q(\eta) > 7/8$ and the condition $q^{[\eta]}$ decides the value of $\dot\tau$,
and
(β) no initial segment of η has the property stated in (α) above.

Note that B is an antichain of T^p, and $B \cap T^{p'} \ne \emptyset$ for every condition $p' \ge p$ such that ${\rm root}(p') = {\rm root}(p)$ (by 1.8). For each $\nu \in B$ fix a condition q_ν witnessing clause (α) (for ν). Now apply 1.11: case (i) there is not possible by what we stated above, so we get a condition q as described in 1.11(ii). It should be clear that it is as required here. $\square$

Theorem 1.13. *Suppose that $p \in \mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$, and $\dot\tau_n$ are $\mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$–names for ordinals ($n<\omega$). Then there are a condition $q \ge p$ and fronts A_n of T^q (for $n<\omega$) such that for each $n<\omega$ and $\nu \in A_n$, the condition $q^{[\nu]}$ decides the value of $\dot\tau_n$.*

Proof. We may assume that p is normal, $k_0 = {\rm lh}({\rm root}(p)) > 4$, $\mu^{\mathbf{F}}(p) > \frac{1}{2}$, and $\mathbf{nor}[t^p_\eta] > 3$ for $\eta \in T^p$. We build inductively a sequence $\langle q_n, A_n : n<\omega\rangle$ such that

(1) $q_n \in \mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$ is a normal condition, ${\rm root}(q_n) = {\rm root}(p)$, $q_n \le q_{n+1}$, $q_0 = p$,
(2) $A_n \subseteq T^{q_{n+1}}$ is a front of $T^{q_{n+1}}$, $(\forall \nu \in A_n)(\exists \eta \in A_{n+1})(\nu \vartriangleleft \eta)$,
(3) if $\nu \in A_n$, then $\mu^{\mathbf{F}}_{q_{n+1}}(\nu) > \frac{7}{8}$, and for each $\eta \in T^{q_{n+1}}$ such that $\nu \trianglelefteq \eta$ we have $\mathbf{nor}[t^{q_{n+1}}_\eta] \ge n+4$,
(4) if ${\rm root}(p) \trianglelefteq \eta \vartriangleleft \nu \in A_n$, then $t^{q_{n+1}}_\eta = t^{q_{n+2}}_\eta$,
(5) for each $\nu \in A_n$, the condition $(q_{n+1})^{[\nu]}$ decides the value of $\dot\tau_n$,
(6) $\mu^{\mathbf{F}}(q_{n+1}) \ge \prod\limits_{\ell=k_0}^{k_0+n}(1-2^{-\ell}) \cdot \mu^{\mathbf{F}}(p)$.

The construction can be carried out by 1.12 (q_1, A_0 are obtained by applying 1.12 to p and $\dot\tau_0$; if q_{n+1}, A_n have been defined, then we apply 1.12 to $\dot\tau_{n+1}$ and $(q_{n+1})^{[\nu]}$ for $\nu \in A_n$; remember 1.5). Next define $q = \langle t^q_\eta : \eta \in T^q\rangle$ so that ${\rm root}(q) = {\rm root}(p)$, each A_n is a front of T^q, and if ${\rm root}(p) \trianglelefteq \eta \trianglelefteq \nu \in A_n$ then $t^q_\eta = t^{q_{n+1}}_\eta$. It is straightforward to check that q is as required in 1.13. $\square$

Corollary 1.14. *Let $(K,\Sigma,\mathbf{F})$ be a strongly finitary nice measured tree creating triple. Then the forcing notion $\mathbb{Q}_4^{\rm mt}(K,\Sigma,\mathbf{F})$ is proper and ω^ω–bounding.*

Let us recall the following definition.

Definition 1.15 (Goldstern [?, Definition 7.17])**.** Let $(\mathbb{P},\leq_{\mathbb{P}})$ be a definable forcing notion, $\mathbb{P}\subseteq\omega^\omega$, and let $\mathrm{epd}_{\mathbb{P}}$ be a relation on $\mathbb{P}\times[\mathbb{P}]^\omega$. We say that $(\mathbb{P},\leq_{\mathbb{P}},\mathrm{epd}_{\mathbb{P}})$ is a Souslin$^+$ proper forcing notion if

1. $\leq_{\mathbb{P}}$ is an analytic subset of $\omega^\omega\times\omega^\omega$, $\mathrm{epd}_{\mathbb{P}}$ is a Σ^1_2 set (both definitions are with a parameter r),
2. for each $(p,A)\in\mathbb{P}\times[\mathbb{P}]^\omega$, $\mathrm{epd}_{\mathbb{P}}(p,A)$ implies that A is predense above p,
3. if $(M,\in)$ is a countable model of ZFC*, $r\in M$ and $p\in\mathbb{P}^M$, then there is a condition $q\in\mathbb{P}$ stronger than p and such that
   $(*)$ if $A\in M$ and $M\models$ " A is predense above p ", then $\mathrm{epd}_{\mathbb{P}}(q,A)$.

Souslin$^+$ proper forcing notions are nep, so the results of [?] apply to them, see also Kellner [?], [?] and Kellner and Shelah [?].

Corollary 1.16. *Let $(K,\Sigma,\mathbf{F})$ be a strongly finitary nice measured tree–creating triple. Let $\mathbb{P}=\mathbb{Q}^{\rm mt}_4(K,\Sigma,\mathbf{F})$ and for $p\in\mathbb{P}$ and $A\in[\mathbb{P}]^\omega$ let*

$\mathrm{epd}_{\mathbb{P}}(p,A)\quad\Leftrightarrow$
there is a front $F\subseteq T^p$ such that $(\forall\eta\in F)(\exists p'\in A)(p'\leq p^{[\eta]})$.

Then $(\mathbb{P},\leq,\mathrm{epd}_{\mathbb{P}})$ is a Souslin$^+$ proper forcing notion.

The arguments for properness (and Souslin$^+$ properness) of the forcing notion $\mathbb{Q}^{\rm mt}_4(K,\Sigma,\mathbf{F})$ is essentially an Axiom A argument. However, to have an explicit representation of what was discussed above in the language of Axiom A, we need a small technical adjustment to our forcing.

Definition 1.17. Let $(K,\Sigma,\mathbf{F})$ be a strongly finitary nice measured tree–creating triple and $p\in\mathbb{Q}^{\rm mt}_4(K,\Sigma,\mathbf{F})$.

1. For $n<\omega$ let
$$B_n(p)=\big\{\eta\in T^p:\mu^{\mathbf{F}}_p(\eta)>\frac{1}{2}\ \ \&\ \ |\{\nu\in T^p:\nu\vartriangleleft\eta\ \&\ \mu^{\mathbf{F}}_p(\nu)>\frac{1}{2}\}|=n\big\}.$$
2. We say that the condition p is *super normal* if it is normal and for each $n<\omega$ the set $B_n(p)$ is a front of T^p.
3. Let $\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})=\{p\in\mathbb{Q}^{\rm mt}_4(K,\Sigma,\mathbf{F}):p$ is super normal $\}$.

Proposition 1.18. *$\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ is a dense subset of $\mathbb{Q}^{\rm mt}_4(K,\Sigma,\mathbf{F})$.*

Proof. It follows from the proof of 1.13 — the condition q constructed there is super normal. $\square$

Definition 1.19. Let $n<\omega$. We define a binary relation $\leq_n$ on $\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ by: $p\leq_n q$ if and only if $(p,q\in\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ and)

(α) $p\leq q$ (in $\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$), $\mathrm{root}(p)=\mathrm{root}(q)$, and
(β) $T[p,B_n(p)]\subseteq T^q$ and $\big(\forall\eta\in T[p,B_n(p)]\big)\big(t^q_\eta=t^p_\eta\big)$, and
(γ) if $\eta\in T^q$ and $\mathbf{nor}[t^q_\eta]\leq n$, then $t^q_\eta=t^p_\eta$,
(δ) if $\eta\in B_n(p)$, then $\mu^{\mathbf{F}}_q(\eta)\geq\big(1-r2^{-n-4}\big)\cdot\mu^{\mathbf{F}}_p(\eta)$, where
$$r=\min\big(\{\mu^{\mathbf{F}}_p(\nu)-\frac{1}{2}:\nu\in T[p,B_n(p)]\ \&\ \mu^{\mathbf{F}}_p(\nu)>\frac{1}{2}\}\big).$$

Proposition 1.20. (1) *For each $n<\omega$, $\leq_n$ is reflexive and $\leq_{n+1}\subseteq\leq_n\subseteq\leq$.*
(2) *If a sequence $\langle p_n:n<\omega\rangle\subseteq\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ satisfies $(\forall n\in\omega)(p_n\leq_n p_{n+1})$, then there is a condition $q\in\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ such that $(\forall n\in\omega)(p_{n+1}\leq_n q)$.*

(3) *If $\mathcal{I} \subseteq \mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ is an antichain, $p \in \mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$, $n<\omega$, then there is a condition $q \in \mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ such that $p \le_n q$ and the set $\{r \in \mathcal{I} : r,q$ are compatible $\}$ is finite.*

(4) *If $p,q,r \in \mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$, $n \in \omega$ and $p \le_{n+1} q \le_{n+1} r$, then $p \le_n r$.*

Remark 1.21. The relations $\le_n$ on $\mathbb{Q}^{\rm sn}_4(K,\Sigma,\mathbf{F})$ are not exactly like those needed to witness Baumgartner's Axiom A (see Baumgartner [**?**, §7]). However, the properties stated in 1.20 are enough to carry out the arguments of [**?**, §7]. We will use this in 4.7.

2. The Forcing

In this section we define a nice, strongly finitary measured tree creating triple $(K^*,\Sigma^*,\mathbf{F}^*)$, and we show several technical properties of it and of the forcing notion $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$. This forcing will be used in the next two sections to show our main results 3.2 and 4.15.

For each $k<\omega$, fix a function $\varphi_k : \omega \longrightarrow \omega$ such that

$$\varphi_k(0) = 2^{k+4} \quad \text{and} \quad \varphi_k(i+1) > \big(2^{2^{k+3}}+1\big)\cdot\varphi_k(i) + \frac{2^{2k+7}}{\log_2(1+2^{-2^{2k+7}})}.$$

Let $N_k = 2^{1+\lfloor \log_2(\varphi_k(k+1))\rfloor}$ (where $\lfloor r\rfloor$ is the integer part of the real number r), and let $\mathbf{H}^*(k) = 2^{N_k}$.

Let K^* consist of tree creatures $t \in \mathrm{LTCR}[\mathbf{H}^*]$ such that

- $\mathbf{dis}[t] = (k_t,\eta_t,n_t,g_t,P_t)$, where $n_t \le k_t < \omega$, $\eta_t \in \prod\limits_{i<k_t}\mathbf{H}^*(i)$, g_t is a partial function from N_{k_t} to 2 such that $|g_t| \le \varphi_{k_t}(k_t-n_t)$, and
$$\emptyset \ne P_t \subseteq \{f \in \mathbf{H}^*(k_t) : g_t \subseteq f\},$$
- $\mathbf{nor}[t] = n_t$,
- $\mathbf{val}[t] = \{\langle \eta_t,\nu\rangle : \eta_t \vartriangleleft \nu \in \prod\limits_{i\le k_t}\mathbf{H}^*(i)\ \&\ \nu(k_t) \in P_t\}$. (So $\mathrm{pos}(t) = P_t$.)

The operation Σ^* is trivial, and for $t \in K^*$:

$$\Sigma^*(t) = \{s \in K^* : \eta_s = \eta_t\ \&\ n_s \le n_t\ \&\ g_t \subseteq g_s\ \&\ P_s \subseteq P_t\}.$$

Finally, for $t \in K^*$ and $\langle r_\nu : \nu \in \mathrm{pos}(t)\rangle \subseteq [0,1]$ we let

$$\begin{aligned} &F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) = \\ &\min\{2^{|h|-N_{k_t}}\cdot \textstyle\sum\{r_\nu : h \subseteq \nu(k_t) \in P_t\} : \ h \text{ is a partial function from } N_{k_t} \text{ to } 2,\\ &\qquad\qquad g_t \subseteq h \ \text{ and } \ |h\setminus g_t| \le 2^{k_t+3}\}. \end{aligned}$$

(So this defines $\mathbf{F}^* = \langle F^*_t : t \in K^*\rangle$.)

It should be clear that $(K^*,\Sigma^*,\mathbf{F}^*)$ is a strongly finitary measured tree creating triple. (And now we are aiming at showing that it is nice, see 1.2.)

Lemma 2.1. *Assume that $t \in K^*$, $\mathbf{nor}[t] > 1$, and g' is a partial function from N_{k_t} to 2 such that $g' \supseteq g_t$ and $|g'\setminus g_t| \le 2^{k_t+3}$. Furthermore, suppose that $r_\nu \in [0,1]$ (for $\nu \in \mathrm{pos}(t)$) are such that*

$$2^{-2^{k_t+3}} \le 2^{|g'|-N_{k_t}} \cdot \sum\{r_\nu : \nu \in \mathrm{pos}(t)\ \&\ g' \subseteq \nu(k_t)\} \stackrel{\rm def}{=} a.$$

Then there is $s \in \Sigma^*(t)$ such that

(α) $\mathbf{nor}[s] = \mathbf{nor}[t]-1$, $g' \subseteq g_s$,

(β) $F^*_s(r_\nu:\nu\in{\rm pos}(s))\geq a\cdot(1-2^{-2^{k_t+3}})$,

(γ) *if h is a partial function from N_{k_s} to 2 such that $g_s\subseteq h$ and $|h\setminus g_s|\leq 2^{k_s+3}$, then*

$$\frac{\sum\{r_\nu:\nu\in{\rm pos}(s)\ \&\ h\subseteq\nu(k_s)\}}{2^{N_{k_s}-|h|}}$$

is in the interval $[F^*_s(r_\nu:\nu\in{\rm pos}(s)),F^*_s(r_\nu:\nu\in{\rm pos}(s))\cdot(1+2^{-2^{k+3}})]$.

Proof. Let $k=k_t$, $n=n_t$.

We try to choose inductively partial functions g_ℓ from N_k to 2 such that

(a) $g'=g_0\subseteq g_1\subseteq\ldots$, $|g_\ell\setminus g'|\leq\ell\cdot 2^{k+3}$,

(b)$_\ell$ $2^{|g_\ell|-N_k}\cdot\sum\{r_\nu:\nu\in{\rm pos}(t)\ \&\ g_\ell\subseteq\nu(k)\}\geq a\cdot(1+2^{-2^{2k+7}})^\ell$.

Note that in (b)$_\ell$, the left hand side expression is not more than 1, so if the inequality holds, then (as $a\geq 2^{-2^{k+3}}$)

$$(\oplus)\qquad \ell\leq\frac{2^{k+3}}{\log_2(1+2^{-2^{2k+7}})}.$$

Consequently, in the procedure described above, we are stuck at some ℓ_0 satisfying ($\oplus$). Let

$$g_s=g_{\ell_0},\quad n_s=n-1,\quad k_s=k,\quad \eta_s=\eta_t,\quad P_s=\{f\in P_t:g_{\ell_0}\subseteq f\}.$$

So this defines s, but we have to check that $s\in K^*$. For this note that

$$|g_s|\leq|g'|+\ell_0\cdot 2^{k+3}\leq\varphi_k(k-n)+2^{k+3}+\frac{2^{2k+6}}{\log_2(1+2^{-2^{2k+7}})}\leq\varphi_k(k-n_s).$$

(So indeed $s\in K^*$, and plainly $s\in\Sigma^*(t)$.) Also note that

$$2^{|g_s|-N_k}\cdot\sum\{r_\nu:\nu\in{\rm pos}(s)\}\geq a\cdot(1+2^{-2^{2k+7}})^{\ell_0}\stackrel{\rm def}{=}a^*\geq a.$$

Now, suppose that $u\subseteq N_k\setminus{\rm dom}(g_s)$, $|u|\leq 2^{k+3}$. Let $h:u\longrightarrow 2$. We cannot use $g_s{}^\frown h$ as g_{ℓ_0+1}, so the condition (b)$_{\ell_0+1}$ fails for it. Therefore

$$b_h\stackrel{\rm def}{=}2^{|g_s|+|h|-N_k}\cdot\sum\{r_\nu:\nu\in{\rm pos}(t)\ \&\ g_s{}^\frown h\subseteq\nu(k)\}<$$
$$a\cdot(1+2^{-2^{2k+7}})^{\ell_0+1}=a^*\cdot(1+2^{-2^{2k+7}}).$$

Claim 2.1.1. *For each $h:u\longrightarrow 2$, we have*

$$b_h\geq a^*\cdot(1-2^{-2^{k+4}}).$$

Proof of the claim. Assume that $h_0:u\longrightarrow 2$ is such that $b_{h_0}<a^*\cdot(1-2^{-2^{k+4}})$. We know that $b_h<a^*\cdot(1+2^{-2^{2k+7}})$ for each $h:u\longrightarrow 2$, so

$$a^*\cdot 2^{N_k-|g_s|}\leq\sum\{r_\nu:\nu\in{\rm pos}(s)\}\leq$$
$$a^*\cdot(1-2^{-2^{k+4}})\cdot 2^{N_k-|g_s|-|u|}+a^*\cdot(1+2^{-2^{2k+7}})\cdot(2^{|u|}-1)\cdot 2^{N_k-|g_s|-|u|}.$$

Hence

$$2^{|u|}\leq(1-2^{-2^{k+4}})+(1+2^{-2^{2k+7}})\cdot(2^{|u|}-1)=$$
$$2^{|u|}(1+2^{-2^{2k+7}})-(2^{-2^{k+4}}+2^{-2^{2k+7}}),$$

and so $2^{-2^{k+4}}\leq 2^{-2^{k+4}}+2^{-2^{2k+7}}\leq 2^{|u|}\cdot 2^{-2^{2k+7}}\leq 2^{2^{k+3}-2^{2k+7}}$, a contradiction. $\square$

Consequently, we get that

$$F^*_s(r_\nu : \nu \in \mathrm{pos}(s)) \geq a^* \cdot (1 - 2^{-2^{k+4}}) \geq a \cdot (1 - 2^{-2^{k+3}}),$$

so s satisfies the demand (β).

But we also know that for each partial function h from N_k to 2, if $g_s \subseteq h$ and $|h \setminus g_s| \leq 2^{k+3}$, then

$$b_h < a^* \cdot (1 + 2^{-2^{2k+7}}) \leq F^*_s(r_\nu : \nu \in \mathrm{pos}(s)) \cdot \tfrac{1+2^{-2^{2k+7}}}{1-2^{-2^{k+4}}} \leq \\ F^*_s(r_\nu : \nu \in \mathrm{pos}(s)) \cdot (1 + 2^{-2^{k+3}}),$$

and thus s satisfies the demand (γ) as well. □

Proposition 2.2. $(K^*, \Sigma^*, \mathbf{F}^*)$ *is a nice (strongly finitary) measured tree creating triple.*

Proof. Clauses $1.2(\alpha, \gamma, \delta)$ should be obvious, so let us check $1.2(\beta)$ only.

Let $t \in K^*$, $k = k_t$, r^0_ν, r^1_ν, r_ν be as in the assumptions of $1.2(\beta)$. So in particular

$$2^{|g_t|-N_k} \cdot \sum\{r_\nu : \nu \in \mathrm{pos}(t)\} \geq F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \geq 2^{-2^k} > 2^{-2^{k+3}}.$$

For $\ell < 2$ let $a_\ell = 2^{|g_t|-N_k} \cdot \sum\{r^\ell_\nu : \nu \in \mathrm{pos}(t)\}$.

First, we consider the case when both a_0 and a_1 are not smaller than $2^{-2^{k+3}}$. Then we may apply 2.1 and get $s_0, s_1 \in \Sigma^*(t)$ such that $\mathbf{nor}[s_\ell] = \mathbf{nor}[t] - 1$, $\mathrm{pos}(s_\ell) \subseteq \{\nu \in \mathrm{pos}(t) : r^\ell_\nu > 0\}$ and

$$c_\ell \stackrel{\mathrm{def}}{=} F^*_{s_\ell}(r^\ell_\nu : \nu \in \mathrm{pos}(s_\ell)) \geq a_\ell \cdot (1 - 2^{-2^{k+3}}).$$

Then

$$c_0 + c_1 \geq (a_0 + a_1) \cdot (1 - 2^{-2^{k+3}}) \geq F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \cdot (1 - 2^{-2^k}),$$

and we are done.

So suppose now that $a_\ell < 2^{-2^{k+3}}$. Then

$$a_{1-\ell} \geq 2^{|g_t|-N_k} \cdot \sum\{r_\nu : \nu \in \mathrm{pos}(t)\} - 2^{-2^{k+3}} \geq 2^{-2^k} - 2^{-2^{k+3}} \geq 2^{-2^{k+3}},$$

and using 2.1 we find $s_{1-\ell} \in \Sigma^*(t)$ such that $\mathbf{nor}[s_{1-\ell}] = \mathbf{nor}[t] - 1$, $\mathrm{pos}(s_{1-\ell}) \subseteq \{\nu \in \mathrm{pos}(t) : r^{1-\ell}_\nu > 0\}$, and

$$\begin{aligned}
&c_{1-\ell} \stackrel{\mathrm{def}}{=} F^*_{s_{1-\ell}}(r^{1-\ell}_\nu : \nu \in \mathrm{pos}(s_{1-\ell})) \geq a_{1-\ell} \cdot (1 - 2^{-2^{k+3}}) \geq \\
&(F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) - 2^{-2^{k+3}}) \cdot (1 - 2^{-2^{k+3}}) = \\
&F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \cdot (1 - 2^{-2^k} + 2^{-2^k} - 2^{-2^{k+3}}) - 2^{-2^{k+3}} + 2^{-2^{k+4}} \geq \\
&F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \cdot (1 - 2^{-2^k}) + 2^{-2^k}(2^{-2^k} - 2^{-2^{k+3}}) - 2^{-2^{k+3}} + 2^{-2^{k+4}} = \\
&F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \cdot (1 - 2^{-2^k}) + 2^{-2^{k+1}} - 2^{-9 \cdot 2^k} - 2^{-2^{k+3}} + 2^{-2^{k+4}} \geq \\
&F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \cdot (1 - 2^{-2^k}).
\end{aligned}$$

□

The following lemma and the proposition are, as a matter of fact, included in 2.6, 2.7. However, we decided that 2.3 and 2.4 could be a good warm-up, and also we will use their proofs later.

Lemma 2.3. *Assume that:*

(i) $t \in K^*$, $\mathbf{nor}[t] > 1$, $k = k_t$, $\gamma \in [0, 1]$,

(ii) $\langle r_\nu :\in \mathrm{pos}(t)\rangle \subseteq [0, 1]$, $a = F^*_t(r_\nu : \nu \in \mathrm{pos}(t))$, $\gamma \cdot a \geq 2^{-6 \cdot 2^k}$,

(iii) *Y is a finite non-empty set,*
(iv) *for $\nu \in \mathrm{pos}(t)$, u_ν is a function from Y to $[0,1]$ such that*

$$\gamma \cdot r_\nu \cdot |Y| \leq \sum\{u_\nu(y) : y \in Y\},$$

(v) *for $y \in Y$ we let*

$$u(y) = \sup\{b : \text{ there is } s \in \Sigma^*(t) \text{ such that } \mathbf{nor}[s] \geq \mathbf{nor}[t] - 1 \text{ and } b \leq F^*_s(u_\nu(y) : \nu \in \mathrm{pos}(s))\}.$$

Then

$$\gamma \cdot a \cdot (1 - 2^{-2^k}) \leq \frac{\sum\{u(y) : y \in Y\}}{|Y|}.$$

Proof. Let $k = k_t$, $N = N_{k_t}$, $g = g_t$.

First note that

$$a = F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \leq 2^{|g|-N} \cdot \sum\{r_\nu : \nu \in \mathrm{pos}(t)\} \leq$$
$$2^{|g|-N} \cdot \tfrac{1}{\gamma} \cdot \tfrac{1}{|Y|} \cdot \sum_{\nu \in \mathrm{pos}(t)} \sum_{y \in Y} u_\nu(y) = \tfrac{1}{\gamma} \cdot \tfrac{1}{|Y|} \cdot \sum_{y \in Y} \Big(2^{|g|-N} \cdot \sum_{\nu \in \mathrm{pos}(t)} u_\nu(y)\Big).$$

Let $C \stackrel{\rm def}{=} \{y \in Y : 2^{|g|-N} \cdot \sum\limits_{\nu \in \mathrm{pos}(t)} u_\nu(y) \geq 2^{-2^{k+3}}\}$. For each $y \in C$ we may use 2.1 to pick $s_y \in \Sigma^*(t)$ such that $\mathbf{nor}[s_y] \geq \mathbf{nor}[t] - 1$ and

$$F^*_{s_y}(u_\nu(y) : \nu \in \mathrm{pos}(s_y)) \geq 2^{|g|-N} \cdot \sum_{\nu \in \mathrm{pos}(t)} u_\nu(y) \cdot (1 - 2^{-2^{k+3}}).$$

Hence,

$$a \leq \tfrac{1}{\gamma} \cdot \tfrac{|Y \setminus C|}{|Y|} \cdot 2^{-2^{k+3}} + \tfrac{1}{\gamma} \cdot \tfrac{1}{|Y|} \cdot \sum_{y \in C} \tfrac{F^*_{s_y}(u_\nu(y) : \nu \in \mathrm{pos}(s_y))}{1 - 2^{-2^{k+3}}} \leq$$
$$\tfrac{1}{\gamma} \cdot 2^{-2^{k+3}} + \tfrac{1}{\gamma} \cdot \tfrac{1}{|Y|} \cdot \tfrac{1}{1 - 2^{-2^{k+3}}} \cdot \sum_{y \in C} u(y).$$

Consequently,

$$(\gamma a - 2^{-2^{k+3}})(1 - 2^{-2^{k+3}}) \leq \frac{\sum\limits_{y \in C} u(y)}{|Y|} \leq \frac{\sum\limits_{y \in Y} u(y)}{|Y|},$$

and hence

$$\gamma a (1 - 2^{-2^k}) \leq \frac{\sum\limits_{y \in Y} u(y)}{|Y|}.$$

□

Proposition 2.4. *The forcing notion $\mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*)$ preserves outer (Lebesgue) measure one.*

Proof. Assume that $A \subseteq \prod\limits_{i<\omega} N_i$ is a set of outer (Lebesgue) measure 1. We are going to show that, in $\mathbf{V}^{\mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*)}$, it is still an outer measure one set.

Let $\dot{T}$ be a $\mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*)$–name for a tree such that $\dot{T} \subseteq \bigcup\limits_{i \in \omega} \prod\limits_{j<i} N_i$ and the Lebesgue measure $m^{\rm Leb}([\dot{T}])$ of the set $[\dot{T}]$ of ω–branches through $\dot{T}$ is positive, and suppose that some condition p forces "$[\dot{T}] \cap A = \emptyset$". Take a condition $q \geq p$ such that

(α) q is special (remember 1.10) and $\mathrm{lh}(\mathrm{root}(q)) = k_0 > 5$, and $\mathbf{nor}[t^q_\eta] > 2$ for all $\eta \in T^q$, and $\mu^{\mathbf{F}^*}(q) > \frac{1}{2}$,

(β) for some $\rho \in \prod\limits_{j<n} N_j$, $n < \omega$, the condition q forces that $m^{\mathrm{Leb}}([(\dot{T})^{[\rho]}]) \cdot \prod\limits_{j<n} N_j \geq \frac{7}{8}$,

(γ) for some $k_0 < k_1 < k_2 < \ldots$, letting $F_i = T^q \cap \prod\limits_{m<k_i} \mathbf{H}^*(m)$, we have that for each $\nu \in F_i$, the condition $q^{[\nu]}$ decides the value of $\dot{T} \cap \prod\limits_{j<n+i} N_j$ (remember 1.13).

Fix $i < \omega$ for a moment, and let $Y_i = \{y \in \prod\limits_{j<n+i} N_j : \rho \trianglelefteq y\}$.

For $\nu \in T[q, F_i]$ and $y \in Y_i$ we let

$$u_\nu(y) = \sup\{\mu^{\mathbf{F}^*}(q') : \begin{array}{l} q' \text{ is a condition stronger than } q \text{ and such that} \\ \mathrm{root}(q') = \nu \ \text{ and } \ (\forall \eta \in T^{q'})(\mathbf{nor}[t^{q'}_\eta] \geq \mathbf{nor}[t^q_\eta] - 1) \\ \text{and } \ q' \Vdash y \in \dot{T}\}. \end{array}$$

Claim 2.4.1. *If $\eta \in T[q, F_i]$, $k_0 \leq \mathrm{lh}(\eta) = k \leq k_i$, then*

$$\frac{7}{8} \cdot \prod_{\ell=k}^{k_i-1} (1 - 2^{-2^\ell}) \cdot |Y_i| \cdot \mu^{\mathbf{F}^*}_q(\eta) \leq \sum \{u_\eta(y) : y \in Y_i\}.$$

[If $k = k_i$, then we stipulate $\prod\limits_{\ell=k}^{k_i-1} (1 - 2^{-2^\ell}) = 1$.]

Proof of the claim. We show it by downward induction on $\eta \in T[q, F_i]$. If $k = \mathrm{lh}(\eta) = k_i$, then $q^{[\eta]}$ decides $\dot{T} \cap Y_i$, and if $q^{[\eta]}$ forces that $y \in \dot{T} \cap Y_i$, then $u_\eta(y) \geq \mu^{\mathbf{F}^*}_q(\eta)$. Hence, by ($\beta$), we have $\frac{7}{8} \cdot |Y_i| \cdot \mu^{\mathbf{F}^*}_q(\eta) \leq \sum\{u_\eta(y) : y \in Y_i\}$.

Let us assume now that $k = \mathrm{lh}(\eta) = k_i - 1$. Apply 2.3 to t^q_η, $\gamma = \frac{7}{8}$, $Y = Y_i$, u_ν defined as before the formulation of the claim, and $r_\nu = \mu^{\mathbf{F}^*}_q(\nu)$ (for $\nu \in \mathrm{pos}(t^q_\eta)$). Note that, as q is special, $\mu^{\mathbf{F}^*}_q(\eta) \geq 2^{-2^{k+1}}$, so $\gamma \cdot F^*_{t^q_\eta}(r_\nu : \nu \in \mathrm{pos}(t^q_\eta)) = \frac{7}{8} \mu^{\mathbf{F}^*}_q(\eta) > 2^{-6 \cdot 2^k}$. Also note that

($*$) $u(y)$ defined as in 2.3(v) is $u_\eta(y)$.

[Why? First suppose that $u(y) < u_\eta(y)$. By the definition of u_η we may find $q' \geq q$ such that $\mathrm{root}(q') = \eta$, $\mathbf{nor}[t^{q'}_\nu] \geq \mathbf{nor}[t^q_\nu] - 1$ for $\nu \in T^{q'}$, and $q' \Vdash y \in \dot{T}$, and $\mu^{\mathbf{F}^*}(q') > u(y)$. Note that $\mu^{\mathbf{F}^*}_{q'}(\nu) \leq u_\nu(y)$ for all $\nu \in \mathrm{pos}(t^{q'}_\eta)$, and thus

$$u(y) < \mu^{\mathbf{F}^*}(q') = F^*_{t^{q'}_\eta}(\mu^{\mathbf{F}^*}_{q'}(\nu) : \nu \in \mathrm{pos}(t^{q'}_\eta)) \leq F^*_{t^{q'}_\eta}(u_\nu(y) : \nu \in \mathrm{pos}(t^{q'}_\eta)).$$

By the definition of $u(y)$, the last expression is $\leq u(y)$, a contradiction. Now suppose $u(y) > u_\eta(y)$. Take $s \in \Sigma^*(t^q_\eta)$ such that $\mathbf{nor}[s] \geq \mathbf{nor}[t^q_\eta] - 1$ and $F^*_s(u_\nu(y) : \nu \in \mathrm{pos}(s)) > u_\eta(y)$; clearly we may request that $u_\nu(y) > 0$ for $\nu \in \mathrm{pos}(s)$. Let $z_\nu < u_\nu(y)$ (for $\nu \in \mathrm{pos}(s)$) be positive numbers such that if $z_\nu \leq r_\nu \leq u_\nu(y)$ for $\nu \in \mathrm{pos}(s)$, then $F^*_s(r_\nu : \nu \in \mathrm{pos}(s)) > u_\eta(y)$ (compare 1.3). Pick conditions q'_ν such that $\mu^{\mathbf{F}^*}(q'_\nu) > z_\nu$, q'_ν as in definition of $u_\nu(y)$, and let q' be such that $\mathrm{root}(q') = \eta$, $t^{q'}_\eta = s$, and $(q')^{[\nu]} = q'_\nu$ for $\nu \in \mathrm{pos}(s)$. Then $\mu^{\mathbf{F}^*}(q') > u_\eta(y)$ giving an easy contradiction.]

Thus we get

$$\frac{7}{8}\cdot\mu^{\mathbf{F}^*}_q(\eta)\cdot(1-2^{-2^{k_i-1}})\cdot|Y_i|\leq\sum\{u_\eta(y):y\in Y_i\},$$

as required.

Now suppose $k_0\leq k=\mathrm{lh}(\eta)<k_i-1$, and we have proved the assertion of the claim for all $\nu\in\mathrm{pos}(t^q_\eta)$. We again apply 2.3, this time to $\gamma=\frac{7}{8}\cdot\prod\limits_{\ell=k+1}^{k_i-1}(1-2^{-2^\ell})$, and t^q_η, u_ν, $r_\nu=\mu^{\mathbf{F}^*}_q(\nu)$ (for $\nu\in\mathrm{pos}(t^q_\eta)$) and Y_i. We note that

$$\tfrac{7}{8}\cdot\prod_{\ell=k+1}^{k_i-1}(1-2^{-2^\ell})\cdot F^*_{t^q_\eta}(r_\nu:\nu\in\mathrm{pos}(t^q_\eta))=\tfrac{7}{8}\cdot\prod_{\ell=k+1}^{k_i-1}(1-2^{-2^\ell})\cdot\mu^{\mathbf{F}^*}_q(\eta)\geq$$
$$\tfrac{7}{8}\cdot(1-2^{1-2^{k+1}})\cdot 2^{-2^{k+1}}\geq 2^{-6\cdot 2^k},$$

so the assumptions of 2.3 are satisfied. Therefore we may conclude that

$$\frac{7}{8}\cdot\prod_{\ell=k+1}^{k_i-1}(1-2^{-2^\ell})\cdot\mu^{\mathbf{F}^*}_q(\eta)\cdot(1-2^{-2^k})\cdot|Y_i|\leq\sum\{u_\eta(y):y\in Y_i\},$$

as needed. This finishes the proof of 2.4.1. $\square$

Applying 2.4.1 to $\eta=\mathrm{root}(q)$ we get

$$\frac{7}{8}\cdot\prod_{\ell=k_0}^{k_i-1}(1-2^{-2^\ell})\cdot\mu^{\mathbf{F}^*}(q)\leq\frac{\sum\big\{u_{\mathrm{root}(q)}(y):y\in Y_i\big\}}{|Y_i|},$$

and hence $\frac{3}{4}\mu^{\mathbf{F}^*}(q)\cdot|Y_i|\leq\sum\{u_{\mathrm{root}(q)}(y):y\in Y_i\}$. Then necessarily

$$\frac{1}{4}|Y_i|\leq|\{y\in Y_i:u_{\mathrm{root}(p)}(y)\geq\frac{1}{4}\mu^{\mathbf{F}^*}(q)\}|$$

(remember $\mu^{\mathbf{F}^*}(q)>\frac{1}{2}$). Let $Z_i=\{y\in Y_i:u_{\mathrm{root}(p)}(y)\geq\frac{1}{4}\mu^{\mathbf{F}^*}(q)\}$ and note that

$$m^{\mathrm{Leb}}\Big(\{x\in\prod_{j<\omega}N_j:x{\restriction}(n+i)\in Z_i\}\Big)\geq\big(4\prod_{j<n}N_j\big)^{-1}.$$

Look at the set $\{x\in\prod\limits_{i<\omega}N_i:(\exists^\infty i<\omega)(x\restriction(n+i)\in Z_i)\}$ – it is a Borel set of positive (Lebesgue) measure, and therefore we may pick $x\in A$ such that $(\exists^\infty i<\omega)(x\restriction(n+i)\in Z_i)$. For each $i<\omega$ such that $x\restriction(n+i)\in Z_i$ choose a condition $q_i\in\mathbb{Q}^{\mathrm{mt}}_4(K^*,\Sigma^*,\mathbf{F}^*)$ such that

- $q_i\geq q$, $\mathrm{root}(q_i)=\mathrm{root}(q)$, $\mu^{\mathbf{F}^*}(q_i)>\frac{1}{8}\mu^{\mathbf{F}^*}(q)$, and
- $(\forall\eta\in T^{q_i})(\mathbf{nor}[t^{q_i}_\eta]\geq\mathbf{nor}[t^q_\eta]-1)$, and
- $q_i\Vdash x\restriction(n+i)\in\dot{T}$.

By König's Lemma (remember (K^*,Σ^*) is strongly finitary) we find an infinite set $I\subseteq\omega$ such that for each $i<j_0<j_1$ from I we have

$$T^{q_{j_0}}\cap\prod_{k<k_i}\mathbf{H}(k)=T^{q_{j_1}}\cap\prod_{k<k_i}\mathbf{H}(k)\quad\text{and}\quad(\forall\eta\in T^{q_{j_0}})(\mathrm{lh}(\eta)<k_i\ \Rightarrow\ t^{q_{j_0}}_\eta=t^{q_{j_1}}_\eta).$$

Let $q^*=\langle s_\eta:\eta\in S\rangle$ be such that $\mathrm{root}(S)=\mathrm{root}(q)$,

$$S=\bigcup_{i\in I}\{\eta\in T^{q_j}:j\in I\ \&\ i<j\ \&\ \mathrm{lh}(\eta)<k_i\},$$

and if $\eta \in S$, then $\mathrm{succ}_S(\eta) = \mathrm{pos}(s_\eta)$ and $s_\eta = t^{q_i}_\eta$ for sufficiently large $i \in I$. It should be clear that $q^* \in \mathbb{Q}^{\mathrm{mt}}_4(K^*, \Sigma^*, \mathbf{F}^*)$ is a condition stronger than q, and it forces that $x \in [\dot{T}] \cap A$, a contradiction. $\square$

Remark 2.5. It follows from 1.16 and the proof of 2.4 that (the definition of) the forcing notion $\mathbb{P} = \mathbb{Q}^{\mathrm{mt}}_4(K^*, \Sigma^*, \mathbf{F}^*)$ satisfies:

($\heartsuit$) For any transitive model N of ZFC*,

$$N \models \text{“ } \mathbb{P} \text{ is a Souslin}^+ \text{ proper forcing notion and it forces that the old reals are of positive outer Lebesgue measure ”.}$$

By Kellner and Shelah [**?**, Corollary 9.4], any CS iteration of forcing notions satisfying ($\heartsuit$) (in particular, a CS iteration of $\mathbb{Q}^{\mathrm{mt}}_4(K^*, \Sigma^*, \mathbf{F}^*)$) preserves the outer measure of sets from the ground model.

Lemma 2.6. *Assume that:*

(i) $t \in K^*$, $\mathbf{nor}[t] > 1$, $k = k_t > 1$, $\gamma \in [0, 1]$,

(ii) $\langle r_\nu :\in \mathrm{pos}(t)\rangle \subseteq [0, 1]$, $a = F^*_t(r_\nu : \nu \in \mathrm{pos}(t))$, $\gamma \cdot a \geq 2^{-6\cdot 2^k}$,

(iii) Y^* *is a finite set,* $Y = Y^* \times N_k$,

(iv) *for* $\nu \in \mathrm{pos}(t)$, u_ν *is a function from* Y *to* $[0, 1]$ *such that*

$$\gamma \cdot r_\nu \cdot |Y| \leq \sum\{u_\nu(y) : y \in Y\},$$

(v) *for* $y = (y_0, y_1) \in Y^* \times N_k$ *and* $\ell < 2$ *we let*

$$u(y, \ell) = \sup\{b : \ \text{there is } s \in \Sigma^*(t) \text{ such that } \mathbf{nor}[s] \geq \mathbf{nor}[t] - 1 \text{ and } (\forall \nu \in \mathrm{pos}(s))(\nu(k)(y_1) = \ell) \text{ and } b \leq F^*_s(u_\nu(y) : \nu \in \mathrm{pos}(s))\}.$$

Then

$$\gamma \cdot a \cdot (1 - 2^{-2^k}) \leq \frac{1}{2 \cdot |Y|} \sum\{u(y, \ell) : y \in Y \ \& \ \ell < 2\}.$$

Proof. Let $k = k_t$, $N = N_k$, $g = g_t$. Note that

$$a = F^*_t(r_\nu : \nu \in \mathrm{pos}(t)) \leq 2^{|g|-N} \cdot \sum\{r_\nu : \nu \in \mathrm{pos}(t)\} \leq$$
$$2^{|g|-N} \cdot \tfrac{1}{\gamma} \cdot \tfrac{1}{|Y|} \cdot \sum_{\nu \in \mathrm{pos}(t)} \sum_{\ell < 2} \Big(\sum\{u_\nu(y_0, y_1) : (y_0, y_1) \in Y \ \& \ \nu(k)(y_1) = \ell\}\Big) =$$
$$\tfrac{1}{\gamma} \cdot \tfrac{1}{2\cdot|Y|} \cdot \sum_{(y_0, y_1, \ell) \in Y \times 2} \Big(2^{|g|-N+1} \cdot \sum\{u_\nu(y_0, y_1) : \nu \in \mathrm{pos}(t) \ \& \ \nu(k)(y_1) = \ell\}\Big).$$

Let C consist of all triples $(y_0, y_1, \ell) \in Y^* \times N \times 2$ such that $y_1 \notin \mathrm{dom}(g)$ and

$$2^{|g|+1-N} \cdot \sum\{u_\nu(y_0, y_1) : \nu \in \mathrm{pos}(t) \ \& \ \nu(k)(y_1) = \ell\} \geq 2^{-2^{k+3}},$$

and fix $(y_0, y_1, \ell) \in C$ for a moment. Let $g' : \mathrm{dom}(g) \cup \{y_1\} \longrightarrow 2$ be such that $g \subseteq g'$ and $g'(y_1) = \ell$. Apply 2.1 (to t, g' and $u_\nu(y_0, y_1)$ for $\nu \in \mathrm{pos}(t)$, $g' \subseteq \nu(k)$) to pick $s = s(y_0, y_1, \ell) \in \Sigma^*(t)$ such that $\mathbf{nor}[s] \geq \mathbf{nor}[t] - 1$, $g' \subseteq g_s$ and

$$\frac{F^*_s(u_\nu(y_0, y_1) : \nu \in \mathrm{pos}(s))}{1 - 2^{-2^{k+3}}} \geq 2^{|g|+1-N} \cdot \sum\{u_\nu(y_0, y_1) : \nu \in \mathrm{pos}(t) \ \& \ g' \subseteq \nu(k)\}.$$

Next note that $\frac{|g|}{N} < 2^{-2^{k+3}}$, so

$$\tfrac{1}{\gamma\cdot|Y|} \cdot \sum_{(y_0, y_1, \ell) \in Y \times 2 \setminus C} \Big(2^{|g|-N} \cdot \sum\{u_\nu(y_0, y_1) : \nu \in \mathrm{pos}(t) \ \& \ \nu(k)(y_1) = \ell\}\Big) \leq$$
$$\tfrac{|g|}{\gamma \cdot N} + \tfrac{1}{\gamma} \cdot 2^{-2^{k+3}} \leq \tfrac{1}{\gamma} \cdot 2^{1-2^{k+3}}.$$

Therefore,

$$a \le \frac{1}{\gamma}\cdot 2^{1-2^{k+3}} + \frac{1}{\gamma}\cdot\frac{1}{2\cdot|Y|}\cdot\sum_{(y_0,y_1,\ell)\in C}\frac{F^*_{s(y_0,y_1,\ell)}(u_\nu(y_0,y_1):\nu\in\mathrm{pos}(s(y_0,y_1,\ell)))}{1-2^{-2^{k+3}}} \le$$

$$\frac{1}{\gamma}\cdot 2^{1-2^{k+3}} + \frac{1}{\gamma}\cdot\frac{1}{2\cdot|Y|}\cdot\frac{1}{1-2^{-2^{k+3}}}\cdot\sum_{(y,\ell)\in C}u(y,\ell).$$

Hence,

$$(\gamma a - 2^{1-2^{k+3}})(1-2^{-2^{k+3}}) \le \frac{1}{|Y\times 2|}\sum_{(y,\ell)\in Y\times 2}u(y,\ell),$$

and therefore, as $\gamma a \ge 2^{-6\cdot 2^k}$ and $k>1$,

$$\gamma a(1-2^{-2^k}) \le \frac{1}{|Y\times 2|}\sum_{(y,\ell)\in Y\times 2}u(y,\ell).$$

$\square$

Let $\dot{W}$ be the canonical $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$–name for the generic real (so $\dot{W}$ is a name for a function in $\prod\limits_{i<\omega}\mathbf{H}^*(i)$ such that $p\Vdash \mathrm{root}(p)\subseteq\dot{W}$). Also, let $\dot{h}$ be a name for the function from $\prod\limits_{i<\omega}N_i$ to 2^ω such that $\dot{h}(x)(i)=\dot{W}(i)\big(x(i)\big)$. Clearly, $\dot{h}$ is (a name for) a continuous function.

Now comes the main property of the forcing notion $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$.

Proposition 2.7. *Suppose that $A\subseteq\prod\limits_{i<\omega}N_i\times 2^\omega$ is a set of outer (Lebesgue) measure 1. Then, in $\mathbf{V}^{\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)}$, the set*

$$\{x\in\prod_{i<\omega}N_i:(x,\dot{h}(x))\in A\}$$

has outer measure 1.

Proof. Assume, towards a contradiction, that $\dot{T}$ is a $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$–name for a tree included in $\bigcup\limits_{k<\omega}\prod\limits_{i<k}N_i$, and $p\in\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$ is a condition such that

$$p\Vdash_{\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)} \text{“ } m^{\rm Leb}([\dot{T}])>0 \text{ and } (\forall x\in[\dot{T}])((x,\dot{h}(x))\notin A) \text{ ”}.$$

(Here, $m^{\rm Leb}$ stands for the product measure on $\prod\limits_{i<\omega}N_i$.) Passing to a stronger condition and shrinking the tree $\dot{T}$ (if necessary) we may assume that

(α) p is special and $\mathrm{lh}(\mathrm{root}(p))=k_0>5$, and $\mathbf{nor}[t^p_\eta]>2$ for all $\eta\in T^p$, and $\mu^{\mathbf{F}^*}(p)>\frac{1}{2}$,

(β) for some $\rho\in\prod\limits_{j<n}N_j$, $n<k_0$, the condition p forces that

$$m^{\rm Leb}([(\dot{T})^{[\rho]}])\cdot\prod_{j<n}N_j\ge\frac{7}{8},$$

(γ) for some $k_0<k_1<k_2<\ldots$, letting $F_i=T^p\cap\prod\limits_{m<k_i}\mathbf{H}^*(m)$, we have that for each $\nu\in F_{i+1}$, the condition $p^{[\nu]}$ decides the value of $\dot{T}\cap\prod\limits_{j<k_i}N_j$.

Fix $i<\omega$ for a moment, and let $Y^{**}_i=\{y\in\prod\limits_{j<k_i} N_j:\rho\vartriangleleft y\}$.

Let $\nu_0\in F_i$, and for $\nu\in T[p^{[\nu_0]},F_{i+1}]$ and $y\in Y^{**}_i$ let

$$u_\nu(y)=\sup\{\mu^{\mathbf{F}^*}(p'):\ \begin{array}{l} p' \text{ is a condition stronger than } p \text{ and such that}\\ \mathrm{root}(p')=\nu \ \text{ and } \ (\forall\eta\in T^{p'})(\mathbf{nor}[t^{p'}_\eta]\geq\mathbf{nor}[t^p_\eta]-1),\\ \text{and } \ p'\Vdash y\in\dot{T}\}.\end{array}$$

So we are at the situation from the proof of 2.4 (with q there replaced by p), and we may use 2.4.1 to conclude that

$$(\circledast)\qquad \tfrac{7}{8}\cdot\prod_{\ell=k_i}^{k_{i+1}-1}(1-2^{-2^\ell})\cdot|Y^{**}_i|\cdot\mu^{\mathbf{F}^*}_p(\nu_0)\leq\sum\{u_{\nu_0}(y):y\in Y^{**}_i\}.$$

Now, for each $\nu\in T[p,F_i]$ we define $u^*_\nu:Y^{**}_i\times 2^{[\mathrm{lh}(\nu),\,k_i)}\longrightarrow[0,1]$ by

$$\begin{array}{l} u^*_\nu(y,\sigma)=\\ \sup\{\mu^{\mathbf{F}^*}(p'):\ \begin{array}{l} p' \text{ is a condition stronger than } p \text{ and such that}\\ \mathrm{root}(p')=\nu \ \text{ and } \ (\forall\eta\in T^{p'})(\mathbf{nor}[t^{p'}_\eta]\geq\mathbf{nor}[t^p_\eta]-1),\\ \text{and } \ p'\Vdash \text{“ } y\in\dot{T} \ \& \ \Big(\forall j\in[\mathrm{lh}(\nu),k_i)\Big)\Big(\dot{W}(j)\big(y(j)\big)=\sigma(j)\Big) \text{ ”}\}.\end{array}\end{array}$$

(If $\nu\in F_i$, so $\mathrm{lh}(\nu)=k_i$, then $2^{[\mathrm{lh}(\nu),\,k_i)}=\{\emptyset\}$ and $u^*_\nu(y,\emptyset)=u_\nu(y)$.)

Claim 2.7.1. *If $\eta\in T[p,F_i]$, $k_0\leq\mathrm{lh}(\eta)=k\leq k_i$, then*

$$\frac{7}{8}\cdot\prod_{\ell=k}^{k_{i+1}-1}(1-2^{-2^\ell})\cdot|Y^{**}_i|\cdot 2^{k_i-k}\cdot\mu^{\mathbf{F}^*}_p(\eta)\leq\sum\{u^*_\eta(y,\sigma):(y,\sigma)\in X^i_\eta\},$$

where $X^i_\eta=Y^{**}_i\times 2^{[\mathrm{lh}(\eta),\,k_i)}$.

Proof of the claim. The proof, by downward induction on η, is similar to that of 2.4.1, but this time we use 2.6.

First note that if $k=k_i$, then our assertion is exactly what is stated in $(\circledast)$. So suppose that $\eta\in T[p,F_i]$, $\mathrm{lh}(\eta)=k<k_i$, and that we have proved our claim for all $\nu\in\mathrm{pos}(t^p_\eta)$. We are going to apply 2.6 to $t=t^p_\eta$, $\gamma=\frac{7}{8}\cdot\prod\limits_{\ell=k+1}^{k_{i+1}-1}(1-2^{-2^\ell})$, $Y^*=\{y\restriction(k_i\setminus\{k\}):y\in Y^{**}_i\}\times 2^{[k+1,\,k_i)}$ (and $Y=Y^*\times N_k$ being interpreted as $Y^{**}_i\times 2^{[k+1,\,k_i)}$), and $r_\nu=\mu^{\mathbf{F}^*}_p(\nu)$, and $u_\nu(y,\sigma)=u^*_\nu(y,\sigma)$ (for $\nu\in\mathrm{pos}(t^p_\eta)$, $(y,\sigma)\in X^i_\nu$), so we have to check the assumptions there. Note that (as p is special)

$$\gamma\cdot F^*_t(r_\nu:\nu\in\mathrm{pos}(t))=\gamma\cdot\mu^{\mathbf{F}^*}_p(\eta)\geq\frac{7}{8}\cdot\prod_{\ell=k+1}^{k_{i+1}-1}(1-2^{-2^\ell})\cdot 2^{-2^{k+1}}>2^{-6\cdot 2^k}$$

(so the demand in 2.6(ii) is satisfied). Also, by the inductive hypothesis, for each $\nu\in\mathrm{pos}(t^p_\eta)$ we have

$$\gamma\cdot|Y^*\times N_k|\cdot r_\nu\leq\sum\{u^*_\nu(y,\sigma):(y,\sigma)\in X^i_\nu\}$$

(so 2.6(iv) holds). Finally note that if $(y,\sigma)\in Y^{**}_i\times 2^{[k+1,\,k_i)}$, $\ell<2$, and $\sigma':[k,k_i)\longrightarrow 2$ is such that $\sigma'(k)=\ell$, $\sigma'\restriction[k+1,k_i)=\sigma$, then $u(y,\sigma,\ell)$ defined by 2.6(v) is $u^*_\eta(y,\sigma')$ (by the same argument as for $(*)$ in the proof of 2.4.1).

So, by 2.6, we may conclude that

$$\frac{7}{8}\cdot\prod_{\ell=k+1}^{k_{i+1}-1}(1-2^{-2^\ell})\cdot\mu^{\mathbf{F}^*}_p(\eta)\cdot(1-2^{-2^k})\cdot 2\cdot|Y^{**}_i|\cdot 2^{k_i-k-1}\leq \sum\{u^*_\eta(y,\sigma'):(y,\sigma')\in X^i_\eta\},$$

as needed. $\square$

In particular, it follows from 2.7.1 that

$$\frac{7}{8}\cdot\prod_{\ell=k_0}^{k_{i+1}-1}(1-2^{-2^\ell})\cdot\mu^{\mathbf{F}^*}(p)\leq\frac{\sum\{u^*_{\mathrm{root}(p)}(y,\sigma):(y,\sigma)\in Y^{**}_i\times 2^{[k_0,k_i)}\}}{|Y^{**}_i|\cdot 2^{k_i-k_0}},$$

and hence

$$\frac{3}{4}\cdot\mu^{\mathbf{F}^*}(p)\leq\frac{\sum\{u^*_{\mathrm{root}(p)}(y,\sigma):(y,\sigma)\in Y^{**}_i\times 2^{[k_0,k_i)}\}}{|Y^{**}_i|\cdot 2^{k_i-k_0}}.$$

Let $\pi:\prod\limits_{j<k_0}N_j\longrightarrow 2^{k_0}$ be such that $\pi(y)(j)=(\mathrm{root}(p)(j))(y(j))$. Now we define:

$$\begin{array}{rcl} Z_i &=& \{(y,\sigma)\in Y^{**}_i\times 2^{[k_0,k_i)}:u^*_{\mathrm{root}(p)}(y,\sigma)\geq\frac{1}{4}\mu^{\mathbf{F}^*}(p)\},\quad\text{and}\\ Z^+_i &=& \{(y,\sigma)\in Y^{**}_i\times 2^{k_i}:\pi(y{\restriction}k_0)=\sigma\restriction k_0\ \&\ (y,\sigma{\restriction}[k_0,k_i))\in Z_i\}.\end{array}$$

Note that $|Z^+_i|=|Z_i|\geq\frac{1}{4}|Y^{**}_i\times 2^{[k_0,k_i)}|\geq\frac{1}{4}\cdot\prod\limits_{j=n}^{k_i}N_j\cdot 2^{k_i-k_0}$, and therefore

$$\frac{|Z^+_i|}{2^{k_i}\cdot\prod\limits_{j<k_i}N_j}\geq\frac{1}{2^{k_0+2}\cdot\prod\limits_{j<k_0}N_j}.$$

Now we may finish like in 2.4: the set

$$\{(x_0,x_1)\in\prod_{j<\omega}N_j\times 2^\omega:(\exists^\infty i<\omega)((x_0{\restriction}k_i,x_1{\restriction}k_i)\in Z^+_i)\}$$

is a Borel set of positive (Lebesgue) measure, so we may choose $(x_0,x_1)\in A$ such that for infinitely many $i<\omega$ we have $(x_0{\restriction}k_i,x_1{\restriction}k_i)\in Z^+_i$. For each such i pick a condition $q_i\geq p$ such that

- $\mathrm{root}(q_i)=\mathrm{root}(p)$, $\mu^{\mathbf{F}^*}(q_i)>\frac{1}{8}\mu^{\mathbf{F}^*}(p)$, and
- $(\forall\eta\in T^{q_i})(\mathbf{nor}[t^{q_i}_\eta]\geq\mathbf{nor}[t^p_\eta]-1)$, and
- $q_i\Vdash$ " $x_0\restriction k_i\in\dot{T}$ and $\Big(\forall j\in[k_0,k_i)\Big)\Big(\dot{W}(j)\big(x_0(j)\big)=x_1(j)\Big)$ ".

By König's Lemma, we may find a condition $q\in\mathbb{Q}^{\mathrm{mt}}_4(K^*,\Sigma^*,\mathbf{F}^*)$ stronger than p, and an infinite set $I\subseteq\omega$ such that

($\otimes$) if $i<j$ are from I, then $i+1<j$ and

$$T^{q_j}\cap\prod_{k<k_{i+1}}N_k=T^q\cap\prod_{k<k_{i+1}}N_k\quad\text{and}\quad(\forall\eta\in T^{q_j})(\mathrm{lh}(\eta)<k_{i+1}\ \Rightarrow\ t^{q_j}_\eta=t^q_\eta).$$

Then clearly $q\Vdash$ " $x_0\in\dot{T}$ & $\dot{h}(x_0)=x_1$ ", a contradiction. $\square$

3. The first model: sup-measurability

To prove the first of our main results, let us start with a reduction of the sup-measurability problem.

Lemma 3.1. *The following conditions are equivalent:*

$(\boxtimes)^1_{\rm sup}$ *Every sup-measurable function $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is Lebesgue measurable.*

$(\boxtimes)^2_{\rm sup}$ *For every non-measurable set $A\subseteq\mathbb{R}\times\mathbb{R}$ there exists a Borel function $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that the set $\{x\in\mathbb{R}:(x,f(x))\in A\}$ is not measurable.*

$(\boxtimes)^3_{\rm sup}$ *For every non-measurable set $A\subseteq 2^\omega\times 2^\omega$ there is a Borel function $f:2^\omega\longrightarrow 2^\omega$ such that the set $\{x\in 2^\omega:(x,f(x))\in A\}$ is not measurable.*

$(\boxtimes)^4_{\rm sup}$ *For every set $A\subseteq\prod\limits_{k<\omega}N_k\times 2^\omega$ of outer measure one and inner measure zero, there is a Borel function $h:\prod\limits_{k<\omega}N_k\longrightarrow 2^\omega$ such that the set*

$$\{x\in\prod_{k<\omega}N_k:(x,h(x))\in A\}$$

is not measurable.

(Here, the sequence $\langle N_k:k<\omega\rangle$ is the one defined at the beginning of the second section.)

Proof. The equivalences $(\boxtimes)^1_{\rm sup}\Leftrightarrow(\boxtimes)^2_{\rm sup}\Leftrightarrow(\boxtimes)^3_{\rm sup}$ are well known (see Balcerzak [**?**, Proposition 1.5]; also compare with the proof of Ciesielski and Shelah [**?**, Corollary 3]).

$(\boxtimes)^4_{\rm sup}\Rightarrow(\boxtimes)^3_{\rm sup}$: Assume $(\boxtimes)^4_{\rm sup}$, and suppose that $A\subseteq 2^\omega\times 2^\omega$ is a non-measurable set. Then we may find a closed set $C\subseteq 2^\omega\times 2^\omega$ of positive Lebesgue measure and such that

- for each $x\in 2^\omega$, the set $\{y\in 2^\omega:(x,y)\in C\}$ is either empty or is of positive Lebesgue measure,
- for every Borel set $D\subseteq C$ of positive measure, both $A\cap D\neq\emptyset$ and $D\setminus A\neq\emptyset$ (that is, both $A\cap C$ and $C\setminus A$ are of full outer measure in C).

By shrinking C if necessary, we may also pick a Borel isomorphism $\psi=(\psi_0,\psi_1): C\longrightarrow\prod\limits_{k<\omega}N_k\times 2^\omega$ such that

- if $(x,y),(x',y')\in C$, then $\psi_0(x,y)=\psi_0(x',y')\ \Leftrightarrow\ x=x'$,
- if $B\subseteq C$ is Borel, then B has measure 0 if and only if its image $\psi[B]$ has measure zero.

Now note that the set $\psi[A]$ has outer measure 1 and inner measure 0 (in $\prod\limits_{k<\omega}N_k\times 2^\omega$), so we may apply $(\boxtimes)^4_{\rm sup}$ to get a Borel function $h:\prod\limits_{k<\omega}N_k\longrightarrow 2^\omega$ such that the set $\big\{x\in\prod\limits_{k<\omega}N_k:(x,h(x))\in\psi[A]\big\}$ is not measurable. Let $B=\{x\in 2^\omega:(\exists y)((x,y)\in C)\}$, and let $f^*:B\longrightarrow 2^\omega$ be defined by

$$(x,f^*(x))=\psi^{-1}\big((\psi_0(x,y),h(\psi_0(x,y)))\big)$$

for some (equivalently: all) y such that $(x,y)\in C$. Easily f^* is a Borel function. Take any Borel extension $f:2^\omega\longrightarrow 2^\omega$ of f^* - it is as required in $(\boxtimes)^3_{\rm sup}$ for A.

$(\boxtimes)^3_{\rm sup}\Rightarrow(\boxtimes)^4_{\rm sup}$: Even easier. (Note that, since all N_k's are powers of 2, we have a very nice measure preserving homeomorphism $\psi^*:\prod\limits_{k<\omega}N_k\longrightarrow 2^\omega$.) $\square$

Theorem 3.2. *It is consistent that every sup-measurable function is Lebesgue measurable.*

Proof. Start with universe $\mathbf{V}$ satisfying CH. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ be countable support iteration such that each iterand $\dot{\mathbb{Q}}_\alpha$ is (forced to be) the forcing notion $\mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*)$ (defined in the second section; of course it is taken in the respective universe $\mathbf{V}^{\mathbb{P}_\alpha}$). It follows from 1.14 (and [**?**, Ch. VI, 2.8D]) that the limit $\mathbb{P}_{\omega_2}$ is proper and ω^ω–bounding. Also it satisfies $\aleph_2$–cc, and consequently the forcing with $\mathbb{P}_{\omega_2}$ does not collapse cardinals nor changes cofinalities (and $\Vdash_{\mathbb{P}_{\omega_2}}$ " $\mathfrak{c} = \aleph_2$ ").

We are going to prove that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{" every sup-measurable function is Lebesgue measurable ".}$$

By 3.1, it is enough to show that $\Vdash_{\mathbb{P}_{\omega_2}} (\boxtimes)^4_{\rm sup}$. To this end suppose that $\dot{A}$ is a $\mathbb{P}_{\omega_2}$–name for a subset of $\prod\limits_{k<\omega} N_k \times 2^\omega$ such that both $\dot{A}$ and its complement are of outer measure one. By a standard argument using $\aleph_2$–cc of $\mathbb{P}_{\omega_2}$ (and the fact that each $\mathbb{P}_\alpha$ for $\alpha < \omega_2$ has a dense subset of size $\aleph_1$), we may find $\delta < \omega_2$ of cofinality ω_1, and a $\mathbb{P}_\delta$–name $\dot{A}_\delta$ such that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{" } \dot{A} \cap \big(\prod_{k<\omega} N_k \times 2^\omega\big)^{\mathbf{V}^{\mathbb{P}_\delta}} = \dot{A}_\delta \text{ ",} \qquad \text{and}$$

$$\Vdash_{\mathbb{P}_\delta} \text{" } \dot{A}_\delta \text{ has outer measure 1 and inner measure 0 ".}$$

Let $\dot{h}$ be the $\mathbb{P}_{\delta+1}$–name for the continuous function from $\prod\limits_{k<\omega} N_k$ to 2^ω added at stage $\delta+1$ by $\dot{\mathbb{Q}}_\delta = (\mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*))^{\mathbf{V}^{\mathbb{P}_\delta}}$ (as defined right before 2.7). Then, by 2.7 (applied to $\dot{A}_\delta$ and to its complement), in $\mathbf{V}^{\mathbb{P}_{\delta+1}}$ the set

$$X_\delta \stackrel{\rm def}{=} \{x \in \prod_{k<\omega} N_k : (x, \dot{h}(x)) \in \dot{A}_\delta\}$$

has outer measure 1 and inner measure 0. Now, in $\mathbf{V}^{\mathbb{P}_{\delta+1}}$ we may use 2.5 to conclude that $\mathbb{P}_{\omega_2}/\mathbb{P}_{\delta+1}$ preserves the Lebesgue outer measure of sets from $\mathbf{V}^{\mathbb{P}_{\delta+1}}$. Consequently,

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{" the set } X_\delta \text{ and its complement have outer measure one ",}$$

finishing the proof. $\square$

Remark 3.3. Note that for the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ to work for the proof of 3.2 we do not need that all iterands are $\mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*)$. It is enough that for some stationary set $Z \subseteq \{\delta < \omega_2 : {\rm cf}(\delta) = \omega_1\}$, for every $\alpha \in Z$, we have $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \mathbb{Q}^{\rm mt}_4(K^*, \Sigma^*, \mathbf{F}^*)$, and that the forcings used in the iteration are such that each $\mathbb{P}_{\omega_2}/\mathbb{P}_{\delta+1}$ preserves non-nullity of sets from $\mathbf{V}^{\mathbb{P}_{\delta+1}}$. So, in particular, we may use in the iteration also other forcing notions satisfying $(\heartsuit)$ of 2.5. This will be used in the next section, where we will add the random forcing "here-and-there".

4. Possibly every real function is continuous on a non-null set

The aim of this section is to show that a slight modification of the iteration from the previous section results in a model in which every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ agrees with a continuous function on a set of positive outer measure. Let us start with a reduction that shows how the tools developed earlier are relevant for our problem.

Proposition 4.1. *Assume:*

(a) *the condition* $(\boxtimes)^3_{\rm sup}$ *of 3.1 holds,*
(b) *for every function* $f^*: 2^\omega \longrightarrow 2^\omega$ *there are functions* f_1, f_2 *and a set* A *such that*
- $A \subseteq 2^\omega$ *and* $f_1 : A \longrightarrow 2^\omega$ *is such that the set*
$$\{(x, f_1(x)) : x \in A\} \subseteq 2^\omega \times 2^\omega$$
has positive outer measure,
- $f_2 : 2^\omega \times 2^\omega \longrightarrow 2^\omega$ *is Borel, and*
- $(\forall x \in A)(f^*(x) = f_2(x, f_1(x)))$.

Then for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ *there is a continuous function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *such that the set* $\{x \in \mathbb{R} : f(x) = g(x)\}$ *has positive outer measure.*

Proof. Assume $f : \mathbb{R} \longrightarrow \mathbb{R}$. Let $\varphi : \mathbb{R} \longrightarrow 2^\omega$ be a Borel isomorphism preserving null sets (see, e.g., [**?**, Thm 17.41]), and let $f^* = \varphi \circ f \circ \varphi^{-1}$. Let f_1, f_2, A be given by the assumption (b) for f^*. Put $A^* = \{(x, f_1(x)) : x \in A\} \subseteq 2^\omega \times 2^\omega$. We know that A^* is a non-null set (and consequently it is non-measurable), so applying $(\boxtimes)^3_{\rm sup}$ we may pick a Borel function $g_0 : 2^\omega \longrightarrow 2^\omega$ such that the set
$$B \stackrel{\rm def}{=} \{x \in A : f_1(x) = g_0(x)\}$$
has positive outer measure, and so does $\varphi^{-1}[B]$. Let $g_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by
$$g_1(x) = \varphi^{-1}\Big(f_2\big(\varphi(x), g_0(\varphi(x))\big)\Big).$$
Clearly g_1 is Borel and for each $x \in \varphi^{-1}[B]$ we have $g_1(x) = f(x)$. Finally, using Lusin's theorem (see, e.g., [**?**, Thm 17.12]) we may pick a continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that the set $\{x \in \varphi^{-1}[B] : g_1(x) = g(x)\}$ is not null (just take g so that it agrees with g_1 on a set of large enough measure). $\square$

The iteration of 3.2 will be changed by adding random reals on a stationary set. So just for uniformity of our notation we represent the random real forcing as $\mathbb{Q}^{\rm mt}_4(K^r, \Sigma^r, \mathbf{F}^r)$. Let $\mathbf{H}^r(i) = 2$ (for $i < \omega$). Let K^r consist of tree creatures $t \in {\rm LTCR}[\mathbf{H}^r]$ such that
- $\mathbf{dis}[t] = (k_t, \eta_t, P_t)$, where $k_t < \omega$, $\eta_t \in \prod\limits_{i<k_t} \mathbf{H}^r(i)$, $\emptyset \neq P_t \subseteq 2$, and
- $\mathbf{nor}[t] = k_t$,
- $\mathbf{val}[t] = \{\langle \eta_t, \nu\rangle : \eta_t \vartriangleleft \nu \in \prod\limits_{i \le k_t} \mathbf{H}^r(i)\ \&\ \nu(k_t) \in P_t\}$.

The operation Σ^r is trivial:
$$\Sigma^r(t) = \{s \in K^r : \eta_s = \eta_t\ \&\ P_s \subseteq P_t\}.$$
For $t \in K^r$ and a sequence $\langle r_\nu : \nu \in {\rm pos}(t)\rangle \subseteq [0,1]$ we let
$$F^r_t(r_\nu : \nu \in {\rm pos}(t)) = \frac{\sum\{r_\nu : \nu \in {\rm pos}(t)\}}{2}.$$
It is easy to check that $(K^r, \Sigma^r, \mathbf{F}^r)$ is a (nice) measured tree creating triple for $\mathbf{H}^r$, and that the forcing notion $\mathbb{Q}^{\rm mt}_4(K^r, \Sigma^r, \mathbf{F}^r)$ is (equivalent to) the random real forcing.

Like in 3.2, we start with universe $\mathbf{V}$ satisfying CH. Let $Z \subseteq \{\delta < \omega_2 : {\rm cf}(\delta) = \omega_1\}$ be a stationary set such that $\{\delta < \omega_2 : {\rm cf}(\delta) = \omega_1\} \setminus Z$ is stationary as well. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\rangle$ be countable support iteration such that
- if $\alpha \in Z$, then $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \mathbb{Q}^{\rm mt}_4(K^r, \Sigma^r, \mathbf{F}^r)$,

- if $\alpha\in\omega_2\setminus Z$, then $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha=\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$.

We are going to show that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{“ every real function is continuous on a non-null set ”},$$

and for this we will show that the assumptions of 4.1 are satisfied in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. First note that $\mathbb{P}_{\omega_2}\Vdash(\boxtimes)^3_{\rm sup}$ (see 3.3; remember 3.1). To show that, in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, the assumption (b) of 4.1 holds, we need to analyze conditions and continuous reading of names in the iteration.

Definition 4.2. Let $(K,\Sigma,\mathbf{F})$ be a measured tree creating triple for $\mathbf{H}$ (say, either $(K^*,\Sigma^*,\mathbf{F}^*)$ defined in the second section, or $(K^r,\Sigma^r,\mathbf{F}^r)$ defined above).

(1) *A finite candidate for* $(K,\Sigma,\mathbf{F})$ (or just for (K,Σ)) is a system $\mathbf{s}=\langle s_\eta:\eta\in S\setminus\max(S)\rangle$ such that
- $S\subseteq\bigcup\limits_{n<\omega}\prod\limits_{i<n}\mathbf{H}(i)$ is a finite tree, $s_\eta\in K\cap{\rm LTCR}_\eta[\mathbf{H}]$ for $\eta\in S\setminus\max(S)$,
- $\max(S)\subseteq\prod\limits_{i<m}\mathbf{H}(i)$ for some $m={\rm ht}(\mathbf{s})$ (we will call this m the height of the candidate $\mathbf{s}$),
- if $\eta\in S\setminus\max(S)$, then ${\rm succ}_S(\eta)={\rm pos}(s_\eta)$.

We may also write ${\rm root}(\mathbf{s})$ for ${\rm root}(S)$ (and call it the root of the candidate $\mathbf{s}$), and write $\max(\mathbf{s})$ for $\max(S)$.

(2) Let ${\rm FC}(K,\Sigma)$ be the family of all finite candidates for (K,Σ).

(3) For candidates $\mathbf{s}^0,\mathbf{s}^1\in{\rm FC}(K,\Sigma)$, we say that $\mathbf{s}^1$ *end–extends* $\mathbf{s}^0$ (in short: $\mathbf{s}^0\preceq_{\rm end}\mathbf{s}^1$) if ${\rm root}(\mathbf{s}^1)={\rm root}(\mathbf{s}^0)$, ${\rm ht}(\mathbf{s}^1)\geq{\rm ht}(\mathbf{s}^0)$ and, letting $\mathbf{s}^\ell=\langle s^\ell_\eta:\eta\in S^\ell\setminus\max(S^\ell)\rangle$, we have $S^0\subseteq S^1$ and $(\forall\eta\in S^0\setminus\max(S^0))(s^0_\eta=s^1_\eta)$.

(4) We say that a condition $p\in\mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma)$ *end–extends* a candidate $\mathbf{s}=\langle s_\eta:\eta\in S\setminus\max(S)\rangle\in{\rm FC}(K,\Sigma)$ if
- ${\rm root}(p)={\rm root}(\mathbf{s})$, $S\subseteq T^p$, and
- $s_\eta=t^p_\eta$ for $\eta\in S\setminus\max(S)$, and
- $\mu^{\mathbf{F}}_p(\nu)>0$ for all $\nu\in\max(S)$.

Definition 4.3. (1) *A finite pre–template* is a tuple

$$\mathbf{t}=\langle w^{\mathbf{t}},\mathbf{k}^{\mathbf{t}},\mathbf{c}^{\mathbf{t}},\bar{\mathcal{Y}}^{\mathbf{t}}\rangle=\langle w,\mathbf{k},\mathbf{c},\bar{\mathcal{Y}}\rangle$$

such that

(α) w is a finite non-empty set of ordinals below ω_2, $w=\{\alpha_0,\ldots,\alpha_n\}$ (the increasing enumeration);
let x_i be r if $\alpha_i\in Z$, and x_i be $*$ if $\alpha_i\in\omega_2\setminus Z$,

(β) $\mathbf{k}:w\longrightarrow\omega$, $\mathbf{c}=\langle c_{\alpha_0},\ldots,c_{\alpha_n}\rangle$, $\bar{\mathcal{Y}}=\langle\mathcal{Y}_{\alpha_0},\ldots,\mathcal{Y}_{\alpha_n}\rangle$ (we treat $\mathbf{c},\bar{\mathcal{Y}}$ as functions with domain w),

(γ) $c_{\alpha_0}\in{\rm FC}(K^{x_0},\Sigma^{x_0})$, ${\rm ht}(c_{\alpha_0})=\mathbf{k}(\alpha_0)$, $\mathcal{Y}_{\alpha_0}=\{\langle s\rangle:s\in\max(c_{\alpha_0})\}$, and for $0<i\leq n$:

(δ) $c_{\alpha_i}:\mathcal{Y}_{\alpha_{i-1}}\longrightarrow{\rm FC}(K^{x_i},\Sigma^{x_i})$ is such that ${\rm ht}(c_{\alpha_i}(\bar\nu))=\mathbf{k}(\alpha_i)$ for each $\bar\nu\in\mathcal{Y}_{\alpha_{i-1}}$,
$\mathcal{Y}_{\alpha_i}=\{\bar\nu^\frown\langle\nu_{\alpha_i}\rangle:\bar\nu=\langle\nu_{\alpha_0},\ldots,\nu_{\alpha_{i-1}}\rangle\in\mathcal{Y}_{\alpha_{i-1}}\ \&\ \nu_{\alpha_i}\in\max(c_{\alpha_i}(\bar\nu))\}$.
(We think of elements of $\mathcal{Y}_{\alpha_i}$ as functions from $\{\alpha_0,\ldots,\alpha_i\}$ with values being sequences in appropriate $\prod\limits_{j<\mathbf{k}(\alpha_\ell)}\mathbf{H}^{x_\ell}(j)$.)

$\mathcal{Y}_{\alpha_n}$ will be also called $\mathcal{Y}_*$ or $\mathcal{Y}^{\mathbf{t}}_*$.

(2) We say that a finite pre–template $\mathbf{t}'$ *properly extends* a pre–template $\mathbf{t}$ (and then we write $\mathbf{t} \preceq \mathbf{t}'$) if

(α) $w^{\mathbf{t}} \subseteq w^{\mathbf{t}'}$, and $(\forall \alpha \in w^{\mathbf{t}})(\mathbf{k}^{\mathbf{t}}(\alpha) \leq \mathbf{k}^{\mathbf{t}'}(\alpha))$, and

(β) let $w^{\mathbf{t}'} = \{\alpha_0, \ldots, \alpha_n\}$ (the increasing enumeration).
If $\ell^* = \min\{i \leq n : \alpha_i \in w^{\mathbf{t}}\}$, then for every $\langle \nu_{\alpha_0}, \ldots, \nu_{\alpha_{\ell^*-1}} \rangle \in \mathcal{Y}^{\mathbf{t}'}_{\alpha_{\ell^*-1}}$ we have $c^{\mathbf{t}}_{\alpha_{\ell^*}} \preceq_{\rm end} c^{\mathbf{t}'}_{\alpha_{\ell^*}}(\nu_{\alpha_0}, \ldots, \nu_{\alpha_{\ell^*-1}})$.
If $\ell > \ell^*$ is such that $\alpha_\ell \in w^{\mathbf{t}}$ and $k < \ell$ is such that α_k is the predecessor of α_ℓ in $w^{\mathbf{t}}$, then for every $\langle \nu_{\alpha_0}, \ldots, \nu_{\alpha_{\ell-1}} \rangle \in \mathcal{Y}^{\mathbf{t}'}_{\alpha_{\ell-1}}$ we have

$$\langle \nu_{\alpha_i} \restriction \mathbf{k}^{\mathbf{t}}(\alpha_i) : i < \ell \ \&\ \alpha_i \in w^{\mathbf{t}} \rangle \in \mathcal{Y}^{\mathbf{t}}_{\alpha_k} \quad \text{and}$$
$$c^{\mathbf{t}}_{\alpha_\ell}(\nu_{\alpha_i} \restriction \mathbf{k}^{\mathbf{t}}(\alpha_i) : i < \ell \ \&\ \alpha_i \in w^{\mathbf{t}}) \preceq_{\rm end} c^{\mathbf{t}'}_{\alpha_\ell}(\nu_{\alpha_0}, \ldots, \nu_{\alpha_{\ell-1}}).$$

(3) For an ordinal $\zeta < \omega_2$ and a finite pre-template $\mathbf{t}$ we define the restriction $\mathbf{t}' = \mathbf{t} \restriction \zeta$ of $\mathbf{t}$ in a natural way: $w^{\mathbf{t}'} = w^{\mathbf{t}} \cap \zeta$, $\mathbf{k}^{\mathbf{t}'} = \mathbf{k}^{\mathbf{t}} \restriction w^{\mathbf{t}'}$, $\mathbf{c}^{\mathbf{t}'} = \mathbf{c}^{\mathbf{t}} \restriction w^{\mathbf{t}'}$ and $\bar{\mathcal{Y}}^{\mathbf{t}'} = \bar{\mathcal{Y}}^{\mathbf{t}} \restriction w^{\mathbf{t}'}$. (Note that $\mathbf{t} \restriction \zeta \preceq \mathbf{t}$.)

(4) We say that finite pre-templates $\mathbf{t}, \mathbf{t}'$ are *isomorphic* if $|w^{\mathbf{t}}| = |w^{\mathbf{t}'}|$, and if $h : w^{\mathbf{t}} \longrightarrow w^{\mathbf{t}'}$ is the order preserving isomorphism, then
- $h[w^{\mathbf{t}} \cap Z] = w^{\mathbf{t}'} \cap Z$, and
- $\mathbf{k}^{\mathbf{t}} = \mathbf{k}^{\mathbf{t}'} \circ h$, $\mathbf{c}^{\mathbf{t}} = \mathbf{c}^{\mathbf{t}'} \circ h$, and $\bar{\mathcal{Y}}^{\mathbf{t}} = \bar{\mathcal{Y}}^{\mathbf{t}'} \circ h$.

We also may say that h *is an isomorphism from* $\mathbf{t}$ *to* $\mathbf{t}'$.

Definition 4.4. By induction on $n = |w^{\mathbf{t}}| - 1$ we define

(a) when a condition $p \in \mathbb{P}_{\omega_2}$ obeys a pre-template $\mathbf{t}$, and
(b) if $w^{\mathbf{t}} = \{\alpha_0, \ldots, \alpha_n\}$, $\bar{\nu} = \langle \nu_{\alpha_0}, \ldots, \nu_{\alpha_n} \rangle \in \mathcal{Y}^{\mathbf{t}}_*$, and $p \in \mathbb{P}_{\omega_2}$ obeys $\mathbf{t}$, then we define a condition $p^{[\mathbf{t},\bar{\nu}]} \in \mathbb{P}_{\omega_2}$ stronger that p.

First consider the case when $n = 0$. Let $\mathbf{t}$ be a pre-template such that $w^{\mathbf{t}} = \{\alpha_0\}$ and let $p \in \mathbb{P}_{\omega_2}$. We say that p obeys $\mathbf{t}$ if

$$p \restriction \alpha_0 \Vdash_{\mathbb{P}_{\alpha_0}} \text{“ } p(\alpha_0) \text{ end extends the candidate } c^{\mathbf{t}}_{\alpha_0} \text{ ”}.$$

If p obeys $\mathbf{t}$ as above, and $\bar{\nu} = \langle \nu_{\alpha_0} \rangle \in \mathcal{Y}^{\mathbf{t}}_{\alpha_0}$, then $p^{[\mathbf{t},\bar{\nu}]}$ is defined as follows:

- $p^{[\mathbf{t},\bar{\nu}]} \restriction (\omega_2 \setminus \{\alpha_0\}) = p \restriction (\omega_2 \setminus \{\alpha_0\})$, and
- $p^{[\mathbf{t},\bar{\nu}]} \restriction \alpha_0 \Vdash_{\mathbb{P}_{\alpha_0}}$ “ $p^{[\mathbf{t},\bar{\nu}]}(\alpha_0) = (p(\alpha_0))^{[\nu_{\alpha_0}]}$ ”.

(Plainly, $p^{[\mathbf{t},\nu]} \in \mathbb{P}_{\omega_2}$; remember the last demand in 4.2(4).) Now, suppose that $w^{\mathbf{t}} = \{\alpha_0, \ldots, \alpha_n\}$ (the increasing enumeration; $n > 0$), and that we have dealt with $n-1$ already. We say that a condition $p \in \mathbb{P}_{\omega_2}$ obeys $\mathbf{t}$ if

- p obeys $\mathbf{t} \restriction \alpha_n$, and
- for every $\bar{\nu} = \langle \nu_{\alpha_0}, \ldots, \nu_{\alpha_{n-1}} \rangle \in \mathcal{Y}^{\mathbf{t}}_{\alpha_{n-1}}$, the condition $p^{[\mathbf{t} \restriction \alpha_n, \bar{\nu}]} \restriction \alpha_n$ forces (in $\mathbb{P}_{\alpha_n}$) that $p(\alpha_n)$ end–extends the candidate $c^{\mathbf{t}}_{\alpha_n}(\bar{\nu})$.

In that case we also define $p^{[\mathbf{t},\bar{\nu}]}$ for $\bar{\nu} = \langle \nu_{\alpha_0}, \ldots, \nu_{\alpha_n} \rangle \in \mathcal{Y}^{\mathbf{t}}_{\alpha_n}$:

- $p^{[\mathbf{t},\bar{\nu}]} \restriction \omega_2 \setminus \{\alpha_n\} = p^{[\mathbf{t} \restriction \alpha_n, \bar{\nu} \restriction \alpha_n]} \restriction \omega_2 \setminus \{\alpha_n\}$,
- $p^{[\mathbf{t},\bar{\nu}]} \restriction \alpha_n \Vdash_{\mathbb{P}_{\alpha_n}}$ “ $p^{[\mathbf{t},\bar{\nu}]}(\alpha_n) = (p(\alpha_n))^{[\nu_{\alpha_n}]}$ ”.

Definition 4.5. (1) *A weak template* is a $\preceq$–increasing sequence $\bar{\mathbf{t}} = \langle \mathbf{t}_n : n < \omega \rangle$ of finite pre-templates such that

$$(\forall \alpha \in \bigcup_{n<\omega} w^{\mathbf{t}_n})(\lim_{n\to\infty} \mathbf{k}^{\mathbf{t}_n}(\alpha) = \infty).$$

(2) We say that *weak templates* $\bar{\mathbf{t}}, \bar{\mathbf{t}}'$ *are isomorphic* if
   - $\mathrm{otp}(\bigcup\limits_{n<\omega} w^{\mathbf{t}_n}) = \mathrm{otp}(\bigcup\limits_{n<\omega} w^{\mathbf{t}'_n})$, and
   - letting $h : \bigcup\limits_{n<\omega} w^{\mathbf{t}_n} \longrightarrow \bigcup\limits_{n<\omega} w^{\mathbf{t}'_n}$ be the order isomorphism, we have that all restrictions $h \restriction w^{\mathbf{t}_n}$ (for $n < \omega$) are isomorphisms from $\mathbf{t}_n$ to $\mathbf{t}'_n$.

   (We will also call the mapping h as above *the isomorphism from* $\bar{\mathbf{t}}$ *to* $\bar{\mathbf{t}}'$.)

(3) A condition $p \in \mathbb{P}_{\omega_2}$ *obeys the weak template* $\bar{\mathbf{t}} = \langle \mathbf{t}_n : n < \omega \rangle$ if $\mathrm{supp}(p) = \bigcup\limits_{n<\omega} w^{\mathbf{t}_n}$ and p obeys each pre-template $\mathbf{t}_n$ (for $n < \omega$).

(4) *A weak template with a name* is a pair $(\bar{\mathbf{t}}, \bar{\tau})$ such that $\bar{\mathbf{t}} = \langle \mathbf{t}_n : n < \omega \rangle$ is a weak template, and $\bar{\tau} = \langle \tau_n : n < \omega \rangle$ is a sequence of functions such that $\tau_n : \mathcal{Y}_*^{\mathbf{t}_n} \longrightarrow 2^n$, and if $\langle \nu_\alpha : \alpha \in w^{\mathbf{t}_{n+1}} \rangle \in \mathcal{Y}_*^{\mathbf{t}_{n+1}}$, then

$$\tau_n(\nu_\alpha \restriction \mathbf{k}^{\mathbf{t}_n}(\alpha) : \alpha \in w^{\mathbf{t}_n}) \vartriangleleft \tau_{n+1}(\nu_\alpha : \alpha \in w^{\mathbf{t}_{n+1}}).$$

(5) Let $(\bar{\mathbf{t}}, \bar{\tau}), (\bar{\mathbf{t}}', \bar{\tau}')$ be weak templates with names. We say that they are isomorphic provided that $\bar{\mathbf{t}}$ and $\bar{\mathbf{t}}'$ are isomorphic, and the isomorphism maps $\bar{\tau}$ to $\bar{\tau}'$. (To be more precise, if h is the isomorphism from $\bar{\mathbf{t}}$ to $\bar{\mathbf{t}}'$, then for each $n < \omega$ it induces a bijection $g_n : \mathcal{Y}_*^{\mathbf{t}_n} \longrightarrow \mathcal{Y}_*^{\mathbf{t}'_n}$; we request that $\tau_n = \tau'_n \circ g_n$.)

(6) Let $(\bar{\mathbf{t}}, \bar{\tau})$ be a weak template with a name, $p \in \mathbb{P}_{\omega_2}$ and let $\dot{\tau}$ be a $\mathbb{P}_{\omega_2}$–name for a real in 2^ω. We say that $(p, \dot{\tau})$ *obeys* $(\bar{\mathbf{t}}, \bar{\tau})$ if
   - the condition p obeys the weak template $\mathbf{t}$, and
   - for each $n < \omega$ and $\bar{\nu} \in \mathcal{Y}_*^{\mathbf{t}_n}$ we have: $p^{[\mathbf{t}_n, \bar{\nu}]} \Vdash_{\mathbb{P}_{\omega_2}} \dot{\tau} \restriction n = \tau_n(\bar{\nu})$.

Lemma 4.6. (1) *There are only countably many isomorphism types of finite pre-templates.*
(2) *There are $\mathfrak{c}$ many isomorphism types of weak templates with names.*

Lemma 4.7. *Suppose that $\dot{\tau}$ is a $\mathbb{P}_{\omega_2}$–name for a real in 2^ω and $p \in \mathbb{P}_{\omega_2}$. Then there is a condition $q \in \mathbb{P}_{\omega_2}$ stronger than p, and a weak template with a name $(\bar{\mathbf{t}}, \bar{\tau})$ such that $(q, \dot{\tau})$ obeys $(\bar{\mathbf{t}}, \bar{\tau})$.*

Proof. Let $\bar{\mathbb{Q}}' = \langle \mathbb{P}'_\alpha, \dot{\mathbb{Q}}'_\alpha : \alpha < \omega_2 \rangle$ be a CS iteration such that
(1) if $\alpha \in Z$, then $\Vdash_{\mathbb{P}'_\alpha} \dot{\mathbb{Q}}'_\alpha = \mathbb{Q}^{\rm sn}(K^r, \Sigma^r, \mathbf{F}^r)$,
(2) if $\alpha \in \omega_2 \setminus Z$, then $\Vdash_{\mathbb{P}'_\alpha} \dot{\mathbb{Q}}'_\alpha = \mathbb{Q}^{\rm sn}(K^*, \Sigma^*, \mathbf{F}^*)$

(where $\mathbb{Q}^{\rm sn}$ is as defined in 1.17). Then $\mathbb{P}'_{\omega_2}$ is a dense subset of $\mathbb{P}_{\omega_2}$ (remember 1.18). For $F \in [\omega_2]^{<\omega}$ and $n \in \omega$ we define a binary relation $\leq_{F,n}$ on $\mathbb{P}'_{\omega_2}$ by

$p \leq_{F,n} q$ if and only if $(p, q \in \mathbb{P}'_{\omega_2}$ and)
$p \leq q$ and $q{\restriction}\alpha \Vdash_{\mathbb{P}'_\alpha}$ " $p(\alpha) \leq^\alpha_n q(\alpha)$ " for each $\alpha \in F$

(where $\leq^\alpha_n$ is a $\mathbb{P}'_\alpha$–name for the binary relation $\leq_n$ on $\dot{\mathbb{Q}}'_\alpha$ defined in 1.19). As we said in 1.21, one can carry out the proofs of Baumgartner [?, §7] for $\leq_{F,n}$, in particular getting the following two claims.

Claim 4.7.1 (Baumgartner [?, Lemma 7.2])**.** *Suppose that a sequence $\langle (p_n, F_n) : n < \omega \rangle$ satisfies*
(a) *$p_n \in \mathbb{P}'_{\omega_2}$, $F_n \in [\omega_2]^{<\omega}$ (for each $n < \omega$), and*
(b) *$p_n \leq_{F_n, n+1} p_{n+1}$, $F_n \subseteq F_{n+1}$ (for each $n < \omega$), and*
(c) *$\bigcup \{F_n : n < \omega\} = \bigcup \{\mathrm{supp}(p_n) : n < \omega\}$.*

Then there is a condition $p \in \mathbb{P}'_{\omega_2}$ such that $p_n \leq_{F_n,n} p$ for all $n < \omega$.

Claim 4.7.2 (Baumgartner [?, Lemma 7.3(c)])**.** *Suppose that $\alpha < \beta \leq \omega_2$, $F \in [\alpha]^{<\omega}$, $n < \omega$ and $p \in \mathbb{P}'_\alpha$. Let $\dot{f}$ be a $\mathbb{P}'_\alpha$–name such that $\Vdash_{\mathbb{P}'_\alpha} \dot{f} \in \mathbb{P}'_{\alpha\beta}$. Then there are $f \in \mathbb{P}_{\alpha\beta}$ and $q' \in \mathbb{P}'_\alpha$ such that $p \leq_{F,n} q$ and $q \Vdash_{\mathbb{P}'_\alpha} \dot{f} = f$.*

Now we may start the actual proof of 4.7. The following observation should be clear.

Claim 4.7.3. *Suppose that $\mathbf{t} = \langle w, \mathbf{k}, \mathbf{c}, \bar{\mathcal{Y}}\rangle$ is a finite pre–template and a condition $p \in \mathbb{P}'_{\omega_2}$ obeys $\mathbf{t}$. Let $N = \max\big(\mathbf{k}(\alpha) : \alpha \in w\big)$ and $F \in [\omega_2]^{<\omega}$ be such that $w \subseteq F$. Then $p \leq_{F,N} q$ implies that q obeys $\mathbf{t}$.*

The main part of the inductive construction of a weak template with a name $(\bar{\mathbf{t}}, \bar{\tau})$ as required in our Lemma will be done by the following claim.

Claim 4.7.4. *Assume that a condition $p \in \mathbb{P}'_{\omega_2}$ obeys a finite pre–template $\mathbf{t} = \langle w, \mathbf{k}, \mathbf{c}, \bar{\mathcal{Y}}\rangle$ and $\dot{a}$ is a $\mathbb{P}'_{\omega_2}$–name for an ordinal. Let $N > \max\big(\mathbf{k}(\alpha) : \alpha \in w\big)$. Then there are a pre–template $\mathbf{t}' = \langle w', \mathbf{k}', \mathbf{c}', \bar{\mathcal{Y}}'\rangle$ and a condition $q \in \mathbb{P}'_{\omega_2}$ such that*

1. *$\mathbf{t} \preceq \mathbf{t}'$, $w = w'$ and $(\forall \alpha \in w)(\mathbf{k}(\alpha) < \mathbf{k}'(\alpha))$, and*
2. *$p \leq_{w,N} q$ and q obeys $\mathbf{t}'$, and*
3. *if $\bar{\nu} \in \mathcal{Y}'_*$, then the condition $q^{[\mathbf{t}',\bar{\nu}]}$ decides $\dot{a}$.*

Proof of the claim. We are going to show the claim by induction on $|w|$. First, let us assume that w is a singleton, say $w = \{\beta\}$. Let $m = N + 5$. It follows from 4.7.2 that we may pick a condition $p_0 \in \mathbb{P}'_{\omega_2}$ and a $\mathbb{P}'_{\beta+1}$–name $\dot{a}_0$ such that

$$p \leq_{w,m} p_0 \quad \text{and} \quad p_0 \Vdash_{\mathbb{P}'_{\omega_2}} \dot{a} = \dot{a}_0.$$

Now, working in $\mathbf{V}^{\mathbb{P}'_\beta}$, we may choose a condition $r \in \dot{\mathbb{Q}}'_\beta$ and an integer $k' > k(\beta)$ such that

- $p_0(\alpha) \leq^\beta_m r$ and
- for each $\nu \in T^r$ with $\mathrm{lh}(\nu) = k'$, for some a'_ν we have $r^{[\nu]} \Vdash \dot{a}_0 = a'_\nu$

(possible by 1.12; remember that $p_0(\alpha)$ is super normal). Next, back in $\mathbf{V}$, pick a condition $q{\restriction}\beta \in \mathbb{P}'_\beta$, a finite candidate $c'(\beta)$ and a system $\langle a_\nu : \nu \in \max(c'(\beta))\rangle$ so that $\mathrm{ht}(c'(\beta)) = \mathbf{k}'(\beta)$ and the condition $q{\restriction}\beta$ forces that $\mathbf{k}'(\beta), q(\beta), \langle a_\nu : \nu \in \max(c'(\beta))\rangle$ are like $k', r, \langle a'_\nu : \nu \in T^r \ \&\ \mathrm{lh}(\nu) = k'\rangle$ above and $q(\beta)$ end extends $c(\beta)$. Let $q = q{\restriction}\beta^\frown\langle q(\beta)\rangle^\frown p_0{\restriction}[\beta+1, \omega_2)$ and let $\mathbf{t}'$ be determined by $w, c'(\beta), k'(\beta)$. It should be clear that they are as required.

Now suppose that $|w| = n + 1$ (and for n we are done). Let $\beta = \max(w)$. We follow the procedure from the case when w is a singleton with small changes at the end only. So let $m = N + 5$. Choose $p_0 \in \mathbb{P}'_{\omega_2}$ and a $\mathbb{P}'_{\beta+1}$–name $\dot{a}_0$ such that

$$p \leq_{w,m} p_0 \quad \text{and} \quad p_0 \Vdash_{\mathbb{P}'_{\omega_2}} \dot{a} = \dot{a}_0.$$

Then, in $\mathbf{V}^{\mathbb{P}'_\beta}$, we may find a condition $r \in \dot{\mathbb{Q}}'_\beta$ and an integer $k' > k(\beta)$ such that

- $p_0(\alpha) \leq^\beta_m r$ and
- for each $\nu \in T^r$ with $\mathrm{lh}(\nu) = k'$, for some a'_ν we have $r^{[\nu]} \Vdash \dot{a}_0 = a'_\nu$

and let $q(\beta)$ be a $\mathbb{P}'_\beta$–name for r as above.

Using the inductive hypothesis (for $w \setminus \{\beta\}$) we may pick a condition $q' \in \mathbb{P}'_\beta$ and a pre-template $\mathbf{t}'' = \langle w'', \mathbf{k}'', \mathbf{c}'', \bar{\mathcal{Y}}''\rangle$ such that

(a) $w''=w\setminus\{\beta\}$, $\mathbf{t}{\restriction}w''\preceq\mathbf{t}''$ and $(\forall\alpha\in w'')(\mathbf{k}(\alpha)<\mathbf{k}''(\alpha))$,
(b) $p_0{\restriction}\beta\leq_{w'',m} q'$ and q' obeys $\mathbf{t}''$, and
(c) if $\bar{\nu}\in\mathcal{Y}''_*$, then the condition $(q')^{[\mathbf{t}'',\bar{\nu}]}$ decides k', $q(\beta)$ up to the level k' and the respective values of a'_ν.

Let $\mathbf{k}'(\beta)$ be an integer larger than all the values forced to k' by conditions of the form $(q')^{[\mathbf{t}'',\bar{\nu}]}$ in (c) above. Now use the inductive hypothesis again to choose a condition $q^+\in\mathbb{P}'_\beta$ and a pre–template $\mathbf{t}^+=\langle w^+,\mathbf{k}^+,\mathbf{c}^+,\bar{\mathcal{Y}}^+\rangle$ such that

(d) $w^+=w''=w\setminus\{\beta\}$, $\mathbf{t}''\preceq\mathbf{t}^+$ and $(\forall\alpha\in w^+)(\mathbf{k}''(\alpha)<\mathbf{k}^+(\alpha))$,
(e) $q'\leq_{w'',m} q^+$ and q^+ obeys $\mathbf{t}^+$, and
(f) if $\bar{\nu}\in\mathcal{Y}^+_*$, then for some finite candidate $c(\bar{\nu})$ of height $\mathbf{k}'(\beta)$ and a sequence $\langle a^{\bar{\nu}}_\eta:\eta\in\max(c(\bar{\nu}))\rangle$ we have

$$(q^+)^{[\mathbf{t}^+,\bar{\nu}]}\Vdash_{\mathbb{P}'_\beta}\text{“ } q(\beta) \text{ end extends } c(\bar{\nu}) \text{ and } q(\beta)^{[\eta]}\Vdash \dot{a}_0=a^{\bar{\nu}}_\eta \text{ ”}.$$

Let $q=q^+{}^\frown\langle q(\beta)\rangle{}^\frown p_0{\restriction}[\beta{+}1,\omega_2)$ and let $\mathbf{t}'=\langle w',\mathbf{k}',\mathbf{c}',\bar{\mathcal{Y}}'\rangle$ be a finite pre–template such that $w'=w$, $\mathbf{k}'{\restriction}w^+=\mathbf{k}^+$ and $\mathbf{k}'(\beta)$ is as chosen earlier, $\mathbf{c}'{\restriction}w^+=\mathbf{c}^+$ and if $\bar{\nu}\in\mathcal{Y}^+_*$ then $c'_\beta(\bar{\nu})$ is the $c(\bar{\nu})$ given by (f) above, and $\bar{\mathcal{Y}}'$ is determined appropriately. It should be clear that $q,\mathbf{t}'$ are as required. □

Now we may easily finish the proof of the lemma. By a repeated use of 4.7.4 with a suitable bookkeeping we may construct a sequence $\langle p_n,F_n,\mathbf{t}_n,m_n:n<\omega\rangle$ such that for each $n<\omega$:

(1) $p_n\in\mathbb{P}'_{\omega_2}$, $F_n\in[\omega_2]^{<\omega}$ and $\mathbf{t}_n$ is a finite pre–template and p_n obeys $\mathbf{t}_n$ and $w^{\mathbf{t}_n}=F_n$, and $m_n=\max(k^{\mathbf{t}_n}(\alpha):\alpha\in w^{\mathbf{t}_n})+7$,
(2) if $\bar{\nu}\in\mathcal{Y}^{\mathbf{t}_n}_*$ then the condition $(p_n)^{[\mathbf{t}_n,\bar{\nu}]}$ decides the value of $\dot{\tau}{\restriction}n$,
(3) $p_n\leq_{F_n,m_n}p_{n+1}$, $\mathbf{t}_n\preceq\mathbf{t}_{n+1}$ and $\big(\forall\alpha\in w^{\mathbf{t}_n}\big)\big(\mathbf{k}^{\mathbf{t}_n}(\alpha)<\mathbf{k}^{\mathbf{t}_{n+1}}(\alpha)\big)$,
(4) $\bigcup\{F_n:n\in\omega\}=\bigcup\{\mathrm{supp}(p_n):n\in\omega\}$.

Finally we use 4.7.1 and 4.7.3. □

Note that there are weak templates $\mathbf{t}$ such that no condition $p\in\mathbb{P}_{\omega_2}$ obeys $\mathbf{t}$ – there could be a problem with norms and/or measures! From all weak templates we will select only those which correspond to conditions in $\mathbb{P}_{\omega_2}$ (and they will be called just *templates*; see 4.11 below).

Definition 4.8. (1) *A cover for a condition* $p\in\mathbb{Q}^{\rm tree}_\emptyset(K^*,\Sigma^*)$ is the condition $q\in\mathbb{Q}^{\rm tree}_\emptyset(K^*,\Sigma^*)$ defined so that $\mathrm{root}(p)=\mathrm{root}(q)$, $q\leq p$ and:
if $\eta\in T^p$, $k=\mathrm{lh}(\eta)$, then $\mathbf{nor}[t^q_\eta]=\mathbf{nor}[t^p_\eta]$, $g_{t^q_\eta}=g_{t^p_\eta}$, and

$$P_{t^q_\eta}=\{f\in\mathbf{H}^*(k):g_{t^q_\eta}\subseteq f\},$$

if $\eta\notin T^p$, $k=\mathrm{lh}(\eta)$, then $g_{t^q_\eta}=\emptyset$, $P_{t^q_\eta}=\mathbf{H}^*(k)$ and $\mathbf{nor}[t^q_\eta]=k$.

(2) Let $p\in\mathbb{Q}^{\rm tree}_\emptyset(K^*,\Sigma^*)$, and let q be the cover of p (note that T^q is a perfect tree). *The covering mapping for* p is the mapping $\mathbf{h}_p:[T^q]\longrightarrow 2^\omega$ defined as follows. First we define a mapping $h_p:T^q\longrightarrow 2^{<\omega}$: we let $h_p(\mathrm{root}(T^q))=\langle\rangle$. Suppose that $h_p(\eta)$ has been defined, $\eta\in T^q$, and say $h_p(\eta)\in 2^n$, $n<\omega$. We note that $|\mathrm{pos}(t^q_\eta)|$ is a power of 2, and thus we may pick $k>n$ such that $|\mathrm{pos}(t^q_\eta)|=|2^{[n,k)}|$. Now, h_p maps $\mathrm{pos}(t^q_\eta)$ onto $\{\nu\in 2^k:h_p(t^q_\eta)\vartriangleleft\nu\}$ (preserving some fixed well-ordering of $\mathcal{H}(\aleph_1)$). Finally we let $\mathbf{h}_p(\rho)=\bigcup\limits_{n<\omega}h_p(\rho\restriction n)$.

(3) *The cover of a condition* $p \in \mathbb{Q}^{\rm tree}_\emptyset(K^r,\Sigma^r)$ is the condition p itself and *the covering mapping* $\mathbf{h}_p : [T^p] \longrightarrow 2^\omega$ is defined by $\mathbf{h}_p(\rho)(n) = \rho(n_0+n)$, where $n_0 = {\rm lh}({\rm root}(p))$ (and $\rho \in [T^p]$, $n<\omega$).

Remark 4.9. (1) The reason for "preserving some fixed well-ordering" in 4.8(2) is that we want that the covering mapping can be read continuously from p: if two conditions p,p' agree up to level n, then also the covers and the covering mappings agree up to that level. (This statement, however, should be interpreted in the right way.)

(2) Suppose that $p \in \mathbb{Q}^{\rm tree}_\emptyset(K^*,\Sigma^*)$ and q is a cover of p. Then $[T^q]$ is a Polish space with the topology generated by $\big\{\big[(T^q)^{[\eta]}\big] : \eta \in T^q\big\}$. It is also equipped with a probability Borel measure m such that for each $\eta \in T^q$ we have

$$m\Big(\big[(T^q)^{[\eta]}\big]\Big) = \prod_{k=n_0}^{{\rm lh}(\eta)-1} 2^{|g^{t^q}_{\eta\restriction k}|-N_k}$$

where $n_0 = {\rm lh}({\rm root}(q))$. Plainly, the covering mapping $\mathbf{h}_p$ is a measure preserving homeomorphism from $[T^q]$ onto 2^ω (where 2^ω carries the standard product measure and topology). The measure m on $[T^q]$ will also be called $m^{\rm Leb}$.

(3) If p,q are as above, $p \in \mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$, then $[T^p]$ is an m–positive closed subset of $[T^q]$, as a matter of fact we have

$$m([T^p]) \geq \mu^{\mathbf{F}}_p({\rm root}(p)) > 0.$$

In 4.10 below we will show a kind of converse.

(4) The parallel statements for the case of $p \in \mathbb{Q}^{\rm tree}_\emptyset(K^r,\Sigma^r)$ and/or $p \in \mathbb{Q}^{\rm mt}_4(K^r,\Sigma^r,\mathbf{F}^r)$ should be clear.

Lemma 4.10. *Suppose that* $p \in \mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$*, and* $q^* \in \mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$ *is a cover for* p*. Let* $C \subseteq [T^p] \subseteq [T^{q^*}]$ *be a closed set of positive Lebesgue measure in* $[T^{q^*}]$*. Then there is a condition* $p^* \in \mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F}^*)$ *stronger than* p *and such that* $[T^{p^*}] \subseteq C$*.*

Proof. For $t \in K^*$ let $F_t : [0,1]^{{\rm pos}(t)} \longrightarrow [0,1]$ be defined by

$$F_t(r_\nu : \nu \in {\rm pos}(t)) = \frac{\sum\{r_\nu : \nu \in {\rm pos}(t)\}}{2^{N_{k_t}-|g_t|}}.$$

This defines a function $\mathbf{F}$ on K^*. Plainly, $(K^*,\Sigma^*,\mathbf{F})$ is a nice measured tree creating pair (we are going to use it to simplify notation only).

Let $T \subseteq T^p$ be a tree such that $\max(T)=\emptyset$ and $C=[T]$. For $\eta \in T$ let $t_\eta \in \Sigma^*(t^p_\eta)$ be such that

$${\rm pos}(t_\eta) = {\rm succ}_T(\eta), \quad \mathbf{nor}[t_\eta] = \mathbf{nor}[t^p_\eta] \quad \text{and} \quad g_{t_\eta} = g_{t^p_\eta}.$$

Let $q = \langle t_\eta : \eta \in T\rangle$. It should be clear that (as C has positive Lebesgue measure) q is a condition in $\mathbb{Q}^{\rm mt}_4(K^*,\Sigma^*,\mathbf{F})$ (note: $\mathbf{F}$, not $\mathbf{F}^*$!). Moreover, possibly shrinking T and C, we may request that

- $\mathbf{nor}[t_\eta] > 2$ for all $\eta \in T$,
- $\mu^{\mathbf{F}}(q) > 1/2$, and $\mu^{\mathbf{F}}_q(\eta) \geq 2^{-2^{{\rm lh}(\eta)+1}}$ for each $\eta \in T$

(remember 1.10, or actually its proof). Let $k_0 = \mathrm{lh}(\mathrm{root}(T))$.

Fix an integer $k > k_0$ for a moment. Let $A = \{\eta \in T : \mathrm{lh}(\eta) = k\}$ (so it is a front of T). For each $\eta \in T[q, A]$, by downward induction, we define $s_\eta \in \Sigma^*(t_\eta)$ and a real $a_\eta \in [0, 1]$ such that

$$(\bigstar)_\eta \qquad a_\eta \geq \prod_{\ell=\mathrm{lh}(\eta)}^{k-1} \left(1 - 2^{-2^{\ell+3}}\right) \cdot \mu^{\mathbf{F}}_q(\eta).$$

If $\eta \in A$, then we let $a_\eta = 1$ (and s_η is not defined).
Suppose that a_ν has been defined for all $\nu \in \mathrm{pos}(t_\eta)$ so that $(\bigstar)_\nu$ holds. Then

$$F_{t_\eta}(a_\nu : \nu \in \mathrm{pos}(t_\eta)) \geq \prod_{\ell=\mathrm{lh}(\eta)+1}^{k-1} \left(1 - 2^{-2^{\ell+3}}\right) \cdot F_{t_\eta}(\mu^{\mathbf{F}}_q(\nu) : \nu \in \mathrm{pos}(t_\eta)) = \prod_{\ell=\mathrm{lh}(\eta)+1}^{k-1} \left(1 - 2^{-2^{\ell+3}}\right) \cdot \mu^{\mathbf{F}}_q(\eta) \geq 2^{-2^{\mathrm{lh}(\eta)+3}}$$

(remember our requests on q). Consequently we may apply 2.1 (for $t = t_\eta$, $r_\nu = a_\nu$ and $g' = g_{t_\eta}$) to pick $s_\eta \in \Sigma^*(t_\eta)$ such that

(α) $\mathbf{nor}[s_\eta] = \mathbf{nor}[t_\eta] - 1$, and

(β) $a_\eta \stackrel{\mathrm{def}}{=} F^*_{s_\eta}(a_\nu : \nu \in \mathrm{pos}(s_\eta)) \geq \left(1 - 2^{-2^{\mathrm{lh}(\eta)+3}}\right) \cdot F_{t_\eta}(a_\nu : \nu \in \mathrm{pos}(t_\eta)) \geq \prod_{\ell=\mathrm{lh}(\eta)}^{k-1} \left(1 - 2^{-2^{\ell+3}}\right) \cdot \mu^{\mathbf{F}}_q(\eta)$.

This completes the choice of s_η's and a_η's. Now we build a system $\langle s^k_\eta : \eta \in S_k \setminus \max(S_k)\rangle$ such that $S_k \subseteq T[q, A]$ is a finite tree, $\mathrm{root}(S_k) = \mathrm{root}(T)$, $s^k_\eta = s_\eta$ and $\mathrm{succ}_{S_k}(\eta) = \mathrm{pos}(s^k_\eta)$ for $\eta \in S_k \setminus \max(S_k)$.

Next, applying König Lemma, we pick an infinite set $I \subseteq \omega$ and a system $p^* = \langle t^{p^*}_\eta : \eta \in T^{p^*}\rangle \in \mathbb{Q}^*_\emptyset(K^*, \Sigma^*)$ such that $\mathrm{root}(T^{p^*}) = \mathrm{root}(T)$ and

$$\eta \in T^{p^*} \ \&\ k_1, k_2 \in I \ \&\ \mathrm{lh}(\eta) < k_1 < k_2 \quad \Rightarrow \quad t^{p^*}_\eta = s^{k_2}_\eta.$$

It follows from our construction that necessarily $p^* \in \mathbb{Q}^{\mathrm{mt}}_4(K^*, \Sigma^*, \mathbf{F}^*)$, and it is a condition stronger than p, and $[T^{p^*}] \subseteq [T] = C$. $\square$

Now we are going to introduce the main technical tool involved in the proof that our iteration is OK. Fix a weak template $\bar{\mathbf{t}} = \langle \mathbf{t}_n : n < \omega\rangle$ for a while. Let $w^{\bar{\mathbf{t}}} = \bigcup\limits_{n<\omega} w^{\mathbf{t}_n}$ and $\zeta_{\bar{\mathbf{t}}} = \mathrm{otp}(w^{\bar{\mathbf{t}}})$, and let $w^{\bar{\mathbf{t}}} = \langle \alpha_\zeta : \zeta < \zeta_{\bar{\mathbf{t}}}\rangle$ (the increasing enumeration). For $\zeta < \zeta_{\bar{\mathbf{t}}}$ let x_ζ be r if $\alpha_\zeta \in Z$, and $*$ if $\alpha_\zeta \notin Z$.

By induction on $\zeta \leq \zeta_{\bar{\mathbf{t}}}$ we define a space $\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta$ and mappings

$$\pi^{\bar{\mathbf{t}}}_\zeta : \mathcal{Z}^{\bar{\mathbf{t}}}_\zeta \longrightarrow \mathbb{Q}^{\mathrm{tree}}_\emptyset(K^{x_\zeta}, \Sigma^{x_\zeta}) \qquad \text{and} \qquad \psi^{\bar{\mathbf{t}}}_\zeta : \mathcal{Z}^{\bar{\mathbf{t}}}_\zeta \longrightarrow (2^\omega)^\zeta.$$

First we let $\mathcal{Z}^{\bar{\mathbf{t}}}_0 = \{\emptyset\}$ and let $\pi^{\bar{\mathbf{t}}}_0(\emptyset) \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K^{x_0}, \Sigma^{x_0})$ is be the unique condition end–extending all $c^{\mathbf{t}_n}_{\alpha_0}$ (for $n < \omega$, $\alpha_0 \in w^{\mathbf{t}_n}$) (and $\psi^{\bar{\mathbf{t}}}_\zeta(\emptyset) = \emptyset$).
Suppose now that $\zeta + 1 \leq \zeta_{\bar{\mathbf{t}}}$ and we have defined $\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta, \pi^{\bar{\mathbf{t}}}_\zeta$ and $\psi^{\bar{\mathbf{t}}}_\zeta$. We let

$$\mathcal{Z}^{\bar{\mathbf{t}}}_{\zeta+1} = \{\bar{z}^\frown\langle z_\zeta\rangle : \bar{z} \in \mathcal{Z}^{\bar{\mathbf{t}}}_\zeta \ \&\ z_\zeta \in [T^{\pi^{\bar{\mathbf{t}}}_\zeta(\bar{z})}] \subseteq \prod_{i<\omega} \mathbf{H}^{x_\zeta}(i)\},$$

and let $\bar{z}^* = \langle z_0, \ldots, z_\zeta\rangle = \bar{z}^\frown\langle z_\zeta\rangle \in \mathcal{Z}^{\bar{\mathbf{t}}}_{\zeta+1}$ (we ignore the first term "$\emptyset$" of the sequence $\bar{z}$). To define $\psi^{\bar{\mathbf{t}}}_{\zeta+1}(\bar{z}^*)$, we let $q \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K^{x_\zeta}, \Sigma^{x_\zeta})$ be the cover of the

condition $\pi^{\bar{\mathbf{t}}}_\zeta(\bar{z})$, and let $\mathbf{h}:[T^q]\longrightarrow 2^\omega$ be the covering mapping for $\pi^{\bar{\mathbf{t}}}_\zeta(\bar{z})$ (see 4.8). Put $\psi^{\bar{\mathbf{t}}}_{\zeta+1}(\bar{z}^*)=\psi^{\bar{\mathbf{t}}}_\zeta(\bar{z})^\frown\langle\mathbf{h}(z_\zeta)\rangle$.
If $\zeta+1<\zeta_{\bar{\mathbf{t}}}$, then we also define $\pi^{\bar{\mathbf{t}}}_{\zeta+1}(\bar{z}^*)$ as the unique condition in $\mathbb{Q}^{\rm tree}_\emptyset(K^{x_{\zeta+1}},\Sigma^{x_{\zeta+1}})$ such that

- if $n<\omega$, $w^{\mathbf{t}_n}=\{\alpha_{\zeta_0},\ldots,\alpha_{\zeta_m}\}$ (the increasing enumeration), and $\zeta_\ell=\zeta+1$, $\ell\leq m$, *then* $\pi^{\bar{\mathbf{t}}}_{\zeta+1}(\bar{z}^*)$ end extends $c^{\mathbf{t}_n}_{\alpha_{\zeta_\ell}}(z_{\zeta_0}\restriction\mathbf{k}^{\mathbf{t}_n}(\alpha_{\zeta_0}),\ldots,z_{\zeta_{\ell-1}}\restriction\mathbf{k}^{\mathbf{t}_n}(\alpha_{\zeta_{\ell-1}}))$.

Suppose now that $\zeta\leq\zeta_{\bar{\mathbf{t}}}$ is a limit ordinal, and that we have defined $\mathcal{Z}^{\bar{\mathbf{t}}}_\xi$, $\pi^{\bar{\mathbf{t}}}_\xi$ and $\psi^{\bar{\mathbf{t}}}_\xi$ for $\xi<\zeta$. We put

$$\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta=\{\langle z_\rho:\rho<\zeta\rangle:(\forall\xi<\zeta)(\langle z_\rho:\rho<\xi\rangle\in\mathcal{Z}^{\bar{\mathbf{t}}}_\xi)\}$$

(again, above, like before and later, we ignore the first term "$\emptyset$" whenever considering elements of $\mathcal{Z}^{\bar{\mathbf{t}}}_\xi$). The mapping $\psi^{\bar{\mathbf{t}}}_\zeta:\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta\longrightarrow(2^\omega)^\zeta$ is such that $\psi^{\bar{\mathbf{t}}}_\zeta(\bar{z})\restriction\xi=\psi^{\bar{\mathbf{t}}}_\xi(\bar{z}\restriction\xi)$ (for $\bar{z}\in\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta$). Also if, additionally, $\zeta<\zeta_{\bar{\mathbf{t}}}$, then for $\bar{z}=\langle z_\rho:\rho<\zeta\rangle\in\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta$ we let $\pi^{\bar{\mathbf{t}}}_\zeta(\bar{z})$ be the unique element of $\mathbb{Q}^{\rm tree}_\emptyset(K^{x_\zeta},\Sigma^{x_\zeta})$ such that

- if $n<\omega$, $w^{\mathbf{t}_n}=\{\alpha_{\zeta_0},\ldots,\alpha_{\zeta_m}\}$ (the increasing enumeration), and $\zeta_\ell=\zeta$, $\ell\leq m$, *then* $\pi^{\bar{\mathbf{t}}}_\zeta(\bar{z})$ end extends $c^{\mathbf{t}_n}_{\alpha_{\zeta_\ell}}(z_{\zeta_0}\restriction\mathbf{k}^{\mathbf{t}_n}(\alpha_{\zeta_0}),\ldots,z_{\zeta_{\ell-1}}\restriction\mathbf{k}^{\mathbf{t}_n}(\alpha_{\zeta_{\ell-1}}))$.

Definition 4.11. Let $\bar{\mathbf{t}}$ be a weak template, and $w^{\bar{\mathbf{t}}}=\bigcup\limits_{n<\omega}w^{\mathbf{t}_n}=\langle\alpha_\zeta:\zeta<\zeta_{\bar{\mathbf{t}}}\rangle$ be the increasing enumeration. Also for $\zeta<\zeta_{\bar{\mathbf{t}}}$ let x_ζ be r if $\alpha_\zeta\in Z$, and $*$ if $\alpha_\zeta\notin Z$. We say that $\bar{\mathbf{t}}$ is *a template* if for every $\zeta<\zeta_{\bar{\mathbf{t}}}$ and $\bar{z}\in\mathcal{Z}^{\bar{\mathbf{t}}}_\zeta$ we have

$$\pi^{\bar{\mathbf{t}}}_\zeta(\bar{z})\in\mathbb{Q}^{\rm sn}_4(K^{x_\zeta},\Sigma^{x_\zeta},\mathbf{F}^{x_\zeta}).$$

Lemma 4.12. (1) *Assume that $p\in\mathbb{P}_{\omega_2}$ and $\dot{\tau}$ is a $\mathbb{P}_{\omega_2}$–name for a real in 2^ω. Then there are a condition $q\in\mathbb{P}_{\omega_2}$ and a template with a name $(\bar{\mathbf{t}},\bar{\tau})$ such that $q\geq p$, $(q,\dot{\tau})$ obeys $(\bar{\mathbf{t}},\bar{\tau})$, $\omega\leq\zeta_{\bar{\mathbf{t}}}<\omega_1$, and for some enumeration $\langle\zeta_n:n<\omega\rangle$ of $\zeta_{\bar{\mathbf{t}}}$ we have:*
($\boxplus$) *for every $n<\omega$ and $\bar{z}\in\mathcal{Z}^{\bar{\mathbf{t}}}_{\zeta_n}$,*

$$\mu^{\mathbf{F}}(\pi^{\bar{\mathbf{t}}}_{\zeta_n}(\bar{z}))\geq\big(1-2^{-n-10}\big),$$

where $\mathbf{F}$ is suitably $\mathbf{F}^r$ or $\mathbf{F}^*$.
[If a template $\bar{\mathbf{t}}$ satisfies ($\boxplus$) for an enumeration $\bar{\zeta}=\langle\zeta_n:n<\omega\rangle$ of $\zeta_{\bar{\mathbf{t}}}$, then we we will say that $\bar{\mathbf{t}}$ behaves well for *$\bar{\zeta}$.]*
(2) *For every template $\bar{\mathbf{t}}$, there is a condition $p\in\mathbb{P}_{\omega_2}$ which obeys $\bar{\mathbf{t}}$.*

Proof. (1) The argument given in in the proof of 4.7 can be easily modified to suit the current lemma (remember 1.8).
(2) Should be clear. □

For a countable ordinal ζ, the space $(2^\omega)^\zeta$ is equipped with the product measure $m^{\rm Leb}$ of countably many copies of 2^ω. We will use the same notation $m^{\rm Leb}$ for this measure in various products (and related spaces), hoping that no real confusion is caused.

Lemma 4.13. *Let $\zeta<\omega_1$. Suppose that $C\subseteq(2^\omega)^\zeta$ is a closed set of positive Lebesgue measure. Then there is a closed set $C^*\subseteq C$ of positive Lebesgue measure such that for each $\xi<\zeta$:*

$(\otimes)^{\xi}_{C^*}$ *for every* $\bar{y}\in(2^\omega)^\xi$*, the set*

$$(C^*)_{\bar{y}}\stackrel{\rm def}{=}\{\bar{y}'\in(2^\omega)^{[\xi,\zeta)}:\bar{y}^\frown\bar{y}'\in C^*\}$$

is either empty or has positive Lebesgue measure (in $(2^\omega)^{[\xi,\zeta)}$*).*

Proof. For a set $X\subseteq(2^\omega)^\zeta$, $\xi<\zeta$, and $\bar{y}\in(2^\omega)^\xi$ we let

$$(X)_{\bar{y}}\stackrel{\rm def}{=}\{\bar{y}'\in(2^\omega)^{[\xi,\zeta)}:\bar{y}^\frown\bar{y}'\in X\}.$$

We may assume that $\zeta\geq\omega$ (otherwise the lemma is easier and actually included in this case). Fix an enumeration $\zeta=\{\zeta_n:n<\omega\}$ such that $\zeta_0=0$, and let

$$e_0=2^{-4}\cdot m^{\rm Leb}(C),\qquad e_{n+1}=2^{-6(n+2)^2}\cdot(e_n)^{n+2}.$$

We are going to define inductively a decreasing sequence $\langle C_n:n<\omega\rangle$ of closed (non-empty) subsets of C such that $C_0=C_1=C$ and

($\circledast$) for each $m<n$ and $\bar{y}\in(2^\omega)^{\zeta_m}$ we have

$$\text{either}\quad (C_n)_{\bar{y}}=\emptyset\qquad\text{or}\quad m^{\rm Leb}((C_n)_{\bar{y}})\geq e_m\cdot\Big(1-\sum_{\ell=m+2}^{n}4^{-\ell}\Big).$$

(Note that ($\circledast$) implies $m^{\rm Leb}(C_n)\geq e_0\cdot(1-\sum\limits_{\ell=2}^{n}4^{-\ell})$; just consider $m=0$.)

Suppose that C_n has been already defined, $n\geq 1$. Let $\{\xi_\ell:\ell\leq\ell^*\}$ enumerate the set

$$\{\zeta_m:m\leq n\ \&\ \zeta_m\leq\zeta_n\}$$

in the increasing order. By downward induction on $0<\ell\leq\ell^*$ we choose open sets $U_\ell\subseteq(2^\omega)^{\xi_\ell}$. So, the set $U_{\ell^*}\subseteq(2^\omega)^{\xi_{\ell^*}}$ is such that (remember $\xi_{\ell^*}=\zeta_n$):

- $(\forall\bar{y}\in(2^\omega)^{\zeta_n}\setminus U_{\ell^*})\big(m^{\rm Leb}((C_n)_{\bar{y}})\geq e_n\big)$,
- $m^{\rm Leb}\big(C_n\cap(U_{\ell^*}\times(2^\omega)^{[\zeta_n,\zeta)})\big)<e_n$.

Now suppose that $U_{\ell^*},\ldots,U_{\ell+1}$ have been already chosen so that

$$m^{\rm Leb}\big(C_n\cap(U_k\times(2^\omega)^{[\xi_k,\zeta)})\big)<\Big(\frac{2^{3n+3}}{e_{n-1}}\Big)^{\ell^*-k}\cdot e_n$$

for each $k\in\{\ell+1,\ldots,\ell^*\}$. Let

$$U=U_{\ell+1}\times(2^\omega)^{[\xi_{\ell+1},\zeta)}\cup\ldots\cup U_{\ell^*}\times(2^\omega)^{[\xi_{\ell^*},\zeta)}.$$

Note that (by our assumptions)

$$m^{\rm Leb}(U\cap C_n)<(\ell^*-\ell)\cdot\Big(\frac{2^{3n+3}}{e_{n-1}}\Big)^{\ell^*-\ell-1}\cdot e_n.$$

Let $\xi_\ell=\zeta_m$ and $A=\{\bar{y}\in(2^\omega)^{\zeta_m}:m^{\rm Leb}\big((C_n\cap U)_{\bar{y}}\big)>\frac{e_m}{2^{2n+2}}\}$. Note that

$$m^{\rm Leb}(A)\cdot\frac{e_m}{2^{2n+2}}<m^{\rm Leb}(C_n\cap U)<(\ell^*-\ell)\cdot\Big(\frac{2^{3n+3}}{e_{n-1}}\Big)^{\ell^*-\ell-1}\cdot e_n,$$

and hence

$$m^{\rm Leb}(A)<\Big(\frac{2^{3n+3}}{e_{n-1}}\Big)^{\ell^*-\ell-1}\cdot\frac{2^{2n+2}}{e_{n-1}}\cdot(\ell^*-\ell)\cdot e_n<\Big(\frac{2^{3n+3}}{e_{n-1}}\Big)^{\ell^*-\ell}\cdot e_n.$$

Pick an open set $U_\ell\subseteq(2^\omega)^{\zeta_m}$ such that $A\subseteq U_\ell$ and $m^{\rm Leb}(U_\ell)<\Big(\frac{2^{3n+3}}{e_{n-1}}\Big)^{\ell^*-\ell}\cdot e_n$.

Finally we let $C_{n+1} = C_n \setminus \bigcup\limits_{\ell=1}^{\ell^*} \big(U_\ell \times (2^\omega)^{[\xi_\ell,\zeta)}\big)$. It is easy to check that C_{n+1} is as required.

After the sets C_n are all constructed we put $C^* = \bigcap\limits_{n<\omega} C_n$. It follows from ($\circledast$) that the demand $(\otimes)^\xi_{C^*}$ is satisfied for each $\xi < \zeta$. $\square$

Theorem 4.14. *In $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, the condition (b) of 4.1 holds.*

Proof. For $\alpha < \omega_2$ let $\dot{x}_\alpha$ be a $\mathbb{P}_\alpha$–name for the generic real added at stage α (so it is a member of 2^ω if $\alpha \in Z$, and a member of $\prod\limits_{k<\omega} \mathbf{H}^*(k)$ if $\alpha \in \omega_2 \setminus Z$).

Suppose that $\dot{f}^*$ is a $\mathbb{P}_{\omega_2}$–name for a function from 2^ω to 2^ω, and $p \in \mathbb{P}_{\omega_2}$.
For each $\delta \in Z$ pick a template with a name $(\bar{\mathbf{t}}^\delta, \bar{\tau}^\delta)$, an enumeration $\bar{\zeta}^\delta = \langle \zeta^\delta_n : n < \omega \rangle$ of $\zeta_{\bar{\mathbf{t}}^\delta} = \mathrm{otp}(w^{\bar{\mathbf{t}}^\delta})$, and a condition $p^\delta \in \mathbb{P}_{\omega_2}$ such that

- $\zeta_{\bar{\mathbf{t}}^\delta} \geq \omega$, $\bar{\mathbf{t}}^\delta$ behaves well for $\bar{\zeta}^\delta$ (see 4.12(1)),
- $p^\delta \geq p$ and $(p^\delta, \dot{f}^*(\dot{x}_\delta))$ obeys $(\bar{\mathbf{t}}^\delta, \bar{\tau}^\delta)$,
- $\delta \in w^{\bar{\mathbf{t}}^\delta}$ and $w^{\bar{\mathbf{t}}^\delta} \setminus (\delta+1) \neq \emptyset$.

Using Fodor Lemma (and 4.6(2)) we find a template with a name $(\bar{\mathbf{t}}, \bar{\tau})$, ordinals $\zeta^* < \zeta_{\bar{\mathbf{t}}} = \mathrm{otp}(w^{\bar{\mathbf{t}}})$ and $\xi < \omega_2$, an enumeration $\bar{\zeta} = \langle \zeta_n : n < \omega \rangle$ of $\zeta_{\bar{\mathbf{t}}}$, and a stationary set $Z^* \subseteq Z$ such that for each $\delta, \delta' \in Z^*$ we have

(i) $(\bar{\mathbf{t}}^\delta, \bar{\tau}^\delta)$ is isomorphic to $(\bar{\mathbf{t}}, \bar{\tau})$ by an isomorphism mapping $\bar{\zeta}^\delta$ to $\bar{\zeta}$, and $\bar{\mathbf{t}} = \langle \mathbf{t}_n : n < \omega \rangle$, $\bar{\tau} = \langle \tau_n : n < \omega \rangle$, and
(ii) $\mathrm{otp}(w^{\bar{\mathbf{t}}^\delta} \cap \delta) = \zeta^*$, $w^{\bar{\mathbf{t}}^\delta} \cap \delta \subseteq \xi$, and $p \in \mathbb{P}_\xi$ and
(iii) $\bar{\mathbf{t}}^\delta \restriction \xi = \bar{\mathbf{t}}^{\delta'} \restriction \xi$.

Let $\dot{A}$ be the $\mathbb{P}_{\omega_2}$–name for the set $\{\dot{x}_\delta : \delta \in Z^* \ \&\ p^\delta \in \Gamma_{\mathbb{P}_{\omega_2}}\}$ and let $\Psi : (2^\omega)^{[\zeta^*+1, \zeta_{\bar{\mathbf{t}}})} \longrightarrow 2^\omega$ be the canonical homeomorphism (induced by a bijective mapping from $\omega \times [\zeta^*+1, \zeta_{\bar{\mathbf{t}}})$ onto ω). Now, in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, we define a mapping $\dot{f}_1 : \dot{A} \longrightarrow 2^\omega$ by:

$$\dot{f}_1(\dot{x}_\delta) = \Psi\Big(\psi^{\bar{\mathbf{t}}^\delta}_{\zeta_{\bar{\mathbf{t}}}}(\dot{x}_\alpha : \alpha \in w^{\bar{\mathbf{t}}^\delta}) \restriction [\zeta^*+1, \zeta_{\bar{\mathbf{t}}})\Big)$$

($\psi^{\bar{\mathbf{t}}^\delta}_{\zeta_{\bar{\mathbf{t}}}}$ is as defined before 4.11). Let $p^* = p^\delta \restriction \delta$ for some (equivalently: all) $\delta \in Z^*$.

Claim 4.14.1.

$$p^* \Vdash_{\mathbb{P}_{\omega_2}} \text{“ the set } \{(x, \dot{f}_1(x)) : x \in \dot{A}\} \text{ has positive outer measure ”.}$$

Proof of the claim. Assume not. Then there are an ordinal ξ^*, a condition q, and a $\mathbb{P}_{\omega_2}$–name $\dot{D}$ such that

- $\xi \leq \xi^* < \omega_2$, $q \in \mathbb{P}_{\xi^*}$, and $q \geq p^*$,
- $\dot{D}$ is a $\mathbb{P}_{\xi^*}$–name for a (Lebesgue) null subset of $(2^\omega)^{[\zeta^*, \zeta_{\bar{\mathbf{t}}})}$, and
- $q \Vdash_{\mathbb{P}_{\omega_2}}$ “ $(\forall \delta \in Z^*)\big(p^\delta \in \Gamma_{\mathbb{P}_{\omega_2}} \ \Rightarrow\ \psi^{\bar{\mathbf{t}}^\delta}_{\zeta_{\bar{\mathbf{t}}}}(\dot{x}_\alpha : \alpha \in w^{\bar{\mathbf{t}}^\delta}) \restriction [\zeta^*, \zeta_{\bar{\mathbf{t}}}) \in \dot{D}\big)$ ”.

(Note that above we use the fact that the forcing used at $\delta \in Z$ is the random real forcing, and the covering mapping at this coordinate is - essentially - the identity. This allows us to replace $\big(\dot{x}_\delta, \psi^{\bar{\mathbf{t}}^\delta}_{\zeta_{\bar{\mathbf{t}}}}(\dot{x}_\alpha : \alpha \in w^{\bar{\mathbf{t}}^\delta}) \restriction [\zeta^*+1, \zeta_{\bar{\mathbf{t}}})\big)$ by $\psi^{\bar{\mathbf{t}}^\delta}_{\zeta_{\bar{\mathbf{t}}}}(\dot{x}_\alpha : \alpha \in w^{\bar{\mathbf{t}}^\delta}) \restriction [\zeta^*, \zeta_{\bar{\mathbf{t}}})$.) Fix any $\delta^* \in Z^*$ larger than ξ^* and let $\langle \alpha_\zeta : \zeta < \zeta^* \rangle$ be the increasing enumeration of $w^{\bar{\mathbf{t}}^{\delta^*}} \cap \delta^*$ and let $\dot{z}_\zeta = \dot{x}_{\alpha_\zeta}$, and $\dot{\bar{z}} = \langle \dot{z}_\zeta : \zeta < \zeta^* \rangle$. Note that the conditions p^{δ^*} and q are compatible. Also, as $\dot{x}_{\delta^*}$ is (a name for) a random real over $\mathbf{V}^{\mathbb{P}_{\delta^*}}$, we have

$$q \Vdash_{\mathbb{P}_{\delta^*+1}} \text{" the set } \dot{B} \stackrel{\rm def}{=} \{\bar{y} \in (2^\omega)^{[\zeta^*+1,\zeta_{\bar{\mathbf{t}}})} : \langle \psi^{\bar{\mathbf{t}}^{\delta^*}}_{\zeta^*+1}(\dot{\bar{z}}{}^\frown\langle \dot{x}_{\delta^*}\rangle)(\zeta^*)\rangle{}^\frown\bar{y} \in \dot{D}\} \text{ is null ".}$$

Using Lemma 4.13, we may pick (a $\mathbb{P}_{\delta^*+1}$–name for) a closed set $\dot{C}^* \subseteq (2^\omega)^{[\zeta^*+1,\zeta_{\bar{\mathbf{t}}})}$ such that the condition q forces (in $\mathbb{P}_{\delta^*+1}$):

- $\dot{C}^* \subseteq \{\psi^{\bar{\mathbf{t}}^{\delta^*}}_{\zeta_{\bar{\mathbf{t}}}}(\bar{z}) \restriction [\zeta^*+1,\zeta_{\bar{\mathbf{t}}}) : \dot{\bar{z}}{}^\frown\langle \dot{x}_{\delta^*}\rangle \vartriangleleft \bar{z} \in \mathcal{Z}^{\bar{\mathbf{t}}^{\delta^*}}_{\zeta_{\bar{\mathbf{t}}}}\}$,
- $\dot{C}^* \cap \dot{B} = \emptyset$, and
- the condition $(\otimes)^{\xi}_{\dot{C}^*}$ of 4.13 holds true for every $\xi \in [\zeta^*+1,\zeta_{\bar{\mathbf{t}}})$.

(For the first demand remember that $\bar{\mathbf{t}}^{\delta^*}$ is well behaving, so the set on the right-hand side has positive Lebesgue measure.) But now, using 4.10, we may inductively build a condition $q' \in \mathbb{P}_{\omega_2}$ stronger than both q and p^{δ^*} (and with the support included in $(\delta^*+1) \cup w^{\bar{\mathbf{t}}^{\delta^*}}$) and such that

$$q' \Vdash_{\mathbb{P}_{\omega_2}} \text{" } \psi^{\bar{\mathbf{t}}^{\delta^*}}_{\zeta_{\bar{\mathbf{t}}}}(\dot{x}_\alpha : \alpha \in w^{\bar{\mathbf{t}}^{\delta^*}}) \restriction [\zeta^*+1,\zeta_{\bar{\mathbf{t}}}) \notin \dot{B} \text{ ",}$$

getting an immediate contradiction. $\square$

Pick any $\delta^* \in Z^*$ and let $\dot{\bar{z}} = \langle \dot{z}_\zeta : \zeta < \zeta^*\rangle$ be as defined in the proof of 4.14.1 above. Let $\dot{E}$ be a $\mathbb{P}_{\delta^*}$–name for the set

$$\{(r_0,r_1) \in 2^\omega \times 2^\omega : \dot{\bar{z}}{}^\frown\langle r_0\rangle \in \mathcal{Z}^{\bar{\mathbf{t}}}_{\zeta^*+1} \text{ and } \psi^{\bar{\mathbf{t}}}_{\zeta^*+1}(\dot{\bar{z}}{}^\frown\langle r_0\rangle){}^\frown\Psi^{-1}(r_1) \in \operatorname{rng}(\psi^{\bar{\mathbf{t}}}_{\zeta^{\bar{\mathbf{t}}}})\}.$$

So $\dot{E}$ is (a name for) a closed subset of $2^\omega \times 2^\omega$. Let $\dot{f}_2$ be a name for a Borel function from $2^\omega \times 2^\omega$ to 2^ω such that

if $(r_0,r_1) \in \dot{E}$, and $\psi^{\bar{\mathbf{t}}}_{\zeta^*+1}(\dot{\bar{z}}{}^\frown\langle r_0\rangle){}^\frown\Psi^{-1}(r_1) = \psi^{\bar{\mathbf{t}}}_{\zeta^{\bar{\mathbf{t}}}}(\langle z_\zeta : \zeta < \zeta^{\bar{\mathbf{t}}}\rangle)$,
then for each $n < \omega$

$$\dot{f}_2(r_0,r_1) \restriction n = \tau_n(z_\zeta \restriction \mathbf{k}^{\mathbf{t}_n}(\zeta) : \zeta \in w^{\mathbf{t}_n})$$

(remember (i)). It should be clear that $\dot{f}_2$ is (a name for) a continuous function and

$$p^* \Vdash_{\mathbb{P}_{\omega_2}} \text{" } (\forall x \in \dot{A})(\dot{f}^*(x) = \dot{f}_2(x,\dot{f}_1(x))) \text{ ",}$$

finishing the proof. $\square$

Corollary 4.15. *It is consistent that*

- *every sup-measurable function is Lebesgue measurable, and*
- *for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ there is a continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that the set $\{x \in \mathbb{R} : f(x) = g(x)\}$ has positive outer measure.*

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