CATEGORICITY OF THEORIES IN $L_{\kappa^*\omega}$, WHEN κ^* IS A MEASURABLE CARDINAL. PART II

SAHARON SHELAH

ABSTRACT. We continue the work of [?] and prove that for λ successor, a λ-categorical theory $\mathbf{T}$ in $L_{\kappa^*,\omega}$ is μ-categorical for every μ,$\mu \leq \lambda$ which is above the $(2^{LS(\mathbf{T})})^+$-beth cardinal.

0. INTRODUCTION

We deal here with the categoricity spectrum of theories $\mathbf{T}$ in the logic: $L_{\kappa^*,\omega}$ with κ^* measurable and more generally, continued the attempts develop classification theory of non elementary classes in particular non forking. Makkai and Shelah [?] dealt with the case κ^* a compact cardinal. So κ^* measurable is too high compared with the hope of dealing with $\mathbf{T} \subseteq L_{\omega_1,\omega}$ (or any $L_{\kappa,\omega}$) but seems quite small compared to the compact cardinal in [?]. Model theoretically a compact cardinal ensures many cases of amalgamation, whereas measurable cardinal ensures no maximal model. We continue [?], Makkai and Shelah [?], Kolman and Shelah [?]; try to imitate [?]; a parallel line of research is [?]. Earlier works are [?], [?], [?]; for later works on the upward Łoś conjecture, look at [?] and [?].

On the situation generally see more [?].

This paper continues the tasks begun in Kolman and Shelah [?]. We use the results obtained there in to advance our knowledge of the categoricity spectrum of theories in $L_{\kappa^*,\omega}$, when κ^* is a measurable cardinal.

The main theorems are proved in section three; section one treats of types and section two describes some constructions.

Note that we may expect to be able to develop better, more informative classification theory, in particular stability theory, for $\mathbf{T} \subseteq L_{\kappa^*,\omega}$ κ^* measurable than without the measurables assumption, and less informative then the case κ^* compact.

The notation follows [?], except in two important details: we reserve κ^* for the fixed measurable cardinal and $\mathbf{T}$ for the fixed λ-categorical theory in $L_{\kappa^*,\omega}$ in a given vocabulary L; κ is any infinite cardinal and T is usually some kind of tree. To recap briefly: $\mathbf{T}$ is a λ- categorical theory in $L_{\kappa^*,\omega}$, $LS(\mathbf{T}) \stackrel{\text{def}}{=} \kappa^* + |\mathbf{T}|$, $\mathcal{K} = \langle K, \preceq_{\mathcal{F}} \rangle$ is the class of models of $\mathbf{T}$, where $\mathcal{F}$ is a fragment of $L_{\kappa^*,\omega}$ satisfying $\mathbf{T} \subseteq \mathcal{F}$, $|\mathcal{F}| \leq \kappa^* + |\mathbf{T}|$, and for M, $N \in K$, $M \preceq_{\mathcal{F}} N$ means that M is an $\mathcal{F}$-elementary submodel of N.

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The principal relevant results from [?] are: $\mathcal{K}_{<\lambda}$ has the amalgamation property (5.5 there), and every member of $K_{<\lambda}$ is nice (5.4 there). But this assumption ($\mathbf{T}$ categorical in λ) or its consequences mentioned above will be mentioned in theorems when used.

Let $(M_1, M_0) \preceq_{\mathcal{F}} (M_3, M_2)$ mean $M_1 \preceq_{\mathcal{F}} M_3$, $M_0 \preceq_{\mathcal{F}} M_2$.

(I_1, I_2) is a Dedekind cut of the linear order I if

$$I = I_1 \cup I_2, \quad I_1 \cap I_2 = \emptyset, \quad \forall x \in I_1 \forall y \in I_2 (x < y).$$

The two sided cofinality of the Dedekind cut (I_1, I_2) of I, $\mathrm{cf}(I_1, I_2)$ is $(\mathrm{cf}(I_1), \mathrm{cf}(I_2^*))$, where I_2^* is the order I_2 inverted. The two sided cofinality of I, $\mathrm{cf}(I, I) = \mathrm{dcf}(I)$ is $(\mathrm{cf}(I^*), \mathrm{cf}(I))$.

Writing proofs we also consider their possible rule in the hopeful classification theory. But we have been always trying to be careful in stating the assumptions.

Note that [?] improves some of the results of [?]; but they do not fully recapture the results on the compact case to the measurable case. E.g. there categoricity in successor λ implies that categoricity start in the relevant Hanf number of omitting types so in general we deduce categoricity in larger cardinals. For a good understanding of this work, the reader is expected to know well [?]. Now it will be helpful to beware of some "black boxes" [?], [?] for less good source and some knowledge of [?] or [?] but usually proofs are repeated.

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1. Knowing the right types

The classical notion of type relates to the satisfaction of sets of formulas in a model. We shall define a post-classical type (following [?], [?] which was followed by Makkai and Shelah [?], or see [?, §0], but here niceness is involved) and use this to define notions of freeness and non-forking appropriate in the context of a λ-categorical theory in $L_{\kappa^*,\omega}$. The definitions try to locate a notion which under the circumstances behave as in [?] and, if you accept some inevitable limitations, succeed.

Context 1.1. $\mathbf{T} \subseteq L_{\kappa^*,\omega}$ in the vocabulary L, $K = \{M : M \text{ a model of } \mathbf{T}\}$, $\preceq_{\mathcal{F}}$ as in the introduction.

$K_\mu = \{M \in K : \|M\| = \mu\}$, $K_{<\kappa} = \bigcup\limits_{\mu<\kappa} K_\mu$, and $\mathcal{K} = (K, \preceq_{\mathcal{F}})$ and we stipulate $K_{<\kappa^*} = \emptyset$, hence, e.g., $K_{<\kappa} = \bigcup\{K_\mu : \mu < \kappa \text{ but } \mu \geq \kappa^*\}$ (Why? Models of cardinality $< \kappa^*$ are the parallel of finite ones for first order logic: such models may have no $\prec_{L_{\kappa^*,\omega}}$ proper extensions, and using our main tool ultrapower we can tell little on them. So instead of excluding them many times, we ignore them always). We let $LS(\mathcal{K}) = |\mathcal{F}| + \kappa^*$.

We assume if $A \subseteq N \in K$, $\|N\| \geq \lambda$, $\mu = |A| \in [\kappa^* + \mathbf{T}, \lambda)$, then for some nice $N \in K_\mu$, $A \subseteq M \preceq_{\mathcal{F}} N$. This is reasonable as by [?, 5.4 p.238] every $M \in K_{<\lambda}$ is nice. The reader may simplify assuming every $M \in K_{<\lambda}$ is nice.

Remember "$M \in K$ is nice" is defined in [?], definitions 3.2, 1.8; nice implies being an amalgamation base in $K_{<\lambda}$ (see 3.7). Here for simplicity we mean "amalgamation" to include the JEP (the joint embedding property).

Definition 1.2. Suppose that $M \in K_{<\lambda}$ is a nice model of $\mathbf{T}$. Define a binary relation, $E_M = E_M^{<\lambda}$, as follows:

$(\bar{a}_1, N_1) E_M (\bar{a}_2, N_2)$ if and only if

for $\ell = 1, 2$, $N_\ell \in K_{<\lambda}$ is nice and $M \preceq_{\mathcal{F}} N_\ell$, $\bar{a}_\ell \in N_\ell$ (i.e., $\bar{a}_\ell$ a finite sequence of members of N_ℓ), and there exist a model N and embeddings h_ℓ such that

$$M \preceq_{\mathcal{F}} N, \quad h_\ell : N_\ell \underset{\mathcal{F}}{\to} N, \quad \mathrm{id}_M = h_1 \restriction M = h_2 \restriction M,$$

and $h_1(\bar{a}_1) = h_2(\bar{a}_2)$.

Remark: This definition, in fact a generalization for amalgamation bases and more general, are important in [**?**], [**?**], [**?**], but here we restrict ourselves to nice models.

Fact 1.3. (1) *E_M is an equivalence relation.*

(2) *Let $M \in K_{<\lambda}$, $M \preceq_{\mathcal{F}} N$, $\bar{a} \in N$, and for $\ell = 1, 2$, $M \cup \bar{a} \subseteq N_\ell \preceq_{\mathcal{F}} N$, $\|N_\ell\| < \lambda$ then $(\bar{a}, N_1) E_M (\bar{a}, N_2)$*

(3) *E_M is preserved by isomorphism.*

Proof. 1) To prove 1.3, let's look at transitivity.

Suppose $(\bar{a}_\ell, N_\ell) E_M (\bar{a}_{\ell+1}, N_{\ell+1})$, $\ell = 1, 2$. Now M, being nice is an amalgamation base in $K_{<\lambda}$ thus there are models N^ℓ and embeddings h_0^ℓ, h_1^ℓ of N_ℓ, $N_{\ell+1}$ over M into N^ℓ, with $h_0^\ell(\bar{a}_\ell) = h_1^\ell(\bar{a}_{\ell+1})$, $\ell = 1, 2$. W.l.o.g., $N^\ell \in K_{<\lambda}$ (by the Downward Loewenheim Skolem Theorem). By assumption N_2 is nice, hence by [**?**, 3.5] is an amalgamation base for $\mathcal{K}_{<\lambda}$, i.e., there is an amalgam $N^* \in K_{<\lambda}$, and embeddings $g_\ell : N^\ell \xrightarrow{\mathcal{F}} N^*$, amalgamating N^1, N^2 over N^2 w.r.t h_1^1, h_0^2. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & & N^* & & \\
 & N^1 \arrow[ur, "g_1"] & & N^2 \arrow[ul, "g_2"'] & \\
\bar{a}_1 \in N_1 \arrow[ur, "h^1_0"] & & \bar{a}_2 \in N_2 \arrow[ul, "h^1_1"'] \arrow[ur, "h^2_0"] & & \bar{a}_3 \in N_3 \arrow[ul, "h^2_1"'] \\
 & & M \arrow[ull, "id"] \arrow[u, "id"] \arrow[urr, "id"'] & &
\end{tikzcd}$$

Just notice now that N^*, $g_1 h^1_0$, $g_2 h^2_1$ witness that $(\bar{a}_1, N_1) E_M (\bar{a}_3, N_3)$, since:

$$g_1 h^1_0(\bar{a}_1) = g_1(h^1_1(\bar{a}_2)) = g_2 h^2_0(\bar{a}_2) = g_2 h^2_1(\bar{a}_3).$$

2), 3) Left to reader. $\square$

Definition 1.4. Suppose that $M, N \in K_{<\lambda}$ are nice, $a \in N$ and $M \preceq_{\mathcal{F}} N$. Then

(1) $\mathrm{tp}(a, M, N)$, the type of a over M in N, is the E_M-equivalence class of (a, N),

$$(a, N)/E_M = \{(b, N^1) : (a, N) E_M (b, N^1)\}.$$

We also say "$a \in N$ realizes p". If $\|N\| \geq \lambda$ define $\mathrm{tp}(\bar{a}, M, N)$ by 1.3(2) (using the hypothesis).

(2) If $M' \preceq_{\mathcal{F}} M \in K_{<\lambda}$, $p \in S(M)$ (see below) is $(a, N)/E_M$, then $p \restriction M' = (a, N)/E_{M'}$.

(3) If $LS(\mathbf{T}) < \kappa \leq \mu \leq \lambda$, we call $M \in K_\mu$ κ-saturated if for every nice $N \preceq_{\mathcal{F}} M$, $||N|| < \kappa$ and $p \in S(N)$, some $\bar{a} \in M$ realizes p (in M so necessarily M is nice) or at least for some nice N', $N \preceq_{\mathcal{F}} N' \preceq_{\mathcal{F}} M$, some $a' \in N'$ realizes p in N'.

(4) $S^m(N) = \{p : p = \mathrm{tp}(\bar{a}, N, N_1)$ for any N_1, $\bar{a}$ satisfying: $N \preceq_{\mathcal{F}} N_1$, $\|N_1\| \leq \|N\| + LS(\mathcal{K})$ and $\bar{a} \in {}^m(N_1)\}$,

$$S(N) = S^{<\omega}(N) = \bigcup_{m<\omega} S^m(N).$$

(5) $\mathbf{T}$ is μ-stable if $N \in K_{\leq\mu} \Rightarrow |S(N)| \leq \mu$.
(6) We say N is μ-universal over M when: $M \preceq_{\mathcal{F}} N$, $N \in K_\mu$ and if $M \preceq_{\mathcal{F}} N' \in K_{\leq\mu}$ *then* there is a $\preceq_{\mathcal{F}}$-embedding of N' into N over M.
(7) We say N is (μ,κ)-saturated over M if there is a $\preceq_{\mathcal{F}}$- increasing continuous sequence $\langle M_i : i < \kappa\rangle$ such that: $M_0 = M$, $N = \bigcup_{i<\kappa} M_i$, $M_i \in K_\mu$ and M_{i+1} is μ-universal over M_i. We say N is saturated over M if for some $\mu \in [LS(\mathbf{T}), \lambda]$, and some $\kappa \leq \mu$, we have: N is (μ,κ)-saturated over M. So (μ,κ)-saturated over M implies universal over M.
(8) We say $\mathcal{K}$ (or $\mathbf{T}$) is stable in μ if for every $M \in K_\mu$, M is nice and $|S(M)| \leq \mu$.

Definition 1.5. We shall write $M_1 \underset{M_0}{\overset{M_3}{\biguplus}} M_2$ to mean:

$$M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} M_3, \quad M_0 \preceq_{\mathcal{F}} M_2 \preceq_{\mathcal{F}} M_3$$

and there exist suitable operation (I, D, G) and an embedding

$$h : M_3 \xrightarrow{\mathcal{F}} \mathrm{Op}(M_1, I, D, G)$$

such that $h \restriction M_1 = \mathrm{id}_{M_1}$ and $\mathrm{Rang}(h \restriction M_2) \subseteq \mathrm{Op}(M_0, I, D, G)$ (remember that $\mathrm{Op}(M, I, D, G)$ is the limit ultrapower of M with respect to (I, D, G); see [**?**, 1.7.4]). We say that M_1, M_2 do not fork in M_3 over M_0 if

$$M_1 \underset{M_0}{\overset{M_3}{\biguplus}} M_2.$$

If

$$M_1 \underset{M_0}{\overset{M_3}{\biguplus}} M_2$$

does not hold, we'll write

$$M_1 \underset{M_0}{\overset{M_3}{\not\biguplus}} M_2$$

and say that M_1, M_2 forks in M_3 over M_0.

Theorem 1.6. (1) *Suppose that*

$$M_1 \underset{M_0}{\overset{M_3}{\biguplus}} M_2 \quad \textit{and} \quad M_2 \underset{M_0}{\overset{M_3}{\not\biguplus}} M_1$$

(failure of $\biguplus$-symmetry) and $M_0 \preceq_{\text{nice}} M_3$.

Let $\mu = \kappa^* + |\mathbf{T}| + ||M_2|| + ||M_1||$. Then for every linear order $(I, <)$ there exists an Ehrenfeucht–Mostowski model $N = EM(I, \Phi)$ with μ (individual) constants $\{\tau_i^0 : i < \mu\}$ and unary function symbols $\{\tau_i^1(x_i) : i < \mu\}$, $\{\tau_i^2(x_i) : i < \mu\}$ such that, for $M = (N \restriction L) \restriction \{\tau_i^0 : i < \mu\}$ (i.e., M is a submodel of N with the same vocabulary as $\mathbf{T}$ and universe $\{\tau_i^0 : i < \mu\}$ i.e., the set of interpretations of these individual constants) and for every $t \in I$, $\ell = 1, 2$,

$$M_t^\ell = (N \restriction L) \restriction \{\tau_i^\ell(x_t) : i < \mu\},$$

one has $M \preceq_{\mathcal{F}} N$, $M^\ell_t \preceq_{\mathcal{F}} N$ *and for* $s \neq t \in I$, $t < s$ *iff* $M^1_t \underset{M}{\overset{N}{\bigcup}} M^2_s$.

(2) *Assume*

(a) $\mu \geq LS(\mathbf{T})$, $M \in K_\mu$ *is nice,*

(b) *for* $\ell = 1, 2$ *we have* Op_ℓ *is defined by* (I_ℓ, D_ℓ, G_ℓ), $f_{\ell,\alpha} \in {}^I M$ *for* $\alpha < \alpha_\ell$ *with* $eq(f_{\ell,\alpha}) \in G_\ell$, *i.e., such that* $eq(f_{\ell,\alpha})/D \in M^I_D | G$,

(c) *for* $\ell = 1, 2$ *we have* $M^\ell_0 = M$, $M^\ell_1 = \mathrm{Op}_\ell(M^\ell_0)$, $M^\ell_2 = \mathrm{Op}_{3-\ell}(M^\ell_1)$, $a^{\ell,1}_\alpha = f_{\ell,\alpha}/D_1 \in (M^\ell_0)^{I_\ell}_{D_\ell} | G_\ell = M^\ell_1$ *and* $a^{l,2}_\beta = f_{3-\ell,\alpha}/D_2 \in (M^\ell_2)^{I_{3-\ell}}_{D_{3-\ell}} | G_{3-\ell} = M^\ell_2$.

Then there are Φ, τ^ℓ_i $(\ell = 0,\ i < \mu$ *or* $\ell \in \{1, 2\}$, $i < \alpha_\ell)$ *such that*

(α) Φ *is a blueprint for E.M. models,* $|L_\Phi| \leq \mu$, L_Φ *the vocabulary of* Φ *so* $L \subseteq L_\Phi$,

(β) *for any linear order* I *we have* $EM(I, \Phi) = EM_L(I, \Phi)$ *is the* L*–reduct of* $EM_{L_\Phi}(I, \Phi)$, *(an* L_Φ*)–model) which is a model of* $\mathbf{T}$ *of cardinality* $\mu + |I|$ *and*

$$I \subseteq J \quad \Rightarrow \quad EM(I, \Phi) \preceq_{\mathcal{F}} EM(J, \Phi),$$

(γ) τ^l_i *are unary function symbols in* L_Φ,

(δ) $EM(\emptyset, \Phi)$ *is* M,

(ε) *for any linear order* I, *and* $s < t$ *in* I *we have: the type which*

(i) $\langle \tau^1_\alpha(x_s) : \alpha < \alpha_1 \rangle^\wedge \langle \tau^2_\beta(x_t) : \beta < \alpha_2 \rangle$ *realizes over* M *in* $EM(I, \Phi)$ *is the same type as* $\langle a^{1,1}_\alpha : \alpha < \alpha_1 \rangle^\wedge \langle a^{1,2}_\alpha : \alpha < \alpha_2 \rangle$ *realizes over* M *in* M^1_2,

(ii) $\langle \tau^1_\alpha(x_t) : \alpha < \alpha_1 \rangle^\wedge \langle \tau^2_\beta(x_s) : \beta < \alpha_2 \rangle$ *realizes over* M *in* $EM(I, \Phi)$ *the same type as* $\langle a^{2,2}_\alpha : \alpha < \alpha_1 \rangle^\wedge \langle a^{2,1}_\beta : \beta < \alpha_2 \rangle$ *realizes over* M *in* M^2_2.

Remark: Note $M_0 \preceq_{\text{nice}} M_3$ is automatic in the interesting case since $M_0 \in K_{<\lambda}$ and every element of $K_{<\lambda}$ is nice by [**?**, 5.4].
On the operations see [**?**].

Proof. (1) W.l.o.g. $\|M_3\| = \mu$. Let M^+_0 be an expansion of M_0 by $\leq LS(\mathbf{T})$ functions such that M^*_0 has Skolem functions for the formulas in $\mathcal{F}$. We know that $M_0 \preceq_{\text{nice}} M_3$. So there is Op^1 such that $M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} \mathrm{Op}^1(M_0)$ and as $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} M_2$ there is Op^2 such that $M_1 \preceq_{\mathcal{F}} M_3 \preceq_{\mathcal{F}} \mathrm{Op}^2(M_1)$, $M_2 \preceq_{\mathcal{F}} \mathrm{Op}^2(M_0)$. Let $\mathrm{Op} = \mathrm{Op}^2 \circ \mathrm{Op}^1$. For each $t \in I$, let $\mathrm{Op}_t = \mathrm{Op}$. Let N be the iterated ultrapower of M_0 w.r.t. $\langle \mathrm{Op}_t : t \in I \rangle$. For each $t \in I$, there is a canonical $\mathcal{F}$-elementary embedding $F_t : \mathrm{Op}_t(M_0) \xrightarrow{\mathcal{F}} N$. Let $M = M_0$, and $M^\ell_t = F_t(M_\ell)$ for $\ell = 1, 2, t \in I$.

For each $t < s$, we can let $M^+_s = \langle \mathrm{Op}_v : v < s \rangle(M_0)$, so $M_0 \preceq_{\mathcal{F}} M^+_t \preceq_{\mathcal{F}} M^+_s \preceq_{\mathcal{F}} \mathrm{Op}^1(M^+_s)$ and we can extend $F_t \restriction M_1$ to an embedding of $\mathrm{Op}^2(M_1)$ into $\mathrm{Op}^2_s(\mathrm{Op}^1_s(M^+_s))$, so $(F_t \restriction M_1) \cup (F_s \restriction M_2)$ can be extended to a $\preceq_{\mathcal{F}}$–embedding of M_3 into N. From the definition of the iterated ultrapower and non forking it follows that for $s \neq t \in I$, $t < s$ implies $M^1_t \underset{M_0}{\overset{N}{\bigcup}} M^2_s$. On the other hand, similarly, if $s,\ t \in I,\ s < t$ then $(F_s \restriction M_1) \cup (F_t \restriction M_2)$ can be extended to an $\preceq_{\mathcal{F}}$-embedding of M_3 into N, and hence by the assumption it follows that $M_t^1 \underset{M_0}{\overset{N}{\biguplus}} M_s^2$.

(2) A similar proof. $\square$

Corollary 1.7. *Assume* $\mathbf{T}$ *categorical in* λ *or just* $I(\lambda, \mathbf{T}) < 2^\lambda$. *Then* $\bigcup\limits_{\mu^+ < \lambda} K_\mu$ *obeys* $\bigcup$*-symmetry, i.e.: for* $M_0, M_1, M_2, M_3 \in \bigcup\limits_{\mu^+<\lambda} K_\mu$,

$$\textit{if} \qquad M_1 \underset{M_0}{\overset{M_3}{\bigcup}} M_2 \qquad \textit{then} \qquad M_2 \underset{M_0}{\overset{M_3}{\bigcup}} M_1 \textit{ holds.}$$

Proof. If $\mu^+ < \lambda$, $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} M_2$ and $M_2 \underset{M_0}{\overset{M_3}{\biguplus}} M_2$, then theorem 1.6 gives the assumptions of the results at the end of section three in [**?**, III] (or better [**?**, III,§3]). These yield a contradiction to the λ-categoricity of $\mathbf{T}$ and even 2^λ pairwise non isomorphic models.

But we give a self contained proof of the needed version $\mathbf{T}$ categorical in λ, allowing ourselves to use the rest of this section (which does not relay on 1.7 except 1.24, really use just 1.16, 1.18, 1.20 here. Let Φ be as in 1.6(2), and wlog as used in 1.18, 1.19. Choose an increasing continuous sequence $\langle I_\alpha : \alpha \leq \mu^+ + 1\rangle$ of linear orders each of cardinality μ^+, $|I_{\alpha+1} \setminus I_\alpha| = \mu^+$, $t^* \in I_{\mu^++1} \setminus I_{\mu^+}$, s_α^+, $s_\alpha^- \in I_{\alpha+1} \setminus I_\alpha$ for $\alpha < \mu$ such that

$$\alpha < \beta \quad \Rightarrow \quad s_\alpha^+ < s_\alpha^- < t^* < s_\beta^+ < s_\beta^-,$$

and s_α^+, s_α^- realize the same Dedekind cut of I_α. Let $M_\alpha = EM(I_\alpha, \Phi)$ for $\alpha \leq \mu^+$, so $\langle M_\alpha : \alpha \leq \mu^+ + 1\rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous, $M_\alpha \in K_{\mu^+}$, $M_{\alpha+1}$ is (μ,μ)-saturated over M_α, $\bar{a}_t = \langle \tau_i^1(x_t) : i\rangle$, $\bar{b}_t = \langle \tau_i^2(x_t) : i\rangle$ for $t \in I_{\mu^++1}$. Easily $\mathrm{tp}(\bar{a}_{s_\alpha^-}, M_\alpha, M_{\mu^++1}) = \mathrm{tp}(\bar{a}_{s_\alpha^+}, M_\alpha, M_{\mu^++1})$ for $\alpha < \mu$ but

$$\mathrm{tp}(\bar{b}_t^{*\wedge}\bar{a}_{s_\alpha^-}, M_\alpha, M_{\mu^++1}) \neq \mathrm{tp}(\bar{b}_t^{*\wedge}\bar{a}_{s_\alpha^+}, M_\alpha, M_{\mu^++1}).$$

We now choose enough sequences of models, first we define a linear order J with set of elements

$$\{t_i : i < \kappa^*\} \cup \{s_\gamma : \gamma < \mu^+ \times (\mu^+ + 1)\}$$

such that

$$i < j \ \&\ \beta < \gamma < \mu^+ \times \mu^+ \quad \Rightarrow \quad t_i < t_j < s_\beta < s_\gamma.$$

For $\alpha \leq \mu^+ + 1$ let $J_\alpha = \{t_i : i < \kappa^*\} \cup \{t_\gamma : \gamma < \mu^+ \times (1+\alpha)\}$, let $J^* = J_{\mu^++1} \setminus J_\mu$. Let $N_\alpha = EM(J_\alpha, \Phi)$. Again $\langle N_\alpha : \alpha \leq \mu^+ + 1\rangle$ is $\preceq_{\mathcal{F}}$–increasing continuous in K_{μ^+}, $N_{\alpha+1}$ is (μ^+, μ^+)-saturated over N_α. Hence there is an isomorphism f^* from $M_{\mu+1}$ onto N_{μ^++1} mapping each M_α onto N_α. Now, $\bar{b}^* = f(\bar{b}_{t^*}) = \langle f(\tau_i^2(x_t)) : i\rangle$ is a sequence of $\leq \mu$ members of $EM(J_{\mu^++1}, \Phi)$, hence for some $\alpha < \mu^+$ we have $\bar{b}^* \subseteq EM_{(J',\Phi)}$ where $J' = \{t_i : i < \kappa^*\} \cup J^* \cup \setminus J_\alpha$. However by [**?**, 2.6] we have $J'_{\mu^+} \underset{J_\alpha}{\overset{J}{\bigcup}} J'$. Hence ([**?**, 2.5])

$$(*) \qquad EM(J'_{\mu^+},\Phi) \bigcup_{EM(J_\alpha,\Phi)}^{EM(J,\Phi)} EM(J',\Phi).$$

Now easily there is an automorphism f of $EM(J_{\mu^+},\Phi)$ over $EM(J_\alpha,\Phi)$ which maps $\bar a_{s^-_\alpha}$ to $\bar a_{s^+_\alpha}$. The Op which witnesses $(*)$ extends f to an automorphism of $\mathrm{Op}(EM(J_{\mu^+},\Phi)$ which is the identity over $EM(J',\Phi)$ continuous. $\square$

It may be helpful, though somewhat vague, to add the remark that $\bigcup$-asymmetry enables one to define order and to build many complicated models; so 1.7 removes a potential obstacle to a categoricity theorem. Note that we could have put 3.11(2) here.

Definition 1.8. Let A be a set. We write $M_1 \bigcup_{M_0}^{M_3} A$ (where $A \subseteq M_3$, $M_0 \preceq_{\mathcal F} M_1 \preceq_{\mathcal F} M_3$) to mean that there exist M_2, M_3' such that $A \subseteq |M_2|$, $M_3 \preceq_{\mathcal F} M_3'$ and $M_1 \bigcup_{M_0}^{M_3'} M_2$. In this situation we say that $A/M_1 = \mathrm{tp}(A, M_1, M_3)$ does not fork over M_0 in M_3.

We will write $M_1 \bigcup_{M_0}^{M_3} a$ to mean $M_1 \bigcup_{M_0}^{M_3} \{a\}$, we then say $\mathrm{tp}(a, M_1, M_3)$ does not fork on M_0.

We write $A_1 \bigcup_{M_0}^{M_3} A_2$ if for some M_3, $M_3 \preceq_{\mathcal F} M_3' \in K_{<\lambda}$, and for some M_1', $A_2 \subseteq M_1' \preceq_{\mathcal F} M_3'$, and $M_1' \bigcup_{M_0}^{M_3'} A_2$.

Remark 1.9. (1) Of particular importance is the case where A is finite. Let us explain the reason. We wish to prove a result of the form:

$(*)$ if $\langle M_i : i \le \delta+1\rangle$ is a continuous $\prec_{\mathcal F}$-chain and $a \in M_{\delta+1}$, then there is $i < \delta$ such that $M_\delta \bigcup_{M_i}^{M_{\delta+1}} a$.

This says roughly that the type $\mathrm{tp}(a, M_\delta, M_{\delta+1})$ is definable over a finite set (or at least in some sense has finite character). In general the former relation is not obtained. However its properties are correct. Hence it will be possible to define the rank of a over M_0, $\mathrm{rk}(a, M_0)$, as an ordinal, so that for large enough M_3, if $M_1 \biguplus_{M_0}^{M_3} a$, then $\mathrm{rk}(a, M_1) < \mathrm{rk}(a, M_0)$.

(2) If A is an infinite set, then we cannot prove $(*)$, in general. For example, suppose that $\langle M_i : i \le \omega\rangle$ is (strictly) increasing continuous, $a_i \in (M_{i+1} \setminus M_i)$ and $A = \{a_i : i < \omega\}$. Then for every $i < \omega$, $\left(\bigcup_{j<\omega} M_j\right) \biguplus_{M_i}^{M_\omega} A$ as the operation Op we use in the definition, increase M_i and increase $\bigcup_{j<\omega} M_j$, but $\mathrm{Op}(M_i) \bigcap \bigcup_{j<\omega} M_j = M_i$. Still we can restrict ourselves to δ of cofinality $> |A|$.

(3) Notice that quite generally speaking, $N_1 \underset{N_0}{\overset{N_3}{\bigcup}} N_2$ implies that $N_1 \cap N_2 = N_0$ (see above).

Definition 1.10. We define

$$\begin{aligned}
&\boldsymbol{\kappa}_\mu(\mathbf{T}) = \\
&\boldsymbol{\kappa}_\mu(\mathcal{K}) = \{\kappa : \ \text{cf}(\kappa) = \kappa \leq \mu \text{ and there exist a continuous } \prec_{\mathcal{F}}\text{-chain} \\
&\qquad\qquad \langle M_i : i \leq \kappa + 1\rangle \subseteq K_{\leq\mu} \text{ and } a \in M_{\kappa+1} \text{ such that} \\
&\qquad\qquad \text{for all } i < \kappa,\ a/M_\kappa \text{ forks over } M_i \text{ in } M_{\kappa+1}\}.
\end{aligned}$$

I.e., for $\kappa \in \boldsymbol{\kappa}_\mu(\mathbf{T})$ there are $\langle M_i \in K_{\leq\mu} : i \leq \kappa + 1\rangle$ and $a \in M_{\kappa+1}$ such that $i < \kappa \ \Rightarrow \ M_\kappa \underset{M_i}{\overset{M_{\kappa+1}}{\biguplus}} a$.

Example 1.11. Fix μ and $\alpha \leq \mu$. Let $({}^\mu\omega, E_\beta)_{\beta<\alpha}$ be the structure with universe

$${}^\mu\omega = \{\eta : \eta \text{ is a function from } \mu \text{ to } \omega\},$$

$\eta E_\beta \nu$ iff $\eta \restriction \beta = \nu \restriction \beta$. Let $\mathbf{T} = Th({}^\mu\omega, E_\beta)_{\beta<\alpha}$. Then

$$\boldsymbol{\kappa}_\mu(\mathbf{T}) = \{\kappa : \text{cf}(\kappa) = \kappa \leq \alpha\}.$$

Why? If $\text{cf}(\kappa) = \kappa \leq \alpha$, then there are M_i $(i \leq \kappa + 1)$, $a \in M_{\kappa+1}$ and $a_i \in (M_{i+1} \setminus M_i)$ for $i < \kappa$ such that $a_i/E_{i+1} \notin M_i$ (that's to say, no element of M_i is E_{i+1}-equivalent to a_i) and aE_ia_i.

Definition 1.12. The class $\mathcal{K} = \langle K, \preceq_{\mathcal{F}}\rangle$ is χ-based iff for every pair of continuous $\prec_{\mathcal{F}}$-chains $\langle N_i \in K_{\leq\chi} : i < \chi^+\rangle$, $\langle M_i \in K_{\leq\chi} : i < \chi^+\rangle$, with $M_i \preceq_{\mathcal{F}} N_i$, there is a club C of χ^+ such that

$$(\forall i \in C)\left(M_{i+1} \underset{M_i}{\overset{N_{i+1}}{\bigcup}} N_i\right).$$

Replacing χ^+ by regular χ we write $(< \chi)$-based. We say synonymously that $\mathbf{T}$ is χ-based.

Definition 1.13. The class $\mathcal{K} = \langle K, \preceq_{\mathcal{F}}\rangle$ has continuous non-forking in (μ, κ) iff

(α) *whenever* $\langle M_i \in K_{\leq\mu} : i \leq \delta\rangle$ is a continuous $\prec_{\mathcal{F}}$-chain, $|\delta| \leq \mu$, $\text{cf}(\delta) = \kappa$,

$$M_0 \preceq_{\mathcal{F}} N_0 \preceq_{\mathcal{F}} N^*,\ M_\delta \preceq_{\mathcal{F}} N^* \quad \text{and} \quad (\forall i < \delta)\left(M_i \underset{M_0}{\overset{N^*}{\bigcup}} N_0\right),$$

then $M_\delta \underset{M_0}{\overset{N^*}{\bigcup}} N_0$;

(β) *whenever* $\langle M_i \in K_{\leq\mu} : i \leq \delta + 1\rangle$, $\langle N_i \in K_{\leq\mu} : i \leq \delta + 1\rangle$ are continuous $\prec_{\mathcal{F}}$-chains, $M_i \preceq_{\mathcal{F}} N_i$, $|\delta| \leq \mu$, $\text{cf}(\delta) = \kappa$ and

$$(\forall i < \delta)\left(M_{\delta+1} \underset{M_i}{\overset{N_{\delta+1}}{\bigcup}} N_i\right),$$

then $M_{\delta+1} \underset{M_\delta}{\overset{N_{\delta+1}}{\bigcup}} N_\delta$.

Again we will mean the same thing by saying that $\mathbf{T}$ has continuous non-forking in (μ,κ).

Our next goal is to show that if $\mathbf{T}$ fails to possess these features for some $\mu<\lambda$ such that $\mu\geq\kappa+LS(\mathcal{K})$, then $\mathbf{T}$ has many models in λ.

Let us recall in this context a further important result from [**?**, II, 3.10]:

Theorem 1.14. *Assume* $\mathbf{T}$ *be a* λ*-categorical theory, or just* $K_{<\lambda}$ *has amalgamation and every* $N\in K_{<\lambda}$ *is nice.*

(1) *Let* $LS(\mathbf{T})<\mu\leq\lambda$, $M\in K_\mu$. *Then TFAE:*
  (A) M *is universal-homogeneous: if* $N\preceq_{\mathcal{F}}M$, $\|N\|<\mu$, $N\preceq_{\mathcal{F}}N'\in K_{<\mu}$, then *there is an* $\mathcal{F}$*-elementary embedding* $g:N'\xrightarrow{\mathcal{F}}M$ *such that* $g\restriction N=\mathrm{id}_N$.
  (B) *If* $N\preceq_{\mathcal{F}}M$, $||N||<\mu$ *and* $p\in S(N)$, then p *is realized in* M, *i.e.,* N *is saturated.*

(2) M *as in (A) or (B) is unique for fixed* $\mathbf{T}$, μ.

(3) *Let* $LS(\mathbf{T})\leq\mu<\lambda$, *and* $\kappa\leq\mu$. *Any two* (μ,κ)*-saturated models are isomorphic (see 1.4(7)).*

(4) *Let* $LS(\mathbf{T})\leq\mu<\lambda$, *and* $\kappa\leq\mu$. *If* N_1,N_2 *are* (μ,κ)*-saturated over* M *then* N_1,N_2 *are isomorphic over* M.

Proof. (1), (2) See [**?**, II 3.10], or better presented [**?**, 0.19].

(3) Easy and exist but we shall prove. Assume N_1,N_2 are (μ,κ)–saturated, hence for $l=1,2$ there is a $\preceq_{\mathcal{F}}$–increasing continuous sequence $\langle M_{l,\alpha}:\alpha<\kappa\rangle$ in K_μ such that $M_{l,\kappa}=N_l$ and $M_{l,\alpha+1}$ is universal over $M_{l,\alpha}$. We now choose by induction on $\alpha\leq\kappa$ a triple $(f_l,M'_{1,\alpha},M'_{2,\alpha})$ such that

(a) for $l\in\{1,2\}$ $M'_{l,\alpha}\in K_\mu$ is $\preceq_{\mathcal{F}}$–increasing continuous with $\alpha<\kappa$,
(b) f_α is an isomorphism from $M'_{1,\alpha}$ onto $M'_{2,\alpha}$ increasing with α,
(c) if α is even $M'_{1,\alpha}=M_{1,\alpha}$ and $M'_{2,\alpha}\preceq_{\mathcal{F}}M_{2,\alpha+1}$,
(d) if α is odd, $M'_{2,\alpha}=M_{2,\alpha}$ and $M'_{1,\alpha}\preceq_{\mathcal{F}}M_{,\alpha+1}$,
(e) if α is a limit ordinal then $M'_{1,\alpha}=M_{1,\alpha}$, $M'_{2,\alpha}=M_{2,\alpha}$.

Using the universality assumptions there is no problem to carry out the induction and f_κ is an isomorphism from $N_1=M_{1,\kappa}$ onto $N_2=N_2$.

(4) Similar to (3) (just let $M=M_{1,0}=M_{2,0}$, $f_0=\mathrm{id}_M$). □

Proposition 1.15. *Assume* $\mathbf{T}$ *is* λ*-categorical or just* $\mathcal{K}_{<\lambda}$ *has amalgamation.*

(1) *If* $LS(\mathbf{T})\leq\mu<\lambda$, $N_0\preceq_{\mathcal{F}}N_1$ *are in* K_μ, *then TFAE*
  (A) N_1 *is* (μ,μ)*-saturated over* N_0,
  (B) *there is a* $\preceq_{\mathcal{F}}$*-increasing continuous* $\langle M_i:i\leq\mu\times\mu\rangle$, *such that:* $M_{\mu\times\mu}=N_1$, $M_0=N$ *and every* $p\in S(M_i)$ *is realized in* M_{i+1}

(2) *Also TFAE for* $\kappa=\mathrm{cf}(\kappa)\leq\mu^+$
  $(A)_\kappa$ N_1 *is* (μ,κ)*-saturated over* N_0,
  $(B)_\kappa$ *there is a* $\preceq_{\mathcal{F}}$*-increasing continuous* $\langle M_i:i\leq\mu\times\kappa\rangle$ *with* $M_{\mu\times\kappa}=N_1$, $M_0=N$ *and every* $p\in S(M_i)$ *is realized in* M_{i+1}

(3) *If* $\mathcal{K}$ *is stable in* μ, $\mu\geq LS(\mathcal{K})$, $\kappa=\mathrm{cf}(\kappa)\leq\mu^+$ then *there is a* (μ,κ)*-saturated model (in fact, over any given model in* K_μ*).*

Proof. (1) Follows from the proof of 1.14(1).

(2), (3) Straightforward. □

Proposition 1.16. *($\mathbf{T}$ categorical in λ)*

(1) *Any $M \in K_\lambda$ is saturated.*
(2) *Every $N \in K_{<\lambda}$ is nice.*
(3) *$K_{<\lambda}$ has $\preceq_{\mathcal{F}}$-amalgamation.*
(4) *If $\mu \in [LS(\mathbf{T}), \lambda)$ and $M \in K_\mu$, then there is $N \in M_\mu$ which is μ-universal over M (see Definition 1.4). $\mathcal{K}$ is stable in μ for $\mu \in [LS(\mathbf{T}), \lambda)$.*
(5) *$\mathcal{K}$ is stable in μ for $\mu \in [LS(\mathbf{T}), \lambda)$.*
(6) *If $\mu \in [LS(\mathbf{T}), \lambda)$, $\kappa \leq \mu$ and $M \in K_\mu$, then there is $N \in K_\mu$ which is (μ, κ)–saturated over M.*

Proof. (1) By the proof of [**?**, 5.4] (for λ-regular easier).

(2) See [**?**, 5.4].

(3) See [**?**, 5.5].

(4) See [**?**, 3.7].

(5) Follows by the two previous parts.
(6) Follows by (3)+(5) and 1.15. □

Intermediate Corollary 1.17. (1) *Suppose that $\mathbf{T}$ is λ-categorical. If $\mu < \lambda$, $\mu > LS(\mathbf{T})$ and $\mathbf{T}$ is not μ-categorical, then there is an unsaturated model $M \in K_\mu$.*
(2) *It now follows that if we show that the existence of an unsaturated model in K_μ implies that of an unsaturated model in K_λ, then λ-categoricity of $\mathbf{T}$ implies μ-categoricity of $\mathbf{T}$.*

Conclusion 1.18. [$\mathbf{T}$ categorical in λ] If I is a linear order, $I = I_1 + I_2$, $|I| < \lambda$ and $J = I_1 + \omega + I_2$ *then* every $p \in S(EM(I))$ is realized in $EM(J)$.

Proof. Clearly $EM(I_1 + \lambda + I_2)$ is in K_λ, and hence is saturated, and hence every $p \in S(EM(I))$ is realized in it, say by a_p, for some finite $w_p \subseteq \lambda$ we have $a_p \in EM(J_1 + w_p + I_2))$, now we use indiscernibility. □

Remark 1.19. By changing Φ we can replace "ω" by "1".

Conclusion 1.20. [$\mathbf{T}$ categorical in λ]

(1) If $J = \bigcup_{\alpha<\mu} I_\alpha$; $|J| = \mu \in [LS(\mathbf{T}), \lambda)$ or $|J| = \mu = \lambda$ & $LS(\mathbf{T}) \leq |I_0| < \lambda$, I_α and increasing continuous, for each α some Dedekind cut of I_α is realized by infinitely many members of $I_{\alpha+1} \setminus I_\alpha$ *then* $EM(J)$ is $(\mu, |I_0|)$-saturated over $EM(I_0)$.
(2) If Φ is "corrected" as in 1.19, $I_0 \subseteq J$, $|J \setminus I_0| = |J| = \mu$, $\mu \in [LS(\mathbf{T}), \lambda)$, or $|J| = \mu = \lambda$ & $LS(\mathbf{T}) \leq |I_0| < \lambda$, *then* $EM(J)$ is $(\mu, |I_0|)$-saturated over $EM(I_0)$ moreover for any $\kappa = \mathrm{cf}(\kappa) \leq \mu$ it is (μ, κ)-saturated.
(3) If $\langle M_i : I \leq \kappa \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous, $M_i \in K_\mu$, M_{i+1} is universal over M_i then M_κ is (μ, θ)-saturated over M_0 for every $\theta \leq \mu$, even $\theta \leq \mu^+$, so $N \in K_\mu$ which is saturated over $M \in M_\mu$ is unique up to isomorphism over M.

Proof. (1), (2) by 1.20+1.15(1).

(3) Follows. □

Proposition 1.21. (1) *Suppose* $\langle N^\ell_i : i \le \alpha \rangle$ *is* $\preceq_{\rm nice}$*-increasing continuous-inuous for* $\ell = 1, 2$, $N^1_i \preceq_{\mathcal F} N^2_i \in K_{<\lambda}$ *and* $N^2_i \underset{N^1_i}{\overset{N^2_{i+1}}{\bigcup}} N^1_{i+1}$ *for each* $i < \alpha$, *then* $N^2_0 \underset{N^1_0}{\overset{N^2_\alpha}{\bigcup}} N^1_\alpha$.

(2) *The monotonicity properties of* $\bigcup$, *i.e.: if* $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} M_2$ *and for some operation* Op *and models* M_1', M_2', M_3' *we have* $M_3 \preceq_{\mathcal F} M_3' \preceq \mathrm{Op}(M_3)$ *and* $M_0 \preceq_{\mathcal F} M_1' \preceq_{\mathcal F} M_1$ *and* $M_0 \preceq_{\mathcal F} M_2' \preceq_{\mathcal F} M_2$, *then* $M_1' \underset{M_0}{\overset{M_3'}{\bigcup}} M_2'$.

(3) *If* $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} A$ *and* $M_0 \preceq_{\mathcal F} M_0' \preceq_{\mathcal F} M_1' \preceq_{\mathcal F} M_1 \preceq_{\mathcal F} M_3' \preceq_{\mathcal F} M_3''$ *and* $M_3 \preceq_{\mathcal F} M_3''$ *and* $A' \subseteq A$, *then* $M_1' \underset{M_0'}{\overset{M_3'}{\bigcup}} A'$.

(4) *Note that by the definition if* $A_1 \underset{N_0}{\overset{N_3}{\bigcup}} A_2$ *and* $N_0 \subseteq N_0' \subseteq A_1$, *and* $N_0' \preceq_{\mathcal F} N_3$, *then* $A_1 \underset{N_0'}{\overset{N_3}{\bigcup}} A_2$ *(the same operation witness this).*

Proof. Use [**?**, 1.11], e.g.:
(1) For each $i < \alpha$ there is Op_i such that $N^1_{i+1} \preceq_{\mathcal F} \mathrm{Op}_i(N^1_i)$, $N^2_{i+1} \preceq_{\mathcal F} \mathrm{Op}_i(N^2_i)$. We can find Op resulting from the iterated $\langle \mathrm{Op}_i : i < \alpha \rangle$. Let $N^*_1 = \mathrm{Op}(N^1_0)$, $N^*_2 = \mathrm{Op}(N^2_0)$, so we can choose by induction on i an $\preceq_{\mathcal F}$-embedding f_i of N^2_i into N^*_2 mapping N^1_i into N^*_1, increasing continuous with i, such that $f_i(N^2_i)$ is included in $\langle \mathrm{Op}_i : i < \alpha \rangle (N^2_0)$. $\square$

Proposition 1.22. *[**T** is* λ*-categorical] If* $M_0 \preceq_{\rm nice} M_1, M_2$ *are in* $K_{<\lambda}$ then *we can find* $M_4 \in K_{<\lambda}$, $M_0 \preceq_{\mathcal F} M_4$ *and* $\preceq_{\mathcal F}$*-embeddings* f_1, f_2 *of* M_1, M_2 *respectively into* M_4 *such that*

(α) $f_1(M_1) \underset{M_0}{\overset{M_4}{\bigcup}} f_2(M_2)$ *and*

(β) $f_2(M_2) \underset{M_0}{\overset{M_4}{\bigcup}} f_1(M_1)$.

Remark 1.23. Note 1.7 deals only with models in $\bigcup\{K_\mu : \mu^+ < \lambda\}$, hence ($\beta$) is not totally redundant.

Proof. If we want to get (α) only, use operation Op such that $\mathrm{Op}(M_0)$ has cardinality $\ge \lambda$, choose $N \preceq_{\mathcal F} \mathrm{Op}(M_0)$, $\|N\| = \lambda$, hence N is saturated hence we can find a $\preceq_{\mathcal F}$-embedding $f_2 : M_2 \to N$, let $N_1 = \mathrm{Op}(M_1)$, so $N \preceq_{\mathcal F} \mathrm{Op}(M_0) \preceq_{\mathcal F} \mathrm{Op}(M_1) = N_1$, and choose $M_4 \prec N_1$, $M_4 \in K_\mu$, $\mu < \lambda$ such that $M_1 \cup \mathrm{Rang} f_2 \subseteq N$. So we have gotten clause (α) and if $\mu^+ < \lambda$ by 1.7 we are done; but as we need the case $\mu^+ = \lambda$ we have to restart the proof.

By "every $N \in K_\lambda$ is saturated" there are an operation Op and $N \in K_\lambda$ such that $M_0 \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} \mathrm{Op}(M_0)$ hence there are M_0^+, M_1^+, M_2^+ in $K_{<\lambda}$ such that:

$(*)_0$ $(M_1^+, M_0^+) \preceq_{\mathcal{F}} \mathrm{Op}(M_1, M_0)$, $(M_2^+, M_0^+) \preceq_{\mathcal{F}} \mathrm{Op}(M_2, M_0)$ and M_0^+ has the form $EM(I_0)$, I_0 a linear order with $|I_0|$ Dedekind cuts with cofinality (κ^*, κ^*). [Note that by 1.20(2) if $|I_0| = \lambda$ then $EM(I_0)$ is saturated and N is saturated, clearly there is I_0 as required.]

Clearly w.l.o.g. the cardinality of I_0 is $< \lambda$. Hence we can find I_1, I_2, I_3 such that: $I_0 \overset{\text{def}}{=} I \subseteq I_1 \subseteq I_3$ $I_0 \subseteq I_2 \subseteq I_3$, $I_1 \cap I_2 = I$, no $t_1 \in I_1 \setminus I_0$, $t_2 \in I_2 \setminus I_0$ realize the same Dedekind cut of I, and every $t \in I_3 \setminus I_0$ realizes a cut of I with cofinality (κ^*, κ^*). Hence $I_0 \subseteq_{\text{nice}} I_\ell$ $(\ell \leq 3)$, moreover $I_1 \underset{I_0}{\overset{I_3}{\bigcup}} I_2$ and $I_2 \underset{I_0}{\overset{I_3}{\bigcup}} I_1$. Hence

$(*)_1$ $EM(I_1) \underset{EM(I_0)}{\overset{EM(I_3)}{\bigcup}} EM(I_2)$ and $EM(I_2) \underset{EM(I_0)}{\overset{EM(I_3)}{\bigcup}} EM(I_1)$.

Also by 1.20(2), wlog $(\ell = 1,2)$ $M_\ell^+ \preceq_{\mathcal{F}} EM(I_\ell)$. So by 1.21(2)

$(*)_2$ $M_1^+ \underset{M_0^+}{\overset{EM(I_3)}{\bigcup}} M_2^+$ and $M_2^+ \underset{M_0^+}{\overset{EM(I_3)}{\bigcup}} M_1^+$.

By $(*)_0 + (*)_2$ and 1.21(1) (for $\alpha = 2$) we get the conclusions. $\square$

Proposition 1.24. *[***T** *is λ-categorical]*

(1) *If $M_1^\ell \underset{M_0^\ell}{\overset{M_3^\ell}{\bigcup}} M_2^\ell$ for $\ell = 1,2$, $M_3^\ell \in K_{<\lambda}$ moreover $\|M_3^\ell\|^+ < \lambda$ and f_k an isomorphism from M_k^1 onto M_k^2 for $k = 0,1,2$ such that $f_0 \subseteq f_1$, $f_0 \subseteq f_2$* then *there is M, $M_3^2 \preceq_{\mathcal{F}} M \in K_{<\lambda}$, $\|M\| = \|M_3^1\| + \|M_3^2\|$ and a $\preceq_{\mathcal{F}}$-embedding f of M_3^1 into M_3^2 extending f_1 and f_2.*

(2) *Assume $M_1^\ell \underset{M_0^\ell}{\overset{M_3^\ell}{\bigcup}} A_2^\ell$ for $\ell = 1,2$ and $A_2^\ell \subseteq M_2^\ell \preceq M_3^\ell$, and $M_3^\ell \in K_{<\lambda}$ moreover $\|M_3^\ell\|^+ < \lambda$, and f_k is an isomorphism from M_k^1 onto M_k^2 for $k = 0,1,2$ such that $f_0 \subseteq f_1$ and $f_0 \subseteq f_2$ and f_2 maps A_2^1 onto A_2^2* then *there is M, $M_2^3 \preceq_{\mathcal{F}} M \in K_{<\lambda}$ such that $\|M\| = \|M_3^1\| + \|M_3^2\|$ and a $\preceq_{\mathcal{F}}$-embedding f of M_3^1 into M_3^2 extending f_1 and $f_2 \restriction A_2^1$.*

(3) *If for $\ell = 1,2$ we have $p_\ell \in S(N)$ does not fork over M (see Definition 1.8), $M \preceq_{\mathcal{F}} N \in K_\mu$, $\mu^+ < \lambda$ and $p_1 \restriction M = p_2 \restriction M$* then $p_1 = p_2$

Remark 1.25. (1) This is uniqueness of non forking amalgamation.

(2) The requirement is $\|M_3^\ell\|^+ < \lambda$ rather than $\|M_3^\ell\| < \lambda$ only because of the use of symmetry, i.e., 1.7.

Proof. (1) Wlog $f_0 = \mathrm{id}$, $M_0^1 = M_0^2$ call it M_0 and $f_1 = \mathrm{id}_{M_1^1}$, $M_1^1 = M_1^2$ call it M_1. By the assumption for some operation Op_ℓ we have $(M_3^\ell, M_2^\ell) \preceq_{\mathcal{F}} \mathrm{Op}_\ell(M_1^\ell, M_0^\ell)$. Let $\mathrm{Op} = \mathrm{Op}_1 \circ \mathrm{Op}_2$, so w.l.o.g. $M_3^\ell \preceq_{\mathcal{F}} \mathrm{Op}(M_1)$, $M_2^\ell \preceq_{\mathcal{F}} \mathrm{Op}(M_0)$. W.l.o.g. $\|\mathrm{Op}(M_0)\| \geq \lambda$ and $\|\mathrm{Op}(M_1)\| \geq \lambda$, so there is N_0, $\bigcup_{\ell=1}^{2} M_2^\ell \subseteq N_0 \preceq_{\mathcal{F}}$

$\mathrm{Op}(M_0)$, such that $\|N_0\| = \lambda$, hence N_0 is saturated hence there is an automorphism g_0 of N_0 such that $g_0 \restriction M_2^1 = f_2$ (so $g_0 \restriction M_0 = \mathrm{id}_{M_0}$). So there is N_2, $\bigcup\limits_{\ell=1}^{2} M_2^\ell \subseteq N_2 \preceq_{\mathcal{F}} N_0$, $\|N_2\|^+ < \lambda$, N_2 closed under g_0, g_0^{-1}. Now there is N_3, $N_0 \cup M_1 \subseteq N_3 \preceq_{\mathcal{F}} \mathrm{Op}(M_1)$, $N_3 \in K_\lambda$, hence N_3 is saturated. So $M_1 \underset{M_0}{\overset{N_3}{\bigcup}} N_2$ and hence $N_2 \underset{M_0}{\overset{N_3}{\bigcup}} M_1$ (by symmetry, i.e., 1.7). Hence for some N_3', $N_3' \preceq_{\mathcal{F}} N_3 \in K_{<\lambda}$ and some automorphism g_1 of N_3' extends $(g_0 \restriction N_2) \cup \mathrm{id}_{M_1}$. [Why? for some Op', $(N_3, M_1) \preceq_{\mathcal{F}} \mathrm{Op}'(N_1, M_0)$ and $\mathrm{Op}'(N_1)$, $\mathrm{Op}'(g_0 \restriction N_2)$ are as required except having too large cardinality, but this can be rectified.]

Clearly we are done.

2), 3) Follow from part (1). $\square$

2. Various constructions

In this section we will attempt to describe some constructions of models of $\mathbf{T}$ relating to the situations in 1.12 and 1.13, i.e., we want to prove there are "many complicated" models of $\mathbf{T}$ when $\mathbf{T}$ is "on the unstable side" of Definition 1.12 or Definition 1.13; they will be use in the proofs in 3.2 — 3.5. May we suggest that on a first reading the reader be content with the perusal of 2.1 and 2.2, leaving the heavier work of 2.2.1 until after section three which contains the model-theoretic fruits of the paper. The construction should be meaningful for the classification problem.

What we actually need are 2.2.1, 2.2.2, 2.2.3

Construction 2.1. First try

Data 2.1.1. Suppose that $\langle M_i \in K_{\leq\mu} : i \leq \kappa+1\rangle$ is a continuous $\preceq_{\text{nice}}$-chain of models of $\mathbf{T}$, $\mu < \lambda$; T is a non empty subset of $({}^{\kappa+1\geq}Ord)$ and

(i) T is closed under initial segments, i.e. if $\eta \in T$ and $\nu \triangleleft \eta$, then $\nu \in T$,
(ii) if $\eta \in T$ and $\ell g(\eta) = \kappa$ then $\eta^\frown\langle 0\rangle \in T$ and for all i, $\eta^\frown\langle 1+i\rangle \notin T$.

Let $\lim_\kappa(T) = \{\eta : \ell g(\eta) = \kappa$ and $\bigwedge\limits_{i<\kappa}(\eta \restriction i \in T)\}$. Let $\{\eta_i : i < i^*\}$ be an enumeration of T such that if $\eta_i \triangleleft \eta_j$ (η_i is an initial segment of η_j), then $i < j$, and if $\eta_i = \nu^\frown\langle\alpha\rangle$, $\eta_j = \nu^\frown\langle\beta\rangle$, $\alpha < \beta$, then $i < j$. For simplicity i^* is a limit ordinal.

First Try 2.1.2. From the data of 2.1.1 we shall build a model N^* with Skolem functions, $N^* \restriction L \in K$, and for $\eta \in T$, $M_\eta^* \subseteq N^*$, $f_\eta : M_{lg(\eta)} \overset{\text{onto}}{\underset{\mathcal{F}}{\mapsto}} M_\eta^* \restriction L$ such that if $\eta_i \triangleleft \eta_j$, then $f_{\eta_i} \subseteq f_{\eta_j}$, and $M_{\eta_i}^* \preceq_{\mathcal{F}^{sk}} M_{\eta_j}^*$, where $\mathcal{F}^{sk} \supseteq \mathbf{T}^{sk}$ is a fragment of $(L^{sk})_{\kappa^*,\omega}$.

Let $M_i^* = Sk(M_i)$ be a Skolemization of M_i for $\mathcal{F}$, increasing ($\subseteq$) with i i.e. for every formula $(\exists y)\varphi(y,\bar{x}) \in \mathcal{F}$ we choose a function $F_{\varphi(y,\bar{x})}^{M_i}$ from M_i to M_i, with $\ell g(\bar{x})$-places such that

$$M_i \models (\exists y)\varphi(y,\bar{a}) \rightarrow \varphi(F_{\varphi(y,\bar{x})}^{M_i}(\bar{a}),\bar{a})$$

and

$$j < i \quad \Rightarrow \quad F_{\varphi(y,\bar{y})}^{M_i} \restriction M_j = F_{\varphi(y,\bar{x})}^{M_j}.$$

Note: we do not require even $M^*_i \prec M^*_{i+1}$.

To achieve this, let us define by induction on $i < i^*$, N^*_i, $M^*_{\eta_i}$ and f_{η_i}. W.l.o.g. $\eta_0 = \langle\,\rangle$ and i limit implies $\lg(\eta_i)$ limit. Let $N^*_0 = M^*_{\eta_0} = Sk(M_0)$, the Skolemization of M_0, $f_{\langle\,\rangle} = \mathrm{id}_{M_0}$. If i is a limit ordinal, let $N^*_i = \bigcup_{j<i} N^*_j$. If i is a successor ordinal and $\ell g(\eta_i) = \alpha+1$, then letting $\eta_j = \eta_i \restriction \alpha$, note that $\eta_j \triangleleft \eta_i$ so $j < i$ and so $M^*_{\eta_j}$ and f_{η_j} are defined. We are assuming $M_\alpha \preceq_{\mathrm{nice}} M_{\alpha+1}$ hence, there is an operator $\mathrm{Op} = \mathrm{Op}_\alpha$ such that $M_{\alpha+1} \preceq_{\mathrm{nice}} \mathrm{Op}(M_\alpha)$. Let $N^*_i = \mathrm{Op}(N^*_{i-1})$, let $\mathrm{Op}(N^*_{i-1}, M_\alpha, f_{\eta_j}) = (N^*_i, \mathrm{Op}(M_\alpha), (\mathrm{Op}(f_{\eta_j}))$, and let $f_{\eta_i} = \mathrm{Op}(f_{\eta_j}) \restriction M_{\ell g(\eta_i)}$ and $M^*_{\eta_i} = \mathrm{Rang}(f_{\eta_i})$. (We can replace N^*_{i+1} by any N' such that $N^*_i \cup M^*_{\eta_i} \subseteq N' \preceq_{\mathcal{F}} N^*_{i+1}$ so preserving $|N^*_i| \le \mu + |i|$). Finally, let $N^* = \bigcup_{i<i^*} N^*_i$. We are left with the case i successor ordinal, $\ell g(\eta_i)$ a limit ordinal; we let $N^*_i = N^*_{i+1}$, $M^*_{\eta_i} = \bigcup_{\nu \triangleleft \eta_i} M^*_\nu$ and $f_{\eta_j} = \bigcup_{\nu\triangleleft\eta_j} f_\nu$.

Explanation: In order to use this construction to prove non-structure results, we intend to use the property: for every $\eta \in \lim_\kappa T$, it is possible to extend $f_\eta = \bigcup_{\alpha<\kappa} f_{\eta\restriction\alpha}$ to an $\mathcal{F}$-elementary embedding f^* of $M_{\kappa+1}$ into N^* *iff* $\eta \in T$.

Let us remark that if for example χ is a strong limit cardinal of cofinality κ^* and $\chi^{<\kappa} \subseteq T \subseteq \chi^{\le\kappa} \cap \{\eta^\wedge\langle 0\rangle : (\exists\alpha < \kappa)\ell g(\eta) = \alpha+1)\}$, then over $\bigcup_{\eta\in\chi^{<\kappa}} M^*_\eta$ for χ parameters there are 2^χ independent decisions. This is not only a reasonable result, it has been shown ([**?**, VIII §1] for χ as above, [**?**, III §5] more generally) that this result is sufficient to prove the existence of many models in every cardinality $\lambda > \mu + LS(\mathbf{T})$.

But to use this construction we have to have some continuity of non forking, which we have not proved. Hence we shall use another variant of the construction

Construction 2.2. We modify the construction of 2.1 to suit our purposes.

Modified Data 2.2.1. Let $\langle M_i \in K_{\le\mu} : i \le \kappa+1\rangle$ be a continuous $\preceq_{\mathrm{nice}}$-chain of models of $\mathbf{T}$, $\|M_{\kappa+1}\| = \mu < \lambda$. Let T be a subset of ${}^{\kappa+1\ge}(Ord)$, $<_{lex}$ be the lexicographic order on T, this is a linear order of T; suppose that T is $\triangleleft$-closed i.e. $(\nu \triangleleft \eta \in T \Rightarrow \nu \in T)$, and if $\eta \in {}^\kappa(Ord) \cap T$, then $\eta^\wedge\langle 0\rangle$ is the unique $<_{lex}$-successor of η in T. For $S \subseteq T$ let $S^{se} = \{\eta \in S : \ell g(\eta) \text{ successor}\}$. Let Op_{i+1} witness $M_i \preceq_{\mathrm{nice}} M_{i+1}$.

We define $\mathrm{Op}_\eta = \mathrm{Op}_{\ell g(\eta)}$ for $\eta \in T^{se}$. We can iterate the operation Op_η with respect to $(T^{se}, <_{lex})$. Also, for each $S \subseteq T$, we can iterate Op_η with respect to $(S^{se}, <_{lex})$. Let us denote the result of this iteration with respect to $(S, <_{lex})$ by Op^S (see [**?**, 1.11]). Note that for any $M \in K$, if $S_1 \subseteq S_2 \subseteq T$, then $M \preceq_{\mathcal{F}} \mathrm{Op}^{S_1}(M) \preceq_{\mathcal{F}} \mathrm{Op}^{S_2}(M) \preceq_{\mathcal{F}} \mathrm{Op}^T(M)$ (by natural embeddings). More formally:

Claim 2.2.2. *There exist operations* Op^S *for* $S \subseteq T$ *such that*

(1) *for every* $S \subseteq T$ *which is* $\triangleleft$*-closed* $M_S = \mathrm{Op}^S(M)$ *is defined, and whenever* $S_1 \subseteq S_2 \subseteq T$*, then* $M_{S_1} \preceq_{\mathcal{F}} M_{S_2}$*; let* $M_\eta = M_{\{\eta\restriction\alpha : \alpha \le lg(\eta)\}}$.

(2) *for* $\eta \in T$*,* h_η *is a surjective* $\prec_{\mathcal{F}}$*-elementary embedding from* $M_{\ell g(\eta)}$ *to* M_η*,* $M_\eta \preceq_{\mathcal{F}} M_{\{\eta\}}$*, and* $\langle h_\eta : \eta \in T\rangle$ *is a* $\triangleleft$*–increasing sequence, i.e.,* $h_\eta \subseteq h_\nu$ *whenever* $\eta \triangleleft \nu$*;*

(3) *for every* $x \in M_T$*, there exists a* $\triangleleft$*-closed* $S \subseteq T$*,* $|S| \le \kappa$ *such that* $x \in M_S$ *(in fact* S *is the union of finitely many branches);*

(4) *for* $\eta \in T$, *letting* $T[\eta] = \{\nu \in T : \neg(\eta \lhd \nu)\}$, $T^{\leq}[\eta] = \{\nu \in T[\eta] : \nu \leq_{lex} \eta\}$, $T^{\geq}[\eta] = \{\nu \in T[\eta] : \eta \leq_{lex} \nu\}$ *(so* $T[\eta] = T^{\leq}[\eta] \cup T^{\geq}[\eta]$*) and* $\alpha < \ell g(\eta)$ *we have* $M_{T^{\leq}[\eta\restriction\alpha]} \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M_\eta$ *so we can replace* M_T *by* $M_{T^{\leq}[\eta]}$ *and* $M_{T^{\leq}[\eta]} \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M_{T^{\geq}[\eta\restriction\alpha]}$ *for* $\alpha < \kappa$;

(5) *if* $\eta \in \lim_\kappa(T)$ *and* $\eta \notin T$, *then* $M_T = \bigcup_{\alpha<\kappa} M_{T[\eta\restriction\alpha]}$.

(6) $\|M_S\| \leq |S| + \|M_{\kappa+1}\|^{\kappa^*} + \sup_{\eta \in S} \|M_{\ell g \eta}\|$.

(7) *for* $\eta \in T \cup \lim_\kappa(T)$, $\langle M_{T[\eta\restriction\alpha]} : \alpha \leq \ell g(\eta)\rangle$ *is* $\preceq_{\mathcal{F}}$*-increasing continuous. Note:* $\langle T_{[\eta\restriction\alpha]} : \alpha \leq \ell g(\eta)\rangle$ *is increasing but generally not continuous however* $\langle T^{se}_{[\eta\restriction\alpha]} : \alpha \leq \ell g(\eta)\rangle$ *is.*

Fact 2.2.3. (1) *By clause (4), if we have the conclusion of 1.7 for models of cardinality* $\leq \mu$ *(and 1.21(1)) then*

(∗) *if* $\|M_{\eta\restriction\alpha}\| \leq \mu$, $M_{\eta\restriction\alpha} \prec_{\mathcal{F}} M' \prec_{\mathcal{F}} M_\eta$, $\|M'\| \leq \mu$, $M_{\eta\restriction\alpha} \prec_{\mathcal{F}} M'' \prec_{\mathcal{F}} M_{T[\eta\restriction\alpha]}$ *and* $\|M''\| \leq \mu$, *then* $M_\eta \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M_{T[\eta\restriction\alpha]}$ *and hence* $M' \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M''$.

(2) *Then in fact one can replace clause (4) above by the weaker condition*

$(4)^-$ $\mu \geq \kappa$ *and for every* $S \subseteq T$ *closed under initial segments, if* $|S| \leq \mu$ *and* $\{\eta \restriction i : i \leq \alpha\} \subseteq S \subseteq T$, *then* $M_\eta \underset{M_{\eta\restriction\alpha}}{\overset{M_T}{\Cup}} M_S$.

(2) by (4).

Short Proof of 2.2.2. As $\langle M_i : i \leq \kappa+1\rangle$ is $\preceq_{\text{nice}}$-increasing continuous by renaming there is $\langle M^*_i : i \leq \kappa+1\rangle$ $\preceq_{\text{nice}}$-increasing continuous, $M^*_0 = M_0$, $M^*_{i+1} = \text{Op}_{i+1}(M^*_i)$, $M_i \preceq_{\mathcal{F}} M^*_i$ and $M^*_i \underset{M_i}{\overset{M^*_{i+1}}{\Cup}} M_{i+1}$ (for $i \leq \kappa$). W.l.o.g. $\|M^*_i\| \leq \|M_i\|^{\kappa^*}$. Let (I_η, D_η, G_η) be a copy of Op_η for $\eta \in T^{se}$ with I_η's pairwise disjoint. Define $I = \Pi\{I_\eta : \eta \in T^{se}\}$, D, G as in the proof of [**?**, 1.11], so every equivalence relation $e \in G$ has a finite subset $w[e] = \{\eta^\ell_0 <_{lex} \ldots <_{lex} \eta^\ell_{n(\ell)-1}\} \subseteq T^{se}$ and $\mathfrak{e}_\ell[e] \in G_{\eta^\ell_e}$ as there. We let $\text{Op}_{T^{se}} = (I, D, G)$, $M_{T^{se}} = \text{Op}_{T^{se}}(M_0)$ and for $S \subseteq T^{se}$ we let

$$M_S = \{x \in M_T : w[eq(x)] \subseteq S\}.$$

Naturally there are canonical maps f^*_η from $M^*_{\ell g\eta}$ onto $M_{\{\nu:\nu\lhd\eta\}}$ and let $M_\eta = f''_\eta(M_{\ell g(\eta)})$. □

Improvement in cardinality 2.2.1.

We can replace $\|M_{\kappa+1}\|^{\kappa^*}$ by $\|M_{\kappa+1}\| + LS(\mathbf{T})$ in part (6) of claim 2.2.2. After choosing $\langle M^*_i : i \leq \kappa+1\rangle$, let M^+_0 be a Skolemization of $M_0 = M^*_0$, $M^*_{i+1} = \text{Op}(M^+_i)$, $M^+_\delta = \bigcup_{i<\delta} M^+_i$. Of course M^T_S ($S \subseteq T$ is $\lhd$-closed) are well defined similarly. Let N_i be the Skolem hull of M_i in M^*_i. For $\eta \in T$ let $N_\eta = f^*_\eta(N_{\ell g\eta})$. Now for any $\lhd$-closed $S \subseteq T$ let

$$N_S = \text{Skolem hull in } M^+_S \text{ of } \cup\{N_\eta : \eta \in S\}.$$

* * *

There are two different ways to carry on the construction (under Data 2.2.1). We will consider each in its turn.

Construction 2.3. Recall that it is possible to iterate the operation Op with respect to the linear order $(T, <_{lex})$ and this iteration can be defined as the direct limit of finite approximations. We shall use different approximations and take the direct limit we obtain the required operation.

Suppose that $w \subseteq T$ is closed with respect to $\triangleleft$, (i.e., initial segment) and is $<_{lex}$-well-ordered. For each approximation w of this kind, the iterated ultrapower $\mathrm{Op}^w(M_0)$ of M_0 with respect to w is defined as a limit ultrapower and there are natural elementary embeddings into this limit. The principal difference is that this limit is a little larger than a limit obtained using only finite approximations. For example, if $\langle \eta_n : n \leq \omega \rangle$ is a $<_{lex}$-increasing sequence, then in $\mathrm{Op}^{\eta_\omega}\left(\dots \mathrm{Op}^{\eta_n}\left(\dots \left(\mathrm{Op}^{\eta_0}(M_0) \right)\right)\right)$, the last operation $\mathrm{Op}^{\eta_\omega}$ adds elements which are dispersed over all $\mathrm{Op}^{\eta_n}\left(\dots \mathrm{Op}^{\eta_0}(M_0)\right)$. (This is of more interest when the sequence has length κ.) Now it is easy to check the symmetry (for $\eta \in {}^\alpha\lambda$, $\alpha < \kappa$) between the $<_{lex}$-successors and $<_{lex}$-predecessors of η.

We define the embeddings h_η for $\eta \in T$ as follows. For $\eta = \langle\, \rangle$, $h_\eta = \mathrm{id} \restriction M_0$. If $\eta = \nu^\wedge\langle i \rangle$, then Op^η acts on $M_\nu = h_\nu[M_{\ell g(\nu)}]$ and we use the commuting diagram:

$$\begin{tikzcd}
\mathrm{Op}^\eta(M_{\ell g(\nu)}) \arrow[r, "\text{canonical}"'] & \mathrm{Op}^\eta(M_\nu) \\
M_{\ell g(\eta)} \arrow[u] \arrow[r, "\text{canonical}"'] & M_\eta \arrow[u, "\text{canonical}"'] \\
M_{\ell g(\nu)} \arrow[u] \arrow[r, "h_\nu"'] & M_\nu \arrow[u]
\end{tikzcd}$$

This completes the construction.

Construction 2.4. In this approach, we employ the generalized Ehrenfeucht-Mostowski models $EM(I, \Phi)$ from chapter VII in [?] or [?]. For this we need to specify the generators of the model and what the types are.

Let M_0^+ be the model obtained from M_0 by adding Skolem functions and individual constants for each element of M_0. We know that there is an operation Op such that, for $i \leq \kappa$, $M_i \preceq_{\mathcal{F}} M_{i+1} \preceq_{\mathcal{F}} \mathrm{Op}(M_i)$. As in [?, 1.7.4] this means that there are I, D and G such that $\mathrm{Op}(M) = \mathrm{Op}(M, I, D, G)$ where I is a non-empty set, D is an ultrafilter on I, and G is a suitable set of equivalence relations on I, i.e.,

(i) if $e \in G$ and e' is an equivalence relation on I coarser than e, then $e' \in G$;
(ii) G is closed under finite intersections;
(iii) if $e \in G$, then $D/e = \{A \subset I/e : \bigcup_{x \in A} x \in D\}$ is a κ^*-complete ultrafilter on I/e.

For each $b \in M_{i+1} \setminus M_i$, let $\langle x_t^b : t \in I\rangle/D$ be the image of b in $\mathrm{Op}(M_i)$. We'll also write $\langle x_t^b : t \in I\rangle/D$ for the canonical image $d(b)$ of $b \in M_i$ in $\mathrm{Op}(M_i)$.

$$\begin{tikzcd}
M_{i+1} \ni b \arrow[r, mapsto] & \langle x_t^b : t \in I\rangle/D \in \mathrm{Op}(M_i) \\
& M_i \arrow[ul] \arrow[u]
\end{tikzcd}$$

We define a model M^+, $M_0^+ \preceq_{L_{\kappa^*,\omega}} M^+$, as follows. M^+ is generated by the set $\{x_\eta^b : b \in M_{i+1} \setminus M_i,\ \eta \in T, \ell g(\eta) = i+1\}$. Note that this set does generate a model since M_0^+ is closed under Skolem functions. Since functions have finite arity, it is enough to specify, for each finite set of the x_η^b, what quantifier-free type it realizes. Since there is monotonicity, we shall obtain indiscernibility as in [?]. The type of a finite set $\langle x_{\eta_\ell}^{b_\ell} : \ell = 1, \dots, n\rangle$ depends on the set $\langle b_1, \dots b_n\rangle$ and the atomic (i.e., quantifier-free) type of $\langle \eta_1, \dots, \eta_n\rangle$ in the model $\langle T, \triangleleft, <_{lex}, \text{"}\eta \restriction i = \nu \restriction i\text{"}\rangle$. Now w.l.o.g. we can allow finite sequence $\bar{b}$ instead of b for $\bar{b} \in M_{i+1} \setminus M_i$ and thus w.l.o.g. $\eta_1, \dots, \eta_n$ is repetition-free, so w.l.o.g. $\eta_1 <_{lex} \eta_2 <_{lex} \dots <_{lex} \eta_n$. Suppose that the lexicographic order $<_{lex}$ on $\{\eta_\ell \restriction \alpha : \alpha \leq \ell g(\eta_\ell) \text{ and } \ell = 1, \dots, n\}$ is a well-order and the sequence $\langle \nu_\zeta : \zeta < \zeta(*)\rangle$ is $\triangleleft$-increasing. We define $N_0 = M_0^+$, $N_{\zeta+1} = \mathrm{Op}(N_\zeta)$, $N_\zeta = \bigcup_{\xi < \zeta} N_\xi$ (for limit ζ). Next, we define $h_{\nu_\zeta} : M_{\ell g(\nu_\zeta)} \underset{\mathcal{F}}{\to} N_{\zeta+1}$, $h_{\nu_\zeta \restriction \beta} \subseteq h_{\nu_\zeta}$. If $\ell g(\nu)$ is a limit ordinal, then $\alpha < \ell g(\nu) \Rightarrow h_{\nu \restriction \alpha}$ is defined and we let $h_\nu = \bigcup_{\alpha < \ell g(\nu)} h_{\nu \restriction \alpha}$. If $\nu_\zeta = \nu_\xi {}^\frown \langle \gamma\rangle$, $i = \langle u_\xi\rangle$, then $M_{\zeta+1} = \mathrm{Op}(M_\zeta, I, D, G)$, identifying elements of M_ζ with their images in the ultrapower. Now define

$$h_{\nu_\zeta}(b) = \begin{cases} d(H_{\nu_\zeta}(b)) & \text{if } b \in M_i, \\ \langle h_{\nu_\zeta}(x^b_t) : t \in I\rangle / D & \text{if } b \in M_{i+1} \setminus M_i, \end{cases}$$

where $d(h_{\nu_\xi}(b))$ is the canonical image of $H_{\nu_\xi}(b)$ in the ultrapower. The type of $\langle x^{b_\ell}_{\eta_\ell} : \ell = 1, \dots, n\rangle$ is defined to be the type of $\langle h_{\eta_\ell}(b_\ell) : \ell = 1, \dots, n\rangle$ in N_ξ.

Remark 2.4.1. It is possible to split the construction into two steps. For $i \le j \le \kappa + 1$, there is an operation $\mathrm{Op}^{i,j}$, $M_i \preceq M_j \preceq = \mathrm{Op}^{i,j}(M_i)$, moving b to $\langle\, {}^{i,j}a^b_t : t \in I\rangle$, $b \in M_j$, ${}^{i,j}a^b_t \in M_i$, with the obvious commutativity and continuity properties. Now the construction is done on a finite tree $\langle \eta_\ell : \ell = 1, \dots, n\rangle$, $\langle \eta_\ell \cap \eta_m : \ell, m < \omega\rangle$. We omit the details of monotonicity.

Notation 2.4.2. Let $M_T = M$ be the Skolem closure. If $S \subseteq T$ is closed with respect to initial segments, let $M_S = Sk_{M_T}(x^b_\eta : \eta \in S, b \in M_{\ell g(\eta)})$ and $M^*_\eta = M_{\{\eta \restriction \alpha : \alpha \le \ell g(\eta)\}}$. Define $h_\eta : M_{\ell g(\eta)} \to M^*_\eta$ by $h_\eta(b) = x^b_{\eta \restriction \tau(\mathbf{T})}$ and $N_\eta = h_\eta[M_\eta]$.

Remark 2.4.3. The construction can be used to get many fairly saturated models. We list the principal properties below.

Fact 2.4.4. *Suppose that $S_\ell \subseteq T$ is closed with respect to initial segments, $S_0 = S_1 \cap S_2$ and*

$$\eta \in S_1 \ \&\ \nu \in S_2 \setminus S_1 \quad \Rightarrow \quad \eta <_{lex} \nu$$

then

$$\begin{array}{ccc} & M_T & \\ M_{S_1} & \bigcup & M_{S_2}. \\ & M_{S_0} & \end{array}$$

Proof. W.l.o.g. S_ℓ is closed, $M_{cl(S_\ell)} = M_{S_\ell}$. Let $S_2 \setminus S_0 = \{\nu_\zeta : \zeta < \zeta(*)\}$ be a list such that $\nu_\zeta < \zeta_\xi \ \Rightarrow\ \zeta < \xi$; let $S^\xi_2 = S_0 \cup \{\nu_\zeta : \zeta < \zeta(*)\}$. Then

(1) $\langle M_{S^\xi_2} : \xi \le \xi(*)\rangle$ is continuous increasing;
(2) $\langle M_{S^\xi_2 \cap S_1} : \xi \le \xi(*)\rangle$ is continuous increasing.
Hence one has
(3) $$\begin{array}{ccc} & M_{S^{\xi+1}_2 \cup S_1} & \\ M_{S^\xi_2 \cup S_1} & \bigcup & M_{S^{\xi+1}_2} \\ & M_{S^\xi_2} & \end{array}$$

This is immediate from the definitions, because $M_{S^{\xi+1}_2 \cup S_1}$ is the Skolem closure of $M_{S^2_\xi \cup S_1} \cup N_{\nu_\xi}$, and so elements of N_{ν_ξ} can be represented as averages. $\square$

3. Categoricity in μ, when $LS(\mathbf{T}) \le \mu < \lambda$

Hypothesis 3.1. Every $M \in K_{<\lambda}$ is nice hence has a $\prec_{\mathcal{F}}$-extension of cardinality λ which is saturated and $\mathcal{K}_{<\lambda}$ has amalgamation.

This section contains the principal theorems of the paper: if $\mathbf{T}$ is λ-categorical, $LS(\mathbf{T}) \le \mu < \lambda$, then $\boldsymbol{\kappa}_\mu(\mathbf{T}) = \emptyset$ when $\mu \in [LS(\mathbf{T}), \lambda)$ and when $LS(\mathbf{T}) \le \chi = \mathrm{cf}(\chi) < \lambda$, $\mathbf{T}$ is χ–based, (and $\mathcal{K}$ does not have (μ, κ)–continuous non forking when $\mu \in [LS(\mathbf{T}), \lambda)$, $\kappa \le \mu$) also there is a saturated model in $\mathcal{K}_\mu = \langle K_\mu, \preceq_{\mathcal{F}}\rangle$ and $\mathbf{T}$ is categorical in every large enough $\mu < \lambda$. However we first deal with some preliminary results, quoting [**?**] for "black boxes" saying during the combinatorial work for "there are many non isomorphic models" extensively.

Theorem 3.2. *Assume the conclusion of 1.7 for every $\mu \leq \mu^*$ (e.g., $\mu^+ < \lambda$) and $\kappa \leq \mu^+$. Suppose that the tree T is as in Claim 2.2.2 and suppose further: $\langle M_i \in K_{\leq\mu} : i \leq \kappa + 1\rangle$ is $\preceq_{\rm nice}$-increasing continuous sequence of members of $K_{\leq\mu}$, such that $\|M_{\kappa+1}\| = \|M_\kappa\|$ and we apply §2 and*

- $(*)_1$ *there is no $\preceq_{\mathcal{F}}$-increasing continuous sequence $\langle N_i \in K_{\leq\mu} : i \leq \kappa\rangle$ such that:*
  - (i) $M_i \preceq_{\mathcal{F}} N_i$,
  - (ii) $M_{\kappa+1} \preceq_{\mathcal{F}} N_\kappa$,
  - (iii) *if $i < j \leq \kappa$ and $\|N_j\| < \mu^*$, then* $N_i \underset{M_i}{\overset{N_j}{\bigcup\!\!\!\!\bigcup}} M_j$.

Then *TFAE for $\eta \in \lim_\kappa(T) \overset{\rm def}{=} \{\eta \in {}^\kappa(Ord) : \bigwedge_{i<\kappa} (\eta \restriction (i+1) \in T)\}$:*

- (α) *There is an $\mathcal{F}$-elementary embedding h from $M_{\kappa+1}$ into M_T such that*
$$\bigcup_{i<\kappa} h_{\eta\restriction i+1} \subseteq h.$$
- (β) *$\eta^\frown\langle 0\rangle \in T$ (equivalently, $\eta \in T$, see 2.2.1).*

Proof. As regards the implication from (β) to (α), so assume $\eta \in T$ and consider the $\mathcal{F}$-elementary embedding $h_{\eta^\wedge\langle 0\rangle}$. Check that $h_{\eta^\wedge\langle 0\rangle}$ is as required in (α). The other direction follows by 2.2.3(1) and $(*)$. That is we are assuming that h exemplify clause (α) but $\eta^\wedge\langle 0\rangle \notin T$, equivalently $\eta \notin T$ and we shall get a contradiction. We let $\eta_\alpha = \eta \restriction \alpha$ for $\alpha \leq \kappa$, and let $T_\alpha = T[\eta]$ hence $\langle M_{T_\alpha} : \alpha \leq \kappa\rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous (see 2.2.2(7)). We can choose by induction on $\alpha \leq \kappa$, a model $N_\alpha \preceq_{\mathcal{F}} M_{T_\alpha}$, $\|N_\alpha\| < \mu^*$, (even $\|N_\alpha\| \leq \|M_\alpha\| + LS(T)$), $M_{\eta\restriction\alpha} \subseteq N_\alpha$ and $N = \bigcup_{\alpha<\kappa} N_\alpha$ include $f(M_{\kappa+1})$. By 2.2.3 we get $N_i \underset{M_i}{\overset{N_j}{\bigcup\!\!\!\!\bigcup}} M_j$ if $i < j \leq \kappa$, $\|M_j\| < \mu^*$, so we have contradict (*). $\square$

Proposition 3.3. *Suppose the conclusion of 1.7 for μ, and $\kappa \leq \mu^+$ and an $\preceq_{\mathcal{F}}$-increasing sequence $\bar{M} = \langle M_i : i \leq \kappa + 1\rangle$ is given with $M_i \in K_{\leq\mu}$ when $i < \mu$, $i \leq \kappa + 1$. Then $\bar{M}$ satisfies $(*)$ of 3.2 if one of the following holds:*

- (α) *there is $a \in M_{\kappa+1}$ such that $i < \kappa \Rightarrow M_\kappa \underset{M_{i+1}}{\overset{M_{\kappa+1}}{\biguplus}} a$, or*
- (β) *$\kappa = {\rm cf}(\kappa) = \mu > LS(\mathbf{T})$ and $\kappa < \lambda$ and $i < \kappa \Rightarrow \|M_i\| < \kappa$, and there is a continuous $\prec_{\mathcal{F}}$-chain $\langle N_i : i \leq \kappa\rangle$, $M_{\kappa+1} = \bigcup_{i\leq\kappa} N_i$, $\kappa = \chi^{{\rm cf}(\kappa)}$,*
$$\bigwedge_{i<\kappa} (N_i \in K_{<\kappa}), \text{ and } E = \{i < \kappa : M_{i+1} \underset{M_i}{\overset{N_\kappa}{\biguplus}} N_i\} \text{ is a stationary subset of } \kappa.$$

Proof. Straight from 3.2, and the monotonicity of $\bigcup\!\!\!\!\bigcup$, that is 1.21(3). $\square$

Remark 3.4. Clause (β) can also be proved using niceness as in the proof of 3.8. This works for any $\kappa < \lambda$. Also we can imitate 2.2.2 but no need arises.

Corollary 3.5. *If $\mathbf{T}$ is a λ-categorical theory*[1], then

[1] or just has $< 2^\lambda$ non isomorphic models in λ

(1) $\mathbf{T}$ *is* χ*-based if* $\chi^+ < \lambda$ *and* $\chi \geq LS(\mathbf{T})$*; also it is* $(< \mu)$*-based if* $\mu = \mathrm{cf}(\mu)$, $LS(T) < \mu$, $\mu < \lambda$*;*
(2) $\boldsymbol{\kappa}_\mu(\mathbf{T}) = \emptyset$ *for every* μ *such that* $\mu^+ < \lambda$ *and* $\mu \geq LS(\mathbf{T})$.

Proof. (1), (2) We use 3.2, 3.3 to contradict λ-categoricity. In the first phrase of (1) let $\mu = \chi$, $\kappa = \chi^+$, in the second let us repeat the proofs (i.e., prove the appropriate variants of 3.2, 3.3 be regular; so κ =cf(κ) and $\kappa^+ < \lambda$.

Case 1: $\lambda^\mu = \lambda$.
By [**?**, III, 5.1] = [**?**, IV, 2.1].

Case 2: λ is regular.
We can find a stationary $W^* \in I[\lambda]$, $W^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ (by [**?**, §1]). Hence, possibly replacing W^* by its intersection with some club of λ, there is W^+, $W^* \subseteq W^+$ and $\langle a_\alpha : \alpha \in W^+\rangle$ such that: $\alpha \in a_\beta$ (so $\beta \in W^+$) implies $\alpha \in W^+$, $a_\alpha = a_\beta \cap a_\alpha$ and $\mathrm{otp}(a_\alpha) \leq \kappa$ and

$$\alpha = \sup a_\alpha \iff \mathrm{cf}(\alpha) = \kappa \iff \alpha \in W^*.$$

Now let η_α enumerate a_α in increasing order (for $\alpha \in W^+$), and for any $W \subseteq W^*$ let

$$T_W = \{\eta_\alpha : \alpha \in W^+ \text{ but } \alpha \notin W^* \setminus W\} \cup \{\eta_\alpha{}^\frown\langle 0\rangle : \alpha \in W\}.$$

Now if W_1, $W_2 \subseteq W$, $W_1 \setminus W_2$ is stationary, then $M_{T_{W_1}}$ cannot be $\preceq_{\mathcal{F}}$-embedded into $M_{T_{W_2}}$(again by [**?**, III, §5] = [**?**, IV §2]).

Case 3: λ singular.
Choose λ', $\lambda > \lambda' = \mathrm{cf}(\lambda') > \mu^+$ and act as in case 2 using λ' instead λ except adding to T_W the set $\{\langle i\rangle : i < \lambda\}$ (to get 2^λ we need more, see in [**?**, IV,VI] on pairwise non isomorphic models). □

Hypothesis 3.6. The conclusion of 3.5 (in addition to 3.1 of course).

Conclusion 3.7. Suppose $\mu \geq LS(\mathbf{T})$, $\mu^+ < \lambda$, $M \in K_\mu$

(1) If $p \in S(M)$ then p is determined by $\{p \restriction N : N \preceq_{\mathcal{F}} M \text{ and } \|N\| = LS(\mathbf{T})\}$
(2) Assume further

$(*)^M_{\{N_t:t\in I\}}$ (a) I is a directed partial order,
(b) $N_t \preceq_{\mathcal{F}} M$,
(c) $I \models t \leq s$ implies $N_t \subseteq N_s$ (hence $N_t \preceq_{\mathcal{F}} N_s$ by clause (b)),
(d) $\bigcup_{t\in I} N_t = M$.

Then

(α) every $p \in S(M)$ is determined by $\{p \restriction N_t : t \in I\}$ which mean just that if $q \in S(M)$ and for every $t \in I$ we have $p \restriction N_t = q \restriction N_t$ then $p = q$,
(β) for some $t \in I$, p does not fork over N_t, $\{p \restriction N_t : t \in I\}$.

Proof. (1) Follows from part (2): We can find $\bar N = \langle N_t : t \in I\rangle$ such that $(*)^M_{\{N_t:t\in I\}}$ holds, $\|N_t\| \leq LS(\mathbf{T})$ and on it use part (2). Why $\bar N$ exists? E.g., as the proof of part (2) which $I = \{\emptyset\}$, $N_\emptyset = M$ and use $\langle N^*_u : u \in I^*\rangle$ for $I^* = ([M]^{<\aleph_0}, \subseteq)$. Now apply part (2).

(2) Easily (and as [**?**, §1]):

($\otimes$) we can choose by induction on $n < \omega$ for every $u \in [M]^n$, $t[u] \in I$ and N^*_u such that: $u \subseteq N^*_u$, $N^*_u \preceq_{\mathcal{F}} N_{t[u]}$, $\|N^*_u\| \leq LS(\mathbf{T})$ and

$$u \subseteq v \in [|M|]^{<\aleph_0} \quad \text{implies} \quad N^*_u \prec N^*_v \text{ and } t[u] \leq_I t[v].$$

For $U \subseteq |M|$ let $N^*_U =: \bigcup\{N^*_u : u \subseteq U \text{ is finite}\}$ (the definitions are compatible). Easily $U_1 \subseteq U_2 \subseteq |M| \Rightarrow N^*_{U_1} \preceq_{\mathcal{F}} N^*_{U_2} \preceq_{\mathcal{F}} M$. Now we prove by induction on $\mu \leq \|M\|$ that:

($**$) if $U \subseteq \|M\|$, $|U| = \mu$, $p \in S(N^*_U)$ then for some $u \in [U]^{<\aleph_0}$, p does not fork over N^*_u.

This is enough for clause (β), as by monotonicity p also does not fork over $N_{t[u]}$. For μ finite this is trivial, for μ infinite then $\mathrm{cf}(\mu) \notin \boldsymbol{\kappa}_{\mu+LS(\mathbf{T})}(\mathbf{T})$ (by 3.5(2)) so ($**$) holds. So we have proved clause (β) and clause (α) follows by 1.24(3), and we are done. $\square$

Theorem 3.8. *Suppose that* $\mathrm{cf}(\kappa) = \kappa \leq \mu < \lambda$ *and* $LS(\mathbf{T}) < \mu$. *Then*

(1) *The* (μ, κ)*-saturated model* M *is saturated (i.e.,* $N \preceq_{\mathcal{F}} M$, $\|N\| < \|M\|$, $p \in S(N) \Rightarrow p$ *realized in* M, *and hence unique). Hence there is a saturated model in* K_μ.

(2) *The union of a continuous* $\preceq_{\mathcal{F}}$*-chain of length* κ *of saturated models from* K_μ *is saturated.*

(3) *In part (1) we can replace saturated by* (μ, μ)*-saturated if* $\mu = LS(\mathbf{T})$.
We can in part (2) replace saturated by μ*-saturated if* $\mu > LS(\mathbf{T})$.

Proof. (1), (2) Suppose that $M = M_\kappa$ and $\langle M_i : i \leq \kappa\rangle$ is a continuous $\preceq_{\mathcal{F}}$-chain of members of K_μ such that for the proof of (1) M_{i+1}, is a universal extension of M_i and for the proof of (2) M_{i+1} is saturated. Let $i \leq j \leq \kappa$. Then $M_i \preceq_{\text{nice}} M_j$ (by [**?**, 5.4], or more exactly by the hypothesis 3.1). So there is an operation $\mathrm{Op}_{i,j}$ such that $M_i \preceq_{\mathcal{F}} M_j \preceq_{\mathcal{F}} \mathrm{Op}_{i,j}(M_i)$. It follows that there is an expansion $M^+_{i,j}$ of M_j by at most $LS(\mathbf{T})$ Skolem functions such that if N is a submodel of $M^+_{i,j}$, then

$$\begin{array}{ccc} & M_j & \\ M_i & \bigcup\!\!\!\!\!\bigcup & N \restriction M_j. \\ & (N \cap M_j \restriction M_i) & \end{array}$$

[Why? as we use operations coming from equivalence relations with $\leq \kappa^*$ classes and $LS(\mathbf{T}) \geq \kappa^*$ by its definition]. More fully, letting $\mathrm{Op}_{i,j}(N) = N^I_D/G$, every element $b \in M_j$ being in $\mathrm{Op}_{i,j}(M_i)$ has a representation as the equivalence class of $\langle x^b_t : t \in I\rangle/D$ under $\mathrm{Op}_{i,j}$, $x^b_t \in M_i$ and $|\{x^b_t : t \in I\}| \leq \kappa^*$. The functions of $M^+_{i,j}$ are the Skolem functions of M_j and M_i and functions F_ζ ($\zeta < \kappa^*$) such that $\{F_\zeta(b) : \zeta < \kappa^*\} \supseteq \{x^b_t : t \in I\}$.]

If $\kappa = \mu$, the theorem is immediate as κ is regular, $\mu > LS(\mathbf{T})$. So we will suppose that $\kappa < \mu$. Suppose $N \preceq M = M_\kappa$, $\|N\| < \mu$ and $p \in S(N)$. Let $\chi =: \|N\| + \kappa + LS(\mathbf{T})$ so $\kappa < \mu$ hence $\kappa^+ < \lambda$. W.l.o.g. there is no N_1, $N \preceq_{\mathcal{F}} N_1 \prec M_\kappa$, $\|N_1\| \leq \chi$ and p_1, $p \subseteq p_1 \in S(N_1)$ such that p_1 forks over N (by 3.3 but not used). If there is $i < \kappa$ such that $N \subseteq M_i$, then p is realized in M_{i+1}. By the choice of the models $M^+_{i,j}$, it is easy to find N' such that $N \preceq N' \preceq M_\kappa$,

$\|N'\| = \chi \stackrel{\text{def}}{=} \|N\| + \kappa + LS(\mathbf{T})$ and, for every $i \leq \kappa$,

$$M_i \underset{M_i \cap N'}{\overset{M_\kappa}{\bigcup}} N'.$$

Now let $N_i = N' \cap M_i$ and note that $N_\kappa = N'$. The sequence $\langle N_i : i \leq \kappa \rangle$ is continuous increasing and there is an extension p' of p in $S(N_\kappa) = S(N')$. Hence there exists $i < \kappa$ such that $(i \leq j < \kappa) \Rightarrow$ (p' does not fork over N_j). If we are proving part (2), then M_{i+1} is saturated but $\|M_i\| = \mu > \kappa = \|N_{i+1}\|$ and hence there is $a \in M_{i+1}$ realizing $p' \restriction N_{i+1}$. But by the non forking relation above we get $tp(a, N', M_\kappa)$ does not fork over N_{i+1}, hence is p', as required. If we are proving part (1), M_{i+1} is universal over M_i hence we can find a saturated model $N^* \preceq_{\mathcal{F}} M_{i+1}$ which contains $M_i \cap N'$. Hence we can find $\langle N^*_\varepsilon : \varepsilon < \chi^+ \rangle$ which is $\preceq_{\mathcal{F}}$-increasing continuous such that: $N_i \preceq_{\mathcal{F}} N^*_\varepsilon \preceq_{\mathcal{F}} M_{i+1}$, $N^*_{\varepsilon+1}$ is a χ-universal extension of N^*_ε and $N^*_0 = M_i \cap N'$, and let $a_\varepsilon \in N^*_\varepsilon$ be such that $\mathrm{tp}(a_\varepsilon, N^*_\varepsilon, N^*_{\varepsilon+1})$ does not fork over $M_i \cap N'$ and extend $p' \restriction (M_i \cap N')$. By 3.5(1), for some ε there is N'_ε, $N' \cup N^*_\varepsilon \subseteq N'_\varepsilon$ and $N^*_\varepsilon \underset{N' \cap M_i}{\overset{N'_\varepsilon}{\bigcup}} N'$, so a_ε realizes p'. (Recall symmetry and uniqueness of extensions).

(3) Similar proof for the second sentence, using 1.20 for the first sentence. $\square$

Remark: Using categoricity we can prove 3.8 also by 1.20(2) (and uniqueness).

Conclusion 3.9. Assume $LS(\mathbf{T}) \leq \kappa < \mu \in (LS(\mathbf{T}), \lambda)$, $M \in K_\mu$ is not κ^+-saturated; let $\langle N^*_u : u \in [|M|]^{<\aleph_0} \rangle$ and N^*_U (for $U \subseteq |M|$) be as in the proof of 3.7(2) (for $I = \{\emptyset\}$, $N_\emptyset = M$. *Then* there is $U \subseteq |M|$, $|U| \leq \kappa$, $p \in S(N^*_U)$, i.e., there are N^+, $N^*_U \preceq_{\mathcal{F}} N^+ \in K_\kappa$, and $a^+ \in N^+$ satisfying $(a^+, N^+)/E_{N^*_U} = p$ such that for no $a \in M$ do we have

$$u \in [U]^{<\aleph_0} \quad \Rightarrow \quad \mathrm{tp}(a, N^*_u, M) = \mathrm{tp}(a^+, N^*_u, N^+).$$

Equivalently: w.l.o.g. $N^+ \cap M = N^*_U$ and we can define N^+_u for $u \in [|N^+|]^{<\aleph_0}$, such that $\langle N^+_u : u \in [|N^+|]^{<\aleph_0} \rangle$ as in the proof of 3.7(2), and $u \in [U]^{<\aleph_0} \Rightarrow N^+_u = N^*_u$ and for no $u_0 \in [|M|]^{<\aleph_0}$, $v_0 \in [|N^+|]^{<\aleph_0}$, $a^+ \in N^*_{v_0}$, and $a \in N^*_{u_0}$ do we have

$$\bigwedge_{u \in [U]^{<\aleph_0}} \mathrm{tp}(a, N^*_u, N^*_{u \cup u_0}) = \mathrm{tp}(a^+, N^+_u, N^+_{u \cup v_0}).$$

Corollary 3.10. (1) *If* $\mathbf{T}$ *is* λ*-categorical and* $LS(\mathbf{T}) < \mu < \lambda$, $LS(\mathbf{T}) \leq \chi$, $\delta(*) = (2^{LS(\mathbf{T})})^+$ *and* $\beth_{\delta(*)}(\chi) \leq \mu$ then *every* $M \in K_\mu$ *is* χ^+*-saturated. In fact for some* $\delta < \delta(*)$ *we can replace* $\delta(*)$ *by* δ.

(2) *If* $\mu = \beth_{(2^\chi)^+ \times \delta}$, δ *a limit ordinal* then $\mathbf{T}$ *is* μ*-categorical.*

Proof. By 3.9 (and 1.17(2), that is 1.17(1)+ 1.14(1)) this problem is translated to an omitting type argument + cardinality of a predicate which holds (see [**?**, VIII §4], [**?**, VII §5] for a parallel result for first order logic, pseudo elementary classes, done independently in 1968 by G. Cudnovskii, J. Keisler and S. Shelah). See more on this in [**?**] and better [**?**]. The translated problem is: for $(\kappa, \lambda_1, \lambda_2)$ consider the question:

$Q(\kappa, \lambda_1, \lambda_2)$ for a vocabulary L^* of cardinality $\leq \kappa$ and set Γ of 1-types (or $< \omega$-types, does not matter), and unary predicate P, does the existence of an L-model

M_1 omitting every $p \in \Gamma$ satisfying $\|M_1\| = \lambda_1 > |P_1^M| \geq \kappa$ implies the existence of an L-model M_2 omitting every $p \in \Gamma$ and satisfying $\|M_2\| = \lambda_2 > |P_2^M| \geq \kappa$.

So by 3.9 we have $Q(LS(\mathbf{T}), \lambda_1, \lambda_2)$, $\mathbf{T}$ categorical in $\lambda = \lambda_1 > LS(\mathbf{T})$ and $\lambda_2 < \lambda_1$ implies $\mathbf{T}$ is categorical in λ_2 (the need for $\lambda_2 < \lambda_1$ is as only over models in $K_{<\lambda}$ we somewhat understand types). $\square$

Proposition 3.11. *[$\mathbf{T}$ categorical in λ]*

(1) *If $\langle M_i : i \leq \delta \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous, $M_i \in K_{<\lambda}$, $p \in S(M_\delta)$* then *for some $i < \delta$, p does not fork over M_i.*

(2) *If $N \in K_{<\lambda}$ and $p, q \in S(N)$ does not fork over M, $M \preceq_{\mathcal{F}} N \in K_{<\lambda}$* then *$p = q \iff p \restriction M = q \restriction M$. Moreover, if $M \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} N^+$, $a \in N^+$ then*

$$N \underset{M}{\overset{N^+}{\biguplus}} a \quad \Leftrightarrow \quad a \underset{M}{\overset{N^+}{\biguplus}} N.$$

(3) *If $M \preceq_{\mathcal{F}} N \in K_{<\lambda}$ and $p \in S(M)$* then *there is $q \in S(N)$ extending p not forking over M.*

(4) *If $M_0 \preceq_{\mathcal{F}} M_1 \preceq_{\mathcal{F}} M_2 \in K_{<\lambda}$, $p \in S(M_2)$, $p \restriction M_{\ell+1}$ does not fork over M_ℓ for $\ell = 0, 1$* then *p does not fork over M_0.*

(5) *If μ, $\delta < \lambda$, $M_i \in K_{\leq \mu}$ for $i < \delta$ is $\preceq_{\mathcal{F}}$-increasing continuous, $p_i \in S(M_i)$, $[j < i \Rightarrow p_j \subseteq p_i]$,* then *there is $p \in S(M_\delta)$ such that $i < \delta \Rightarrow p_i \subseteq p_\delta$.*

Proof. (1) Otherwise we can find N, $M_\delta \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} \mathrm{Op}(M_\delta)$, $N \in K_\lambda$ such that $N \preceq_{\mathcal{F}} N^* =: \bigcup\limits_{i<\delta} \mathrm{Op}(M_i)$. So N is saturated so let $a \in N$ realizes p; so for some i, $a \in \mathrm{Op}(M_i)$ and let $N_i' \preceq_{\mathcal{F}} Op(M_i)$ be such that $M_i \cup \{a\} \subseteq N_i'$ clearly $M_\delta \underset{M_i}{\overset{N^*}{\biguplus}} N_i'$. Hence $M_\delta \underset{M_i}{\overset{N^*}{\biguplus}} a$, and hence, by part (2), $\mathrm{tp}(a, M_\delta, N^*)$ does not fork over M_i, so it is $\neq p$.

(2) The first sentence follows from the second. If the second fails then we can contradict stability in $\|N\|$ (holds by 1.16(5)), by a proof just as in 1.6(2).

(3) We can find an operation Op, $\|\mathrm{Op}(M)\| \geq \lambda$, so in $\mathrm{Op}(M)$ some $\bar{a}$ realizes p so $q = \mathrm{tp}(\bar{a}, N, \mathrm{Op}(N))$ is as required.

(4) By part (3) there is $q \in S(M_2)$ such that $q \restriction M_0 = p$ and q does not fork over M_0. Now by 1.21(3) usually and part (2) of the present proposition in general the type $q \restriction M_1$ does not fork over M_0 hence by 1.24(3) $q \restriction M_1 = p \restriction M_1$, and hence by the same argument $q = p$.

(5) *Case 1:* $\mathrm{cf}(\delta) > \aleph_0$.
For every limit $\alpha < \delta$ for some $i < \alpha$ we have p_α does not fork over M_i. By Fodor's lemma, for some $i < \delta$ and stationary $S \subseteq \delta$ we have

$$j \in S \quad \Rightarrow \quad p_j \text{ does not fork over } M_i.$$

So the stationarization of p_i in $S(M_\delta)$ (which exists by 1.22 or use part (3)) is as required.
Case 2: $\mathrm{cf}(\delta) = \aleph_0$.
So w.l.o.g. $\delta = \omega$. Here chasing arrows (using amalgamation) suffices. $\square$

Lemma 3.12. *In $K_{<\lambda}$ we can define $\mathrm{rk}(\mathrm{tp}(a, M, N))$ with the right properties. I.e.,*

(A) *If $M \prec_{\mathcal{F}} N \in K_{<\lambda}$, $\bar{a} \subseteq N$, $M \in \bigcup_{\mu^+<\lambda} K_\mu$, $p = \mathrm{tp}(\bar{a}, M, N)$ then*

$\mathrm{rk}(p) \geq \alpha$ *iff* *for every $\beta < \alpha$ there are p', M' such that $M \prec_{\mathcal{F}} M' \in \bigcup_{\mu^+<\lambda} K_\mu$ $p' \in S(M'), p' \restriction M = p$ and $\mathrm{rk}(p') \geq \beta$ and p' forks over M.*

(B) *For every M, N, $\bar{a}$, p as above $\mathrm{rk}(p)$ is an ordinal.*

(C) *If $M_1 \prec_{\mathcal{F}} M_2 \in \bigcup_{\mu^+<\lambda} K_\mu$ and $p_2 \in S(M_2)$, then $\mathrm{rk}(p_2 \restriction M_1) \geq \mathrm{rk}(p_2)$ and equality holds iff p_2 does not fork over M_1 and then $p_2 \restriction M_1$ (and M_2) determines p_2.*

(D) *If $\langle M_i : i \leq \delta\rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous, $M_i \in \bigcup_{\mu^+<\lambda} K_\mu$ and $p_\delta \in S(M_\delta)$ then for some $i < \delta$ we have:*

$$j \in [i, \delta] \quad \Rightarrow \quad \mathrm{rk}(p_\delta) = \mathrm{rk}(p_\delta \restriction M_j).$$

Proof. Straightforward and used little; in fact by 3.11 we can use $K_{<\lambda}$ instead $\bigcup_{\mu^+<\lambda} K_\mu$. □

Lemma 3.13. *Assume $\mu \geq LS(\mathbf{T})$, $\mu^+ < \lambda$. If $M \in K_\mu$ is saturated (for $\mu = LS(\mathbf{T})$ means (μ, μ)-saturated), and $p \in S(M)$* then *there are N, a such that $N \in K_\mu$ is saturated, $a \in N$, $\mathrm{tp}(a, M, N) = p$ and N is isolated over $M \cup \{a\}$ (where we say that N is isolated over $M \cup \{a\}$ when $M \preceq_{\mathcal{F}} N$, $a \in N \in K_{<\lambda}$ and: if $N \preceq_{\mathcal{F}} N^+ \in K_{<\lambda}$ and $M \preceq_{\mathcal{F}} M^* \preceq_{\mathcal{F}} N^+$, and $\mathrm{tp}(a, M, N^+)$ does not fork over M then $M^* \underset{M}{\overset{N^+}{\bigcup}} N$).*

Remark As in [?, Ch.V] (or Makkai and Shelah [?, 4.22]) because we have 3.5(1) (by 3.6).

Proof. We can find $\langle M'_n : n < \omega\rangle$, $M'_n \in K_\mu$ is saturated, M_{n+1} is saturated over M'_n, hence by the definition $\bigcup_{n<\omega} M'_n$ is $(\mu, \aleph_0)$-saturated over M'_n hence is saturated so by 3.8 wlog it is M, so by 3.11(1), p does not fork over M'_n for some n, by remaining, p does not fork over M'_0; note also that by 3.8, M is saturated over M'_0. We try to choose by induction on $\alpha < \mu^+$, (M_α, N_α) such that

(a) $M_\alpha \in K_\mu$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(b) $N_\alpha \in K_\mu$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(c) M_α, N_α are saturated, $M_\alpha \preceq_{\mathcal{F}} N_\alpha$,
(d) $M_0 = M$, $a \in N_0$, $\mathrm{tp}(a, M_0, N_0)$ is p,
(e) if $\alpha = \beta + 1$, β successor, then $M_{\beta+1}$ is $(\lambda, \aleph_0)$–saturated over M_β,
(f) if $\alpha = \beta + 1$, β successor, then $N_{\beta+1}$ is $(\lambda, \aleph_0)$–saturated over N_β,
(g) $\mathrm{tp}(a, M_\alpha, N_\alpha)$ does not fork over M_0,
(h) $M_{\alpha+1} \underset{M_\alpha}{\overset{N_{\alpha+1}}{\uplus}} N_\alpha$ if α is a limit ordinal.

For $\alpha = 0$ just choose (M_0, N_0) to satisfy clauses (c) for $\alpha = 0$ and (d); and let, e.g., $(M_1, N_1) = (M_0, N_0)$. For $\alpha = \beta + 2$ just satisfy clause (e)+(f) (and $M_\alpha \preceq_{\mathcal{F}} N_\alpha$ in

K_μ), possible by 1.22 + 1.16(6). For α limit take unions (the result are saturated by the definition, and clause (g) holds by 3.5(2)). Lastly for $\alpha = \beta + 1$, β limit, if there are no such M_α, N_α then N_β is isolated over $M_\beta \cup \{a\}$.

Now both M_β and $M = M_0 = \bigcup\limits_{n<\omega} M'_n$ are saturated over M'_0, and hence there is an isomorphism f from M_β onto M which is the identity over M'_0. By uniqueness of non forking extensions, f maps $\mathrm{tp}(a, M_\beta, N_\beta)$ to p. Renaming we have f is the identity and letting $N = N_\beta$ we have gotten the desired conclusion. But if we succeed to carry out the induction we get a contradiction to 3.6; so we are done.

Note that for a limit ordinal β, the model M_β is $(\mu, \mathrm{cf}(\mu))$-saturated over M_γ for any $\gamma < \beta$ and N_β is $(\mu, \mathrm{cf}(\mu))$–saturated over N_γ for any $\gamma < \beta$. $\square$

Proposition 3.14. *If $M \preceq_{\mathcal{F}} N$ are in K_μ, $\mu \geq LS(\mathbf{T})$, $\mu^+ < \lambda$, and $a \in N \setminus M$, then we can find saturated $M', N' \in K_\mu$ such that $M \preceq_{\mathcal{F}} M' \preceq_{\mathcal{F}} N'$, $N \preceq_{\mathcal{F}} N'$, $\mathrm{tp}(a, M', N')$ does not fork over M'; and N' is isolated over $M' \cup \{a\}$ and M' is saturated over M, N' is saturated over N.*

Proof. Contained in the the proof of 3.13. $\square$

Proposition 3.15. *If $\mu \in [LS(\mathbf{T}), \lambda)$, $M \in K_\mu$ is saturated and $p \in S(M)$* then *for some saturated $N \in K_\mu$, $M \preceq_{\mathcal{F}} N$, $a \in N$ $\mathrm{tp}(\bar{a}, M, N)) = p$ and N is locally isolated over $M \cup \{a\}$ which means:*

($\boxtimes$) *$M \preceq_{\mathcal{F}} N \in K_{<\lambda}$, $a \in N$ and if $N \preceq_{\mathcal{F}} N^+ \in K_\lambda$, $M \preceq_{\mathcal{F}} M^* \preceq_{\mathcal{F}} N^+$, $M^* \in K_{<\lambda}$ and* $\mathrm{tp}(a, M^*, N^+)$ *does not fork over M ($\preceq_{\mathcal{F}} M^*$) and $A \subseteq M^*$ is finite, then $A \underset{M}{\overset{N^+}{\bigcup}} N$.*

Proof. Usually we can use 3.14. A problem arises only if $\mu^+ = \lambda$. We can find $\langle M'_i : i \leq \mu \rangle$ which is $\preceq_{\mathcal{F}}$-increasing continuous, $\|M'_i\| = |i| + LS(\mathbf{T})$, $M'_\mu = M$, M'_i is saturated, M'_{i+1} universal over M'_i and p does not fork over M_0.

Now choose by induction on $i \leq \mu$, (M_i, N_i, a) such that:

(a) $M_0 = M'_0$,
(b) $\|M_i\| = \|N_i\| = |i| + LS(\mathbf{T})$,
(c) for i non limit (M_i, N_i, a) is as in 3.13 (with $|i| + LS(\mathbf{T})$ instead of μ), that is, N_i is isolated over $M_i \cup \{a\}$,
(d) $\mathrm{tp}(a, M_0, N_0) = p \restriction M'_0$,
(e) $\langle M_i : i \leq \mu \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(f) $\langle N_i : i \leq \mu \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous,
(g) $\mathrm{tp}(a, M_{i+1}, N_{i+1})$ does not fork over M_i (hence is the stationarization of $\mathrm{tp}(a, M_0, N_0) = p \restriction M'_0$, that is does not fork over $M'_0 = M_0$),
(h) M_{i+1} is saturated over M_i and N_{i+1} is saturated over N_i,
(i) $M_i \preceq_{\mathcal{F}} N_i$.

There is no problem, so as M_μ is saturated and in K_μ, $M_0 = M'_0$ has cardinality $< \mu$ and uniqueness of nonforking extensions (3.11), w.l.o.g., $M_\mu = M$. For any candidates N^+, A,M^*, as in the definition of "N is locally isolated over $M \cup \{a\}$" assume toward contradiction that $N \underset{M}{\overset{N^+}{\uplus}} A$; as A is finite, by 3.11(1), for some $i < \mu$, the type $\mathrm{tp}(A, M, N^+)$ does not fork over M_i, and for some $j < \mu$ the type

$\mathrm{tp}(A, N, N^+)$ does not fork over N_j. W.l.o.g., $i = j$ is a successor ordinal and $\mathrm{tp}(A \cup \{a\}, M)$ does not fork over M_{i-1}. So as $N \underset{M}{\overset{N^+}{\uplus}} A$, necessarily $\mathrm{tp}(A, N_i, N^+)$ forks over M_i, hence (by clause (c) above), $a \underset{M_i}{\overset{N^+}{\uplus}} A$. But M and M_i are by the construction saturated over M_{i-1}, and hence there is an isomorphism f from M_i onto M which is the identity over M_{i-1}. So by using uniqueness of does not fork, it maps $\mathrm{tp}(A \cup \{a\}, M_{i-1}, N^+)$ to $\mathrm{tp}((A \cup \{a\}, M, N^+)$ and hence $a \underset{M}{\overset{N^+}{\uplus}} A$ (by 1.21(4)). Thus we get $a \underset{M}{\overset{M^*}{\uplus}} M^*$, contradiction to the choice of N^+, A, M^*.

Alternatively repeat the proof of 3.13 using 3.11(2)'s second sentence. $\square$

Theorem 3.16. *Assume λ is a successor cardinal, i.e., $\lambda = \lambda_0^+$. Then $\mathbf{T}$ is categorical in every $\mu \in [\beth_{(2^{LS(\mathbf{T})})^+}, \lambda)$ (really for some $\mu_0 < \beth_{(2^{LS(\mathbf{T})})^+}$, $\mu \in [\mu_0, \lambda)$ suffices).*

Proof. As in [**?**]. By 3.10, for some $\mu_1 < \beth_{(2^{LS(\mathbf{T})})^+}$ every $M \in K_{[\mu_1, \lambda]}$ is $LS(\mathbf{T})^+$-saturated. Let $\mu \in [\mu_1, \lambda)$, and assume $M \in K_\mu$ is not saturated, so for some $\kappa \in (LS(\mathbf{T}), \mu)$ the model M is κ-saturated not κ^+-saturated. Let p, $\langle N^*_u : u \in [|M|]^{<\aleph_0} \rangle$, U, N^+, $\langle N^+_u : u \in [|N^+|]^{\aleph_0} \rangle$ be as in 3.9. Let $U_0 = U$. W.l.o.g. $N^*_{U_0}$ is saturated, p does not fork over $N^*_{u^*}$, $u^* \in [U]^{<\aleph_0}$ finite, $\mathrm{rk}(p)$ minimal under the circumstances. Now let $b \in M \setminus N^*_{U_0}$, so there is M^+ satisfying $M \prec_{\mathcal{F}} M^+ \in K_\mu$ such that $N_1 \preceq_{\mathcal{F}} M^+$ which is μ-isolated over $N^*_{U_0} \cup \{b\}$. By defining more N^*_u w.l.o.g. $N_1 = N^*_{U_1}$. So $\mathrm{tp}(b, N^*_{U_0}, M)$, and p are orthogonal (see [**?**, Ch.V]). Now we deal with orthogonal types and we continue as [**?**]: define a $\prec_{\mathcal{F}}$-chain M^*_i $(i < \lambda)$ of saturated models of cardinality λ_0 all omitting some fixed $p \in S(M^*_0)$. $\square$

Discussion 3.17. (1) Below $\beth_{(2^{LS(\mathbf{T})})^+}$.

A problem is what occurs in $[LS(\mathbf{T}), \beth_{(2^{LS(\mathbf{T})})^+}]$. As $\mathbf{T}$ is not necessarily complete, for any ψ and $\mathbf{T}$ we can consider $\mathbf{T}' \overset{\text{def}}{=} \{\psi \to \varphi : \varphi \in \mathbf{T}\}$, if $\neg\psi$ has a model in μ iff $\mu < \mu^*$, we get such examples where categoricity can start "late". So we may consider $\mathbf{T}$ complete in $L_{\kappa^*,\omega}$. Hart and Shelah [**?**] bound our possible improvement but we may want larger gaps, a worthwhile direction.

If $|\mathbf{T}| < \kappa^*$ we may look at what occurs in large enough $\mu < \kappa^*$.

(2) Below λ.

If λ is a limit cardinal we get only 3.11, this is a more serious issue. The problem is that we can get μ-saturated not saturated model in K_{μ^+}, so we get for $M \in K_\mu$ saturated, two orthogonal types p, $q \in S(M)$ (not realized in M). We want to build a prime model over $M \cup$(a large indiscernible set for p). Clearly $\mathcal{P}^-(n)$-diagrams are called for.

(3) Above λ.

In some sense we know every model is saturated: if $M \in K_{>\lambda}$, $N \preceq_{\mathcal{F}} M$, $\|N\| < \lambda$, $p \in S(N)$ then $\dim(p, N, M) = \|M\|$, i.e., if $N \preceq_{\mathcal{F}} N^+ \preceq_{\mathcal{F}} M$ and: $\|N^+\| < \|M\|$ when λ is successor, or $\beth_{(2^{LS(\mathbf{T})})^+}(\|N^+\|)$ when λ is a limit cardinal.

Another way to say it: the stationarization of p over N^+ is realized. But is every $q \in S(N^+)$ a stationarization of some $p \in S(N')$, $N' \preceq_{\mathcal{F}} N^+$, $\|N'\| \leq LS(\mathbf{T})$? We can find $N_0 \preceq_{\mathcal{F}} N^+$, $\|N_0\| \subseteq (\mathbf{T})$, such that: $[N_0 \preceq_{\mathcal{F}} N_1 \leq N^+$ & $\|N_1\| \leq LS(\mathbf{T}) \Rightarrow q \restriction N_1$ does not fork over $N_0]$, we can get it for $\|N_1\| < \mu$, but does it hold for $N_1 = N^+$? A central point is

$(*)$ Does K satisfy amalgamation?

Again it seems that $\mathcal{P}^-(n)$-systems are called for. See more in [**?**].

(4) If $|\mathbf{T}| < \kappa^*$ we can do better, as $\mathrm{Op}(EM(I, \Phi)) = EM(\mathrm{Op}(I), \Phi)$, will discuss elsewhere.

Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel, and, Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA