A Note on Extensions of Infinitary Logic*

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Abstract

We show that a strong form of the so called Lindström's Theorem [4] fails to generalize to extensions of $L_{\kappa\omega}$ and $L_{\kappa\kappa}$: For weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to κ. With an additional set-theoretic assumption, there is no strongest extension of $L_{\kappa\kappa}$ with the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to $< \kappa$.

By a well-known theorem of Lindström [4], first order logic $L_{\omega\omega}$ is the strongest logic which satisifies the compactness theorem and the downward Löwenheim-Skolem theorem. For weakly compact κ, the infinitary logic $L_{\kappa\omega}$ satisfies both the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to κ. In [1] Jon Barwise pointed out that $L_{\kappa\omega}$ is not maximal with respect to these properties, and asked what is the strongest logic based on a weakly compact cardinal κ which still satisfies the (κ, κ)-compactness property and some other natural conditions suggested by κ. We prove (Corollary 5) that for weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with

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the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to κ. This shows that there is no extension of $L_{\kappa\omega}$ which would satisify the most obvious generalization of Lindström's Theorem. A stronger result (Theorem 11) is proved under an additional assumption.

We use the notation and terminology of [2, Chapter II] as much as possible. We will work with concrete logics such as first order logic $L_{\omega\omega}$, infinitary logic $L_{\kappa\lambda}$ and their extensions $L_{\omega\omega}(\{Q_i : i \in I\})$ and $L_{\kappa\lambda}(\{Q_i : i \in I\})$ by generalized quantifiers. Therefore it is not at all critical which definition of a logic one uses as long as these logics are included and some basic closure properties are respected. We use $\mathcal{L} \leq \mathcal{L}'$ to denote the sublogic relation. Let $\mathcal{P}$ be a property of logics. A logic $\mathcal{L}^*$ is *strongest extension of* $\mathcal{L}$ *with* $\mathcal{P}$, if

1. $\mathcal{L} \leq \mathcal{L}^*$,

2. $\mathcal{L}^*$ has property $\mathcal{P}$,

and whenever $\mathcal{L}$ is a sublogic $\mathcal{L}'$ and $\mathcal{L}'$ has property $\mathcal{P}$, then $\mathcal{L}' \leq \mathcal{L}^*$.

Let $\mathcal{L}$ be a logic and κ and λ infinite cardinals. $\mathcal{L}$ is (κ, λ)-*compact* if for all $\Phi \subseteq \mathcal{L}$ of power κ, if each subset of Φ of cardinality $< \lambda$ has a model, then Φ has a model. $\mathcal{L}$ is κ-*compact* if it is (κ, ω)-compact. $\mathcal{L}$ is *weakly* κ-*compact* if $\mathcal{L}$ is (κ, κ)-compact. $\mathcal{L}$ is *fully compact* if it is κ-compact for all κ. $\mathcal{L}$ has the *Löwenheim-Skolem property down to* κ, denoted by $\mathrm{LS}(\kappa)$ if every $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $\leq \kappa$. If every sentence $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $< \kappa$, we say that $\mathcal{L}$ satisfies $\mathrm{LS}(< \kappa)$. The order-type of the well-ordering R is denoted by $\mathrm{otp}(R)$.

Theorem 1 *[4] The logic $L_{\omega\omega}$ is strongest extension of $L_{\omega\omega}$ with $\aleph_0$-compactness and $LS(\aleph_0)$.*

Let C be a class of cardinals. Let

$$\mathfrak{A} \models \mathrm{Q}^{\mathrm{cf}}_C xy\phi(x, y, \vec{z}) \iff \mathrm{otp}(\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}) \in C.$$

By [9], $L_{\omega\omega}(\mathrm{Q}^{\mathrm{cf}}_C)$ is always fully compact. For C an interval we use the notation $\mathrm{Q}^{\mathrm{cf}}_{[\kappa,\lambda)}$ and $\mathrm{Q}^{\mathrm{cf}}_{[\kappa,\lambda]}$.

Proposition 2 *There is no strongest κ-compact extension of $L_{\omega\omega}$. In fact:*

1. *there are fully compact logics $\mathcal{L}_n$, $n < \omega$, such that $\mathcal{L}_n \leq \mathcal{L}_{n+1}$ for all $n < \omega$, but no $\aleph_0$-compact logic extends each $\mathcal{L}_n$.*

2. *There is an $\aleph_0$-compact logic $\mathcal{L}_1$ and a fully compact logic $\mathcal{L}_2$ such that no $\aleph_0$-compact logic extends both $\mathcal{L}_1$ and L_2.*

Proof. Let $\mathcal{L}_n = L_{\omega\omega}(\{\mathrm{Q}^{\mathrm{cf}}_{[\aleph_\omega,\infty]}\} \cup \{\mathrm{Q}^{\mathrm{cf}}_{\aleph_l} : l < n\})$. By [9], each $\mathcal{L}_n$ is fully compact. Clearly, no $\aleph_0$-compact logic can extend each $\mathcal{L}_n$.

For the second claim, let $\mathcal{L}_1$ be the logic $L_{\omega\omega}(Q_1)$, where Q_1 is the quantifier "there exists uncountable many" introduced by Mostowski [8]. This logic is $\aleph_0$-compact [3], see [2, Chapter IV] for more recent results. Let $\mathcal{L}_2$ be the logic $L_{\omega\omega}(Q_B)$, where Q_B is the quantifier "there is a branch" introduced by Shelah [10]. More exactly,

$$Q_B xytuM(x)T(y)(t \leq u)$$

if and only if $\leq_T$ is a partial order of $T \subseteq M$ and there are D,$\leq_D$, f and B such that:

1. $\leq_D$ is a total order of $D \subseteq M$
2. $f : \langle T, \leq_T\rangle \to \langle D, \leq_D\rangle$ is strictly increasing
3. $\forall s \in D \exists p \in T(f(p) = s)$
4. $B \subseteq T$ is totally ordered by $\leq_T$
5. $\forall b \in B((p \in T \& p \leq_T b) \to p \in B)$
6. $\forall s \in D \exists b \in B(s \leq_D f(b))$.

The reader is referred to [10] for a proof of the full compactness of $\mathcal{L}_2$.

Suppose there were an $\aleph_0$-compact logic $\mathcal{L}$ containing both $\mathcal{L}_1$ and $\mathcal{L}_2$ as a sublogic. It is easy to see that the class of countable well-orders can be expressed as a relativized pseudoelementary class in $\mathcal{L}$. This contradicts $\aleph_0$-compactness of $\mathcal{L}$. $\square$

Lauri Hella pointed out that by elaborating the proof of claim (2) of the above proposition, we can make $\mathcal{L}_1$ fully compact. It was proved in [11] that, assuming GCH, there is no strongest extension of $\mathcal{L}_{\omega\omega}$ which is $\aleph_0$-compact. Our proof of (1) of the above proposition essentially occurs in a note, based on a suggestion of Paolo Lipparini, added after Theorem 8 of [11].

Proposition 3 *Suppose $\kappa > \aleph_0$. There is no strongest extension of $L_{\kappa^+\omega}$ with $LS(\kappa)$*

Proof. Let $\mathcal{L}_1 = L_{\kappa^+\omega}(\mathrm{Q}^{\mathrm{cf}}_{\aleph_0})$ and $\mathcal{L}_2 = L_{\kappa^+\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. By using standard arguments with elementary chains of submodels, it is easy to see that both $\mathcal{L}_1$ and $\mathcal{L}_2$ have LS(κ), but the consistent sentence

$$R \text{ is a linear order with no last element } \wedge$$

$$\neg \mathrm{Q}^{\mathrm{cf}}_{\aleph_0} xyR(x,y) \wedge \neg \mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]} xyR(x,y)$$

has no models of size $\leq \kappa$. □

It was proved in [11] that there is no strongest extension of $\mathcal{L}_{\omega\omega}$ with LS(ω).

Lemma 4 *Suppose κ is weakly compact. Then $L_{\kappa\omega}(Q^{cf}_{\{\aleph_0\}})$ and $\mathcal{L}_{\kappa\omega}(Q^{cf}_{[\aleph_1,\kappa]})$ are weakly κ-compact. Moreover, if $\kappa > \omega$, these logics satisfy LS(κ).*

Proof. The claim concerning LS(κ) is proved with a standard elementary chain argument. We prove the weak compactness of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. The case of $L_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{\{\aleph_0\}})$ is similar, but easier. For this end, suppose T is a set of sentences of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ and $|T| = \kappa$. We may assume $T \subseteq \kappa$. If $\alpha < \kappa$, then we assume that there is a model $\mathfrak{M}_\alpha \models T \cap \alpha$. In view of LS($\kappa$), it is not a loss of generality to assume that $\mathfrak{M}_\alpha = \langle \kappa, R_\alpha \rangle$, where $R_\alpha \subseteq \kappa \times \kappa$. Let $R(\alpha,\beta,\gamma) \iff R_\alpha(\beta,\gamma)$. By weak compactness there is a transitive M of cardinality κ such that

$$\langle H(\kappa), \epsilon, T, R \rangle \prec_{L_{\kappa\kappa}} \langle M, \epsilon, T^*, R^* \rangle$$

and $\kappa \in M$. Let $\mathfrak{M} = \langle M, S \rangle$, where $S(x,y) \iff R^*(\kappa, x, y)$. We claim that $\mathfrak{M} \models T$. We need only worry about the cofinality-quantifier. Cofinalities $< \kappa$ can be expressed in $L_{\kappa\kappa}$, so they are preserved both ways. Therefore also cofinality κ is preserved, and no other cofinalities can occur as the models have cardinality κ. □

Since the logics $L_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{\aleph_0}))$ and $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ cannot both be a sublogics of a logic with LS(κ), we get from the above lemma:

Corollary 5 *Suppose $\kappa > \omega$ is weakly compact. Then there is no strongest weakly κ-compact extension of $L_{\kappa\omega}$ with LS(κ).*

The logic $L_{\kappa\omega}$ actually satisfies the property LS($< \kappa$) which is stronger than LS(κ). To prove a result like the above corollary for the property

LS($<\kappa$) we have to work a little harder. At the same time we extend the proof to extensions of $L_{\kappa\kappa}$. Here the cofinality quantifiers $\mathrm{Q}^{\mathrm{cf}}_C$ will not help as $\mathrm{Q}^{\mathrm{cf}}_{\{\lambda\}}$ is definable in $L_{\kappa\kappa}$ for $\lambda<\kappa$. Therefore we use more refined order-type quantifiers.

Definition 6 *Let $L_{\kappa\lambda}(Q)$ denote the formal extension of $L_{\kappa\lambda}$ by the generalized quantifier symbol $Qxy\phi(x,y,\vec{z})$. If $\mathcal{Y}$ is a class of ordinals, we get a logic $L_{\kappa\lambda}(Q,\mathcal{Y})$ from $L_{\kappa\lambda}(Q)$ by defining the semantics by*

$$\mathfrak{A}\models Qxy\phi(x,y,\vec{c}) \iff otp(\{\langle a,b\rangle : \mathfrak{A}\models\phi(a,b,\vec{c})\})\in\mathcal{Y}.$$

If $\phi\in L_{\kappa\lambda}(Q,\mathcal{Y})$ and $\mathfrak{A}\models\phi$, we say that $\mathfrak{A}\models\phi$ holds in the $\mathcal{Y}$-interpretation.

If $\mathfrak{A}$ is a model, then

$$o(\mathfrak{A},\mathcal{Y},\kappa,\lambda)$$

is the supremum of all $\mathrm{otp}(\{\langle a,b\rangle : \mathfrak{A}\models\phi(a,b,\vec{c})\})$ where $\phi\in L_{\kappa\lambda}(\mathcal{Y})$, $\vec{c}\in A^{<\lambda}$ and $\{\langle a,b\rangle : \mathfrak{A}\models\phi(a,b,\vec{c})\}$ is well-ordered.

Lemma 7 *Suppose $\kappa\geq\lambda$, $\phi\in L_{\kappa\lambda}(Q)$, $\mathfrak{A}$ is a model, $\vec{a}\in A^{<\lambda}$, and $\mathcal{Y}'\cap o(\mathfrak{A},\mathcal{Y},\kappa,\lambda)=\mathcal{Y}$. Then $\mathfrak{A}\models\phi(\vec{a})$ in the $\mathcal{Y}$-interpretation if and only if $\mathfrak{A}\models\phi(\vec{a})$ in the $\mathcal{Y}'$-interpretation.*

Proof. This is a straightforward induction of the length of the formula ϕ.$\square$

Lemma 8

1. *Suppose $\kappa>\omega$, $\phi\in L_{\kappa\kappa}(Q)$, and ϕ has a model $\mathfrak{A}$ in the $\mathcal{Y}$-interpretation. Then there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq 2^\kappa$ and $\mathcal{Y}'\subseteq(2^\kappa)^+$ such that $\mathcal{Y}'\cap\kappa=\mathcal{Y}$ and $\mathfrak{B}\models\phi$ in the $\mathcal{Y}'$-interpretation.*
2. *Suppose $\kappa=\kappa^{<\kappa}$, $T\subseteq L_{\kappa\kappa}(Q)$, $|T|\leq\kappa$ and T has a model $\mathfrak{A}$ in the $\mathcal{Y}$-interpretation. Then for all $\xi<\kappa^+$ there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq\kappa$ and $\mathcal{Y}'\subseteq\kappa^+$ such that $\mathcal{Y}\cap\xi=\mathcal{Y}'\cap\xi$ and $\mathfrak{B}\models T$ in the $\mathcal{Y}'$-interpretation.*

Proof. We may assume $|A|\geq 2^\kappa$. Let us expand $\mathfrak{A}$ by

1. A well-ordering $\prec$ the order-type of which exceed all the order-types of well-orderings definable by subformulas of ϕ with parameters in A.
2. A new predicate P which contains those elements d of A for which $\mathrm{otp}(\{\langle a,b\rangle : a\prec b\prec d\})\in\mathcal{Y}$

3. A prediacte F which codes an isomorphism from each well-ordering, definable by a subformula of ϕ with parameters in A, onto an initial segment of $\prec$.

Let $\langle \mathfrak{A}, \prec, P, F\rangle$ be the expanded structure and $\langle \mathfrak{B}, \prec^*, P^*, F^*\rangle$ an $L_{\kappa\kappa}$-elementary substructure of it of cardinality $\leq 2^\kappa$. Let

$$\mathcal{Y}' = \{\text{otp}(\{\langle a, b\rangle \in B^2 : a \prec^* b \prec^* d\} : d \in P^*\}$$

It is easy to see that $\mathfrak{B} \models \phi$ in the $\mathcal{Y}'$-interpretation. The second claim is proved similarly. $\Box$

Let π be a canonical bijection from triples of ordinals to ordinals such that $\pi[\kappa^3] = \kappa$ for each infinite cardinal κ. We say that a pair (δ_1, Z_1) *codes* a pair (δ_2, Z_2) if δ_1, δ_2 are ordinals, $Z_1 \subseteq \delta_1$, $Z_2 \subseteq \delta_2$ and there is a bijection $f : \delta_2 \to \delta_1$ such that

1. δ_1 is closed under π
2. $\pi(0, \alpha, \beta) \in Z_1 \iff f(\alpha) < f(\beta)$
3. $\pi(1, 0, \alpha) \in Z_1 \iff f(\alpha) \in Z_2$.

Suppose κ is an uncountable cardinal. The *weakly compact ideal* on κ is the ideal of subsets of κ generated by the sets $\{\alpha : (H(\alpha), \epsilon, A \cap H(\alpha)) \models \neg\phi\}$, where $A \subseteq H(\kappa)$ and ϕ is a Π^1_1-sentence such that $(H(\alpha), \epsilon, A \cap H(\alpha)) \models \phi$.

Definition 9 *A cardinal κ satisfies $\diamondsuit$(WC) if it is weakly compact and there is a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ such that*

1. *$A_\alpha \subseteq \alpha$ for $\alpha < \kappa$.*
2. *$(\forall A \subseteq \kappa)(\{\lambda < \kappa : A_\lambda = A \cap \lambda\} \in \mathcal{I}^+)$, where $\mathcal{I}$ is the weakly compact ideal on κ.*

We make the following remarks without proof: If κ is measurable $> \omega$, then κ satisfies $\diamondsuit$(WC). If κ is weakly compact $> \omega$, then there is a generic extension which preserves all cardinals and in which κ satisfies $\diamondsuit$(WC). If V=L, then every weakly compact cardinal $> \omega$ satisfies $\diamondsuit$(WC).

Theorem 10 *Suppose $\kappa > \omega$ satisfies $\diamondsuit$(WC) and $2^\kappa = \kappa^+$. Then there is no strongest extension of $L_{\kappa\kappa}$ for which κ is weakly compact and which has LS($< \kappa$).*

Proof. We shall construct two sets $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \kappa^+$ such that the logics $L_{\kappa\kappa}(Q, \mathcal{Y}^i)$ are weakly κ-compact for and these logics satisfy LS($< \kappa$), but no logic containing both $L_{\kappa\kappa}(Q, \mathcal{Y}^1)$ and $L_{\kappa\kappa}(Q, \mathcal{Y}^2)$ satisfies LS($< \kappa$). The sets $\mathcal{Y}^i$ are constructed by induction together with ordinals $\xi^i_\alpha < \kappa^+$ such that:

$$\begin{array}{ll}
\mathcal{Y}^i = \bigcup_{\alpha<\kappa^+} \mathcal{Y}^i_\alpha & \\
\mathcal{Y}^i_0 = \emptyset & \xi^i_0 = 0 \\
\mathcal{Y}^i_\alpha = \mathcal{Y}^i_\beta \cap \xi^i_\alpha & \text{for } \alpha < \beta \\
\xi^i_\alpha \leq \xi^i_\beta & \text{for } \alpha < \beta \\
\mathcal{Y}^i_\nu = \bigcup_{\alpha<\nu} \mathcal{Y}^i_\alpha, & \xi^i_\nu = \bigcup_{\alpha<\nu} \xi^i_\alpha, \text{ for } \nu = \cup\nu \\
\mathcal{Y}^1_\alpha \cap \mathcal{Y}^2_\alpha = \emptyset & \text{for } \alpha < \kappa
\end{array}$$

First we define $\mathcal{Y}^i_\alpha$ for $\alpha < \kappa$ in such a way that $L_{\kappa\kappa}(\mathcal{Y}^i)$ will in the end have the property LS($< \kappa$).

Let S_1, S_2 be a partition of the set of cardinals $< \kappa$ into two stationary sets. Let $\{\phi^i_\nu : \nu \in S_i\}$ list all $L_{\kappa\kappa}(Q)$-sentences so that each sentence is listed as ϕ^i_ν for stationary many $\nu \in S_i$.

Suppose $\alpha = \lambda + 1$ and $\xi^i_\lambda = \lambda$. Suppose $\lambda \in S_i$.

Case 1. Suppose that (λ, A_λ) codes some pair (ξ, Z). In this case we let

$$\mathcal{Y}^i_\alpha = \mathcal{Y}^i_\lambda \cup (Z \setminus \lambda), \xi^i_\alpha = \xi$$

$$\mathcal{Y}^{3-i}_\alpha = \mathcal{Y}^{3-i}_\lambda.$$

Case 2. Otherwise we let $\xi^i_\alpha = \lambda$, $\mathcal{Y}^i_\alpha = \mathcal{Y}^i_\lambda$, $\mathcal{Y}^{3-1}_\alpha = \mathcal{Y}^{3-i}_\lambda$.

Suppose then $\alpha = \lambda + 2$, $\xi^i_\lambda = \lambda \in S_i$ and we have defined $\xi^i_{\lambda+1}$ and $\mathcal{Y}^i_{\lambda+1}$.

Case 3. The sentence ϕ^i_λ has a model in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \subseteq \kappa^+$ with $\mathcal{Y} \cap \xi^i_{\lambda+1} = \mathcal{Y}^i_{\lambda+1}$. By Lemma 8 part 2, ϕ^i_λ has a model $\mathfrak{A}$ of cardinality $< \kappa$ in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \subseteq \kappa$ of cardinality $< \kappa$ with $\mathcal{Y} \cap \xi^i_{\lambda+1} = \mathcal{Y}^i_{\lambda+1}$. Let μ be minimal such that $\phi^i_\lambda \in \mathcal{L}_{\mu\mu}(\mathcal{Y})$. Let $\xi^i_{\lambda+2} = o(\mathfrak{A}, \mathcal{Y}, \mu, \mu)$ and $\mathcal{Y}^i_{\lambda+2} = \mathcal{Y}$. Let $\mathcal{Y}^{3-i}_{\lambda+2} = \mathcal{Y}^{3-i}_{\lambda+1}$.

Case 4. Otherwise $\xi^i_{\lambda+1} = \xi^i_\lambda$, $\mathcal{Y}^i_\alpha = \mathcal{Y}^i_{\lambda+1}$, $\mathcal{Y}^{3-1}_\alpha = \mathcal{Y}^{3-i}_{\lambda+1}$.

Finally for all other $\alpha \leq \kappa$ we let ξ^i_α and $\mathcal{Y}^i_\alpha$ be defined canonically.

This ends the construction of $\mathcal{Y}^i_\alpha$ for $\alpha \leq \kappa$. Note that $\mathcal{Y}^1_\kappa \cap \mathcal{Y}^2_\kappa = \emptyset$. Moreover, if ϕ^i_ν has a model in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \supseteq \mathcal{Y}^i_\kappa$, then, by construction, ϕ^i_ν has a model of cardinality $< \kappa$ in the $\mathcal{Y}^i_\kappa$-interpretation.

Let $\mathcal{Y}^i_{\kappa+1} = \mathcal{Y}^i_\kappa \cup \{\kappa\}$ and $\xi^i_{\kappa+1} = \kappa + 2$. Next we shall define $\mathcal{Y}^i_\alpha$ and ξ^i_α for $\kappa + 1 < \alpha < \kappa^+$. For this, let $\langle T_\alpha : \kappa < \alpha < \kappa^+\rangle$ enumerate all $L_{\kappa\kappa}(Q)$-theories of cardinality $\leq \kappa$ in a language of cardinality $\leq \kappa$ which satisfy the condintion that every subset of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i_\kappa$-interpretation. Here we use the assumption $2^\kappa = \kappa^+$. We may assume $T_\alpha \subseteq \kappa$ for all α.

Suppose $\mathcal{Y}^i_\beta$ and ξ^i_β have been defined for $\beta < \alpha$. If $\alpha = \cup\alpha$, $\mathcal{Y}^i_\alpha$ and ξ^i_α are defined canonically. So assume $\alpha = \beta + 1$. Let $T : H(\kappa) \to H(\kappa)$ be the function $T(a) = T_\beta \cap a$. If $a \in H(\kappa)$, then $T(a)$ has a model $\mathfrak{B}_a$ in the $\mathcal{Y}^i_\kappa$-interpretation. By construction, we may assume $\mathfrak{B}_a \in H(\kappa)$. Let $B : H(\kappa) \to H(\kappa)$ be the function $B(a) = \mathfrak{B}_a$. Let $Z \subseteq \kappa$ code $(\xi^i_\beta, \mathcal{Y}^i_\beta)$. By $\diamondsuit$(WC), $W = \{\lambda < \kappa : A_\lambda = Z \cap \lambda\} \in \mathcal{I}^+$, where $\mathcal{I}$ is the weakly compact ideal on κ. Let $A : \kappa \to H(\kappa)$ be the function $A(\alpha) = A_\alpha$. By the definition of $\mathcal{I}$, there are a transitive set M and Y^*, R^* such that

$$\langle H(\kappa), \epsilon, A, W, \mathcal{Y}^i_\kappa, B, T\rangle \prec_{\kappa\kappa} \langle M, \epsilon, A^*, W^*, Y^*, B^*, T^*\rangle$$

and $\kappa \in W^*$. Now $A^*(\kappa) = Z$ and, by construction, $Y^* \cap \xi^i_\beta = \mathcal{Y}^i_\beta$

It is clear now that $B(\kappa)$ is a model of T_α in the Y^*-interpretation. By Lemma 8 there is a model $\mathfrak{B}$ of cardinality $\leq \kappa$ of T_β in the Y^{**}-interpretation for some Y^{**} with $Y^{**} \cap \xi^i_\beta = \mathcal{Y}^i_\beta$. Let $\xi^i_\alpha = o(\mathfrak{B}, Y^{**}, \kappa, \kappa)$ and $\mathcal{Y}^i_\alpha = Y^{**} \cap \xi^i_\alpha$.

Finally, let $\mathcal{Y}^i = \bigcup_{\alpha<\kappa^+} \mathcal{Y}^i_\alpha$.

Claim 1. $L_{\kappa\kappa}(\mathcal{Y}^i)$ satisfies the LS($< \kappa$)-property.

Suppose ϕ is a sentence of $L_{\kappa\kappa}(\mathcal{Y}^i)$ with a model. Let $\lambda \in S_i$ such that $\xi^i_\lambda = \lambda$ and $\phi^i_\lambda = \phi$. By the construction of $\mathcal{Y}^i_{\lambda+2}$ there is a model of ϕ of cardinality $< \kappa$.

Claim 2. $L_{\kappa\kappa}(\mathcal{Y}^i)$ is weakly κ-compact.

Suppose $T \subseteq L_{\kappa\kappa}(\mathcal{Y}^i)$ is given and every subset of T_α of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i$-interpretation. Then $T = T_\alpha$ for some α. By construction, every subset of T_α of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i \cap \kappa$-interpretation. Thus the definition of $\mathcal{Y}^i_\alpha$ is made so that T_α has a model $\mathfrak{B}$ in the $\mathcal{Y}$-interpretation for some $\mathcal{Y}$ such that $\mathcal{Y} \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa) = \mathcal{Y}^i \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa)$. Thus by Lemma 7, $\mathfrak{B} \models T_\alpha$ in the $\mathcal{Y}^i$-interpretation. The Claim is proved.

We can now finish the proof of the theorem. In a logic in which both the quantifier $Q_{\mathcal{Y}^1}$ and $Q_{\mathcal{Y}^2}$ are definable, we can say that the order-type of a well-ordering is in $\mathcal{Y}^1 \cap \mathcal{Y}^2$. Thus such a logic cannot satisfy LS($< \kappa$). $\square$

It is interesting to note that a proof like above would not be possible for the following stronger Löwenheim-Skolem property: A *filter-family* is a family $\mathcal{F} = (\mathcal{F}(A))_{A \neq \emptyset}$, where $\mathcal{F}(A)$ is always a filter on the set A. Luosto [6] defines the concept of a (κ^+, ω)-*neat* filter family. We will not repeat the definition here, its elements are closure under bijections, fineness, κ^+-completeness, normality and upward relativizability (all defined in [6]). Suppose $\mathcal{L}$ is a logic of the form $L_{\kappa\lambda}(\vec{Q})$ for some sequence $\vec{Q}$ of generalized quantifiers. We say that $\mathcal{L}$ has the $\mathcal{F}, \kappa$-*persistency property*, if for all models $\mathfrak{A}$ and $B \in \mathcal{F}(A)$, we have $\mathfrak{A} \restriction B \prec \mathfrak{A}$. Luosto proves that if $\mathcal{L}_1$ and $\mathcal{L}_2$ both satisfy the $\mathcal{F}, \kappa$-persistency property, then there is $\mathcal{L}_3$ such that $\mathcal{L}_1 \leq \mathcal{L}_3$, $\mathcal{L}_2 \leq \mathcal{L}_3$ and $\mathcal{L}_3$ satisfies the $\mathcal{F}, \kappa$-persistency property. Lipparini [5] proves a similar result for families of limit ultrafilters related closely to compactness.

Tapani Hyttinen pointed out that the assumption $2^\kappa = \kappa^+$ is not needed in Theorem 10, if κ is assumed to be measurable.

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