IT IS CONSISTENT WITH ZFC THAT B_1-GROUPS ARE NOT B_2

SAHARON SHELAH AND LUTZ STRÜNGMANN

Abstract. Both, B_1-groups and B_2-groups are natural generalizations of finite rank Butler groups to the infinite rank case and it is known that every B_2-group is a B_1-group. Moreover, assuming $V = L$ it was proven that the two classes coincide. Here we demonstrate that it is undecidable in ZFC whether or not all B_1-groups are B_2-groups. Using Cohen forcing we prove that there is a model of ZFC in which there exists a B_1-group that is not a B_2-group.

1. Introduction

The study of Butler groups, both in the finite and in the infinite rank case, is a most active area of Abelian Group Theory. There are several challenging problems which require deep insight into the theory of Butler groups and the available methods as well as the development of new machinery. The finite rank case is closely related to the study of representations of finite posets while the infinite rank case has its own special flavor. During the last years more and more the connection between infinite rank Butler groups and infinite combinatorics was discovered and led to numerous interesting results. In this paper we discuss one of the long-standing problems, namely whether or not all B_1-groups are B_2-groups, and show that its solution is independent of ZFC. It is known that any B_2-group is a B_1-group and moreover, assuming Goedel's universe of constructibility the two classes coincide. In contrast to this result we will show, using Cohen forcing, that there is a model of ZFC in which there exists a B_1-group that is not a B_2-group.

In the following all groups are abelian. Our terminology is standard and maps are written on the left. If H is a subgroup of a torsion-free group G then the purification of H in G is denoted by H_*. For notations and basic facts we refer to [11] for abelian groups, [18] and [21] for forcing and [9] or [17] for set-theory. Moreover, the interested reader may look at [2] for a survey on finite rank Butler groups and at [3], [12] for surveys on infinite rank Butler-groups.

Since our problem comes from abelian group theory the authors tried to make the paper

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accessible for non set-theorists. Hence the involved set-theory (forcing) is explained in detail although the methods are very standard.

2. Infinite rank Butler groups

In this section we recall the definitions of B_1-groups and B_2-groups as they were given by Bican-Salce in [6]. Both classes contain the class of finite rank Butler-groups (pure subgroups of completely decomposable groups of finite rank) first studied by Butler in [4]. Let us begin with the notion of a balanced subgroup.

A pure subgroup A of the torsion-free group G is said to be a *balanced* subgroup if every coset $g+A$ $(g \in G)$ contains an element $g+a$ $(a \in A)$ such that $\chi(g+a) \geq \chi(g+x)$ for all $x \in A$, where $\chi(g)$ denotes the characteristic of an element $g \in G$. Such an element is called *proper with respect to* A and $\chi(g)$ denotes the characteristic of an element g in the given group G.

An exact sequence $0 \to A \to G \to C \to 0$ is *balanced exact* if the image of A in G is a balanced subgroup of G. Hunter [16] discovered that the equivalence classes of balanced extensions of a group H by a group G give rise to a subfunctor $\mathrm{Bext}^1(H,G)$ of $\mathrm{Ext}^1(H,G)$ and hence homological algebra is applicable. Thus for a balanced exact sequence

$$(*) \qquad 0 \to A \to G \to C \to 0$$

and a group H we obtain the two long exact sequences

$$0 \to \mathrm{Hom}(C,H) \to \mathrm{Hom}(G,H) \to \mathrm{Hom}(A,H) \to \mathrm{Bext}^1(C,H) \to \mathrm{Bext}^1(G,H) \to$$
$$\to \mathrm{Bext}^1(A,H) \to \mathrm{Bext}^2(C,H) \to \cdots$$

and

$$0 \to \mathrm{Hom}(H,A) \to \mathrm{Hom}(H,G) \to \mathrm{Hom}(H,C) \to \mathrm{Bext}^1(H,A) \to \mathrm{Bext}^1(H,G) \to$$
$$\to \mathrm{Bext}^1(H,C) \to \mathrm{Bext}^2(H,A) \to \cdots$$

It is routine to check that balanced-exactness of the sequence $(*)$ is equivalent to the following property: for every rank one torsion-free group R, every homomorphism $R \to C$ can be lifted to a map $R \to G$, i.e. every rank one torsion-free group is projective with respect to $(*)$. Thus the following lemma is easily established.

Lemma 2.1. *Let*

$$0 \to A \to G \xrightarrow{\varphi} C \to 0$$

be a balanced exact sequence. Then this sequence is locally invertible, i.e. for any element $c \in C$ there exists a homomorphism $\psi_c : \langle c \rangle_* \to G$ such that $\varphi\psi_c = id_{\langle c \rangle_*}$.

We now come to the definitions of B_1-groups and B_2-groups.

Definition 2.2. *A torsion-free abelian group B is called*

(1) a B_1-group if $\mathrm{Bext}^1(B,T) = 0$ for all torsion groups T;

(2) *a B_2-group if there exists a continuous well-ordered ascending chain of pure subgroups,*

$$0 = B_0 \subset B_1 \subset \cdots \subset B_\alpha \subset \cdots \subset B_\lambda = B = \bigcup_{\alpha<\lambda} B_\alpha$$

such that $B_{\alpha+1} = B_\alpha + G_\alpha$ for every $\alpha < \lambda$ for some finite rank Butler group G_α; i.e. B_α is decent in $B_{\alpha+1}$ in the sense of Albrecht-Hill [1]*;*

(3) *finitely Butler if every finite rank pure subgroup of B is a Butler-group.*

Due to Bican-Salce [6] the three definitions are equivalent for countable torsion-free groups.

Theorem 2.3 ([6])**.** *For a countable torsion-free abelian group B the following are equivalent:*

(1) *B is finitely Butler;*
(2) *B is a B_2-group;*
(3) *B is a B_1-group.*

Without any restriction to the cardinality we have in general:

Theorem 2.4 ([6])**.** *B_2-groups of any rank are B_1-groups.*

It turned out that the converse implication in the above theorem couldn't be proved without any additional set-theoretic assumptions. There are some partial results in ZFC characterizing the B_2-groups among the B_1-groups but none of them is really satisfactory. The following was shown by Fuchs and Rangaswamy independently.

Lemma 2.5 ([13], [20])**.** *Suppose that $0 \to H \to C \to G \to 0$ is a balanced-exact sequence where C is a B_2-group and H and G are B_1-groups. If one of H and G is a B_2-group, then so is the other.*

An attempt to characterize the B_2-groups in a homological way is the following theorem due to Fuchs.

Theorem 2.6 ([13])**.** *If B is a B_2-group, then $\mathrm{Bext}^i(B,T) = 0$ for all $i \geq 1$ and for all torsion groups T.*

Assuming the continuum hypothesis Rangaswamy was able to show that also the converse holds and in some cases Fuchs could even remove CH.

Theorem 2.7 ([13], [20])**.** *The following are true:*

(1) *Assuming CH a torsion-free group B is a B_2-group if and only if $\mathrm{Bext}^1(B,T) = \mathrm{Bext}^2(B,T) = 0$ for all torsion groups T.*

(2) *A torsion-free group B of cardinality $\aleph_n$ (for some integer $n \geq 1$) is a B_2-group if and only if* $\mathrm{Bext}^i(B, T) = 0$ *for all $i \leq n+1$ and all torsion groups T.*

It was natural to ask whether $\mathrm{Bext}^2(B, T)$ is always zero for a torsion-free group B and a torsion group T but Magidor-Shelah [19] proved that this is not the case even assuming the generalized continuum hypothesis GCH. That CH was relevant in many papers was explained by Fuchs who showed the following theorem.

Theorem 2.8 ([13])**.** *In any model of ZFC, the following are equivalent:*

(1) $\mathrm{Bext}^2(G, T) = 0$ *for all torsion-free groups G and torsion groups T;*

(2) *CH holds and balanced subgroups of completely decomposable groups are B_2-groups.*

One of the most interesting and main results in the theory of infinite rank Butler groups is the following final theorem of this section proved by Magidor and Fuchs.

Theorem 2.9 ([14])**.** *Assuming $V = L$ every B_1-group is a B_2-group.*

We will show in this paper that the conclusion of the last theorem does not hold in ZFC but is independent of ZFC.

3. The forcing

In this section we will explain the forcing notion we are going to use to construct our B_1 group H which fails to be B_2. The reader who is familiar with forcing, especially with adding Cohen reals may skip this section. Most results are well-known and basic. For unexplained notations and further results on forcing we refer to Kunen's book [18] or to the more advanced first author's book [21].

Let M be any countable transitive model of ZFC and assume of course that the set theory ZFC is consistent. The aim of forcing is to extend M to a new model which still satisfies ZFC but which has additional properties which we are interested in.

A *forcing notion* $\mathbb{P} \in M$ is just a non empty, preordered set $(\mathbb{P}, \leq, 0_{\mathbb{P}})$, where $0_{\mathbb{P}}$ is the minimal element of $\mathbb{P}$, hence $0_{\mathbb{P}} \leq p$ for all $p \in \mathbb{P}$. Note that we don't require that $p \leq q$ and $q \leq p$ imply $q = p$. If two elements $p, q \in \mathbb{P}$ have no common upper bound, i.e. there is no $t \in \mathbb{P}$ such that $p \leq t$ and $q \leq t$, then we say that p and q are *incompatible* and write $p \perp q$. If a common upper bound exists we call the elements *compatible*. We now want to add to M a subset S of $\mathbb{P}$ to construct a transitive set $M[S]$ which is a model of ZFC with the same ordinals as M such that $M \subseteq M[S]$ and $S \in M[S]$. Those sets S are called generic.

Definition 3.1. *Let $D \subseteq \mathbb{P}$, $S \subseteq \mathbb{P}$ and $p \in \mathbb{P}$. Then*

(1) *D is called* dense *in $\mathbb{P}$ if for any $q \in \mathbb{P}$ there is an element $t \in D$ such that $q \leq t$;*

(2) *D is* dense above p *if for any $q \in \mathbb{P}$ such that $p \leq q$ there exists an element $t \in D$ such that $q \leq t$;*

(3) S is called $\mathbb{P}$-generic over M *if the following hold:*
   (a) for all $q, r \in S$ there exists $t \in S$ such that $q \leq t$ and $r \leq t$, i.e. all elements of S are compatible in S;
   (b) if $q \in S$ and $t \leq q$ for some $t \in \mathbb{P}$ then also $t \in S$;
   (c) $S \cap D \neq \emptyset$ for every dense subset D of $\mathbb{P}$ which is in M.

A first observation is that a generic set S intersects non-trivially also sets which are "dense above p" in many cases.

Lemma 3.2. *Let $D \subseteq \mathbb{P}$ and S be $\mathbb{P}$-generic over M. Then*

(1) Either $S \cap D \neq \emptyset$ or there exists $q \in S$ such that for all $r \in D$ we have $r \perp q$;
(2) If $p \in S$ and D is dense above p, then $S \cap D \neq \emptyset$.

Proof. See [18, Lemma 2.20]. ∎

If S is $\mathbb{P}$-generic over M (or, for short, generic), then the existence of the model $M[S]$ with the desired properties follows from the Forcing Theorem (see [21]). $M[S]$ is the smallest transitive model of ZFC that contains M and S. We don't want to recall the construction of $M[S]$ but we would like to mention the following facts. Since we want to prove theorems in $M[S]$ we would like to know the members of $M[S]$ but we can not have full knowledge of them inside M since this would cause these sets to be in M already. If S is in M then $M[S]$ gives nothing new, so we have to assume that S is not in M and this is the case in general as the following lemma shows.

Lemma 3.3. *Let S be $\mathbb{P}$-generic over M. If $\mathbb{P}$ satisfies the following condition*

$$\forall\, p \in \mathbb{P}\ \exists\, q,\ r \in \mathbb{P} \text{ such that } p \leq q, p \leq r \text{ and } q \perp r \tag{3.1}$$

then $S \notin M$.

Proof. See [18, Lemma 2.4]. ∎

Nevertheless, every element p of $\mathbb{P}$ can be a member of a generic set.

Lemma 3.4. *Let $p \in \mathbb{P}$. Then there is a subset S which is $\mathbb{P}$-generic over M such that $p \in S$.*

Proof. See [18, Lemma 2.3]. ∎

Although we don't know the generic set S we assume that we have some prescription for building the members of $M[S]$ out of M and S. These prescriptions are called $\mathbb{P}$-*names*, usually denoted by τ, and their *interpretation in* $M[S]$ is $\tau[S]$. For the exact definition of $\mathbb{P}$-names and their interpretation we refer again to Kunen's book [18] but let us mention that the Strengthened Forcing Theorem (see [21]) shows that

$$M[S] = \{\tau[S] : \tau \in M \text{ and } \tau \text{ is a } \mathbb{P}\text{-name }\}.$$

If we are talking about the $\mathbb{P}$-name of a special object H from $M[S]$ without specifying S then we will write $\tilde{H}$ instead of H to avoid confusion but if H is already in M, then we omit the tilde. Any sentence of our forcing language uses the $\mathbb{P}$-names to assert something about $M[S]$ but the truth or falsity of a sentence ψ in $M[S]$ depends on S in general. If $p \in \mathbb{P}$, then we write $p \Vdash \psi$ and say p *forces* ψ to mean that for all S which are $\mathbb{P}$-generic over M, if $p \in S$, then ψ is true in $M[S]$. If $0_{\mathbb{P}} \Vdash \psi$ then we just write $\Vdash_{\mathbb{P}} \psi$ which means that for any generic S the sentence ψ is true in $M[S]$ since $0_{\mathbb{P}}$ is always contained in S. Hence the elements of $\mathbb{P}$ provide partial information about objects in $M[S]$ but not all information and if $p \leq q$ then q contains more information than p. It may be decided in M whether or not $p \Vdash \psi$ and whenever a sentence ψ is true in $M[S]$ then there is $p \in S$ such that $p \Vdash \psi$.

We now turn to the forcing of adding Cohen reals. Therefore let κ be an uncountable cardinal. We put

$$\begin{aligned}\mathbb{P} &= \{p \mid p \text{ is a function from a finite subset of } \kappa \times \omega \text{ to } 2\} \\ &= \{p \mid p : \mathrm{dom}(p) \longrightarrow 2,\ \ \mathrm{dom}(p) \text{ a finite subset of } \kappa \times \omega\}\end{aligned}$$

The partial ordering of $\mathbb{P}$ is given by set theoretic inclusion, i.e. two functions p and q satisfy $p \leq q$ if and only if q extends p as a function. This forcing is called *"adding κ Cohen reals"* and the elements of $\mathbb{P}$ can obviously be regarded as functions from κ to ${}^{<\omega}2$ which we will do in the sequel.

The next lemma shows why the forcing is called adding κ Cohen reals.

Lemma 3.5. $\Vdash_{\mathbb{P}}$ *"There are at least κ reals".*

Proof. See [21, Chapter I, Lemma 3.3]. ∎

We will give the κ Cohen reals $\mathbb{P}$-names, say $\tilde{\eta}_\alpha$ for $\alpha < \kappa$ and state some basic properties of the Cohen reals. Note that a real is a function from ω to $2 = \{0, 1\}$.

Lemma 3.6. *The following hold for $\alpha, \beta < \kappa$:*

(1) $\Vdash_{\mathbb{P}}$ *"There are infinitely many $n \in \mathbb{N}$ such that $\tilde{\eta}_\alpha(n) = \tilde{\eta}_\beta(n) = 1$";*
(2) $\Vdash_{\mathbb{P}}$ *"There are infinitely many $n \in \mathbb{N}$ such that $\tilde{\eta}_\alpha(n) = \tilde{\eta}_\beta(n) = 0$";*
(3) $\Vdash_{\mathbb{P}}$ *"There are infinitely many $n \in \mathbb{N}$ such that $\tilde{\eta}_\alpha(n) \neq \tilde{\eta}_\beta(n)$".*

Proof. The proof of this fact is standard using a densitiy argument. ∎

Moreover, we have three more important facts.

Lemma 3.7. *The following hold for $\mathbb{P}$.*

(1) $\mathbb{P}$ satisfies the c.c.c. condition, i.e. $\mathbb{P}$ has no uncountable subset of pairwise incompatible members;
(2) $\mathbb{P}$ preserves cardinals and cofinalities, i.e. if λ is a cardinal in M, then λ is also a cardinal in $M[S]$ with the same cofinality;
(3) $\Vdash_{\mathbb{P}}$ " $2^{\aleph_0} \geq \lambda$". In particular, if $\lambda^{\aleph_0} = \lambda$ in M, then $\Vdash_{\mathbb{P}}$ " $2^{\aleph_0} = \lambda$".

Proof. See [21, Chapter I, Lemma 3.8], [21, Chapter I, Theorem 4.1] and [18, Theorem 5.10]. ∎

Finally we would like to remark that our notation is the "Jerusalem style" of forcing notation like in [21] but differs from the notation for example in [18]. In our partial order $p \leq q$ means that q contains more information than p does and not vice versa.

4. Our B_1 group H

Let M be a countable transitive model of ZFC in which the generalized continuum hypothesis holds, i.e. $2^\kappa = \kappa^+$ for all infinite cardinals κ. Moreover, let $\kappa \geq \aleph_4$ be regular and let $\mathbb{P}$ be the forcing of adding κ Cohen reals. As we have seen in the last section, $\mathbb{P}$ preserves cardinals and cofinalities and $2^{\aleph_0} = \kappa$ in $M[S]$ for every generic S. Let $\tilde{\eta}_\alpha$ denote the Cohen reals for $\alpha < \kappa$ and let M^* be a model extending M, obtained by Cohen forcing, e.g. $M^* = M[S]$ for some fixed S.

Inside M^* we choose independent elements

$$\{x_n : n < \omega\} \text{ and } \{y_\alpha : \alpha < \kappa\}$$

and fix a countable set of natural prime numbers

$$\{p_n \in \Pi : n < \omega\}$$

such that $p_n < p_m$ for $n < m$.

Definition 4.1. *Let $W = \bigoplus_{n<\omega} \mathbb{Q}x_n \oplus \bigoplus_{\alpha<\kappa} \mathbb{Q}y_\alpha$ be the rational vector space and let $F = \bigoplus_{n<\omega} \mathbb{Z}x_n \oplus \bigoplus_{\alpha<\kappa} \mathbb{Z}y_\alpha$ be the free abelian group generated by the x_n's and y_α's. In M^* we define H as the subgroup of W generated by $F \cup \{p_n^{-1}(y_\alpha - x_n) : \alpha < \kappa, n < \omega, \eta_\alpha(n) = 1\}$ and let $\tilde{H}$ be its $\mathbb{P}$-name.*

We can now state our Main Theorem.

Main Theorem 4.2. *In the model M^* the group H is a B_1-group but not a B_2-group. Hence it is consistent with ZFC that B_1-groups need not be B_2-groups.*

The proof of the Main Theorem 4.2 will be divided into two parts. The first part is to show that H is a B_1-group which will be done in this section. Section 5 will then consist of proving that H is not B_2.

Theorem 4.3. *In the model M^* the group H is a B_1-group.*

The proof of Theorem 4.3 takes the rest of this section and consists of several steps.

Proof. (of Theorem 4.3) To prove that H is a B_1-group we have to show that $\mathrm{Bext}(H,T) = 0$ for any torsion group T. Hence let

$$0 \longrightarrow \tilde{T} \xrightarrow{id} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{H} \longrightarrow 0 \tag{4.2}$$

be forced to be a balanced exact sequence in M^* with $T = \tilde{T}[S]$ torsion. Thus there exists $r^* \in \mathbb{P}$ such that

$$r^* \Vdash \text{"}0 \longrightarrow \tilde{T} \xrightarrow{id} \tilde{G} \xrightarrow{\tilde{\varphi}} \tilde{H} \longrightarrow 0 \text{ is balanced exact."}$$

We now work in M^* and choose preimages $g_\alpha \in G$ of y_α under φ for all $\alpha < \kappa$. Similarly let $\bar{x}_n \in G$ be a preimage for x_n under φ for $n < \omega$. Moreover, let

$$A_\alpha = \{n < \omega : \eta_\alpha(n) = 1\}$$

for $\alpha < \kappa$.

It is our aim to show that the balanced exact sequence (4.2) is forced to split, hence it is enough to prove that the homomorphism φ is right-invertible, i.e. we have to find $\psi : H \longrightarrow G$ such that $\varphi\psi = id_H$. Therefore it is necessary to find preimages of the generators of H in G such that equations satisfied in H also hold in G. We need the following definition.

Definition 4.4. *Let $\alpha < \kappa$ and $t \in T$ arbitrary. Then the set $R_{\alpha,t}$ is defined as*

$$R_{\alpha,t} = \{n \in A_\alpha : g_\alpha - t - \bar{x}_n \textit{ is not divisible by } p_n\}.$$

We will now use a purely group theoretic argument to show that if for every $\alpha < \kappa$ there is a $t_\alpha \in T$ such that R_{α,t_α} is finite then φ is invertible.

Lemma 4.5. *Let $\alpha < \kappa$ and let $t \in T$ such that $R_{\alpha,t}$ is finite. Then there exists $t_\alpha \in T$ such that $R_{\alpha,t_\alpha} = \emptyset$.*

Proof. Since $R_{\alpha,t}$ is finite we may assume without loss of generality that $R_{\alpha,t}$ has minimal cardinality. Assume that $R_{\alpha,t}$ is not empty and fix $n \in R_{\alpha,t}$. By the primary decomposition theorem we decompose T as

$$T = T_{p_n} \oplus T'$$

where T_{p_n} denotes the p_n-primary component of T. Since $n \in A_\alpha$ it follows that p_n divides $y_\alpha - x_n$, hence there exists $z \in G$ such that

$$\varphi(z) = p_n^{-1}(y_\alpha - x_n).$$

Thus

$$(g_\alpha - t - \bar{x}_n) - p_n z \in T = T_{p_n} \oplus T'$$

and therefore there exist $t_0 \in T_{p_n}$ and $t_1 \in T'$ such that

$$(g_\alpha - t - \bar{x}_n) - p_n z = t_0 + t_1.$$

Since T' is divisible by p_n we can write $t_1 = p_n t_2$ for some $t_2 \in T'$. Hence

$$(g_\alpha - t - \bar{x}_n) - p_n(z - t_2) = t_0.$$

We let $t' = t + t_0$ and will show that $R_{\alpha,t'}$ has smaller cardinality than $R_{\alpha,t}$ - a contradiction. By the choice of t' we have

$$(g_\alpha - t' - \bar{x}_n) = g_\alpha - t - t_0 - \bar{x}_n = p_n(z - t_2)$$

and hence $n \notin R_{\alpha,t'}$. But on the other side, if $m \notin R_{\alpha,t}$, then p_m divides $(g_\alpha - t - \bar{x}_m)$ and thus p_m divides $(g_\alpha - (t' - t_0) - \bar{x}_m)$. Since $p_n \neq p_m$ it follows that p_m divides t_0 and therefore p_m divides $(g_\alpha - t' - \bar{x}_m)$. Hence $m \notin R_{\alpha,t'}$ showing that $R_{\alpha,t'}$ is strictly smaller than $R_{\alpha,t}$. This finishes the proof. ∎

Lemma 4.6. *Assume that for every $\alpha < \kappa$ there exists $t_\alpha \in T$ such that R_{α,t_α} is finite. Then φ is invertible and hence the sequence (4.2) is forced to split.*

Proof. By Lemma 4.5 we may assume without loss of generality that for every $\alpha < \kappa$ the set R_{α,t_α} is empty. Thus for each $n \in A_\alpha$ we can find $z_{\alpha,n} \in G$ such that

$$p_n z_{\alpha,n} = g_\alpha - \bar{x}_n - t_\alpha.$$

We now define a homomorphism $\psi : H \longrightarrow G$ as follows:

(1) $\psi(x_n) = \bar{x}_n$ $(n < \omega)$;
(2) $\psi(y_\alpha) = g_\alpha - t_\alpha$ $(\alpha < \kappa)$;
(3) $\psi(p_n^{-1}(y_\alpha - x_n)) = z_{\alpha,n}$ $(\alpha < \kappa,\ n \in A_\alpha)$.

We leave to the reader to check that (1), (2) and (3) induce a well-defined homomorphism $\psi : H \longrightarrow G$ satisfying $\varphi\psi = id_H$. ∎

(Continuation of the proof of Theorem 4.3) Up to now we haven't used any forcing but we have worked in the model M^*. By Lemma 4.6 it remains to find for every $\alpha < \kappa$ an element $t_\alpha \in T$ such that the set R_{α,t_α} is finite. Here we use the forcing.

We define for $\alpha \neq \beta < \kappa$ the pure subgroup $H_{\alpha,\beta} = \langle y_\beta - y_\alpha \rangle_*$ of H. Since the sequence (4.2) is forced to be balanced exact Lemma 2.1 shows that there exist homomorphisms

$$\psi_{\alpha,\beta} : H_{\alpha,\beta} \longrightarrow G \text{ such that } \varphi\psi_{\alpha,\beta} = id_{H_{\alpha,\beta}}.$$

Let $h_{\alpha,\beta} = \psi_{\alpha,\beta}(y_\beta - y_\alpha) \in G$, hence

$$t_{\alpha,\beta} = h_{\alpha,\beta} - (g_\beta - g_\alpha) \in T.$$

Since T is a torsion group we can find $m_{\alpha,\beta} < \omega$ such that

$$\mathrm{ord}(t_{\alpha,\beta}) = m_{\alpha,\beta}.$$

Let $\tilde{m}_{\alpha,\beta}$ and $\tilde{g}_\alpha, \tilde{g}_\beta$ be $\mathbb{P}$-names for $m_{\alpha,\beta}$ and g_α, g_β, respectively. We can now easily show

Fact 4.7. $r^* \Vdash$ *" If $n > \tilde{m}_{\alpha,\beta}$, then p_n divides $(\tilde{g}_\beta - \tilde{g}_\alpha)$ for $n \in A_\alpha \cap A_\beta$ "*

Proof. If $n > m_{\alpha,\beta}$, then $p_n > m_{\alpha,\beta}$ follows since the primes p_m are increasing. Therefore $\gcd(p_n, m_{\alpha,\beta}) = 1$ and thus p_n divides $(h_{\alpha,\beta} - (g_\beta - g_\alpha))$. Moreover, $h_{\alpha,\beta} = \psi_{\alpha,\beta}(y_\beta - y_\alpha)$ is divisible by p_n since $n \in A_\alpha \cap A_\beta$. Hence p_n divides $(g_\beta - g_\alpha)$. ∎

Now let $r^* \leq r_{\alpha,\beta} \in \mathbb{P}$ be such that $r_{\alpha,\beta}$ forces the value $m_{\alpha,\beta}$ to $\tilde{m}_{\alpha,\beta}$, i.e.

$$r_{\alpha,\beta} \Vdash \text{ " } \tilde{m}_{\alpha,\beta} = m_{\alpha,\beta} \text{ " }.$$

Without loss of generality we assume that $\beta \in \mathrm{dom}(r_{\alpha,\beta})$ for all α, β. Since all elements of $\mathbb{P}$ are functions from κ to 2 with finite domain, we may write for some $n_{\alpha,\beta} < \omega$

$$\mathrm{dom}(r_{\alpha,\beta}) = \{\gamma_{(\alpha,\beta,0)}, \cdots, \gamma_{(\alpha,\beta,n_{\alpha,\beta})}\} \subset \kappa,$$

where $\gamma_{(\alpha,\beta,i)} < \gamma_{(\alpha,\beta,j)}$ if $i < j \leq n_{\alpha,\beta}$. We would like to apply the Δ-Lemma to the functions $r_{\alpha,\beta}$ to obtain a Δ-system but unfortunately the functions $r_{\alpha,\beta}$ depend on two variables. This forces us to do the Δ-Lemma 'by hand'. For this we use the Erdös-Rado Theorem (see [10]).

First we define a coloring on 4-tuples in $\aleph_4$. Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \aleph_1$ such that $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ and let

$$c(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$$

consist of the following entries in an arbitrary but fixed order:

(i) n_{α_0,α_1};
(ii) m_{α_0,α_1};
(iii) $r_{\alpha_0,\alpha_1}(\gamma_{(\alpha_0,\alpha_1,j)} : j \leq n_{\alpha_0,\alpha_1})$;
(iv) $(tv(\gamma_{(\alpha_{n_1},\alpha_{n_2},n_3)} < \gamma_{(\alpha_{m_1},\alpha_{m_2},m_3)}) : n_1,n_2,m_1,m_2 < 4; n_3 < n_{\alpha_{n_1},\alpha_{n_2}}; m_3 < n_{\alpha_{m_1},\alpha_{m_2}})$.

Recall that tv denotes the *truth-value* of the inequality. The above coloring is a coloring with ω colors and thus we may apply the Erdös-Rado Theorem. Note that we are working in our model M in which GCH holds by assumption. Hence we have

$$\aleph_4 \longrightarrow (\aleph_1)^4_{\aleph_0}$$

which is exactly what we need to apply the Erdös-Rado Theorem. We obtain an increasing chain of c-homogeneous elements

$$\Gamma = \{\alpha_\epsilon : \epsilon < \omega_1\}$$

which means that whenever $\alpha_{\epsilon_1}, \alpha_{\epsilon_2}, \alpha_{\epsilon_3}, \alpha_{\epsilon_4} \in \Gamma$ such that $\alpha_{\epsilon_1} < \alpha_{\epsilon_2} < \alpha_{\epsilon_3} < \alpha_{\epsilon_4}$, then

$$c(\alpha_{\epsilon_1}, \alpha_{\epsilon_2}, \alpha_{\epsilon_3}, \alpha_{\epsilon_4}) = c^*$$

for a fixed color c^*. Let this particular color consist of the following entries:

(I) n^*;
(II) m^*;
(III) $(k_1, \cdots, k_{n^*})$ $(k_i \in \{0,1\})$;
(IV) $(l_1, \cdots, l_{16^2(n^*)^2})$ $(l_i \in \{\text{Yes, No}\})$.

Let us first explain what the homogenity implies. Let $\alpha_{\epsilon_1}, \alpha_{\epsilon_2} \in \Gamma$ such that $\epsilon_1 < \epsilon_2$, then (I) ensures that the domain of $r_{\alpha_{\epsilon_1},\alpha_{\epsilon_2}}$ has size n^*. Moreover, (II) says that $r_{\alpha_{\epsilon_1},\alpha_{\epsilon_2}}$ forces the value m^* to $m_{\alpha_{\epsilon_1},\alpha_{\epsilon_2}}$ and (III) implies that the image of $r_{\alpha_{\epsilon_1},\alpha_{\epsilon_2}}$ is uniquely determined. Finally (IV) ensures that if we take another pair $\alpha_{\epsilon_3}, \alpha_{\epsilon_4} \in \Gamma$ such that $\epsilon_3 < \epsilon_4$, then the relationship between the elements of the domains of $r_{\alpha_{\epsilon_1},\alpha_{\epsilon_2}}$ and $r_{\alpha_{\epsilon_3},\alpha_{\epsilon_4}}$ is fixed.

In the sequel we need to be above all the "trouble", hence we may increase m^* without loss of generality such that m^* is greater than or equal to length$(r_{\alpha_\epsilon,\alpha_\rho}(\gamma_{\alpha_\epsilon,\alpha_\rho,e}))$ for all $\epsilon < \rho < \omega_1$ and $e \leq n^*$. We can now approach to the Δ-Lemma.

Definition 4.8. *For $\alpha_\epsilon \in \Gamma$ we define*

(1) $u_{\alpha_\epsilon} = \mathrm{dom}(r_{\alpha_\epsilon,\alpha_{\epsilon+1}}) \cap \mathrm{dom}(r_{\alpha_\epsilon,\alpha_{\epsilon+2}})$;
(2) $u^* = \bigcap_{\epsilon<\omega_1} u_{\alpha_\epsilon}$;
(3) $s_\epsilon = r_{\alpha_\epsilon,\alpha_{\epsilon+1}} \restriction_{u_{\alpha_\epsilon}} = r_{\alpha_\epsilon,\alpha_{\epsilon+2}} \restriction_{u_{\alpha_\epsilon}}$.

We have to explain why (3) in Definition 4.8 is well-defined. This follows from homogenity since (IV) implies that for $\gamma \in u_{\alpha_\epsilon}$ we have $r_{\alpha_\epsilon,\alpha_{\epsilon+1}}(\gamma) = r_{\alpha_\epsilon,\alpha_{\epsilon+2}}(\gamma)$. We are now ready to show the following lemma, our version of the Δ-system. Note that if we talk about a Δ-system of functions then we mean that the corresponding domains of the functions form a Δ-system.

Lemma 4.9. *For $\alpha_\epsilon, \alpha_\rho \in \Gamma$ such that $\epsilon < \rho$ we have*

$$u_{\alpha_\epsilon} \cap u_{\alpha_\rho} = u^*.$$

Hence the functions s_ϵ ($\alpha_\epsilon \in \Gamma$) form a Δ-system with root u^*. Moreover, for fixed $\epsilon < \omega_1$ the functions $r_{\alpha_\epsilon,\alpha_\rho}$ ($\epsilon < \rho < \omega_1$) form a Δ-system with root u_{α_ϵ}.

Proof. Let $\alpha_\epsilon, \alpha_\rho \in \Gamma$ be such that $\epsilon < \rho$. Clearly we have $u^* \subseteq u_{\alpha_\epsilon} \cap u_{\alpha_\rho}$ by Definition 4.8. It remains to show the converse inclusion. Therefore let $\gamma \in u_{\alpha_\epsilon} \cap u_{\alpha_\rho}$ and choose $\tau < \omega_1$ arbitrary. We have to prove that γ lies in u_{α_τ}.

If $\tau = \epsilon$ or $\tau = \rho$, then we are done.

If $\tau \geq \epsilon+1$, then $c(\alpha_\epsilon, \alpha_{\epsilon+1}, \alpha_\tau, \alpha_{\tau+1}) = c^*$ by homogenity. Since $\gamma \in \mathrm{dom}(r_{\alpha_\epsilon,\alpha_{\epsilon+1}})$ we can find $i \leq n^*$ such that $\gamma = \gamma_{(\alpha_\epsilon,\alpha_{\epsilon+1},i)}$ and similarly $\gamma = \gamma_{(\alpha_\rho,\alpha_{\rho+1},j)}$ for some $j \leq n^*$. It follows now that

$$tv(\gamma_{(\alpha_\epsilon,\alpha_{\epsilon+1},i)} < \gamma_{(\alpha_\rho,\alpha_{\rho+1},j)}) = \text{ No and } tv(\gamma_{(\alpha_\rho,\alpha_{\rho+1},j)} < \gamma_{(\alpha_\epsilon,\alpha_{\epsilon+1},i)}) = \text{ No.}$$

Hence there exists by homogenity $k \leq n^*$ such that

$$tv(\gamma_{(\alpha_\epsilon,\alpha_{\epsilon+1},i)} < \gamma_{(\alpha_\tau,\alpha_{\tau+1},k)}) = \text{ No and } tv(\gamma_{(\alpha_\tau,\alpha_{\tau+1},k)} < \gamma_{(\alpha_\epsilon,\alpha_{\epsilon+1},i)}) = \text{ No.}$$

Thus $\gamma = \gamma_{(\alpha_\tau,\alpha_{\tau+1},k)} \in \mathrm{dom}(r_{\alpha_\tau,\alpha_{\tau+1}})$. Similarly it follows that $\gamma \in \mathrm{dom}(r_{\alpha_\tau,\alpha_{\tau+2}})$ and hence $\gamma \in u_{\alpha_\tau}$.

If $\tau < \epsilon + 1$, then we use similar arguments to those above to prove that $\gamma \in u_{\alpha_\tau}$.

Thus we have shown that $\gamma \in u_{\alpha_\tau}$ for any $\tau < \omega_1$ and therefore $\gamma \in u^*$.

The same kind of arguments show that also the functions $r_{\alpha_\epsilon,\alpha_\rho}$ ($\epsilon < \rho < \omega_1$) form a Δ-system with root u_{α_ϵ} for fixed $\epsilon < \omega_1$. ■

It is now easy to see by a pigeon-hole argument that we may assume without loss of generality (and we will assume this in the sequel) that all the functions from the Δ-systems in Lemma 4.9 coincide on their root.

(Continuation of the proof of Theorem 4.3) The following definition now makes sense.

Definition 4.10. *For $\epsilon < \rho < \omega_1$ and a generic $S \subseteq \mathbb{P}$ we define*

(*I*) $s^* = s_\epsilon \restriction_{u^*} = s_\epsilon \restriction_{(u_{\alpha_\epsilon} \cap u_{\alpha_\rho})}$;
(*II*) $\tilde{Y} = \{\alpha_\tau : s_\tau \in S\}$.

We can now show that s^* is strong enough to force that $\tilde{Y}$ has cardinality $\aleph_1$.

Fact 4.11. $s^* \Vdash$ " $|\tilde{Y}| = \aleph_1$ "

Proof. Let $\tilde{Y}' = \{s_\epsilon : \alpha_\epsilon \in \Gamma \backslash \tilde{Y}\}$ and assume that s^* does not force $\tilde{Y}$ to be of size $\aleph_1$. Then $|\tilde{Y}'| = \aleph_1$. We will show that this set is dense above s^*. Therefore let $f \geq s^*$, then dom(f) is a finite subset of κ and $u^* \subseteq$ dom(f). We choose $s_\epsilon \in \tilde{Y}'$ such that dom$(s_\epsilon)\backslash u^*$ is disjoint to dom$(f)\backslash u^*$. This is possible since by Lemma 4.9 the s_ϵ's form a Δ-system, hence

$$\mathrm{dom}(s_\tau)\backslash u^* \cap \mathrm{dom}(s_\beta)\backslash u^* = \emptyset$$

for $\beta \neq \tau$. Now, f and s_ϵ are compatible and thus $\tilde{Y}'$ is dense above s^*. Therefore $\tilde{Y}' \cap S \neq \emptyset$ by Lemma 3.2 - a contradiction. ■

We are almost done and prove the following statement.

Fact 4.12. $s^* \Vdash$ " *If* $\alpha_\epsilon, \alpha_\rho \in \tilde{Y}$ *and* $n \in A_{\alpha_\epsilon} \cap A_{\alpha_\rho} \backslash [0, m^*]$ *then* p_n *divides* $\tilde{g}_{\alpha_\epsilon} - \tilde{g}_{\alpha_\rho}$ ".

Proof. Let $s^* \leq s$ be such that

$$s \Vdash \text{"} n \in A_{\alpha_\epsilon} \cap A_{\alpha_\rho} \backslash [0, m^*] \text{"}.$$

Without loss of generality we may assume that s also forces truth values to $\alpha_\epsilon \in \tilde{Y}$ and $\alpha_\rho \in \tilde{Y}$. If one of them is No, then we are done and hence let us assume that both are Yes. We will show that there exists $\gamma < \omega_1$ such that

(I) $\gamma > \epsilon$;
(II) $\gamma > \rho$;
(III) dom$(r_{\alpha_\epsilon, \alpha_\gamma})\backslash u_{\alpha_\epsilon} \cup$ dom$(r_{\alpha_\rho, \alpha_\gamma})\backslash u_{\alpha_\rho} \cup \{\alpha_\gamma\} \cup u_{\alpha_\gamma} \backslash u^*$ is disjoint to dom(s).

Obviously we can choose $\gamma > \epsilon, \rho$ such that dom(s) is disjoint to $\{\alpha_\gamma\}$, so all we have to ensure is that also dom$(r_{\alpha_\epsilon, \alpha_\gamma})\backslash u_{\alpha_\epsilon} \cup$ dom$(r_{\alpha_\rho, \alpha_\gamma})\backslash u_{\alpha_\rho} \cup u_{\alpha_\gamma} \backslash u^*$ is disjoint to dom(s). For this we prove that the three sets

(1) $\{\gamma < \omega_1 :$ dom$(r_{\alpha_\epsilon, \alpha_\gamma})\backslash u_{\alpha_\epsilon}$ is not disjoint to dom$(s)\}$;
(2) $\{\gamma < \omega_1 :$ dom$(r_{\alpha_\rho, \alpha_\gamma})\backslash u_{\alpha_\rho}$ is not disjoint to dom$(s)\}$;
(3) $\{\gamma < \omega_1 : u_{\alpha_\gamma} \backslash u^*$ is not disjoint to dom$(s)\}$.

are bounded in ω_1. Let us start with (1). By Lemma 4.9 we know that for each $\epsilon < \omega_1$ the domains $\{\mathrm{dom}(r_{\alpha_\epsilon, \alpha_\gamma}) : \epsilon < \gamma < \omega_1\}$ form a Δ-system with root u_{α_ϵ}, hence $\{\mathrm{dom}(r_{\alpha_\epsilon, \alpha_\gamma})\backslash u_{\alpha_\epsilon} : \epsilon < \gamma < \omega_1\}$ is a set of pairwise disjoint sets. Since dom(s) is a finite set $\{\gamma < \omega_1 : \mathrm{dom}(r_{\alpha_\epsilon,\alpha_\gamma})\backslash u_{\alpha_\epsilon}$ is not disjoint to $\mathrm{dom}(s)\}$ must be bounded in ω_1. Similarly $\{\gamma < \omega_1 : \mathrm{dom}(r_{\alpha_\rho,\alpha_\gamma})\backslash u_{\alpha_\rho}$ is not disjoint to $\mathrm{dom}(s)\}$ is bounded in ω_1. Finally, again by Lemma 4.9 the sets $\{u_{\alpha_\gamma} : \gamma < \omega_1\}$ form a Δ-system with root u^* and so also $\{\gamma < \omega_1 : u_{\alpha_\gamma}\backslash u^*$ is not disjoint to $\mathrm{dom}(s)\}$ is bounded in ω_1.

For this γ we are able to prove that there is s^+ such that

(i) $s^+ \geq s$;
(ii) $s^+ \Vdash$ "$\tilde{\eta}_{\alpha_\gamma}(n) = 1$";
(iii) $s^+ \geq r_{\alpha_{\epsilon,\gamma}}$;
(iv) $s^+ \geq r_{\alpha_{\rho,\gamma}}$.

Since n was chosen large enough which means that η_{α_γ} has length less than or equal to m^* and hence less than or equal to n, there is, once we know that we can satisfy (i), (iii) and (iv), also some $s^+ \geq s$ satisfying all conditions (i), (ii), (iii) and (iv). Thus we only have to satisfy conditions (i), (iii) and (iv) and for this it is obviously enough to show that the three functions s, $r_{\alpha_\epsilon,\alpha_\gamma}$ and $r_{\alpha_\rho,\alpha_\gamma}$ are compatible. Assume that $r_{\alpha_\epsilon,\alpha_\gamma}$ and $r_{\alpha_\rho,\alpha_\gamma}$ are incompatible, then by induction we obtain that $r_{\alpha_\epsilon,\alpha_{\omega_1}}$ and $r_{\alpha_\rho,\alpha_{\omega_1}}$ are incompatible. Hence for $\tau < \sigma < \omega_1$ we have that $r_{\alpha_\tau,\alpha_{\omega_1}}$ and $r_{\alpha_\sigma,\alpha_{\omega_1}}$ are incompatible which contradicts the c.c.c. condition of our forcing. Therefore $r_{\alpha_\epsilon,\alpha_\gamma}$ and $r_{\alpha_\rho,\alpha_\gamma}$ are compatible. Finally s and $r_{\alpha_\epsilon,\alpha_\gamma}$ (and similarly s and $r_{\alpha_\rho,\alpha_\gamma}$) are compatible since by the choice of γ we have $\mathrm{dom}(s) \cap \mathrm{dom}(r_{\alpha_\epsilon,\alpha_\gamma}) = u_{\alpha_\epsilon}$.

Now $s^+ \Vdash$ "p_n divides $\tilde{g}_{\alpha_\epsilon} - \tilde{g}_{\alpha_\gamma}$" and $s^+ \Vdash$ "p_n divides $\tilde{g}_{\alpha_\gamma} - \tilde{g}_{\alpha_\rho}$" and therefore $s^+ \Vdash$ "p_n divides $\tilde{g}_{\alpha_\epsilon} - \tilde{g}_{\alpha_\rho}$" as claimed. ∎

Finally we have to prove another fact.

Fact 4.13. $s^* \Vdash$ " *For every* $\beta < \kappa$ *there exists* $m_\beta < \omega$ *such that* $\forall m_\beta < n \in A_\beta$ *and* $\alpha_\epsilon \in \tilde{Y}$ *such that* $n \in A_{\alpha_\epsilon}$ *we have* p_n *divides* $\tilde{g}_\beta - \tilde{g}_{\alpha_\epsilon}$ ".

Let us first show how Fact 4.13 implies Theorem 4.2, i.e. using Fact 4.13 we prove that the set $R_{\beta,0}$ is contained in $[0, m_\beta) \cap \mathbb{Z}$ for all $\beta < \kappa$ and hence finite after modifying the choice of the preimages $\bar{x}_n$ of x_n $(n < \omega)$ slightly (which doesn't has any effect on what we have done so far). Choose $\bar{x}_n \in G$ such that

(i) $\varphi(\bar{x}_n) = x_n$;
(ii) if $n > m^*$ and $\alpha \in Y$ such that $\eta_\alpha(n) = 1$, then p_n divides $g_\alpha - \bar{x}_n$.

For example if $n > m^*$ we choose $\alpha \in Y$ such that $\eta_\alpha(n) = 1$; Let $k_n \in G$ such that $\varphi(k_n) = 1/p_n(y_\alpha - x_n)$ and put $\bar{x}_n = p_n k_n + g_\alpha$. Then clearly (i) and (ii) are satisfied.

Now Fact 4.13 ensures that $R_{\beta,0}$ is contained in $[0, m_\beta)$ for if $m_\beta < n \in A_\beta$ choose $\alpha_\epsilon \in Y$ such that $\eta_{\alpha_\epsilon}(n) = 1$ (the one which was used when choosing the $\bar{x}_n$'s), then we have by the choice of $\bar{x}_n$ that p_n divides $g_{\alpha_\epsilon} - \bar{x}_n$ and by Fact 4.13 we have p_n divides $g_\beta - g_{\alpha_\epsilon}$ and hence p_n divides $g_\beta - \bar{x}_n$. Thus $n \notin R_{[\beta,o)}$ and $R_{[\beta,0)} \subseteq [0, m_\beta)$ follows.

Therefore the proof of Fact 4.13 finishes the proof of Theorem 4.3.

Proof. *(of Fact 4.13)* Fix $\beta < \kappa$ and let $s^+ < s$ be such that s forces $n \in A_\beta$. For every $\epsilon < \omega_1$ we choose (if possible) t^ϵ_β in the generic set such that

(i) $s \leq t^\epsilon_\beta$;
(ii) $t^\epsilon_\beta \Vdash$ "$\alpha_\epsilon \in \tilde{Y}$";
(iii) $t^\epsilon_\beta \Vdash$ "$\tilde{m}_{\alpha_\epsilon,\beta} = m^\sharp_{\alpha_\epsilon,\beta}$" for some $m^\sharp_{\alpha_\epsilon,\beta} \in \mathbb{N}$.

Note that it is sufficient to find one ϵ such that

$$(*) \qquad p_n \text{ divides } g_\beta - g_{\alpha_\epsilon},$$

for then we can use Fact 4.12 to get the conclusion for any α_ρ such that $n \in A_{\alpha_\rho}$. If we have one t^ϵ_β satisfying (ii) and (iii), then it forces $(*)$ for $n > m^\sharp_{\alpha_\epsilon,\beta}$. So we first ensure (ii) and (iii) and then we use that there is an uncountable subset S_β of ω_1 such that $\{t^\epsilon_\beta : \epsilon \in S_\beta\}$ is a Δ-system to ensure (i) where we put $m_\beta = m^\sharp_{\alpha_\epsilon,\beta}$ which can be chosen fixed for the Δ-system. ■

5. Why H fails to be B_2

To complete the proof of our Main Theorem 4.2 we show in this section that the group H from Definition 4.1 can not be a B_2-group in M^*.

Theorem 5.1. *In the model M^* the group H can not be a B_2-group.*

Proof. Towards contradiction assume that H is a B_2-group, hence has a B_2-filtration

$$H = \bigcup_{\alpha<\kappa} H_\alpha.$$

Recall that a B_2-filtration is a smooth ascending chain of pure subgroups H_α such that for every $\alpha < \kappa$ $H_{\alpha+1} = H_\alpha + B_\alpha$ for some finite rank Butler group B_α. Recall that a *cub* in κ is a subset C of κ such that

(i) C is *closed* in κ, i.e. for all $C' \subseteq C$, if $\sup C' < \kappa$, then $\sup C' \in C$;
(ii) C is *unbounded* in κ, i.e. $\sup C' = \kappa$.

The proof of the following lemma is standard (see [9, II.4.12]) but for the convenience of the reader we include it briefly.

Lemma 5.2. *The set* $C = \{\delta < \kappa \mid H_\delta = \langle x_n, y_\beta : n < \omega, \beta < \delta \rangle_*\}$ *is a cub in* κ.

Proof. First we show that C is closed in κ. Therefore let $C' = \{\delta_i \mid i \in I\}$ be a subset of C such that $\sup C' < \kappa$. If we put $\gamma = \sup C'$, then clearly

$$H_\gamma = \bigcup_{i \in I} H_{\delta_i} = \langle x_n, y_\beta : n < \omega, \beta < \gamma \rangle_*$$

and hence $\gamma \in C$.

It remains to show that C is unbounded. Therefore assume that C is bounded by $\delta^* < \kappa$, i.e. $\delta \leq \delta^*$ for all $\delta \in C$. We will show that there exists $\delta^* < \gamma < \kappa$ such that $H_\gamma = \langle x_n, y_\beta : n < \omega, \beta < \gamma \rangle_*$, hence $\gamma \in C$ - a contradiction.

Let $\rho_1 = \delta^*$ and put

$$E_{\rho_1} = \langle x_n, y_\beta : n < \omega, \beta < \rho_1 \rangle_* .$$

Now choose $\rho_1 \leq \alpha_1 < \kappa$ such that $E_{\rho_1} \subseteq H_{\alpha_1}$. If $\alpha_1 \notin C$ then choose $\alpha_1 \leq \rho_2 < \kappa$ such that

$$H_{\alpha_1} \subseteq E_{\rho_2} = \langle x_n, y_\beta : n < \omega, \beta < \rho_2 \rangle_* .$$

Continuing this way we obtain a sequence of groups E_{ρ_i} and H_{α_i} such that

$$E_{\rho_i} \subseteq H_{\alpha_i} \subseteq E_{\rho_{i+1}}$$

for all $i < \omega$. Let $\gamma = \sup\{\rho_i : i < \omega\} = \sup\{\alpha_i : i < \omega\}$. Then

$$H_\rho = \bigcup_{i<\omega} H_{\alpha_i} = \bigcup_{i<\omega} E_{\rho_i} = E_\gamma = \langle x_n, y_\beta : n < \omega, \beta < \gamma \rangle_*$$

and hence $\gamma \in C$. This finishes the proof. ∎

(Continuation of the proof of Theorem 5.1) Now let $\delta \in C$ be such that $\delta > \aleph_1$ and w.l.o.g. let δ be a limit ordinal. This is possible since C is a cub by Lemma 5.2. Note that $y_\delta \notin H_\delta$ but we have the following lemma.

Lemma 5.3. *There exists* $n^* < \omega$ *and a sequence of ordinals* $\delta \leq \alpha_1 \leq \alpha_2 \cdots \leq \alpha_{n^*} < \kappa$ *such that*

$$\langle H_\delta + \mathbb{Z} y_\delta \rangle_* \subseteq \sum_{m \leq n^*} B_{\alpha_m} + H_\delta.$$

Proof. We induct on $\alpha \geq \delta$ to show the even stronger statement that for any $L \subseteq_* H_\alpha$, L of finite rank, there exist $n^* < \omega$ and $\delta \leq \alpha_1 \leq \alpha_2 \cdots \leq \alpha_{n^*} < \kappa$ such that

$$\langle H_\delta + L \rangle_* \subseteq \sum_{m \leq n^*} B_{\alpha_m} + H_\delta.$$

If $\alpha = \delta$, then we are done choosing $n^* = 1$ and $\alpha_1 = \alpha$.

If $\alpha > \delta$ is a limit ordinal, then $L \subseteq_* H_\alpha$ implies $L \subseteq_* H_\beta$ for some $\delta \leq \beta < \alpha$. Hence we are done by induction hypothesis.

If $\alpha = \beta + 1$, let $H_\alpha = H_\beta + B_\beta$ and let $L = \langle l_1, \cdots, l_k \rangle_*$. We can find representations

$$l_i = h_{\beta,i} + b_{\beta,i}$$

for all $1 \leq i \leq k$ where $h_{\beta,i} \in H_\beta$ and $b_{\beta,i} \in B_\beta$. We put

$$L_\beta = \langle h_{\beta,i}, (B_\beta \cap H_\beta) : 1 \leq i \leq k \rangle_* \subseteq H_\beta$$

which is a pure subgroup of finite rank of H_β. An easy calculation shows that $L \subseteq L_\beta + B_\beta$.

The induction hypothesis implies that there exist $n < \omega$ and $\delta \leq \alpha_1 \leq \alpha_2 \cdots \leq \alpha_n$ such that

$$\langle H_\delta + L_\beta \rangle_* \subseteq \sum_{m \leq n} B_{\alpha_m} + H_\delta.$$

Another calculation shows that this implies

$$\langle H_\delta + L \rangle_* \subseteq \sum_{m \leq n} B_{\alpha_m} + B_\beta + H_\delta.$$

This finishes the proof. ■

(Continuation of the proof of Theorem 5.1) By Lemma 5.3 we can choose $n^* < \omega$ and $\delta \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n^*}$ such that

$$\langle H_\delta + \mathbb{Z} y_\delta \rangle_* \subseteq \sum_{m \leq n^*} B_{\alpha_m} + H_\delta.$$

For every $m \leq n^*$ we choose a finite set $W_m \subset \kappa$ and an integer $n_m < \omega$ such that

$$B_{\alpha_m} \subseteq \left\langle \sum_{\gamma \in W_m} \mathbb{Z} y_\gamma + \sum_{i \leq n_m} \mathbb{Z} x_i \right\rangle_* .$$

Collecting all these generators and letting $W = \bigcup_{m \leq n^*} W_m$ and $k = \max\{n_m : m \leq n^*\}$ we obtain

$$\langle H_\delta + \mathbb{Z} y_\delta \rangle_* \subseteq B + H_\delta \tag{5.3}$$

where $B = \left\langle \sum_{\gamma \in W} \mathbb{Z} y_\gamma + \sum_{i \leq k} \mathbb{Z} x_i \right\rangle_*$.

Now choose $\beta < \delta \backslash W$ and let $n \geq k$ be such that

$$n \in A_\beta \cap A_\delta \backslash \bigcup_{\gamma \in W, \gamma \neq \delta} A_\gamma.$$

Note that this choice is possible by the following densitiy argument similar to the proof of Lemma 3.6. Let $p \in \mathbb{P}$ and write $p = (p_{\alpha_0}, p_{\alpha_1}, \cdots, p_{\alpha_{m-1}})$, where each $p_{\alpha_i} \in {}^{<\omega}2$

($\alpha_i < \kappa$). Without loss of generality we may assume that $W \subseteq \{\alpha_i : i < m\}$. Put $\beta = \max(W)+1$ and let $n = \max\{\max(\mathrm{dom}(p_{\alpha_i})) : i < m\}+1$. Now we extend each p_{α_i} to q_{α_i} by putting $q_{\alpha_i}(n) = 0$. Moreover, we let $q_\beta(n) = 1$. Then $q = (q_{\alpha_0}, \cdots, q_{\alpha_{m-1}}, q_\beta)$ is stronger than p and therefore forces what we need. It is now straightforward to see that $p_n^{-1}(y_\delta - y_\beta)$ is an element of $\langle H_\delta + \mathbb{Z}y_\delta\rangle_*$ but it is not an element of $B + H_\delta$ contradicting equation (5.3). This finishes the proof of Theorem 5.1 and therefore the proof of our Main Theorem 4.2. ■

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Department of Mathematics, Hebrew University, 91904 Jerusalem, Israel, and Rutgers University, Newbrunswick, NJ U.S.A., e-Mail: Shelah@math.huji.ac.il,

Department of Mathematics, Hebrew University, 91904 Jerusalem, Israel, e-mail: lutz@math.huji.ac.il