On embedding models of arithmetic of cardinality $\aleph_1$ into reduced powers

Juliette Kennedy*
Department of Mathematics
University of Helsinki
Helsinki, Finland

Saharon Shelah†
Institute of Mathematics
Hebrew University
Jerusalem, Israel

September 26, 2020

Abstract

In the early 1970's S.Tennenbaum proved that all countable models of $PA^- + \forall_1 - Th(\mathbb{N})$ are embeddable into the reduced product $\mathbb{N}^\omega/\mathcal{F}$, where $\mathcal{F}$ is the cofinite filter. In this paper we show that if M is a model of $PA^- + \forall_1 - Th(\mathbb{N})$, and $|M| = \aleph_1$, then M is embeddable into $\mathbb{N}^\omega/D$, where D is any regular filter on ω.

1 Preliminaries

Let LA, the language of arithmetic, be the first order language with non-logical symbols $+, \cdot, 0, 1, \leq$. $\mathbb{N}$ denotes the standard LA structure. We shall be concerned with the following theories: The theory $\forall_1$-$Th(\mathbb{N})$, defined as the set of all universal formulas true in the standard LA structure $\mathbb{N}$. Henceforth we refer to theories satisfying $\forall_1$-$Th(\mathbb{N})$ as Diophantine correct. We will also refer to the theory PA^-, which consists of the following axioms ($x < y$ abbreviates $x \leq y \wedge x \neq y$):

1.) $\forall x, y, z((x + y) + z = x + (y + z))$

*Research partially supported by grant 40734 of the Academy of Finland.
†Research partially supported by the United States-Israel Binational Science Foundation. Publication number [728]

2.) $\forall x, y(x + y = y + x)$
3.) $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
4.) $\forall x, y(x \cdot y = y \cdot x)$
5.) $\forall x, y, z(x \cdot (y + z) = x \cdot y + x \cdot z)$
6.) $\forall x((x + 0 = x) \wedge (x \cdot 0 = 0))$
7.) $\forall x(x \cdot 1 = x)$
8.) $\forall x, y, z((x < y \wedge y < z) \rightarrow x < z)$
9.) $\forall x\ x \leq x$
10.) $\forall x, y(x < y \vee x = y \vee y < x)$
11.) $\forall x, y, z(x < y \rightarrow x + z < y + z)$
12.) $\forall x, y, z(0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z)$
13.) $\forall x, y(x < y \rightarrow \exists z(x + z = y))$
14.) $0 < 1 \wedge \forall x(x > 0 \rightarrow x \geq 1)$
15.) $\forall x(x \geq 0)$

Thus the theory PA^- is the theory of nonnegative parts of discretely ordered rings. For interesting examples of these models see [2].

2 The Countable Case

We begin by presenting the two embedding theorems of Stanley Tennenbaum, which represent countable models of PA^- by means of sequences of real numbers. We note that Theorem 1 follows from the $\aleph_1$-saturation of the structure $\mathbb{N}^\omega/\mathcal{F}$ ([1]), however we present Tennenbaum's construction as it constructs the embeddings directly:

We consider first the reduced power (of LA structures) $\mathbb{N}^\omega/\mathcal{F}$, where $\mathcal{F}$ is the cofinite filter in the boolean algebra of subsets of $\mathbb{N}$. Let $\mathbb{A}$ be the standard LA structure with domain all nonnegative real algebraic numbers. We also consider the reduced power $\mathbb{A}^\omega/\mathcal{F}$.

If f is a function from $\mathbb{N}$ to $\mathbb{N}$, let $[f]$ denote the equivalence class of f in $\mathbb{N}^\omega/\mathcal{F}$. We use a similar notation for $\mathbb{A}^\omega/\mathcal{F}$. When no confusion is possible, we will use f and $[f]$ interchangeably.

Theorem 1 *(Tennenbaum) Let M be a countable Diophantine correct model of PA^-. Then M can be embedded in $\mathbb{N}^\omega/\mathcal{F}$.*

Proof. Let $m_1, m_2, \ldots$ be the distinct elements of M. Let $P_1, P_2, \ldots$ be all polynomial equations over $\mathbb{N}$ in the variables $x_1, x_2, \ldots$ such that $M \models$

$P_i(x_1/m_1, x_2/m_2, \ldots)$. Each system of equations $P_1 \wedge \cdots \wedge P_n$ has a solution in M. Thus, by Diophantine correctness, there is a sequence of natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models (P_1 \wedge \cdots \wedge P_n)(x_1/v_1(n), x_2/v_2(n), \ldots).$$

Note that if the variable x_i does not appear in $P_1 \wedge \cdots \wedge P_n$, then the choice of $v_i(n)$ is completely arbitrary. Our embedding $h : M \longrightarrow \mathbb{N}^\omega/\mathcal{F}$ is given by:

$$m_i \longmapsto [\lambda n.v_i(n)].$$

In the figure below, the i-th row is the solution in integers to $P_1 \wedge \cdots \wedge P_n$, and the i-th column "is" $h(m_i)$.

	m_1	m_2	$\ldots$	m_n	$\ldots$
P_1	$v_1(1)$	$v_2(1)$	$\ldots$	$v_n(1)$	$\ldots$
P_2	$v_1(2)$	$v_2(2)$	$\ldots$	$v_n(2)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
P_n	$v_1(n)$	$v_2(n)$	$\ldots$	$v_n(n)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Note that if m_e is the element 0^M of M, then the polynomial equation $x_e = 0$ appears as one of the P's. It follows that, for n sufficiently large, $v_e(n) = 0$. Thus $h(0^M)$ is the equivalence class of the zero function. Similarly, h maps every standard integer of M to the class of the corresponding constant function.

We show that h is a homomorphism. Suppose $M \models m_i + m_j = m_k$. Then the polynomial $x_i + x_j = x_k$ must be one of the P's, say P_r. If $n \geq r$, then by construction $v_i(n) + v_j(n) = v_k(n)$. Hence $\mathbb{N}^\omega/\mathcal{F} \models h(m_i) + h(m_j) = h(m_k)$, as required. A similar argument works for multiplication. Suppose $M \models m_i \leq m_j$. By an axiom of PA^-, for some k, $M \models m_i + m_k = m_j$. Thus, as we have shown, $h(m_i) + h(m_k) = h(m_j)$. It follows from the definition of the relation $\leq$ in $\mathcal{N}$ that $h(m_i) \leq h(m_j)$.

To see that h is one to one, suppose that $m_i \neq m_j$. Since in models of PA^- the order relation is total, we may assume that $m_i < m_j$. Again by the axioms of PA^-, we can choose m_k such that $m_i + m_k + 1 = m_j$. As we have shown, $\mathbb{N}^\omega/\mathcal{F} \models h(m_i) + h(m_k) + h(1) = h(m_j)$. Since h(1) is the class of the constant function 1, it follows that $h(m_i) \neq h(m_j)$. $\Box$

Corollary 2 *Let M be a countable model of the $\forall_1$-$Th(N)$. Then M can be embedded in $\mathcal{N}$.*

Proof. The models of the $\forall_1$-$Th(N)$ are precisely the substructures of models of $Th(N)$. Thus, M extends to a model of PA^-, which can be embedded in $\mathcal{N}$ as in Theorem 1. □

Before turning to the theorem for the non-Diophantine correct case, we observe first that the given embedding depends upon a particular choice of enumeration $m_1, m_2, \ldots$ of M, since different enumerations will in general produce different polynomials. We also note that different choices of solution yield different embeddings. Also, as we shall see below, we need *not* restrict ourselves to Diophantine formulas: we can carry out the construction for LA formulas of any complexity which hold in M.

We state the non-Diophantine correct case of the theorem:

Theorem 3 *(Tennenbaum) Let M be a countable model of PA^-. Then M can be embedded in $\mathbb{A}^\omega/\mathcal{F}$.*

Proof. Given an enumeration $m_1, m_2, \ldots$ of M, we form conjunctions of polynomial equations P_n exactly as before. We wish to produce solutions of $P_1 \wedge \cdots \wedge P_n$ in the nonnegative algebraic reals for each n. We proceed as follows: The model M can be embedded in a real closed field F by a standard construction. (Embed M in an ordered integral domain, then form the (ordered) quotient field, and then the real closure.) Choose k so large that $x_1, \ldots, x_k$ are all the variables that occur in the conjunction $P_1 \wedge \cdots \wedge P_n$. The sentence $\exists x_1 \ldots x_k (P_1 \wedge \cdots \wedge P_n \wedge x_1 \geq 0 \wedge x_2 \geq 0 \cdots \wedge x_k \geq 0)$ is true in M, hence in F. It is a theorem of Tarski that the theory of real closed fields is complete. Thus, this same sentence must be true in the field of real algebraic numbers. This means we can choose nonnegative algebraic real numbers $v_1(n), v_2(n) \ldots$ satisfying the conjunction $P_1 \wedge \cdots \wedge P_n$. Let $h : M \longrightarrow \mathbb{A}^\omega/\mathcal{F}$ be given by

$$m_i \longmapsto [\lambda n. v_i(n)].$$

The proof that h is a homomorphism, and furthermore an embedding, proceeds exactly as before, once we note that the equivalence classes all consist of nonnegative sequences of real algebraic numbers. □

Remark 4 *Under any of the embeddings given above, if $M \models PA^-$ then nonstandard elements of M are mapped to equivalence classes of functions tending to infinity. Why? If f is a function in the image of M, and f does not tend to infinity, then choose an integer k such that f is less than k infinitely often. Since $M \models PA^-$, either $[f] \leq [k]$ or $[k] \leq [f]$. The second alternative contradicts the definition of $\leq$ in $\mathcal{N}$. Hence $[f] \leq [k]$, i.e., $[f]$ is standard.*

Remark 5 *Let F be a countable ordered field. Then F is embedded in $\mathbb{R}^\omega/\mathcal{F}$, where $\mathbb{R}$ is the field of real algebraic numbers. The proof is mutatis mutandis the same as in Theorem 3, except that due to the presence of negative elements we must demonstrate differently that the mapping obtained is one to one. But this must be the case, since every homomorphism of fields has this property.*

Remark 6 *For any pair of LA structures A and B satisfying PA^-, if A is countable and if A satisfies the $\forall_1$-$Th(B)$ then there is an embedding of A into $B^\omega/\mathcal{F}$. In particular, if M is a model of PA^-, then every countable extension of M satisfying the $\forall_1$-$Th(M)$ can be embedded in $M^\omega/\mathcal{F}$.*

Remark 7 *Given Theorem 1, one can ask, what is a necessary and sufficient condition for a function to belong to a model of arithmetic inside $\mathcal{N}$? For a partial solution to this question, see [3].*

3 The Uncountable Case

We now show that some of the restrictions of Theorem 1 can be to some extent relaxed, i.e. we will prove Theorem 1 for models M of cardinality $\aleph_1$ and with an arbitrary regular filter D in place of the cofinite filter. We note that for filters D on ω for which B^ω/D is $\aleph_1$-saturated, where B is the two element Boolean algebra, this follows from the result of Shelah in [5], that the reduced power $\mathbb{N}^\omega/D$ is $\aleph_1$-saturated.

Our strategy is similar to the strategy of the proof of Theorem 1, in that we give an inductive proof on larger and larger initial segments of the elementary diagram of M. However we must now consider formulas whose variables are taken from a set of $\aleph_1$ variables. This requires representing each ordinal $\alpha < \omega$ in terms of finite sets u_n^α, which sets determine the variables handled at each stage of the construction. The other technicality we require is the use of the following function $g(x, y)$, which bounds the size of the formulas handled at each stage of the induction.

Let $h(n, m)$ = the total number of non-equivalent Diophantine formulas $\phi(x_1, \ldots x_m)$ of length $\leq n$. Define

$$
\begin{aligned}
g(n, n) &= h(n, 0) \\
g(n, m-1) &= 2 + h(g(n, m), m) \cdot (g(n, m) + 3), \text{ for } m \leq n.
\end{aligned}
$$

Theorem 8 *Let M be a model of PA^- of cardinality $\aleph_1$ which is Diophantine correct and let D be a regular filter on ω. Then M can be embedded in $\mathbb{N}^\omega/D$.*

Proof. Let $M = \{a_\alpha : \alpha < \omega_1\}$. Let $\{A_n\}_{n\in\omega}$ be a family witnessing the regularity of D. We define, for each $\alpha < \omega_1$ a function $f_\alpha \in \mathbb{N}^\mathbb{N}$. Our embedding is then $a_\alpha \mapsto [f_\alpha]$. We need first a lemma:

Lemma 9 *There exists a family of sets u_n^α, with $\alpha < \omega_1$, and $n \in \mathbb{N}$, such that for each n, α*

(i) $|u_n^\alpha| < n + 1$

(ii) $\alpha \in u_n^\alpha \subseteq u_{n+1}^\alpha$

(iii) $\bigcup_n u_n^\alpha = \alpha + 1$

(iv) $\beta \in u_n^\alpha \Rightarrow u_n^\beta = u_n^\alpha \cap (\beta + 1)$

(v) $\lim_{n\to\infty} \frac{|u_n^\alpha|}{n+1} = 0$

Suppose we have the lemma and suppose we have defined f_β for all $\beta < \alpha$. We choose $f_\alpha \in \mathbb{N}^\mathbb{N}$ componentwise, i.e. we choose $f_\alpha(n)$ for each n separately so that $f_\alpha(n)$ satisfies the following condition:

$(*)_{\alpha,n}$**:** If $\phi = \phi(..., x_\beta, ...)_{\beta\in u_n^\alpha}$ is a Diophantine formula, such that the length of ϕ is $\leq g(n, |u_n^\alpha|)$, then

$$M \models \phi(..., a_\beta, ...)_{\beta\in u_n^\alpha} \Rightarrow \mathbb{N} \models \phi(..., f_\beta(n), ...)_{\beta\in u_n^\alpha}.$$

(A formula ϕ is said to be Diophantine if it has the form

$$\exists x_0, \ldots, \exists x_{n-1}(t_1(x_0, \ldots, x_{n-1}) = t_2(x_0, \ldots, x_{n-1})),$$

where t_1 and t_2 are LA-terms.) Now suppose $\alpha = 0, n > 0$. By (ii) and (iii), $u_n^0 = \{0\}$ for all n. We claim that $(*)_{0,n}$ holds for each n. To see this, fix n and let

$$\Phi \quad = \quad \{\phi(x) \mid M \models \phi(a_0), \text{where } \phi \text{ is Diophantine with } |\phi| \leq g(n,1)\}.$$

This is – up to equivalence – a finite set of formulas. Also, $M \models \exists x \bigwedge\{\phi(x) \mid \phi \in \Phi\}$ and therefore by Diophantine correctness $\mathbb{N} \models \exists x \bigwedge\{\phi(x) \mid \phi \in \Phi\}$. If k witnesses this formula, set $f_0(n) = k$. Clearly now $(*)_{0,n}$ holds.

Now assume $(*)_{\beta,n}$ holds for all $\beta < \alpha$ and for each n. We choose $f_\alpha(n)$ for each n as follows. Fix $n < \omega$ and let

$$\begin{aligned} \Phi \quad = \quad & \{\phi(x_0, x_1, \ldots x_k) \mid M \models \phi(a_\alpha, ..., a_\beta, ...)_{\beta \in u_n^\alpha \setminus \{\alpha\}}), \\ & \text{where } \phi \text{ is Diophantine and } |\phi| \leq g(n, |u_n^\alpha|)\}, \end{aligned}$$

for $k = |u_n^\alpha| - 1$ (the case that $|u_n^\alpha| \leq 1$ reduces to the previous case). This is again a finite set of formulas, up to equivalence. Now

$$M \models \bigwedge\{\phi(a_\alpha, \langle a_\beta\rangle_{\beta \in u_n^\alpha \setminus \{\alpha\}})|\phi \in \Phi\}$$

and therefore

$$M \models \exists x \bigwedge\{\phi(x, \langle a_\beta\rangle_{\beta \in u_n^\alpha \setminus \{\alpha\}})|\phi \in \Phi\}.$$

Let $\phi'(x_1, \ldots, x_k) \equiv \exists x \bigwedge\{\phi(x, x_1, \ldots, x_k)|\phi \in \Phi\}$. Note that

$$|\phi'| \leq 2 + h(g(n, |u_n^\alpha|), |u_n^\alpha|) \cdot (g(n, |u_n^\alpha|) + 3) = g(n, |u_n^\alpha| - 1).$$

Now let $\gamma = \max\ (u_n^\alpha \setminus \{\alpha\})$. Then $u_n^\gamma = u_n^\alpha \setminus \{\alpha\}$, by (iv) of the lemma. Since $(*)_{\gamma,n}$ holds for each n and we know that $M \models \phi'(\langle a_\beta\rangle_{\beta \in u_n^\gamma})$ then we know by the induction hypothesis that $\mathbb{N} \models \phi'(\langle f_\beta(n)\rangle_{\beta \in u_n^\gamma})$. Thus $\mathbb{N} \models \exists x \bigwedge\{\phi(x, ..., f_\gamma(n), ...)_{\beta \in u_n^\gamma})|\phi \in \Phi\}$. Let k witness this formula and set $f_\alpha(n) = k$. Then $(*)_{\alpha,n}$ holds with $a_\alpha \mapsto [\langle f_\alpha(n)\rangle]$.

This mapping is an embedding of M into $\mathbb{N}^\omega/D$. To see this, suppose $M \models \phi(a_\alpha, a_\beta, a_\gamma)$ where ϕ is the formula $x_0 + x_1 = x_2$. Let $n = n_0$ be large enough so that $|\phi| \leq g(n_0, |u^\delta_{n0}|)$, where δ is chosen so that $u^\delta_{n_0}$ contains α, β, γ. Then for all $n \geq n_0$, $(*)_{\delta,n}$ holds and $\mathbb{N} \models f_\alpha(n) + f_\beta(n) = f_\gamma(n)$, for all $n \geq n_0$. Let A_{n_1} be an element of the chosen regular family of D such that $A_{n_1} \cap \{0, \ldots, n_0\} = \emptyset$. Then since $f_\alpha(n) + f_\beta(n) = f_\gamma(n)$, for $n \geq n_0$, this holds also for $n \in A_{n_1}$. The proof that multiplication is preserved is the same so we omit it. Finally, as in the countable case, we note that our mapping is one-to-one and hence an embedding.

We now prove the lemma, by induction on α. Let $u^0_n = \{0\}$ for all $n < \omega$.

Case 1. α is a successor ordinal, i.e. $\alpha = \beta + 1$. Let n_0 be such that $n \geq n_0$ implies $\frac{|u^\beta_n|}{n} < \frac{1}{2}$. then we set

$$u^\alpha_n = \begin{cases} \{\alpha\} & n < n_0 \\ u^\beta_n \cup \{\alpha\} & n \geq n_0. \end{cases}$$

Then (i), (ii) and (iii) are trivial. Proof of (iv): Suppose $\gamma \in u^\alpha_n = u^\beta_n \cup \{\alpha\}$.

Case 1.1. $\gamma = \alpha$. Then

$$\begin{aligned} u^\gamma_n &= u^\alpha_n \\ &= u^\alpha_n \cap \alpha + 1 \\ &= u^\alpha_n \cap (\gamma + 1). \end{aligned}$$

Case 1.2. $\gamma \in u^\beta_n$. Then by the induction hypothesis

$$\begin{aligned} u^\gamma_n &= u^\beta_n \cap (\gamma + 1) \\ &= u^\alpha_n \cap (\gamma + 1), \end{aligned}$$

since $\gamma \in u^\beta_n$ implies $\gamma \leq \beta$.

Proof of (v):

$$\lim_{n\to\infty} \frac{|u^\alpha_n|}{n} \leq \lim_{n\to\infty} \frac{|u^\beta_n| + 1}{n} = 0,$$

by the induction hypothesis.

Case 2. δ is a limit ordinal > 0. Let δ_n be an increasing cofinal ω-sequence converging to δ, for all $\delta < \omega_1$. Let us choose natural numbers $n_0, n_1, \ldots$ such that $\delta_i \in u_{n_i}^{\delta_{i+1}}$ and $n \geq n_{i+1}$ implies $n_i \cdot |u_n^{\delta_{i+1}}| < n$. Now we let

$$u_n^\alpha = u_n^{\delta_i} \cup \{\alpha\}, \text{ if } n_i \leq n < n_{i+1}.$$

To prove (iv), let $\gamma \in u_n^\alpha$. We wish to show that $u_n^\gamma = u_n^\alpha \cap (\gamma + 1)$. Suppose $n_i \leq n < n_{i+1}$. Then $u_n^\alpha = u_n^{\delta_i} \cup \{\alpha\}$. But then $\gamma \in u_n^\alpha$ implies $\gamma \in u_n^{\delta_i} \cup \{\alpha\}$. If $\gamma = \alpha$, then as before, $u_n^\gamma = u_n^\alpha \cap (\gamma + 1)$. So suppose $\gamma \in u_n^{\delta_i}$. Then

$$u_n^\gamma = u^{\delta_i} \cap (\gamma + 1) = u_n^\alpha \cap (\gamma + 1).$$

Finally, we prove (iii): Let $\gamma \in u_n^\delta$, δ a limit. Let n_i be such that $u_n^\alpha = u_{n_i}^{\delta_i} \cup \{\alpha\}$. We have $u_{n_i}^{\delta_i} = u_{n_i}^{\delta_{i+1}} \cap (\delta_i + 1)$. Therefore $\gamma \in u_{n_i}^{\delta_{i+1}} \cup \{\delta\} = u_{n+1}^\delta$. To prove (v), let $\epsilon > 0$ and choose i so that $\epsilon \cdot n_i > 2$. We observe that $n_i \leq n$ implies

$$\lim_{n \to \infty} \frac{|u_n^\alpha|}{n} = \lim_{n \to \infty} \frac{|u_n^{\delta_i}|}{n} + \frac{1}{n} < \frac{1}{n_i} + \frac{1}{n} < \epsilon.$$

□

In [4] models of higher cardinality are considered, and embedding theorems are obtained under a set theoretic assumption.

References

[1] B. Jónsson and P. Olin, Almost direct products and saturation, Compositio Math., 20, 1968, 125–132

[2] R. Kaye. *Models of Peano Arithmetic.* Oxford Logic Guides. Oxford: Oxford University Press, 1991.

[3] J. Kennedy. On embedding models of arithmetic into reduced powers. Ph.D. thesis, City University of New York Graduate Center, 1996.

[4] J. Kennedy and S. Shelah, On regular reduced products, to appear.

[5] S. Shelah, For what filters is every reduced product saturated?, Israel J. Math., 12, 1972, 23–31