CONSTRUCTING BOOLEAN ALGEBRAS FOR CARDINAL INVARIANTS

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Abstract. We construct Boolean Algebras answering some questions of J. Donald Monk on cardinal invariants. The results are proved in ZFC (rather than giving consistency results). We deal with the existence of superatomic Boolean Algebras with "few automorphisms", with entangled sequences of linear orders, and with semi-ZFC examples of the non-attainment of the spread (and hL, hd).

Annotated Content

§1 *A superatomic Boolean Algebra with fewer automorphisms than endomorphisms.*
We prove in ZFC that for some superatomic Boolean Algebra $\mathbb{B}$ we have $\mathrm{Aut}(\mathbb{B}) < \mathrm{End}(\mathbb{B})$. This solves [**?**, Problem 76, p.291] of Monk.

§2 *A superatomic Boolean Algebra with fewer automorphisms than elements*
We prove in ZFC that for some superatomic Boolean Algebra $\mathbb{B}$, we have $\mathrm{Aut}(\mathbb{B}) < |\mathbb{B}|$. This solves [**?**, Problem 80, p.291] of Monk.

§3 *On entangledness*
We prove that if $\mu < \kappa \le \chi < \mathrm{Ded}(\mu)$ and $2^\mu < \lambda$, and κ is regular, and $\lambda \le \mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\chi)$ (see Definition 3.2), then $\mathrm{Ens}(\kappa,\lambda)$, i.e., there is an entangled sequence of λ linear orders each of cardinality κ. The reader may think of the case

$$\mu = \aleph_0, \quad \kappa = \mathrm{cf}(\chi) < \chi = 2^\mu = 2^{<\kappa} < 2^\kappa, \quad \text{and} \quad \lambda = \chi^+.$$

Note that the existence of entangled linear orders is connected to the problem whether always $\prod\limits_{i<\theta} \mathrm{Inc}(\mathbb{B}_i)/D \ge \mathrm{Inc}(\prod\limits_{i<\theta} \mathbb{B}_i/D)$ for an ultrafilter D on θ. We rely on quotations of some pcf results.

§4 *On attainment of spread*
We construct Boolean Algebras with the spread not obtained under ZFC + "GCH is violated strongly enough, even just for regular

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1991 *Mathematics Subject Classification.* 03E04, 03G05, 03E05, 03E10.
Key words and phrases. Set theory, Boolean algebras, pcf, cardinal invariants of Boolean algebras, automorphisms, endomorphisms, attainment of spread, semi–ZFC answers.
I would like to thank Alice Leonhardt for the beautiful typing.
This research was partially supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. Publication 641.

cardinals"; so the consistency strength is ZFC. We consider this a semi–ZFC answer.

1. A superatomic Boolean Algebra with fewer automorphisms than endomorphisms

Rubin has proved that if $\diamondsuit_{\lambda^+}$, then there is a superatomic Boolean algebra with few automorphisms. We give here a construction in ZFC.

We use some notions of [?], they can be found in [?]; in particular $\mathbf{J}_\theta[\mathfrak{a}] = \mathbf{J}_{<\theta}[\mathfrak{a}] + (\mathfrak{a} \setminus \mathfrak{b}_\theta[\mathfrak{a}])$. For this section we assume

Hypothesis 1.1. (a) $\bar\lambda = \langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals larger than δ; let $\mathfrak{a} = \{\lambda_i : i < \delta\}$.

(b) $\lambda_0 > 2^{|\delta|}$, or at least $\lambda_0 > |\mathrm{pcf}(\mathfrak{a})|$.

The main combinatorial point of our construction is given by the following observation.

Proposition 1.2. *There are sequences* $\langle \bar f^\theta : \theta \in \mathrm{pcf}(\mathfrak{a}) \rangle$ *and* $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a}) \rangle$ *such that*

(a) $\bar f^\theta = \langle f^\theta_\alpha : \alpha < \theta \rangle \subseteq \prod \mathfrak{a}$ *is a* $<_{\mathbf{J}_\theta[\mathfrak{a}]}$*–increasing cofinal sequence,* $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a}) \rangle$ *is a generating sequence,*

(b) $f^\theta_\alpha \restriction (\mathfrak{a} \setminus \mathfrak{b}_\theta[\mathfrak{a}])$ *is constantly zero,*

(c) *if* $\theta_1 < \theta_2$, $\alpha_2 < \theta_2$ *then*
$$f^{\theta_2}_{\alpha_2} \restriction (\mathfrak{b}_{\theta_1}[\mathfrak{a}] \cap \mathfrak{b}_{\theta_2}[\mathfrak{a}]) \in \{f^{\theta_1}_{\alpha_1} \restriction (\mathfrak{b}_{\theta_1}[\mathfrak{a}] \cap \mathfrak{b}_{\theta_2}[\mathfrak{a}]) : \alpha_1 < \theta_1\}.$$

(d) *for* $\theta \in \mathrm{pcf}(\mathfrak{a})$ *and* $\lambda \in \mathfrak{b}_\theta[\mathfrak{a}]$, $f^\theta_\alpha(\lambda)$ *is a limit ordinal* $> \sup(\lambda \cap \mathfrak{a})$.

(e) *if* $\theta_1 < \theta_2$*, both in* $\mathrm{pcf}(\mathfrak{a})$*, then there are* $n < \omega$, $\sigma_1, \ldots, \sigma_n \le \theta$ *(all from* $\mathrm{pcf}(\mathfrak{a})$*) such that* $\mathfrak{b}_{\theta_1}[\mathfrak{a}] \cap \mathfrak{b}_{\theta_2}[\mathfrak{a}] = \bigcup\limits_{k=1}^{n} \mathfrak{b}_{\sigma_k}[\mathfrak{a}]$.

Proof. Let $\mathfrak{a}' = \mathrm{pcf}(\mathfrak{a})$, so $|\mathfrak{a}'| < \min(\mathfrak{a}')$ and $\mathrm{pcf}(\mathfrak{a}') = \mathfrak{a}'$ (by [?, Ch.I, 1.11]). We can find a generating sequence $\langle \mathfrak{b}_\theta[\mathfrak{a}'] : \theta \in \mathfrak{a}' \rangle$ (by [?, Ch.VIII, 2.6]), and hence a closed smooth one (by [?, Ch.I, 3.8(3)]). Now repeat the proof of [?, Ch.II, 3.5] or see [?]. Note that "smooth" means
$$\sigma \in \mathfrak{b}_\theta[\mathfrak{a}'] \quad \Rightarrow \quad \mathfrak{b}_\sigma[\mathfrak{a}'] \subseteq \mathfrak{b}_\theta[\mathfrak{a}'],$$
"closed" means $\mathfrak{b}_\theta[\mathfrak{a}] = \mathrm{pcf}(\mathfrak{b}_\theta[\mathfrak{a}]) \cap \mathfrak{a}$; together clause (e) follows. $\square$

Definition 1.3. Let $\langle \bar f^\theta : \theta \in \mathrm{pcf}(\mathfrak{a}) \rangle$ and $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a}) \rangle$ be sequences given by 1.2 (so they satisfy the demands (a)–(e) there).

(1) For $\ell \in \{0,1\}$, $\theta \in \mathrm{pcf}(\mathfrak{a})$ and $\alpha \le \theta$ we define the Boolean ring $\mathcal{B}^\ell_{\theta,\alpha}$ of subsets of $\sup(\mathfrak{a})$. We do this by induction on θ, and for each θ by induction on α as follows.

(a) If $\theta = \min(\mathfrak{a})$, $\alpha = 0$, then $\mathcal{B}^\ell_{\theta,\alpha}$ is the Boolean ring (of subsets of $\sup(\mathfrak{a})$) generated by
$$\{[\sup(\mathfrak{a} \cap \lambda), \gamma) : \lambda \in \mathfrak{a},\ \sup(\mathfrak{a} \cap \lambda) < \gamma < \lambda\},$$
that is the closure of the above family under $x \cap y$, $x \cup y$, $x - y$.

(b) If $\theta \in \mathrm{pcf}(\mathfrak{a}) \setminus \{\min(\mathfrak{a})\}$, then let $\mathcal{B}^\ell_{\theta,0} = \bigcup\limits_{\sigma\in\theta\cap\mathrm{pcf}(\mathfrak{a})} \mathcal{B}^\ell_{\sigma,\sigma}$.

(c) If $\theta \in \mathrm{pcf}(\mathfrak{a})$, $\alpha < \theta$ is a limit ordinal then $\mathcal{B}^\ell_{\theta,\alpha} = \bigcup\limits_{\beta<\alpha} \mathcal{B}^\ell_{\theta,\beta}$.

(d) If $\theta \in \mathrm{pcf}(\mathfrak{a})$, $\alpha = \theta$, then $\mathcal{B}^\ell_{\theta,\alpha}$ is the Boolean ring generated by

$$\bigcup_{\beta<\alpha} \mathcal{B}^\ell_{\theta,\beta} \cup \Big\{ \bigcup_{\lambda\in\mathfrak{b}_\theta[\mathfrak{a}]} [\sup(\lambda\cap\mathfrak{a}),\lambda)\Big\}.$$

(e) If $\theta \in \mathrm{pcf}(\mathfrak{a})$, $\alpha = \beta+1 < \theta$, then

(i) $\mathcal{B}^0_{\theta,\alpha}$ is the Boolean ring generated by

$$\mathcal{B}^1_{\theta,\beta} \cup \Big\{ \bigcup_{\lambda\in\mathfrak{b}_\theta[\mathfrak{a}]} [\sup(\mathfrak{a}\cap\lambda), f^\theta_\beta(\lambda))\Big\},$$

(ii) $\mathcal{B}^1_{\theta,\alpha}$ is the Boolean ring generated by

$$\mathcal{B}^0_{\theta,\alpha} \cup \Big\{ \bigcup_{\lambda\in\mathfrak{b}_\theta[\mathfrak{a}]} [\sup(\mathfrak{a}\cap\lambda), f^\theta_\beta(\lambda)+1)\Big\}.$$

(2) We let

$$\begin{array}{lcll} x^0_{\theta,\beta} & = & \bigcup\limits_{\lambda\in\mathfrak{b}_\theta[\mathfrak{a}]} [\sup(\mathfrak{a}\cap\lambda), f^\theta_\beta(\lambda)) & \text{for } \theta\in\mathrm{pcf}(\mathfrak{a}),\ \beta<\theta, \\ x^1_{\theta,\beta} & = & \bigcup\limits_{\lambda\in\mathfrak{b}_\theta[\mathfrak{a}]} [\sup(\mathfrak{a}\cap\lambda), f^\theta_\beta(\lambda)+1) & \text{for } \theta\in\mathrm{pcf}(\mathfrak{a}),\ \beta<\theta, \\ y_\theta & = & \bigcup\limits_{\lambda\in\mathfrak{b}_\theta[\mathfrak{a}]} [\sup(\lambda\cap\mathfrak{a}),\lambda) & \text{for } \theta\in\mathrm{pcf}(\mathfrak{a}), \\ z_\alpha & = & [\sup(\lambda\cap\mathfrak{a}),\alpha) & \text{for } \alpha\in[\sup(\lambda\cap\mathfrak{a}),\lambda),\ \lambda\in\mathfrak{a}. \end{array}$$

(3) $\mathbb{B}^\ell_{\theta,\alpha}$ is the Boolean algebra of subsets of $\sup(\mathfrak{a})$ generated by $\mathcal{B}^\ell_{\theta,\alpha}$, and $\mathbb{B}^\ell$ stands for the Boolean Algebra of subsets of $\sup(\mathfrak{a})$ generated by $\mathcal{B}^\ell_{\max\mathrm{pcf}(\mathfrak{a}),\max\mathrm{pcf}(\mathfrak{a})}$.

[After we shall note that $\mathbb{B}^0 = \mathbb{B}^1$ (in 1.4) we can write $\mathbb{B}^0 = \mathbb{B} = \mathbb{B}^1$.]

Proposition 1.4. (1) *$\mathcal{B}^\ell_{\theta,\theta}$ is increasing in θ, and for a fixed θ, $\mathcal{B}^\ell_{\theta,\alpha}$ is increasing in α and is actually a Boolean ring of subsets of* $\sup(\mathfrak{a})$.

(2) *$\mathcal{B}^m_{\theta,\alpha}$ is the Boolean ring generated by*

$$\begin{array}{l} \{y_\sigma : \sigma\in\mathrm{pcf}(\mathfrak{a})\cap\theta \ \ \textit{or} \ \ \sigma=\alpha=\theta\} \cup \{z_\alpha : \alpha<\sup(\mathfrak{a})\} \cup \\ \{x^\ell_{\sigma,\alpha} : \sigma<\theta,\ \sigma\in\mathrm{pcf}(\mathfrak{a}),\ \alpha<\sigma,\ \ell<2\} \cup \\ \{x^\ell_{\theta,\beta} : \beta+1<\alpha \ \&\ \ell<2 \ \textit{or}\ \beta+1=\alpha \ \&\ \ell\le m\}. \end{array}$$

(3) *If α is zero or limit $\le \theta \in \mathrm{pcf}(\mathfrak{a})$, then $\mathcal{B}^0_{\theta,\alpha} = \mathcal{B}^1_{\theta,\alpha}$ and $\mathbb{B}^0_{\theta,\alpha} = \mathbb{B}^1_{\theta,\alpha}$.*

(4) *If $(\theta_1,\alpha_1,\ell_1) \le_{\ell ex} (\theta_2,\alpha_2,\ell_2)$ and $a_i \in \mathcal{B}^{\ell_i}_{\theta_i,\alpha_i}$ for $i=1,2$, then $a_1\cap a_2 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$.*

(5) *If $\ell_i \in \{0,1\}$, $\alpha_i \le \theta_i \in \mathrm{pcf}(\mathfrak{a})$ for $i=1,2$ and*

$$\theta_1<\theta_2 \vee (\theta_1=\theta_2 \ \&\ \alpha_1<\alpha_2) \vee (\theta_1=\theta_2 \ \&\ \alpha_1=\alpha_2 \ \&\ \ell_1<\ell_2),$$

then $\mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$ *is an ideal of* $\mathcal{B}^{\ell_2}_{\theta_2,\alpha_2}$.

Proof. (1)–(3) Straightforward.

(4) First note that it is enough to show the assertion under an additional demand that a_1, a_2 are among the generators of the Boolean rings $\mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}, \mathcal{B}^{\ell_2}_{\theta_2,\alpha_2}$, respectively, as listed in part (2).

Case 1: One of a_1, a_2 is z_α for some $\alpha < \sup(\mathfrak{a})$.
Then the other is either y_θ, or z_β, or $x^m_{\theta,\beta}$, and in all cases the intersection $a_1 \cap a_2$ is either empty or it is $z_{\alpha'}$ for some $\alpha' \le \alpha$. Hence $a_1 \cap a_2 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$.

Case 2: $a_1 = y_{\theta'}$, $a_2 = y_{\theta''}$ for some $\theta', \theta'' \in \mathrm{pcf}(\mathfrak{a})$.
If $\theta'' \le \theta'$ then, as $a_1 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$, we easily get $a_2 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$ and thus the intersection $a_1 \cap a_2$ is in this Boolean ring.

So we may assume that $\theta' < \theta''$. It follows from 1.2(e) that there are $\sigma_1, \ldots, \sigma_n \le \theta'$ such that

$$\mathfrak{b}_{\theta'}[\mathfrak{a}] \cap \mathfrak{b}_{\theta''}[\mathfrak{a}] = \bigcup_{k=1}^{n} \mathfrak{b}_{\sigma_k}[\mathfrak{a}].$$

Then $a_1 \cap a_2 = y_{\sigma_1} \cup \ldots \cup y_{\sigma_n}$ and $y_{\sigma_1}, \ldots, y_{\sigma_n} \in \mathcal{B}^{\ell}_{\theta_1,\alpha_1}$, so we are done.

Case 3: $a_1 = y_{\theta'}$, $a_2 = x^m_{\theta'',\beta}$ for some $\theta', \theta'' \in \mathrm{pcf}(\mathfrak{a})$, $m < 2$, $\beta < \theta''$.
If $\theta'' \le \theta'$ then $a_2 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$ and we are done; so assume that $\theta' < \theta''$. It follows from 1.2(c) that then $f^{\theta''}_\beta \restriction (b_{\theta'}[\mathfrak{a}] \cap b_{\theta''}[\mathfrak{a}]) = f^{\theta'}_\alpha \restriction (b_{\theta'}[\mathfrak{a}] \cap b_{\theta''}[\mathfrak{a}])$ for some $\alpha < \theta'$. Like in Case 2, one shows that $y_{\theta'} \cap y_{\theta''} \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$; also $x^m_{\theta',\alpha} \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$. But now $a_1 \cap a_2 = x^m_{\theta',\alpha} \cap (y_{\theta'} \cap y_{\theta''}) \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$.

Case 4: $a_1 = x^m_{\theta',\beta}$, $a_2 = y_{\theta''}$ for some $\theta', \theta'' \in \mathrm{pcf}(\mathfrak{a})$, $\beta < \theta'$, $m < 2$.
If $\theta'' < \theta'$ then $a_2 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$, and if $\theta'' = \theta'$ then $a_1 \cap a_2 = a_1$. So we may assume $\theta' < \theta''$. If $\mathfrak{b}_{\theta'}[\mathfrak{a}] \subseteq \mathfrak{b}_{\theta''}[\mathfrak{a}]$, then clearly $a_1 \cap a_2 = a_1$ and we are done, so suppose otherwise. Then, using 1.2(e), we find $\sigma_1, \ldots, \sigma_n < \theta'$ such that $\mathfrak{b}_{\theta'}[\mathfrak{a}] \cap \mathfrak{b}_{\theta''}[\mathfrak{a}] = \bigcup\limits_{k=1}^{n} \mathfrak{b}_{\sigma_k}[\mathfrak{a}]$. Since, in this case, all σ_k are smaller than θ' and $x^m_{\theta',\beta} \cap y_{\theta''} = \bigcup\limits_{k=1}^{n} y_{\sigma_k} \cap x^m_{\theta',\beta}$, we easily conclude $a_1 \cap a_2 \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$.

Case 5: $a_1 = x^{m'}_{\theta',\beta'}$, $a_2 = x^{m''}_{\theta'',\beta''}$ for some $\theta', \theta'' \in \mathrm{pcf}(\mathfrak{a})$, $\beta' < \theta'$, $\beta'' < \theta''$ and $m', m'' < 2$.
If $(\theta'', \beta'', m'') \le_{\ell ex} (\theta', \beta', m')$ then we are easily done.

If $\theta'' = \theta'$, $\beta'' = \beta'$ and $0 = m' < m'' = 1$, then clearly $a_1 \cap a_2 = a_1$.

Assume that $\theta'' = \theta'$, $\beta' < \beta''$. Then, by 1.2(a), we find $\mu_0, \ldots, \mu_{k-1} \in \theta' \cap \mathrm{pcf}(\mathfrak{a})$ such that

$$\{\mu \in \mathfrak{a} : f^{\theta''}_{\beta''}(\mu) \le f^{\theta'}_{\beta'}(\mu)\} \subseteq \bigcup_{j<k} \mathfrak{b}_{\mu_j}[\mathfrak{a}] \cup (\mathfrak{a} \setminus \mathfrak{b}_{\theta'}).$$

Then clearly

$$a_1 \cap a_2 = \Big(\bigcup_{j<k} a_1 \cap a_2 \cap y_{\mu_j}\Big) \cup \Big(a_1 \setminus \bigcup_{j<k} y_{\mu_j}\Big).$$

Also, for $j<k$, we have

$$y_{\mu_j} \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1} \quad \text{and} \quad a_1 \cap a_2 \cap y_{\mu_j} = (a_1 \cap y_{\mu_j}) \cap (a_2 \cap y_{\mu_j}),$$

and the sets $a_1 \cap y_{\mu_j}$ and $a_2 \cap y_{\mu_j}$ are in $\mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$ by (suitably applied) case 3. So we can easily finish.

The only remaining possibility is $\theta' < \theta''$. By 1.2(a) we may pick $\gamma < \theta'$ such that

$$f^{\theta''}_{\beta''} \restriction (\mathfrak{b}_{\theta''}[\mathfrak{a}] \cap \mathfrak{b}_{\theta'}[\mathfrak{a}]) = f^{\theta'}_{\gamma} \restriction (\mathfrak{b}_{\theta''}[\mathfrak{a}] \cap \mathfrak{b}_{\theta'}[\mathfrak{a}]).$$

Then $a_1 \cap a_2 = x^{m'}_{\theta',\beta'} \cap x^{m''}_{\theta',\gamma} \cap y_{\theta'} \cap y_{\theta''}$. By the discussion above we know that $x^{m'}_{\theta',\beta'} \cap x^{m''}_{\theta',\gamma} \in \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$. Now, if $y_{\theta'} \subseteq y_{\theta''}$ then $a_1 \cap a_2 = x^{m'}_{\theta',\beta'} \cap x^{m''}_{\theta',\gamma}$ and we are done. Otherwise, $y_{\theta'} \cap y_{\theta''} \in \mathcal{B}^0_{\theta',0} \subseteq \mathcal{B}^{\ell_1}_{\theta_1,\alpha_1}$ (compare Case 2), and again we easily get the required conclusion.

(5) Follows. $\square$

Proposition 1.5. (1) *$\mathcal{B}^{\ell}_{\theta,\alpha}$ is a superatomic Boolean ring with $\{\{\gamma\} : \gamma < \sup(\mathfrak{a})\}$ as the set of atoms.*
(2) *$\mathbb{B}^{\ell}_{\theta,\alpha}$ is a superatomic Boolean algebra, in particular $\mathbb{B}^{\ell}$ is.*
(3) *If $\alpha, \beta < \theta \in \mathrm{pcf}(\mathfrak{a})$ and $\gamma = \omega^\beta$ (ordinal exponentiation; so $\gamma < \theta$ and $\alpha + \gamma < \theta$),* then *the rank of $x^0_{\theta,\alpha+\gamma} - x^0_{\theta,\alpha}$ is $\geq \beta$.*

Proof. 1) Straight by induction on θ and for a fixed θ by induction on $\alpha \leq \theta$ using 1.4(5).

2) Follows.

3) Easy by induction on β. $\square$

Proposition 1.6. (1) *The algebra $\mathbb{B}$ has exactly* $\sup(\mathfrak{a})$ *atoms, so*

$$|\mathrm{Atom}(\mathbb{B}^{\ell})| = \sup(\mathfrak{a}).$$

(2) $|\mathbb{B}| = \max \mathrm{pcf}(\mathfrak{a})$.
(3) $|\mathrm{Aut}(\mathbb{B})| \leq 2^{\sup(\mathfrak{a})}$.

Proof. Parts 1), 2) should be clear. Part 3) holds as the algebra $\mathbb{B}$ has $\sup(\mathfrak{a})$ atoms by part (1) (and two distinct automorphisms of $\mathbb{B}$ differ on an atom). $\square$

Proposition 1.7. *The algebra $\mathbb{B}$ has $2^{\max \mathrm{pcf}(\mathfrak{a})}$ endomorphisms.*

Proof. Let $Z \subseteq \max\mathrm{pcf}(\mathfrak{a})$. We define an endomorphism $T_Z \in \mathrm{End}(\mathbb{B})$ by describing how it acts on the generators. We let:

$$\begin{array}{lll} T_Z(z_\alpha) & = & z_\beta \quad \text{if } \beta \text{ is maximal such that } \beta \le \alpha < \beta+\omega \text{ and:} \\ & & \beta = 0 \text{ or } \beta \text{ limit or } \alpha = \beta \in \bigcup\limits_{\lambda\in\mathfrak{a}} [\sup(\mathfrak{a}\cap\lambda), \sup(\mathfrak{a}\cap\lambda)+\omega), \\ T_Z(y_\theta) & = & y_\theta, \\ T_Z(x^0_{\theta,\alpha}) & = & x^0_{\theta,\alpha}, \end{array}$$

$$T_Z(x^1_{\theta,\alpha}) = \begin{cases} x^0_{\theta,\alpha} & \text{if } \theta < \max\mathrm{pcf}(\mathfrak{a}), \\ x^0_{\theta,\alpha} & \text{if } \theta = \max\mathrm{pcf}(\mathfrak{a}),\ \alpha \notin Z, \\ x^1_{\theta,\alpha} & \text{if } \theta = \max\mathrm{pcf}(\mathfrak{a}),\ \alpha \in Z. \end{cases}$$

One easily checks that the above formulas correctly define an element of $\mathrm{End}(\mathbb{B})$. Clearly $Z_1 \neq Z_2$ implies $T_{Z_1} \neq T_{Z_2}$ and we are done. $\square$

So we can answer (in ZFC) Monk's question [**?**, Problem 76, pages 259, 291].

Conclusion 1.8. Assume that μ is a strong limit singular cardinal, and $\mathrm{cf}(\mu) > \aleph_0$ (or just $\mathrm{pp}^+(\mu) = (2^\mu)^+$, so most of those with $\mathrm{cf}(\mu) = \aleph_0$ are OK) and $\mu < \kappa = \mathrm{cf}(\kappa) \le 2^\mu < 2^\kappa$ (always such μ exists and for each such μ such κ exists). *Then* there is a superatomic Boolean Algebra $\mathbb{B}$ such that:

(a) $|\mathbb{B}| = \kappa$,
(b) $|\mathrm{Atom}(\mathbb{B})| = \mu$,
(c) $|\mathrm{Aut}(\mathbb{B})| \le 2^\mu$,
(d) $|\mathrm{End}(\mathbb{B})| = 2^\kappa$.

Proof. We can find $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu$ such that $|\mathfrak{a}| = \mathrm{cf}(\mu)$ and $\kappa = \max\mathrm{pcf}(\mathfrak{a})$. Why? We know

$$2^\mu = \mu^{\mathrm{cf}(\mu)} = \mathrm{cov}(\mu, \mu, (\mathrm{cf}(\mu))^+, 2) \ge \mathrm{cov}(\mu, \mu, \mathrm{cf}(\mu)^+, \mathrm{cf}(\mu)),$$

and now we use [**?**, Ch.II, 5.4] when $\mathrm{cf}(\mu) > \aleph_0$; see [**?**, 6.5] for references on the $\mathrm{cf}(\mu) = \aleph_0$ case). $\square$

2. A superatomic Boolean Algebra with fewer automorphisms than elements

Monk has asked ([**?**, Problem 80, p.291,260]) if there may be a superatomic Boolean Algebra $|\mathbb{B}|$ with "few" (i.e., $< |\mathbb{B}|$) automorphisms. Remember that $\mathrm{Aut}(\mathbb{B}) \ge |\mathrm{Atom}(\mathbb{B})|$ if $|\mathrm{Atom}(\mathbb{B})| \neq 1$.

In this section we answer this question by showing that, in ZFC, there is a superatomic Boolean Algebra $\mathbb{B}$ with $\mathrm{Aut}(\mathbb{B}) < |\mathbb{B}|$. Moreover, there are such Boolean Algebras in many cardinals.

For our construction we assume the following:

Hypothesis 2.1. (α) μ is a strong limit singular cardinal of cofinality $\aleph_0$,
(β) $\lambda = 2^\mu$, $\kappa \le \lambda$,

(γ) T is a tree with κ levels, $\leq \lambda$ nodes and the number of its κ-branches is $\chi > \lambda$, and T has a root.

Note that there are many μ as in clause (α) of 2.1, and then we can choose $\lambda = 2^\mu$ and, e.g., $\kappa = \min\{\kappa : 2^\kappa > \lambda\}$, $T = {}^{\kappa>}2$.

Theorem 2.2. *There is a superatomic Boolean Algebra $\mathbb{B}$ such that:*

$$|\mathbb{B}| = \chi \quad \textit{and} \quad |\mathrm{Atom}(\mathbb{B})| = |T| + \mu \leq |\mathrm{Aut}(\mathbb{B})| \leq \lambda.$$

Proof. Let $T^+ = T \cup \lim_\kappa(T)$, so $|T^+| = \chi$. Let

$$\mathcal{F} = \big\{f : \ f \text{ is a one-to-one function, } \mathrm{Dom}(f) \subseteq T \times \mu, \\ |\mathrm{Dom}(f)| = \mu, \ \mathrm{Rang}(f) \subseteq T \times \mu \setminus \mathrm{Dom}(f)\big\}.$$

Clearly $|\mathcal{F}| \leq |T \times \mu|^\mu \leq \lambda^\mu = (2^\mu)^\mu = 2^\mu = \lambda$.

Let $x_{(t,\alpha)} = x_{t,\alpha} = \{(t,\alpha)\}$ (for $t \in T$ and $\alpha < \mu$) and let $z_t = \{(s,\alpha) : s <_T t \text{ and } \alpha < \mu\}$ (for $t \in T^+$). (Note that if t_1, t_2 are immediate successors of s, then $z_{t_1} = z_{t_2}$; also the family $\{z_t : t \in T^+\}$ is closed under intersections.)

Claim 2.2.1. *There is a family $\mathcal{A}_2 \subseteq [T \times \mu]^{\aleph_0}$ such that*

(a) *if $y', y'' \in \mathcal{A}_2$ are distinct, then $y' \cap y''$ is finite,*

(b) *if $t \in T^+$ and $y \in \mathcal{A}_2$, then*

$$y \subseteq z_t \ \vee \ |y \cap z_t| < \aleph_0,$$

(c) *if $Y \in [T \times \mu]^\mu$, and f is a one-to-one function from Y to $T \times \mu \setminus Y$ (so $f \in \mathcal{F}$), then there is $y \in \mathcal{A}_2$ such that $f[y]$ is almost disjoint from every member of $\mathcal{A}_2$.*

Proof of the Claim. List $\mathcal{F}$ as $\{g_\alpha : \alpha < \lambda\}$. By induction on $\alpha < \lambda$ we choose y_α, y'_α such that:

(α) y_α, y'_α are disjoint countable subsets of $T \times \mu$,

(β) y_α, y'_α are almost disjoint to any $y'' \in \{y_\beta, y'_\beta : \beta < \alpha\}$,

(γ) $y_\alpha \subseteq \mathrm{Dom}(g_\alpha)$, $y'_\alpha = g_\alpha[y_\alpha]$,

(δ) if $t \in T^+$ then either $y_\alpha \subseteq z_t$ or $|y_\alpha \cap z_t| < \aleph_0$.

So assume y_β, y'_β for $\beta < \alpha$ have been defined. Pick an increasing sequence $\langle \mu_n : n < \omega\rangle$ of regular cardinals such that $\mu = \sum\limits_{n<\omega} \mu_n$ and $2^{\mu_n} < \mu_{n+1}$.

Choose pairwise disjoint sets $Y_n \in [\mathrm{Dom}(g_\alpha)]^{\mu_n}$ (for $n < \omega$). We may replace Y_n by any $Y'_n \in [Y_n]^{\mu_n}$, and even $Y'_n \in [Y_{k_n}]^{\mu_n}$ with a strictly increasing sequence $\langle k_n : n < \omega\rangle$.

Let $Y_n = \{(t^n_i, \alpha^n_i) : i < \mu_n\}$ be an enumeration (with no repetitions). Without loss of generality:

- the sequence $\langle \mathrm{level}(t^n_i) : i < \mu_n\rangle$ is constant or strictly increasing,
- the sequence $\langle \alpha^n_i : i < \mu_n\rangle$ is constant or strictly increasing, and
- for each $n < \omega$, for some truth value $\mathbf{t}_n$ we have

$$(\forall i < j < \mu_n)(\text{truth value } (t^n_i <_T t^n_j) \equiv \mathbf{t}_n).$$

[Why? E.g. use $\mu_{n+1} \to (\mu_n)^2_2$]. Cleaning a little more we may demand that

- for $n \neq m$, for some truth value $\mathbf{t}_{m,n}$,

$$(\forall i < \mu_n)(\forall j < \mu_m)(\text{truth value } (t^n_i <_T t^m_j) = \mathbf{t}_{m,n}).$$

[Why? E.g. use polarized partition relations.] Using Ramsey's theorem applied to the partition $F(m,n) = \mathbf{t}_{m,n}$ (and replacing $\langle \mu_n : n < \omega \rangle$ by an ω–subsequence), without loss of generality:

$$\begin{aligned} \text{either:} \quad & \text{for some } t \in T^+, \text{ for every } \eta \in \prod_{n<\omega} \mu_n \text{ we have} \\ & \{(t^\ell_{\eta(\ell)}, \alpha^\ell_{\eta(\ell)}) : \ell < \omega\} \subseteq z_t, \\ \text{or:} \quad & \text{for every } t \in T^+ \text{ and } \eta \in \prod_{n<\omega} \mu_n \text{ we have} \\ & |\{(t^\ell_{\eta(\ell)}, \alpha^\ell_{\eta(\ell)} : \ell < \omega)\} \cap z_t| \le 1. \end{aligned}$$

Next we choose $\{(t_\eta, \beta_\eta) : \eta \in \prod\limits_{\ell<n} \mu_\ell\} \subseteq Y_n$ (no repetitions) and for each $\eta \in \prod\limits_{n<\omega} \mu_n$ we consider

$$y_\eta = \{(t_{\eta\restriction\ell}, \beta_{\eta\restriction\ell}) : \ell < \omega\}, \quad y'_\eta = g_\alpha[y_\eta]$$

as candidates for y_α, y'_α, respectively. Clause (α) holds as $\mathrm{Rang}(g_\alpha) \cap \mathrm{Dom}(g_\alpha) = \emptyset$, clauses (γ) and (δ) are also trivial. So only clause (β) may fail. Each $\beta < \alpha$ disqualifies at most $2^{\aleph_0}$ of the η's, i.e., of the pairs (y_η, y'_η). So only $\le |\alpha| \times 2^{\aleph_0} < \lambda = |{}^\omega\mu|$ of the η's are disqualified, so some are OK, and we are done. This finishes the proof of the Claim. $\square$

Let $\mathcal{A}_2$ be a family given by 2.2.1 and let

$$\mathcal{A}_0 = \{\{x\} : x \in T \times \mu\} \qquad \text{and} \qquad \mathcal{A}_1 = \{z_t : t \in T^+\}$$

Our Boolean algebra $\mathbb{B}$ is the Boolean Algebra of subsets of $T \times \mu$ generated by $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$.

Claim 2.2.2. *The algebra $\mathbb{B}$ is superatomic.*

Proof of the Claim. Clearly, the family $I = \{b \in \mathbb{B} : \mathbb{B} \restriction b \text{ is superatomic}\}$ is an ideal in $\mathbb{B}$. Plainly $x_{(t,\alpha)} \in I$ for $(t, \alpha) \in T \times \mu$. Now, by induction on $\alpha \le \kappa$ we prove that if $t \in T^+$ is of level α, then $\mathbb{B} \restriction z_t$ is superatomic.
If $\alpha = 0$, then $z_t = \emptyset$ and this is trivial.
If $\alpha = \beta + 1$ and t is the immediate successor of s, then (as $\mathbb{B} \restriction z_s$ is superatomic by the induction hypothesis) it is enough to prove that $\mathbb{B} \restriction (z_t - z_s)$ is superatomic. Now, $\mathbb{B} \restriction (z_t - z_s)$ is the Boolean Algebra of subsets of $\{s\} \times \mu$, generated by

$$\{\{(s, \alpha)\} : \alpha < \mu\} \cup \{y \cap (\{s\} \times \mu) : y \in \mathcal{A}_2\},$$

and we are done by 2.2.1(a).
If α is a limit ordinal and $\mathrm{cf}(\alpha) = \aleph_0$, then $(\mathbb{B} \restriction z_t)/\mathrm{id}_{\mathbb{B}}(\{z_{t\restriction\beta} : \beta < \alpha\})$ is a

Boolean Algebra generated by its atoms

$$\{y \cap z_t : y \in \mathcal{A}_2 \text{ and } y \leq z_t \ \& \ \bigwedge_{s<t} \neg y \leq z_s\}$$

(remember 2.2.1(a+b)), and thus $z_t \in I$. If α is limit of uncountable cofinality the same conclusion is even more immediate.

So $\{z_t : t \in T^+\} \subseteq I$, and $\mathbb{B}/\mathrm{id}_{\mathbb{B}}(\{z_t : t \in T^+\})$ is a Boolean Algebra generated by its set of atoms which is included in

$$\{y \in \mathcal{A}_2 : \neg(\exists t)(y \leq z_t)\}$$

(by 2.2.1(a)). Hence we conclude that $\mathbb{B}$ is superatomic. $\square$

Claim 2.2.3. (1) *$\mathrm{Atom}(\mathbb{B}) = \{x_{(t,\alpha)} : (t,\alpha) \in T \times \mu\}$, so $\mathbb{B}$ has $|T| + \mu$ atoms, and $|\mathbb{B}| = \chi$.*

(2) *$|\mathrm{Aut}(\mathbb{B})| \leq 2^\mu$; moreover for every $f \in \mathrm{Aut}(\mathbb{B})$*

$$|\{(t,\alpha) \in T \times \mu : f(x_{(t,\alpha)}) \neq x_{(t,\alpha)}\}| < \mu.$$

Proof of the Claim. (1) Easy.

(2) Clearly the second statement implies the first. So let $f \in \mathrm{Aut}(\mathbb{B})$ and suppose that f moves at least μ atoms. Then there is $g \in \mathcal{F}$ such that $f(x_{(t,\alpha)}) = x_{g(t,\alpha)}$ for all $(t,\alpha) \in \mathrm{Dom}(g)$. But, by 2.2.1(c), there is $y \in \mathcal{A}_2$ such that $y \subseteq \mathrm{Dom}(g)$ and $g[y]$ is almost disjoint to every member of $\mathcal{A}_2$. An easy contradiction. $\square$

$\square$

Remark 2.3. (1) As $\mathrm{End}(\mathbb{B}) \geq |\mathbb{B}|$ this gives an example for [**?**, Problem 76], too. Still the first example works in more cardinals and is different.

(2) With a little more work we can guarantee that the number of one-to-one endomorphisms of $\mathbb{B}$ is $\leq 2^\mu$.

(3) Alternatively, for the proof of 2.2.1 we can use $\mu^{\aleph_0} = 2^\mu$ almost disjoint subsets of $\mathrm{Dom}(g_\alpha)$, say $\langle y_{\alpha,i} : i < 2^\mu \rangle$; for each i choose $y'_{\alpha,i} \in [y_{\alpha,i}]^{\aleph_0}$ such that it satisfies clause (b) of 2.2.1 (exists by Ramsey theorem), so for some i we have: $y_{\alpha,i}$, $y'_{\alpha,i} =: g_\alpha[y_{\alpha,i}]$ are almost disjoint to y_β, y'_β for $\beta < \alpha$. So $y_{\alpha,i}, y'_{\alpha,i}$ are as required.

3. On entangledness

Definition 3.1. (1) A sequence $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \alpha^* \rangle$ of linear orders is κ-entangled if:

(a) each $\mathcal{I}_\alpha$ is a linear order of cardinality $\geq \kappa$, and

(b) if $n < \omega$, $\alpha_1 < \ldots < \alpha_n < \alpha^*$, and $t^\ell_\zeta \in \mathcal{I}_{\alpha_\ell}$ for $\ell \in \{1,\ldots,n\}$, $\zeta < \kappa$ are such that $\zeta \neq \xi \ \Rightarrow \ t^\ell_\zeta \neq t^\ell_\xi$, then for any $w \subseteq \{1,\ldots,n\}$ we may find $\zeta < \xi < \kappa$ such that:

$$\ell \in w \ \Rightarrow \ \mathcal{I}_{\alpha_\ell} \models t^\ell_\zeta < t^\ell_\xi \qquad \text{and} \qquad \ell \in \{1,\ldots,n\} \setminus w \ \Rightarrow \ \mathcal{I}_{\alpha_\ell} \models t^\ell_\xi < t^\ell_\zeta.$$

If κ is omitted we mean: $\kappa = \min\{|\mathcal{I}_\alpha| : \alpha < \alpha^*\}$.

(2) $\mathrm{Ens}(\kappa,\lambda)$ is the statement asserting that there is an entangled sequence $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda\rangle$ of linear orders each of cardinality κ.

Definition 3.2. (1) For an ideal J on κ we let

$$\mathbf{U}_J(\chi) =: \min\{|\mathcal{A}| : \begin{array}{l}\mathcal{A} \subseteq [\chi]^\kappa \text{ and}\\ (\forall f \in {}^\kappa\chi)(\exists A \in \mathcal{A})(\{i<\kappa : f(i) \in A\} \in J^+)\}.\end{array}$$

(2) $\mathrm{Ded}^+(\mu) =: \min\{\theta$: there is no linear order with θ elements and density $\le \mu$ $\}$.

Theorem 3.3. *Assume that $\mu < \kappa < \chi < \mathrm{Ded}^+(\mu)$, $2^\mu < \lambda$, κ is regular and $\lambda \le \mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\chi)$ (see Definition 3.2). Then $\mathrm{Ens}(\kappa,\lambda)$ by a sequence $\langle \mathcal{I}_\alpha : \alpha<\lambda\rangle$ of linear orders of cardinality κ and density μ (see Definition 3.1).*

Proof. Let $\mathcal{J}$ be a dense linear order of cardinality χ with a dense subset $\mathcal{J}^*$ of cardinality μ. Without loss of generality the set of elements of $\mathcal{J}$ is χ and of $\mathcal{J}^*$ is μ. Let u^i_ζ (for $i<\kappa$, $\zeta<\chi$) be pairwise distinct members of $\mathcal{J}$, and let $\bar{u} = \langle u^i_\zeta : i<\kappa, \zeta<\chi\rangle$. For $f \in {}^\kappa\chi$ let $\mathcal{I}_f = \{u^i_{f(i)} : i<\kappa\}$.

Main Claim 3.3.1. *If $n<\omega$, $f_0,\dots,f_{n-1} \in {}^\kappa\chi$ and $\bar{\mathcal{I}} = \langle \mathcal{I}_{f_\ell} : \ell < n\rangle$ is entangled, then we can find $\mathcal{A} \subseteq [\chi]^\kappa$ such that $|\mathcal{A}| \le 2^\mu$ and:*

($\oplus$) *if $f \in {}^\kappa\chi$ and $(\forall A \in \mathcal{A})(\forall^* i<\kappa)(f(i) \notin A)$, then $\bar{\mathcal{I}}{}^\frown\langle \mathcal{I}_f\rangle$ is entangled*
  ($\forall^*$ means "for every large enough").

Proof of the Claim. Assume $f_0,\dots,f_{n-1} \in {}^\kappa\chi$ and $\bar{\mathcal{I}} = \langle \mathcal{I}_{f_\ell} : \ell<n\rangle$ is entangled.

Let

$$\mathcal{F} = \{f \in {}^\kappa\chi : \bar{\mathcal{I}}{}^\frown\langle \mathcal{I}_f\rangle \text{ is not entangled }\}.$$

For each $f_n = f \in \mathcal{F}$ we fix $w^f \subseteq \{0,\dots,n\}$ and $t^{\ell,f}_j \in \mathcal{I}_{f_\ell}$ (for $\ell \le n$, $j<\kappa$) with no repetitions witnessing that $\bar{\mathcal{I}}{}^\frown\langle\mathcal{I}_f\rangle$ is not entangled. Next we fix a model $N_f \prec (\mathcal{H}(\beth_8^+(\chi)), \in, <^*)$ such that $\mu+1 \subseteq N_f$, $\|N_f\| = \mu$, $\{\bar{\mathcal{I}}, \mathcal{I}_f, \mathcal{J}, f\} \in N_f$ and $\bar{t}^f = \langle t^{\ell,f}_j : \ell \le n, j<\kappa\rangle \in N_f$. Note that for $i<\kappa$ we have:

(i) $t^{\ell,f}_i \notin N_f$ whenever $i \notin N_f$,
(ii) $x \in N_f$ & $|x|<\kappa$ & $\sup(N_f\cap\kappa) \le i < \kappa \quad\Rightarrow\quad i \notin x$.

Now we define a relation E on $\mathcal{F}$ letting for $f,g \in \mathcal{F}$:

$$f\,E\,g \quad\text{if and only if}\quad \begin{array}{ll}(\alpha) & w^f = w^g,\\ (\beta) & N_f \cap \chi = N_g \cap \chi,\\ (\gamma) & (\forall \ell \le n)(\forall j \in N_f\cap\kappa)(t^{\ell,f}_j = t^{\ell,g}_j).\end{array}$$

Note that E is an equivalence relation on $\mathcal{F}$, and there are at most 2^μ E–equivalence classes. Therefore, in order to show 3.3.1, it is enough that for each E–equivalence class g/E we define a set $Y_{g/E} \in [\chi]^\kappa$ such that:

($\boxtimes$) if $f \in g/E$ then $\neg(\forall^* i)(f(i) \notin Y_{g/E})$.

Then, letting $\mathcal{A} = \{Y_{g/E} : g \in \mathcal{F}\}$ we will get a family as required in 3.3.1.

So let $g \in \mathcal{F}$, $w^* = w^g$ and let $i^* = \sup(N_g \cap \kappa)$.
For $i < \kappa$, and a sequence $\bar{t} = \langle t^\ell : \ell < n\rangle \in \prod\limits_{\ell<n} \mathcal{I}_{f_\ell}$ we let

$$Y^i_{\bar{t}} = \big\{f(j) : f \in g/E \ \&\ (\forall \ell < n)(t^{\ell,f}_i = t^\ell) \ \&\ j < \kappa \ \&\ u^j_{f(j)} = t^{n,f}_i\big\}.$$

We claim that

(iii) if $i > i^*$ (but $i < \kappa$) and $\bar{t} \in \prod\limits_{\ell<n} \mathcal{I}_{f_\ell}$, then $|Y^i_{\bar{t}}| \leq 1$.

Why? Assume toward contradiction that $f_1(j_1), f_2(j_2)$ are two distinct members of $Y^i_{\bar{t}}$, $f_1, f_2 \in g/E$, $t^{\ell,f_m}_i = t^\ell$ (for $\ell < n$ and $m = 1,2$) and $t^{n,f_1}_i = u^{j_1}_{f_1(j_1)} \neq u^{j_2}_{f_2(j_2)} = t^{n,f_2}_i$. Pick disjoint intervals $(a^1,b^1),(a^2,b^2)$ of $\mathcal{J}^*$ such that $t^{n,f_m}_i \in (a^m,b^m)$ and $[t^\ell \neq t^{n,f_m}_i \ \Rightarrow\ t^\ell \notin (a^m,b^m)]$ (for $m = 1,2$ and $\ell < n$). Without loss of generality, if $n \in w^*$ then $b^1 < a^2$, else $b^2 < a^1$. We can also pick $a_\ell, b_\ell \in \mathcal{J}^*$ (for $\ell < n$) such that $a_\ell < t^\ell < b_\ell$ and:

- if $t^\ell \neq t^{\ell'}$ then $(a_\ell,b_\ell) \cap (a_{\ell'},b_{\ell'}) = \emptyset$,
- if $t^\ell \neq t^{n,f_m}_i$ then $(a_\ell,b_\ell) \cap (a^m,b^m) = \emptyset$.

Now, we are going to show that

(iii)* if $w \subseteq n$, $m \in \{1,2\}$, and $i_0 \in N_{f_m} \cap \kappa$, and $a^+_\ell \in \mathcal{J}^* \cap [a_\ell,t^\ell)$ and $b^+_\ell \in \mathcal{J}^* \cap (t^\ell,b_\ell]$ (for $\ell < n$),
then we can find $j \in N_{f_m} \cap \kappa \setminus i_0$ such that
$(*)_j$ $t^{n,f_m}_j \in (a^m,b^m)$, and

$$\ell \in w \ \Rightarrow\ a^+_\ell < t^{\ell,f_m}_j < t^\ell \qquad \text{and} \qquad \ell \in n \setminus w \ \Rightarrow\ t^\ell < t^{\ell,f_m}_j < b^+_\ell.$$

So assume that (iii)* fails, so there is no $j \in N_m \cap \kappa \setminus i_0$ such that $(*)_j$ holds. First note that then also there is no $j' < i$ (but $j' > i_0$) satisfying $(*)_{j'}$. [Why? Suppose $(*)_{j'}$ holds true and choose $a^*_\ell, b^*_\ell \in \mathcal{J}^*$ such that

$$\begin{array}{lcll} \ell \in w & \Rightarrow & a^+_\ell = a^*_\ell < t^{\ell,f_m}_{j'} < b^*_\ell < t^\ell & \qquad\text{and}\\ \ell \in n \setminus w & \Rightarrow & t^\ell < a^*_\ell < t^{\ell,f_m}_{j'} < b^*_\ell = b^+_\ell. \end{array}$$

The set

$$Z = \{j \in \kappa \setminus i_0 : (\forall \ell < n)(a^*_\ell < t^{\ell,f_m}_j < b^*_\ell) \ \&\ a_m < t^{\ell,f_m}_j < b_m\}$$

is non-empty (as witnessed by j') and it belongs to the model N_{f_m}. Picking any $j' \in Z \cap N_{f_m}$ provides a witness for (iii)* (so we get a contradiction).]
Next, the set

$$Z_0 =: \{j < \kappa : (\forall \ell < n)(t^{\ell,f_m}_j \in (a^+_\ell, b^+_\ell)) \ \&\ t^{n,f_m}_j \in (a^m,b^m)\}$$

belongs to N_{f_m} and i belongs to it. But $i > i^*$, so necessarily Z_0 has cardinality κ (remember (ii)). Let

$$Z_1 =: \{j \in Z_0 \setminus i_0 : (\exists j_1 < j)(j_1 \in Z_0 \ \&\ (\forall \ell < n)(t^{\ell,f_m}_{j_1} < t^{\ell,f_m}_j \equiv \ell \in w))\}.$$

By the assumption that (iii)* fails (and the discussion above) we have $i \notin Z_1$. But again $Z_1 \in N_{f_m}$, so $|Z_0 \setminus Z_1| = \kappa$. Since the sequence $\bar{\mathcal{I}}$ is entangled, we can find $j_1 < j_2$ in $Z_0 \setminus Z_1$ such that $(\forall \ell < n)(t^{\ell,f_m}_{j_1} < t^{\ell,f_m}_{j_2} \equiv \ell \in w)$. But then j_1 witnesses $j_2 \in Z_1$, a contradiction.

Now we are going to use (iii)* twice to justify (iii). First we apply (iii)* for $w =: w^*$, $i_0 = 0$, $m = 1$ with $a^+_\ell = a_\ell$, $b^+_\ell = b_\ell$ getting $j_1 \in N_{f_1} \cap \kappa$ such that

$$\ell \in w^* \quad \Rightarrow \quad a_\ell < t^{\ell,f_1}_{j_1} < t^\ell, \qquad \text{and}$$

$$\ell \in n \setminus w^* \quad \Rightarrow \quad t^\ell < t^{\ell,f_1}_{j_1} < b_\ell,$$

and $t^{n,f_1}_{j_1} \in (a^1, b^1)$. Next we choose $a^+_\ell, b^+_\ell \in \mathcal{J}^*$ (for $\ell < n$) such that

$$\begin{array}{rcll} \ell \in w^* & \Rightarrow & t^{\ell,f_1}_{j_1} < a^+_\ell < t^\ell & \text{and } \ b^+_\ell = b_\ell, \\ \ell \in n \setminus w^* & \Rightarrow & t^\ell < b^+_\ell < t^{\ell,f_1}_{j_1} & \text{and } \ a^+_\ell = a_\ell. \end{array}$$

Then we again apply (iii)*, this time for $w =: w^*$, $m = 2$, $i_0 = j_1 + 1$ and a^+_ℓ, b^+_ℓ chosen above, getting $j_2 \in N_{f_2} \cap \kappa \setminus j_1$ such that, in particular, $(\forall \ell < n)(t^{\ell,f_2}_{j_2} \in (a^+_\ell, b^+_\ell))$ and $t^{\ell,f_2}_{j_2} \in (a^2, b^2)$. Then clearly

$$(\forall \ell \le n)(t^{\ell,f_1}_{j_1} < t^{\ell,f_2}_{j_2} \equiv \ell \in w^*),$$

and $j_1 < j_2$ both are in $N_{f_1} \cap \kappa = N_{f_2} \cap \kappa$. Since f_1, f_2 are E-equivalent we know that $t^{\ell,f_1}_{j_1} = t^{\ell,f_2}_{j_1}$ (for $\ell \le n$), so we may get a contradiction with the choice of $\bar{t}^{f_2}$ and we finish the proof of (iii).

Now we let

$$Y_{g/E} = \bigcup \{ Y^i_{\bar{t}} : i^* < i < \kappa \ \& \ \bar{t} \in \prod_{\ell < n} \mathcal{I}_{f_\ell} \}.$$

It follows from (iii) that $|Y_{g/E}| \le \kappa$. Clearly, for each $f \in g/E$ the set $\{j < \kappa : f(j) \in Y_{g/E}\}$ is of size κ. Hence $Y_{g/E}$ is as required in ($\boxtimes$) and this finishes the proof of 3.3.1. $\square$

Continuation of the proof of 3.3: Now we can construct the entangled sequence of linear orders as required in the theorem. For this, by induction on $\alpha < \lambda$, we choose functions $f_\alpha \in {}^\kappa\chi$ such that:

($\otimes_\alpha$) the sequence $\langle \mathcal{I}_{f_\beta} : \beta < \alpha \rangle$ is entangled.

Note that if $\alpha \le \lambda$ is limit and f_β have been chosen for $\beta < \alpha$ so that ($\otimes_\beta$) holds (for $\beta < \alpha$), then also ($\otimes_\alpha$) holds. Let $f_0 \in {}^\kappa\chi$ be any function; note that ($\otimes_1$) holds true as κ is $> \mu$ which is the density of $\mathcal{J}$, so in $\mathcal{J}$ there is no monotonic sequence of length μ^+.

Suppose we have defined $f_\beta \in {}^\kappa\chi$ for $\beta < \alpha$ so that ($\otimes_\alpha$) holds true. Let $\langle \bar{\beta}^\zeta : \zeta < \alpha^* \rangle$ list all the sequences $\langle \beta_\ell : \ell < n \rangle \subseteq \alpha$ such that $n < \omega$ and $\bigwedge\limits_{\ell_1 \ne \ell_2} \beta_{\ell_1} \ne \beta_{\ell_2}$. Let $\bar{\beta}^\zeta = \langle \beta(\zeta, \ell) : \ell < n_\zeta \rangle$. Clearly without loss of generality

$$\alpha^* = |\alpha| \ \vee \ (\alpha < \omega \ \& \ \alpha^* < \omega).$$

For each $\zeta < \alpha^*$ we apply 3.3.1 to $f_{\beta(\zeta,0)}, \ldots, f_{\beta(\zeta,n_\zeta-1)}$ to get a family $\mathcal{A}^\zeta \subseteq [\chi]^\kappa$ as there (so in particular $|\mathcal{A}^\zeta| \leq 2^\mu$). There is $f_\alpha \in {}^\kappa\chi$ such that

$$(\forall \zeta < \alpha^*)(\forall A \in \mathcal{A}^\zeta)(\forall^* i < \kappa)(f_\alpha(i) \notin A).$$

Why? Otherwise $\bigcup\limits_{\zeta<\alpha^*} \mathcal{A}^\zeta$ exemplifies

$$\mathbf{U}_{J^{\rm bd}_\kappa}(\chi) \leq |\bigcup_{\zeta<\alpha^*} \mathcal{A}^\zeta| \leq (|\alpha|+\aleph_0)\cdot\sup\{|\mathcal{A}^\zeta| : \zeta<\alpha^*\} \leq (|\alpha|+\aleph_0)\times 2^\mu < \lambda.$$

Now, with f_α chosen as above, $(\otimes_{\alpha+1})$ holds true. $\square$

Remark 3.4. Theorem 3.3 should be compared with:

(a) [**?**, Ch.II, 4.10E], see AP2 there on history. There we got only Ens_2.
(b) [**?**, §2], but there the density is higher.

Conclusion 3.5. (1) Let κ be an uncountable regular cardinal $\leq 2^{\aleph_0}$, $\kappa < \chi \leq 2^{\aleph_0}$, and $\mathbf{U}_{J^{\rm bd}_\kappa}(\chi) > 2^{\aleph_0}$ (e.g., $\chi = 2^{\aleph_0}$, $\mathrm{cf}(\chi) = \kappa < \chi$). Then there is an entangled sequence of length $\mathbf{U}_{J^{\rm bd}_\kappa}(\chi)$ of linear orders of cardinality κ.

(2) Assume μ is a strong limit singular cardinal, $\mu < \kappa = \mathrm{cf}(\kappa) < \chi \leq 2^\mu$ and $\mathbf{U}_{J^{\rm bd}_\kappa}(\chi) > 2^\mu$ (e.g., $\chi = 2^\mu$, $\mathrm{cf}(\chi) = \kappa < \chi$). Then there is an entangled sequence of length $\mathbf{U}_{J^{\rm bd}_\kappa}(\chi)$ of linear orders of cardinality κ.

4. On attainment of spread

In this section we are interested in the following question

Question 4.1. Let λ be a singular cardinal.

(1) Is there a Boolean algebra $\mathbb{B}$ such that $s^+(\mathbb{B}) = \lambda$, e.g., in the following sense:
there is no sequence $\langle a_\alpha : \alpha < \lambda\rangle \subseteq \mathbb{B} \setminus \{0\}$ such that each a_α is not in the ideal generated by

$$I_\alpha = \{a_\beta : \beta \neq \alpha\},$$

but for each $\mu < \lambda$ there is such a sequence?

(2) We can ask also/alternatively for $\mathrm{hd}^+(\mathbb{B}) = \lambda$ (and/or $\mathrm{hL}^+(\mathbb{B}) = \lambda$) defined similarly using $\{a_\beta : \beta < \alpha\}$ (and/or $\{a_\beta : \beta > \alpha\}$, respectively).

For the discussion of the attainment properties of spread we refer the reader to [**?**, p. 175]; the attainment of hd, hL is discussed, e.g., in [**?**, p. 198, p. 191]. Forcing constructions for different attainment properties for hd and hL are presented in [**?**].

Theorem 4.2. (1) *Assume that μ is a strong limit singular cardinal,*

$$\aleph_0 < \mathrm{cf}(\mu) < \mu < \mathrm{cf}(\lambda) < \lambda \leq 2^\mu.$$

Then

($\boxtimes_\lambda$) *there is a Boolean Algebra $\mathbb{B}$ satisfying:*

(i) $|\mathbb{B}| = \lambda = s(\mathbb{B})$,

(ii) *$s(\mathbb{B})$ is not obtained (i.e., $s^+(\mathbb{B}) = \lambda$),*

(iii) *moreover* $\mathrm{hd}^+(\mathbb{B}) = \mathrm{hL}^+(\mathbb{B}) = \lambda$.

(2) *Assume that*

($\otimes_2$) (a) $\mu < \mathrm{cf}(\lambda) < \lambda$,

(b) *$\langle \lambda_i : i < \delta \rangle$ is a (strictly) increasing sequence of regular cardinals with limit μ,*

(c) *J is an ideal on δ extending J^{bd}_δ, $A \in J^+$, $\delta \setminus A \in J^+$,*

(d) *$\langle g_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ is a $<_{J \restriction A}$–increasing $<_{J \restriction A}$–cofinal sequence of members of $\prod\limits_{i \in A} \lambda_i$, and $\langle h_\alpha : \alpha < \lambda \rangle$ is a sequence of distinct members of $\prod\limits_{i \in \delta \setminus A} \lambda_i$ such that*

$$j < \delta \quad \Rightarrow \quad |\{h_\alpha \restriction j, g_\beta \restriction j : \alpha < \lambda,\ \beta < \mathrm{cf}(\lambda)\}| < \lambda_j.$$

Then ($\boxtimes_\lambda$) holds.

(3) *Assume that*

($\otimes_3$) (a) $\mu < \mathrm{cf}(\lambda) < \lambda$,

(b) *$\langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals $< \mu$,*

(c) *J is an ideal on δ extending J^{bd}_δ, $A \subseteq \delta$, $A \in J^+$ and $\delta \setminus A \in J^+$,*

(d) *$g_\alpha \in \prod\limits_{i<\delta} \lambda_i$ for $\alpha < \lambda$ are pairwise distinct,*

(e) *among $\{g_\alpha \restriction A : \alpha < \lambda\}$ we can find an $<_{J \restriction A}$–increasing cofinal sequence of length $\mathrm{cf}(\lambda)$,*

(f) $|\{g_\alpha \restriction i : \alpha < \lambda\}| = \lambda_i$.

Then ($\boxtimes_\lambda$) holds.

Proof. 1) We shall prove that the assumptions of part (2) hold.

As $\mathrm{cf}(\mu) > \aleph_0$, we know (by [**?**, Ch.VIII, §1]) that there is a sequence $\langle \lambda_i : i < \mathrm{cf}(\mu) \rangle$ such that

$$\mu > \lambda_i = \mathrm{cf}(\lambda_i) > |\prod_{j<i} \lambda_j| \quad \text{and} \quad \mathrm{tcf}(\prod_{i<\mathrm{cf}(\mu)} \lambda_i / J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}) = \mathrm{cf}(\lambda).$$

Let $\langle g_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ be an increasing cofinal sequence in $(\prod\limits_{i<\mathrm{cf}(\mu)} \lambda_i, <_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}})$. Let $h_\alpha \in \prod\limits_i \{\lambda_{2i+1} : i < \kappa\}$ (for $\alpha \in [\mathrm{cf}(\lambda), \lambda)$) be just such that $h_\alpha \notin \{h_\beta : \beta < \alpha\}$, so $A =: \{2i : i < \mathrm{cf}(\mu)\}$, $\langle g_\alpha \restriction A : \alpha < \mathrm{cf}(\lambda) \rangle$, $\langle h_\alpha \restriction (\kappa \setminus A) : \alpha < \lambda \rangle$ are as required ($\otimes_2$).

2) Let $\langle \chi_i : i < \mathrm{cf}(\lambda)\rangle$ be an increasing continuous sequence of cardinals such that

- $\lambda = \sum\limits_{i<\mathrm{cf}(\lambda)} \chi_i$,
- $\chi_0 = 0$, $\mathrm{cf}(\lambda) < \chi_1$ and each χ_{i+1} is regular.

For $\alpha < \lambda$ let $j(\alpha) < \mathrm{cf}(\lambda)$ be such that $\alpha \in [\chi_{j(\alpha)}, \chi_{j(\alpha)+1})$ and let $f_\alpha \in \prod\limits_{i<\delta} \lambda_i$ be such that:

$$f_\alpha \restriction A = g_{j(\alpha)} \qquad \text{and} \qquad f_\alpha \restriction (\delta \setminus A) = h_\alpha.$$

Now for $n \geq 1$ we define a Boolean Algebra $\mathbb{B}_n$ (each $\mathbb{B}_n$ will be an example):

it is generated by $\{x_\alpha : \alpha < \lambda\}$ freely except:

($\circledast$) *if* $i \in A$, $\nu_k \in \prod\limits_{i'<i} \lambda_{i'}$, $\nu_k{}^\frown\langle\gamma_{k,\ell}\rangle \vartriangleleft f_{\alpha_{k,\ell}}$ (for $k < m$, $\ell \leq 2n+1$), and $w \subseteq m$, and

$$\begin{aligned} \ell < 2n \;\&\; k \in m \setminus w \;&\Rightarrow\; \gamma_{k,\ell} < \gamma_{k,\ell+1}, \\ k \in w \;&\Rightarrow\; \alpha_{k,n} = \alpha_{k,2n+1}, \end{aligned}$$

and there are no repetitions in the sequence $\langle \nu_k : k < m\rangle$, and $\mathbf{t}_k \in \{0,1\}$,

then $\bigcap\limits_{k<m} x^{\mathbf{t}_k}_{\alpha_{k,n}} \leq \bigcup\limits_{\substack{\ell\neq n,\\ \ell\leq 2n+1}} \bigcap\limits_{k<m} x^{\mathbf{t}_k}_{\alpha_{k,\ell}}$,

where $x^{\mathbf{t}}$ is x if $\mathbf{t} = 1$, and $-x$ if $\mathbf{t} = 0$.

Claim 4.2.1. $s^+(\mathbb{B}_n) \leq \lambda$, $\mathrm{hd}^+(\mathbb{B}_n) \leq \lambda$, $\mathrm{hL}^+(\mathbb{B}_n) \leq \lambda$.

Proof of the Claim. Assume toward contradiction that the sequence $\langle a_\beta : \beta < \lambda\rangle \subseteq \mathbb{B}_n \setminus \{0\}$ exemplifies the failure. Without loss of generality, $a_\beta = \bigcap\limits_{\ell<m_\beta} x^{\mathbf{t}(\beta,\ell)}_{\alpha(\beta,\ell)}$, where $\ell < m < m_\beta \;\Rightarrow\; \alpha(\beta,\ell) \neq \alpha(\beta,m)$. For each $i < \mathrm{cf}(\lambda)$ we choose $S_i \subseteq [\chi_i, \chi_{i+1})$, and $\varepsilon_i(*) < \delta$, $m^i < \omega$, $\mathbf{t}[i,\ell] \in \{0,1\}$, $j[i,\ell] < \mathrm{cf}(\lambda)$ (for $\ell < m^i$) such that (note that we can permute $\langle \alpha(\beta,\ell) : \ell < m_\beta\rangle$):

(i) S_i is unbounded in χ_{i+1},

(ii) for all $\beta \in S_i$ we have

$$m_\beta = m^i \;\&\; (\forall \ell < m^i)(\mathbf{t}(\beta,\ell) = \mathbf{t}[i,\ell] \;\&\; j(\alpha(\beta,\ell)) = j[i,\ell]),$$

(iii) $\langle\langle \alpha(\beta,\ell) : \ell < m^i\rangle : \beta \in S_i\rangle$ is a Δ–system with heart $\langle \alpha[i,\ell] : \ell < k^i\rangle$, so

$$\begin{aligned} \beta \in S_i \;\&\; \ell < k^i \;&\Rightarrow\; \alpha(\beta,\ell) = \alpha[i,\ell], \qquad \text{and} \\ \alpha(\beta_1,\ell_1) = \alpha(\beta_2,\ell_2) \;&\Rightarrow\; (\beta_1 = \beta_2 \;\&\; \ell_1 = \ell_2) \vee (\ell_1 = \ell_2 < k^i), \end{aligned}$$

(iv) for $\beta \in S_i$, there are no repetitions in the sequence $\langle f_{\alpha(\beta,\ell)} \restriction \varepsilon_i(*) : \ell < m^i\rangle$ and it does not depend on β,

(v) for every $\beta^* \in S_i$ and $\varepsilon < \delta$ the set

$$\{\beta \in S_i : (\forall \ell < m^i)(f_{\alpha(\beta,\ell)} \restriction \varepsilon = f_{\alpha(\beta^*,\ell)} \restriction \varepsilon)\}$$

is unbounded in S_i.

Note that necessarily

(vi) $j[i,\ell] \geq i$ for $\ell \in [k^i, m^i)$.

Next pick a set $S \in [\mathrm{cf}(\lambda)]^{\mathrm{cf}(\lambda)}$ such that:

(α) for all $i \in S$ we have $m^i = m^*$, $k^i = k^*$, $\mathbf{t}[i,\ell] = \mathbf{t}[\ell]$, $\varepsilon_i(*) = \varepsilon(*)$,

(β) $\langle\langle \alpha[i,\ell] : \ell < k^*\rangle : i \in S\rangle$ is a Δ–system with heart $\langle \alpha(\ell) : \ell < \ell^*\rangle$, so

$$i \in S \ \&\ \ell < \ell^* \quad \Rightarrow \quad \alpha[i,\ell] = \alpha(\ell),$$

(γ) also $\langle\langle j[i,\ell] : \ell < m^*\rangle : i \in S\rangle$ is a Δ–system with heart $\langle j(\ell) : \ell \in w^*\rangle$, where $w^* \subseteq m^*$.

Note that then $\ell^* \subseteq w^* \subseteq k^*$ (the first inclusion is a consequence of (β), the second one follows from (vi)).

Also by further shrinking of the sets S_i (for $i < \mathrm{cf}(\lambda)$) and S we may require that

(A) if $i_1 < i_2$ are from S, then $j[i_1,\ell] < i_2$ (for $\ell < m^*$),

(B) if $i_1 \neq i_2$ are from S and $\beta_1 \in S_{i_1}$ and $\beta_2 \in S_{i_2}$, then

$$\{\alpha(\beta_1,\ell) : \ell < m^*\} \cap \{\alpha(\beta_2,\ell) : \ell < m^*\} \subseteq \{\alpha(\ell) : \ell < \ell^*\},$$

(C) if $i_1 \in S$, $\gamma_1 \in S_{i_1}$, then

$$(\forall \xi < \delta)(\exists^{\mathrm{cf}(\lambda)} i \in S)(\exists^{\chi_{i+1}} \gamma \in S_i)(\forall \ell < m^*)(f_{\alpha(\gamma,\ell)} \restriction \xi = f_{\alpha(\gamma_1,\ell)} \restriction \xi).$$

Choose $\gamma_i \in S_i$ for $i \in S$. Look at $\bar{f}^i = \langle f_{\alpha(\gamma_i,\ell)} : \ell < m^*\rangle$.

We can (as in [**?**, Ch.II, 4.10A]) find $\varepsilon < \delta$ and $\bar{f} = \langle f_0, \ldots, f_{m^*-1}\rangle$ such that $\varepsilon \in A$, $\varepsilon > \varepsilon(*)$ and:

($*$) for every $\zeta < \lambda_\varepsilon$ there is $i \in S$ such that:

$$(\forall \ell < m^*)\big(f_{\alpha(\gamma_i,\ell)} \restriction \varepsilon = f_\ell \restriction \varepsilon\big) \ \text{ and } \ (\forall \ell \in m^* \setminus w^*)\big(f_{\alpha(\gamma_i,\ell)}(\varepsilon) > \zeta\big).$$

So we can choose inductively ζ_k, i_k (for $k \leq 2n$) such that $i_k \in S$, $\zeta_k < \lambda_\varepsilon$, and

$$(\forall \ell < m^*)\big(f_{\alpha(\gamma_{i_k},\ell)} \restriction \varepsilon = f_\ell \restriction \varepsilon\big) \ \text{ and } \ (\forall \ell \in m^* \setminus w^*)\big(\zeta_k < f_{\alpha(\gamma_{i_k},\ell)}(\varepsilon) < \zeta_{k+1}\big).$$

Note that, as $\varepsilon \in A$, we have

$$(\forall \ell \in w^*)\big(f_{\alpha(\gamma_{i_k},\ell)}(\varepsilon) = g_{j(\alpha(\gamma_{i_k},\ell))}(\varepsilon) = g_{j(\ell)}(\varepsilon)\big)$$

for each $k \leq 2n$. It follows from clause (v) above that we may pick $\gamma \in S_{i_n} \setminus \{\gamma_{i_n}\}$ such that $(\forall \ell < m^*)(f_{\alpha(\gamma,\ell)} \restriction \varepsilon = f_{\alpha(\gamma_{i_n},\ell)} \restriction \varepsilon)$. By our choices, $\alpha(\gamma_{i_n},\ell) = \alpha(\gamma,\ell)$ for $\ell < k^*$ (so in particular for $\ell \in w^*$). Now, by the definition of $\mathbb{B}_n$, we clearly have $a_{\beta_n} \leq \bigcup\limits_{\substack{\ell \neq n,\\ \ell \leq 2n+1}} a_{\beta_\ell}$, where $\beta_\ell = \gamma_{i_\ell}$ for $\ell \leq 2n$ and $\beta_{2n+1} = \gamma$, finishing the proof for s.

Now for hd, hL use clause (C) above. $\square$

Claim 4.2.2. *$s^+(\mathbb{B}_n) > \chi_{i+1}$, more specifically $\{x_\alpha : \alpha \in [\chi_i, \chi_{i+1})\}$ are independent as ideal generators.*

Proof of the Claim. Let $\alpha^* \in [\chi_i, \chi_{i+1})$. We define a function $h_{\alpha^*} = h : \{x_\alpha : \alpha < \lambda\} \longrightarrow \{0,1\}$ by:

$$h(x_\alpha) = \begin{cases} 1 & \text{if } \alpha = \alpha^*, \\ 0 & \text{if } \ell g(f_\alpha \cap f_{\alpha^*}) \in \delta \setminus A, \\ 1 & \text{if } \ell g(f_\alpha \cap f_{\alpha^*}) \in A,\ f_\alpha(\ell g(f_\alpha \cap f_{\alpha^*})) > f_{\alpha^*}(\ell g(f_\alpha \cap f_{\alpha^*})), \\ 0 & \text{if } \ell g(f_\alpha \cap f_{\alpha^*}) \in A,\ f_\alpha(\ell g(f_\alpha \cap f_{\alpha^*})) < f_{\alpha^*}(\ell g(f_\alpha \cap f_{\alpha^*})). \end{cases}$$

We claim that the function h respects the equations in the definition of $\mathbb{B}_n$. To show this suppose that $i \in A$, $\mathbf{t}_k \in \{0,1\}$, $\nu_k \in \prod\limits_{i'<i} \lambda_{i'}$, $\nu_k \vartriangleleft f_{\alpha_{k,\ell}}$ (for $k < m$, $\ell \leq 2n+1$) and $w \subseteq m$ are as in the assumptions of $(\circledast)$. Now we consider three cases.

Case 1: $f_{\alpha^*} \restriction i \notin \{\nu_k : k < m\}$.
Then, by the way h is defined, $h(x_{\alpha_{k,n}}) = h(x_{\alpha_{k,\ell}})$ for each $\ell \leq 2n+1$ and $k < m$. Hence easily

$$\bigcap_{k<m} h(x_{\alpha_{k,n}})^{\mathbf{t}_k} = \bigcup_{\substack{\ell \neq n,\\ \ell \leq 2n+1}} \bigcap_{k<m} h(x_{\alpha_{k,\ell}})^{\mathbf{t}_k},$$

and we are done.

Case 2: $f_{\alpha^*} \restriction i = \nu_{k^*}$, $k^* \in m \setminus w$.
Thus $f_{\alpha_{k^*,0}}(i) < \ldots < f_{\alpha_{k^*,n}}(i) < \ldots < f_{\alpha_{k^*,2n}}(i)$ and $h(x_{\alpha_{k^*,0}}) = h(x_{\alpha_{k^*,n}})$ or $h(x_{\alpha_{k^*,2n}}) = h(x_{\alpha_{k^*,n}})$. Let ℓ^* be 0 in the first case and $2n$ in the second. Note that also for $k < m$, $k \neq k^*$ we have $h(x_{\alpha_{k,\ell}}) = h(x_{\alpha_{k,n}})$ for all $\ell \leq 2n+1$. Hence

$$\bigcap_{k<m} h(x_{\alpha_{k,\ell^*}})^{\mathbf{t}_k} = \bigcap_{\substack{k \neq k^*,\\ k<m}} h(x_{\alpha_{k,n}})^{\mathbf{t}_k} \cap h(x_{\alpha_{k^*,\ell^*}})^{\mathbf{t}_{k^*}} = \bigcap_{k<m} h(x_{\alpha_{k,n}})^{\mathbf{t}_k},$$

and we are done.

Case 3: $f_{\alpha^*} \restriction i = \nu_{k^*}$, $k^* \in w$.
Thus $\alpha_{k^*,n} = \alpha_{k^*,2n+1}$ (so $h(x_{\alpha_{k^*,n}}) = h(x_{\alpha_{k^*,2n+1}})$) and also for $k < m$, $k \neq k^*$ we have $h(x_{\alpha_{k,n}}) = h(x_{\alpha_{k,2n+1}})$. Hence

$$\bigcap_{k<m} h(x_{\alpha_{k,n}})^{\mathbf{t}_k} = \bigcap_{k<m} h(x_{\alpha_{k,2n+1}})^{\mathbf{t}_k},$$

and we are done.

Consequently the function h can be extended to a homomorphism $\hat{h}$ from $\mathbb{B}_n$ to $\{0,1\}$. Clearly $h(x_{\alpha^*}) = 1$ and $h(x_\alpha) = 0$ for all $\alpha \in [\chi_i, \chi_{i+1}) \setminus \{\alpha^*\}$. (Remember $f_\alpha \restriction A = g_i$ for $\alpha \in [\chi_i, \chi_{i+1})$, and hence

$$\text{if } \ \alpha \neq \beta \in [\chi_i, \chi_{i+1}) \ \text{ then } \ \ell g(f_\alpha \cap f_\beta) \in \delta \setminus A. \ )$$

Thus we are done. $\square$

3) We can get the assumptions of part (2). $\square$

Remark 4.3. (1) We cannot really prove in ZFC that there is a Boolean Algebra $\mathbb{B}$ such that $s^+(\mathbb{B})$ is singular ($\equiv s(\mathbb{B})$ singular not obtained) as $s^+(\mathbb{B})$ cannot be strong limit singular.

(2) Note that the demand $(\exists\mu)[\mu < \mathrm{cf}(\lambda) < \lambda < 2^\mu]$ is necessary by [**?**]. The construction is like the one in [**?**, §7]. Earlier see [**?**, 4.14].

(3) Of course, the proof of 4.2(2) shows that we have the respective result for finite variants s_m of spread, as well as for $\mathrm{hd}_m, \mathrm{hL}_m$ (if $m \geq 3$, i.e., $m = 2n+1$). We refer the reader to [**?**, §1] for the definitions of these cardinal invariants (see also [**?**] for discussion and some independence results on s_m; more relevant results can be found in [**?**]).

So we can give examples to 4.1 if we can have for $(\boxtimes_\lambda)$ of 4.2.

Proposition 4.4. (1) *If κ is strong limit singular cardinal, $2^\kappa \geq \aleph_{\kappa^+}$, then we have examples of λ, $\kappa < \mathrm{cf}(\lambda) < \lambda \leq 2^\kappa$ with $(\boxtimes_\lambda)$ (of 4.2), e.g., $\lambda = \aleph_{\kappa^+}$!*

(2) *If $\delta(*) = (2^\kappa)^{+(\kappa^{+4})}$, $\aleph_{\delta(*)} \leq 2^{\kappa^+}$, then also there is $\lambda \in (2^\kappa, 2^{\kappa^+})$, $\mathrm{cf}(\lambda) \in [2^\kappa, (2^\kappa)^{+\kappa^{+4}})$ as needed in 4.2(3), and hence $(\boxtimes_\lambda)$.*

(3) *If κ is inaccessible (possibly weakly) $\delta(*) = (2^{<\kappa})^{+\kappa^{+4}}$ and $\aleph_{\delta(*)} < 2^\kappa$ then we can find $\lambda \in [2^{<\kappa}, 2^\kappa)$, $\mathrm{cf}(\lambda) \in [2^{<\kappa}, (2^{<\kappa})^{+\kappa^{+4}})$, as in 4.2(2), and hence $(\boxtimes_\lambda)$ holds.*
Similarly if κ is a singular cardinal, or a successor cardinal by part (2).

(4) *E.g., if $\aleph_{\aleph_{\omega+1}} \leq 2^{\aleph_0}$, then for $\lambda = \aleph_{\aleph_{\omega+1}}$ we have an example for this cardinal. Generally, if $\mu > \mathrm{cf}(\mu) = \aleph_0$, $\mathrm{cf}(\lambda) = \mu^+$ and $\lambda \leq 2^{\aleph_0}$, then there is an example in λ.*

Proof. 1) Should be clear. (Note that $\mathrm{pp}^+(\kappa) \geq \aleph_{\kappa^++1}$ by [**?**, 5.9, p.408]).

2) First, for some club C of κ^{+4} (for $\alpha \in C \Rightarrow \alpha$ limit) we have

$(*)_1 \qquad \delta \in C \ \&\ \mathrm{cf}(\delta) \leq \kappa \quad \Rightarrow \quad \mathrm{pp}((2^\kappa)^{+\delta}) < (2^\kappa)^{+\min(C \setminus (\delta+1))}$.

(By [**?**, §4]). Hence (again by [**?**])

$(*)_2 \qquad \delta \in C \ \&\ \mathrm{cf}(\delta) \leq \kappa \quad \Rightarrow \quad ((2^\kappa)^{+\delta})^\kappa < (2^\kappa)^{+\min(C \setminus (\delta+1))}$.

We can for any $\delta \in \mathrm{acc}(C)$ with $\mathrm{cf}(\delta) = \kappa^+$ do the following: we can find a strictly increasing sequence $\langle \lambda_i : i < \kappa^+ \rangle$ of regular cardinals with limit $(2^\kappa)^{+\delta}$, $2^\kappa < \lambda_i$, $\mathrm{tcf}(\prod\limits_{i<\kappa^+} \lambda_i)/J^{\mathrm{bd}}_{\kappa^+} = (2^\kappa)^{+\delta+1}$ (if we assume $\mathrm{pp}((2^\kappa)^{+\delta}) > (2^\kappa)^{+\delta+1}$ we can find more examples).

Note:

$$j < \kappa^+ \quad \Rightarrow \quad \prod_{i<j} \lambda_i < (2^\kappa)^{+\delta}$$

(by $(*)_2$), as the ideal is $J^{\mathrm{bd}}_{\kappa^+}$; without loss of generality

$(*)_3$ $\prod_{i<j} \lambda_i < \lambda_j$.

So let $\langle g'_\alpha : \alpha < (2^\kappa)^{+\delta+1} \rangle$ be $<_{J^{\mathrm{bd}}_{\kappa^+}}$–increasing and cofinal in $\prod_{i<\kappa^+} \lambda_i$ and let $A = \{2i : i < \kappa^+\}$.

Now assume $2^{\kappa^+} \geq \lambda > \mathrm{cf}(\lambda) = (2^\kappa)^{+\delta+1}$; such λ exists by the assumption. We can find $h_\alpha \in {}^{\kappa^+}2$ (for $\alpha \in [(2^\kappa)^{+\delta+1}, \lambda)$) with no repetitions.

Note $|\{g'_\alpha \restriction i, h'_\alpha \restriction i : \alpha < \lambda\}| \leq |\prod_{j<i} \lambda_j|$, which has cardinality $< \lambda_i$. So we can apply Theorem 4.2(3).

3), 4) Same. $\square$

Discussion 4.5. (1) If Cardinal Arithmetic is too close to GCH ($2^\kappa < \aleph_{\kappa^+}$ for every κ), no example exists as by [**?**], ZFC $\models 2^{\mathrm{cf}(s^+(\mathbb{B}))} > |\mathbb{B}|$. [Why? If $\mathbb{B}$ is a counterexample, let $\lambda = s^+(\mathbb{B}) = s(\mathbb{B})$ (bring a counterexample); clearly λ is a limit cardinal, so $2^{\mathrm{cf}(\lambda)} > |\mathbb{B}| \geq \lambda \geq \aleph_{\mathrm{cf}(\lambda)}$, a contradiction.]

If Cardinal Arithmetic is far enough from GCH (even just for regulars), then there is an example.

I consider it a semi-ZFC answer — see [**?**] and [**?**].

(2) There are some variants of problem 4.1 related to various versions of the (equivalent) definitions of $s, \mathrm{hd}, \mathrm{hL}$. For s all versions are equivalent [**?**, p. 175]. Concerning hd, hL see the discussion of the attainment relations for the equivalent definitions of hd in [**?**, p. 196, 197] and of hL in [**?**, p.191]. On the remaining cases see also in [**?**, §4].

Problem 4.1. Does $\aleph_{\omega_1} < 2^{\aleph_0}$ imply that an example for $\lambda = \aleph_{\omega_1}$ exists?

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