ON THE EXISTENCE OF RIGID $\aleph_1$–FREE ABELIAN GROUPS OF CARDINALITY $\aleph_1$

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1. Introduction

An abelian group is said to be $\aleph_1$–free if all its countable subgroups are free. A crucial special case of our main result can be stated immediately.
Indecomposable $\aleph_1$–free abelian groups of cardinality $\aleph_1$ do exist.
The first example of any $\aleph_1$–free group which is not free is the Baer–Specker group $\mathbb{Z}^\omega$, which is the cartesian product of countably many copies of the group $\mathbb{Z}$ of integers, known for almost sixty years; cf. Baer [1] or [14, p.94]. Assuming CH, this group of cardinality $2^{\aleph_0} = \aleph_1$ is an example of a non–free abelian group of cardinality $\aleph_1$. Under the same set–theoretic assumption of the continuum hypothesis it can be shown that any countable ring R with free additive group can be realized as the endomorphism ring of an $\aleph_1$–free abelian group G of cardinality $\aleph_1$. The chronologically earlier realization theorem of this kind uses the weak diamond prediction principle which follows from $2^{\aleph_0} < 2^{\aleph_1}$, cf. Devlin and Shelah [6] for the weak diamond, Shelah [28] for the case End $G = \mathbb{Z}$ and Dugas, Göbel [7] for the case $R =$ End G and extensions to larger cardinals. Using, what is called Shelah's Black Box, the existence of $\aleph_1$–free groups G with $|G| = 2^{\aleph_0}$ also follows from Corner, Göbel [5] using Dugas, Göbel [7] and combinatorial fine tuning from Shelah [29].

Without the assumption of CH, the existence of non–free, $\aleph_1$–free groups of cardinality $\aleph_1$ follows from a more general result by Griffith [18], Hill [21], Eklof [11], Mekler [24] and Shelah in Eklof [12, p.82,Theorem 8.8]. By an induction it can be shown, that there are $\aleph_n$–free groups, non–free of cardinality $\aleph_n$. The non–abelian case is due to Higman [19, 20].

By Shelah's singular compactness theorem it is known that λ–free abelian groups of cardinality λ do not exist if λ is singular, e.g. if $\lambda = \aleph_\omega$, cf. Eklof, Mekler [13]. Hence induction breaks down and it is more complicated to show the existence of λ–free, non–free abelian groups of cardinality $\lambda > \aleph_\omega$. This question is investigated in Magidor, Shelah [23] and we just refer to this paper and restrict ourselves to cardinals $\lambda \leq 2^{\aleph_0}$ again, and we will focus on $\lambda = \aleph_1$. Only very little is known about algebraic properties of $\aleph_1$–free groups of cardinality $\aleph_1$, see Eklof [11] and Eklof, Mekler [13]. Shelah's construction [27] (see also [30]) of groups also mentioned in [12, 13] which are not separable was refined in Eda [10] prove the existence of

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an $\aleph_1$–free group G of cardinality $\aleph_1$ such that Hom $(G, \mathbb{Z}) = 0$, a result derived independently but later by Corner, Göbel [5]. Counterexamples for Kaplansky's test problems among $\aleph_1$–free groups of cardinality $\aleph_1$ are given recently in Göbel, Goldsmith [17], realizing rings modulo some large ideal, see also [16]. Moreover, $\aleph_1$–separable groups of cardinality $\aleph_1$ serving as counterexamples of Kaplansky's test problems were constructed in [31]. These results about $\aleph_1$–free groups become special cases of our quite satisfying main theorem.

Main Theorem 4.1 *If R is a ring with R^+ free and $|R| < \lambda \leq 2^{\aleph_0}$, then there exists an $\aleph_1$–free abelian group G of cardinality λ with* End $G = R$.

We have identified R with endomorphisms acting on the R–module G by scalar multiplication. This result has many applications. If $R = \mathbb{Z}$, we derive the existence of $\aleph_1$–free abelian groups of cardinality $\aleph_1$, a result which was unknown.
If Γ is any abelian semigroup, then we use Corner's ring R_Γ, implicitly discussed in Corner, Göbel [4], and constructed for particular $\Gamma's$ in [3] with special idempotents (expressed below), with free additive group and $|R_\Gamma| = \max\{|\Gamma|, \aleph_0\}$. If $|\Gamma| < 2^{\aleph_0}$, we may apply the main theorem and find a family of $\aleph_1$–free abelian groups $G_\alpha (\alpha \in \Gamma)$ of cardinality $\aleph_1$ such that for $\alpha, \beta \in \Gamma$,

$$G_\alpha \oplus G_\beta \cong G_{\alpha+\beta} \text{ and } \quad G_\alpha \cong G_\beta \text{ if and only if } \quad \alpha = \beta.$$

Observe that this induces all kinds of counterexamples to Kaplansky's test problems for suitable $\Gamma's$. If we consider Corner's ring in [2], see Fuchs [15, p.145], then it is easy to see that R^+ is free and $|R| = \aleph_0$. The particular idempotents in R and our main theorem provide the existence of an $\aleph_1$–free superdecomposable group of cardinality $\aleph_1$, which was unknown as well. Recall that a group is superdecomposable if any non–trivial summand decomposes into a proper direct sum.
Finally, we remark that as the reader might suspect, it is easy to replace G in Theorem 4.1 by a rigid family of 2^λ such groups with only the trivial homomorphism between distinct members. The main theorem cannot be generalized, replacing $\aleph_1$ by another cardinal. In Section 5 we will show that there are many models of ZFC (e.g. assuming MA and $\aleph_2 < 2^{\aleph_0}$) in which no $\aleph_2$–free group of cardinality $< 2^{\aleph_0}$ has endomorphism ring $\mathbb{Z}$; it is even possible that all such groups are separable and the best one can do now is a realization theorem of the form End $G = R \oplus$ Ines G with Ines $G \neq 0$ an ideal containing all endomorphisms of finite rank.
This is in contrast to the result [7], that under $\diamond_\lambda$ any countable ring R with R^+ free is of the form $R \cong$ End G for all uncountable regular, not weakly compact cardinal $\lambda = |G| > |R|$ such that G is λ–free. In particular, *the existence of indecomposable $\aleph_2$–free groups of cardinality $\aleph_2$ or the existence of such groups with endomorphism ring $\mathbb{Z}$ is undecidable.*

2. The building blocks, $\aleph_1$–free modules with a distinguished cyclic submodule

Let R be a ring of cardinality $|R| < 2^{\aleph_0}$ such that R^+ is a free abelian group. In view of Pontrjagin's theorem we say that an R–module is $\aleph_1$–free if any subgroup of finite rank is contained in a free R–submodule.

We have the immediate application of Pontrjagin's theorem [14, p.93, Theorem 19.1.].

Observation 2.1. *If M is $\aleph_1$–free as R–module and R^+ is free, then M is $\aleph_1$–free as abelian group, this means all countable subgroups are free.*

Remark 2.2. *If U is a finitely generated submodule of an $\aleph_1$–free R–module M of infinite rank and M/U is flat, then M/U is an $\aleph_1$–free R–module as well.*

Proof If S/U is a subgroup of finite rank in M/U, then S_*/U denotes its purification and S_* is a pure subgroup of finite rank in M, hence it is contained in a free R–submodule F of M. Moreover, we find a finitely generated summand F' of the R–module F with $S_* \subseteq F'$ and F/F' is R–free. Also F'/U is flat because M/U is flat and F'/U can be finitely presented by

$$F'' \to F' \to F'/U \to 0$$

for some finitely generated free module F'' mapping onto $U \subseteq F'$. Hence F'/U is projective by Rotman [25, p. 90, 91], and $F/U \cong F/F' \oplus F'/U$ is projective. Finally we may assume that F/U has infinite rank and F/U is free by a well–known argument of Kaplansky's, cf. [17], for instance. Hence M/U must be an $\aleph_1$–free R–module.

Recall that Remark 2.2 does not hold if U is not finitely generated. Consider a free resolution of any torsion–free abelian group A which is not $\aleph_1$–free: $0 \to U \to M \to A \to 0$. By Remark 2.2 in particular quotients of $\aleph_1$–free groups modulo pure, cyclic subgroups are $\aleph_1$–free again.

Next we will construct particular $\aleph_1$–free R–modules A with distinguished cyclic submodules cR.
First we will fix some more notation. Let $\mathcal{P}$ be a family of $2^{\aleph_0}$ almost disjoint infinite subsets of an infinite set of primes. At present, we choose a fixed $X \in \mathcal{P}$ with an enumeration $X = \{p_n : n \in \omega\}$ without repetitions. Let $T = {}^{\omega>}2$ denote the tree of all finite branches $\eta : n \to 2$, $n < \omega$, where $\ell(\eta) = n$ denotes the length of the branch η. The branch of length 0 is denoted by $\perp = \emptyset \in T$ and we also write $\eta = (\eta \restriction n-1){}^\wedge\eta(n-1)$. Finally ${}^\omega 2 = Br(T)$ denotes all infinite branches $\eta : \omega \to 2$ and clearly $\eta \restriction n \in T$ for all $\eta \in Br(T)$, $n \in \omega$.

Let λ be an infinite cardinal $\leq 2^{\aleph_0}$ and $Y \subseteq Br(T)$ with $|Y| = \lambda$ and $|R| < \lambda$. Then V' will denote the vector space over the rationals $\mathbb{Q}$ with basis $T \cup Y$. Finally R becomes a vector space by $R \otimes_{\mathbb{Z}} \mathbb{Q} = \hat{R}$ and $V = V' \otimes_{\mathbb{Q}} \hat{R}$ is a vector space of dimension λ. We now select an R–submodule $A \subseteq V$ which is generated by T together with elements

$$\eta_0 = \eta,\ \eta_{n+1} = \frac{1}{p_n}(\eta_n + \eta \restriction n + \eta(n) \perp) \in V \qquad (X)$$

defined inductively for all $\eta \in Y$, $n \in \omega$. Hence

$$A = A_X = A_{XY} = \langle \sigma R, \eta_n R : \sigma \in T, \eta \in Y, n \in \omega \rangle \subset V$$

depends on $X \in \mathcal{P}$ and $Y \subseteq Br(T)$. The required cyclic R–submodule is $\perp R$. We will show that $(A, \perp R)$ belongs to the category of modules we are interested in, i.e. the following Lemma holds.

Lemma 2.3. *Let $(A, \perp R)$ be the pair of R–modules defined above, let $B = \langle T \rangle$ and ${}^- : A \to A/B$ be the canonical homomorphism. Then we have*
(a) B is a free R–module and $A/B = \bigoplus_{\eta \in Y} \bar{\eta}(\bar{X} \otimes_{\mathbb{Z}} R)$ with $\bar{X} \subseteq \mathbb{Q}$ of characteristic $\chi : \omega \to 2$ with support X.

(b) A is an $\aleph_1$–free R–module.
(c) $A/\perp R$ is an $\aleph_1$–free R–module.

Proof (a) Clearly $B = \bigoplus_{\sigma\in T}\sigma R$ and if $g \in A$, then we use (X) to find $k \in \omega$ and finite sets $T_1 \subseteq T$, $Y_1 \subseteq Y$ with

$$g = \sum_{\eta\in Y_1}\eta_k g_\eta + \sum_{\sigma\in T_1}\sigma g_\sigma$$

for some $g_\eta, g_\sigma \in R$. Using (X) again, we have

$$g \equiv \sum_{Y_1}\eta\frac{g_\eta}{q_k} \mod B$$

where $q_k = \prod_{i=1}^{k-1} p_i$ by the enumeration in X and $\frac{g_\eta}{q_k} \in \bar{X}\otimes R$. Clearly $\{\bar\eta : \eta \in Y\}$ is $\mathbb{Q}\otimes R$–independent and hence $\bar{X}\otimes R$–independent and (a) follows.
(b) Obviously $|A| = |Y| = \lambda$. Next we show that
(*) any finite subset of A lies in a submodule U which is free and pure in A.
For any finite subset E of A we can find some $n \in \omega$ and a finite subset $Y_0 \subseteq Y$ such that

$$E \subseteq U = \langle \sigma R,\ \eta_n R : \sigma \in T,\ \ell(\sigma) < n,\ \eta \in Y_0\rangle.$$

Obviously U is freely generated by the elements σ, η_n. In order to show that U is pure in A, consider $g \in A$ and $m \in \mathbb{N}$ minimal with $gm = u \in U$.
We may write

$$g = \sum_{\eta\in Y_1}\eta_k g_\eta + \sum_{\sigma\in T_1}\sigma g_\sigma \text{ and } \quad u = \sum_{\eta\in Y_2}\eta_n u_\eta + \sum_{\sigma\in T_2}\sigma u_\sigma$$

with $g_\eta, g_\sigma, u_\eta, u_\sigma \in R$ and $k = k(\eta)$ minimal for each $\eta \in Y_1$. Since $gm = u$ we have $Y_1 = Y_2$ and $\eta_n u_\eta = \eta_k g_\eta m$ for all $\eta \in Y_1$ from (a). If $k < n$ for some $\eta \in Y_1 = Y_2$, then we can reduce Y_1 to a smaller set $Y_1\backslash\{\eta\}$ by the observation $\eta_k g_\eta \in U$ and $\eta_k g_\eta m = \eta_n u_\eta$ and $g \in U$ follows by induction. We derive $k \geq n$ for all $\eta \in Y_1$, and suppose $k > n$ for some η.
We have $p_{k-1}|q = \prod_{i=n}^{k-1} p_i$ and minimality of m requires p_{k-1} does not divide m. On the other hand $g_\eta m = q u_\eta$ and $p_{k-1}|q$ hence $p_{k-1}|g_\eta$ which contradicts minimality of $k = k(\eta)$. We derive $k = n$ for all η and g decomposes into a Y–part $g_Y \in U$ with $g_Y m = \sum_{Y_2}\eta_n u_\eta$ and a T–part $g_T \in B$ with $g_T m = \sum_{T_1}\sigma g_\sigma$. However $g_T \in U$, hence $g = g_Y + g_T \in U$ as well and U is pure in A, i.e. (*) holds. Finally A is an $\aleph_1$–free R–module by the argument in Remark 2.2 and Pontrjagin's collection of a direct sum of projective modules, see Fuchs [14, p.93, Theorem 19.1.]. Now (b) and also (c) follow from (*).

Observation 2.4. *If $(A, \perp R)$ is as above, then A and $A/\perp R$ are $\aleph_1$–free abelian groups with $R \subseteq \mathrm{End}\ A,\ R \subseteq \mathrm{End}\ (A/\perp R)$ identifying $r = r\cdot id$ for all $r \in R$.*

Observation 2.4 is immediate from Observation 2.1 and Lemma 2.3, which is all we need in Section 3.
Moreover we will require enough splitting in A which is established by the following

Proposition 2.5. *Let $(A, \perp R)$ be as above, where $A = A_X$, $X \neq P \in \mathcal{P}$ and $\bar{P} = \mathbb{Z}_P$ the obvious localization at P. Then $A_X \otimes R_P$ is a free R_P–module with $\perp$ a basis element, where $R_P = \mathbb{Z}_P \otimes_{\mathbb{Z}} R$ is the localization of R at P.*

Proof Recall that

$$A_X = \langle \sigma R,\ \eta_n R : \sigma \in T,\ \eta \in Y,\ n \in \omega \rangle.$$

Moreover $X \cap P$ is finite by our choice of $\mathcal{P}$. We find $k \in \omega$ such that $\{p_n \in X : n \geq k\} \cap P = \emptyset$. Now we claim that

$$T \cup \{\eta_k : \eta \in Y\}$$

is a basis of the R_P–module $A_X \otimes R_P$. Note that $\perp \in T$ and Proposition 2.5 will follow.

The set $M = T \cup \{\eta_k : \eta \in Y\}$ is clearly independent over $\mathbb{Q} \otimes_{\mathbb{Z}} R$ in V and hence freely generates the R_P–submodule

$$U = \bigoplus_{m \in M} mR_P = F \otimes R_P \subseteq A_X \otimes R_P$$

with $F = \bigoplus_{m \in M} mR$. It remains to show $U = A_X \otimes R_P$.

The submodule $F \subset A_X$ induces a natural sequence

$$0 \to F \to A_X \to A_X/F \to 0$$

of R–modules, where A_X/F is generated by $\{\eta_n + F : \eta \in Y, n > k\}$, see Lemma 2.3(a). Using (X) we derive $p_{n-1} \cdot \ldots \cdot p_{k+1}\eta_n \equiv \eta_k \equiv 0 \mod F$ where the enumeration of primes is taken in X. These primes belong to $\{p_n \in X : n \geq k\}$ and cannot belong to P by our choice of k. We observe that A_X/F is a P'–group in the well–known sense, that A_X/F is torsion and the order of elements is a product of primes in P', the complement of P. On the other hand R_P is P'–divisible, hence $(A_X/F) \otimes R_P = 0$. Using flatness of R_P the above sequence becomes

$$0 \to F \otimes R_P \to A_X \otimes R_P \to (A_X/F) \otimes R_P \to 0$$

and $A_X/F \otimes R_P = 0$ forces $A_X \otimes R_P = F \otimes R_P$ as desired.

3. Repeating the building blocks

Let R, $\mathcal{P}$ and $|R| < \lambda \leq 2^{\aleph_0}$ be as in Section 2. Then we enumerate $\mathcal{P} = \{X_a : \alpha < \lambda\}$ without repetition and it is easy to find a family $\mathcal{F} = \{L_\alpha \subset \omega : \alpha < \lambda\}$ of infinite, almost disjoint subsets L_α of ω without repetitions. Since $Br(T) = {}^{\omega}2$ and $|{}^{\omega}2| = 2^{\aleph_0}$, we can also find a family $\{Y_\alpha \subset Br(T) : \alpha < \lambda\}$ of sets Y_α of branches with the following additional properties

(b1) $|Y_\alpha| = \lambda$ for all $\alpha < \lambda$.

(b2) Y_α has λ branch points above every level:
If $\eta \in Y_\alpha$ and $n \in \omega$, there are λ distinct branches $\nu \in Y_\alpha$ with $\eta \restriction n = \nu \restriction n$.

(b3) The length of a branch point of branches in Y_α is in L_α :
If $\nu \neq \eta \in Y_\alpha$, then $\ell(\nu \cap \eta) \in L_\alpha$.

We use these three families to enumerate a family of R–modules A_{XY} constructed in Section 2 defining $A_\alpha = A_{X_\alpha Y_\alpha}$ for all $\alpha < \lambda$. Moreover we denote $R_{X_\alpha} = R_\alpha$ the localization of R at the primes X_α from Section 2.

Inductively we define an ascending, continuous chain of R–modules G_α $(\alpha < \lambda)$ with distinguished cyclic submodules $c_\alpha R \subset G_\alpha$ for non–limit ordinals $\alpha < \lambda$. The module we are interested in will then be the R–module $G = G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$. If $\alpha = 0$, let $G_0 = \bigoplus_{\nu<\lambda} e_\nu R$ be free R–module of rank λ, which is also a free abelian group of rank λ because R^+ is free of rank $< \lambda$. We will choose elements $c_\alpha \in G_\alpha$ for non–limit ordinals α subject to the following conditions

(c1) $G_\alpha/c_\alpha R$ is an $\aleph_1$–free R–module

(c2) If $c \in G$ and G/cR is an $\aleph_1$–free R–module, then $|\{\alpha < \lambda : c = c_\alpha\}| = \lambda$.
The extension $G_{\alpha+1}$ will be constructed such that condition (c1) ensures that G is $\aleph_1$–free and (c2) can easily be arranged by an enumeration of elements $c \in G_\alpha$ with G_α/cR $\aleph_1$–free with $|\alpha|$ repetitions for all $\alpha < \lambda$. If $\alpha = 0$, then for (c1) we may choose a basic element c_0 and we do not care for (c2).
If $c_\nu \in G_\nu$ are defined for all $\nu < \alpha$ and α is a limit, then $G_\alpha = \bigcup_{\nu<\alpha} G_\nu$ by continuity and it remains to construct G_α from $c_\beta \in G_\beta$ for $\alpha = \beta + 1$. From our choice (c1) of c_β we know that $G_\beta/c_\beta R$ is an $\aleph_1$–free R–module. We consider a pushout diagram. There exists a (unique) pushout R–module G_α with the well–known pushout mapping properties [14, p.52] or [25] in case of R-modules.

$$\begin{array}{ccc} c_\beta R & \longrightarrow & G_\beta \\ \big\downarrow & & \big\downarrow \\ A_\beta & \longrightarrow & G_\alpha \end{array}$$

The first row is the canonical embedding and the first column is an embedding by the identification $c_\beta = \perp$. By the pushout property we now may assume that
(p_α) $\qquad G_\alpha = A_\beta + G_\beta$ and $A_\beta \cap G_\beta = c_\beta R$
hence $G_\alpha/c_\beta R \cong G_\beta/c_\beta R \oplus A_\beta/\perp R$. The construction of G is complete.

First we will discuss freeness properties of G.

Lemma 3.1. *G is an $\aleph_1$–free R–module of cardinality λ.*

Proof If $G = \bigcup_{\alpha<\aleph_1} G_\alpha$ as above, then we only have to show that G_α is $\aleph_1$–free for any α which we prove by induction. Since G_0 is free we consider $\alpha > 0$ and assume that all G_β $(\beta < \alpha)$ are $\aleph_1$–free. If $\alpha = \beta + 1$, then $G_\alpha = A_\beta + G_\beta$ and (p_α) holds, hence

$$G_\alpha/c_\beta R \cong G_\beta/c_\beta R \oplus A_\beta/\perp R.$$

The right hand side is $\aleph_1$–free by Lemma 2.3 and assumption on the choice of c_β. However, if $G_\alpha/c_\beta R$ is $\aleph_1$–free, then G_α must be $\aleph_1$–free as well.
If α is a limit ordinal, then any subgroup of finite rank in $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ is a subgroup of G_β for some $\beta < \alpha$ and $\aleph_1$–freeness follows.
The following observation plays a role in our next proposition, which provides splittings of G coming from Proposition 2.5 and is based on

$R_\alpha \cap R_\beta$ is divisible by all primes not in $X_\alpha \cap X_\beta$ which is finite for $\alpha \neq \beta$.

Proposition 3.2. *If $G = \bigcup_{\alpha<\lambda} G_\alpha$ is the R–module above, then $G_\alpha \otimes R_\beta$ is a free R_β–module for all $\alpha \leq \beta < \lambda$.*

Proof If $\alpha < \beta$, then $(G_{\alpha+1} \otimes R_\beta)/(G_\alpha \otimes R_\beta) = (G_{\alpha+1}/G_\alpha) \otimes R_\beta$ because R_β is a flat R–module. We also have $G_{\alpha+1}/G_\alpha = A_\alpha/c_\alpha R$ by the pushout property $(p_{\alpha+1})$ and $(A_\alpha/c_\alpha R) \otimes R_\beta$ is a free R_β–module by $\alpha \neq \beta$ and Proposition 2.5. We derive that $(G_{\alpha+1} \otimes R_\beta)/(G_\alpha \otimes R_\beta)$ is a free R_β–module, hence projective and the rest follows inductively by an obvious basis collection. Taking into account that $G_0 \otimes R_\beta$ is a free R_β–module, the same holds for $G_\alpha \otimes R_\beta$.

Proposition 3.3. *With the notation as above we have*
(a) $A_\beta \otimes R_\beta$ is a direct summand of $G_{\beta+1} \otimes R_\beta$
(b) $G_{\beta+1} \otimes R_\beta$ is a direct summand of $G \otimes R_\beta$
(c) $A_\beta \otimes R_\beta$ is a direct summand of $G \otimes R_\beta$.

Proof Obviously (c) follows from (a) and (b) and it remains to show the first two assertions.

(a) Observe that $G_\beta \otimes R_\beta$ is free by Proposition 3.2 and we may write $G_\beta \otimes R_\beta = c_\beta R_\beta \oplus H_\beta$ as R_β–modules by our choice of c_β. The pushout property $(p_{\beta+1})$ gives $G_{\beta+1} \otimes R_\beta = H_\beta \oplus (A_\beta \otimes R_\beta)$ and (a) follows.
(b) Inductively we will find an ascending, continuous chain of complements C_α of $G_{\beta+1} \otimes R_\beta$ in $G_\alpha \otimes R_\beta$ for $\beta+1 \leq \alpha \leq \lambda$ and C_λ will verify (b). If $\alpha = \beta+1$, then $C_\alpha = 0$ and if α is a limit ordinal between $\beta+1$ and λ and all $C_\gamma(\gamma < \alpha)$ are defined, then $C_\alpha = \bigcup_{\gamma<\alpha} C_\gamma$ is already defined by continuity and C_α is a complement of $G_{\beta+1} \otimes R_\beta$ in $G_\alpha \otimes R_\beta$ indeed, because $G_\gamma \otimes R_\beta$ $(\gamma \leq \alpha)$ is continuous at α as well. It remains to define C_α for $\alpha = \gamma+1$ where C_γ is given. We are in the case $\alpha > \beta+1$, hence $\gamma > \beta$ and $\gamma \neq \beta$ follows. From Proposition 2.5 we see that $c_\beta R_\beta =\perp R_\beta$ is a summand of the free R_β–module $A_\gamma \otimes R_\beta$ and we may write $A_\gamma \otimes R_\beta = c_\gamma R_\beta \oplus D_\gamma$. Obviously $C_\alpha = C_\gamma \oplus D_\gamma$ is a complement of $G_{\beta+1} \otimes R_\beta$ in $G_\alpha \otimes R_\beta$ by the pushout property $(p_{\beta+1})$.

4. Proof of the Main Theorem

The main result of this paper is the following

Theorem 4.1. *If R is a ring with R^+ free and $|R| < \lambda \leq 2^{\aleph_0}$, then there exists an $\aleph_1$–free abelian group G of cardinality λ with* End $G = R$.

Remark: G will be the R–module constructed in Section 3 and we have identified $r \in R$ with $r \cdot id_G$.

Proof From Lemma 3.1 we have an R–module G of cardinality λ which is $\aleph_1$–free as R–module, hence $\aleph_1$–free as abelian group. Moreover $R \subseteq$ End G by our identification and we must show that
$\varphi \in$ End $G \backslash R$ does not exist.
Such a homomorphism φ has a unique extension $\hat{\varphi} : G \otimes R_\beta \to G \otimes R_\beta$ because $\hat{\varphi} = \varphi \otimes id$ extends and $G \otimes R_\beta/G = (G \otimes R_\beta)/(G \otimes R) \cong R_\beta/R$ being torsion forces uniqueness.
If $c_\alpha\varphi \in c_\alpha R$ for all $\alpha < \lambda$, then $c_\alpha\varphi = c_\alpha r_\alpha$ for some $r_\alpha \in R$. If $\alpha < \lambda$ is fixed, we can choose an element $c \in G$ (even in G_0) such that G/cR is an $\aleph_1$–free R–module $cR \oplus\ c_\alpha R$ is a direct sum and $G/(c+c_\alpha)R$ is an $\aleph_1$–free R–module as well. There exist some $\gamma, \delta < \lambda$ with $c = c_\gamma$ and $c + c_\alpha = c_\delta$. We have
$c_\gamma r_\gamma + c_\alpha r_\alpha = c_\gamma\varphi + c_\alpha\varphi = (c_\gamma + c_\alpha)\varphi = c_\delta\varphi = c_\delta r_\delta = c_\gamma r_\delta + c_\alpha r_\delta$
and $r_\gamma = r_\delta = r_\alpha$ follows. We find a uniform $r \in R$ such that $c_\alpha\varphi = c_\alpha r$ for all $\alpha < \lambda$. However, G is generated by the set $\{c_\alpha : \alpha < \lambda\}$, hence $\varphi = r$ which was excluded.
There exists $\alpha < \lambda$ such that $c_\alpha\varphi \notin c_\alpha R$. We also find $\gamma > \alpha$ such that $c_\alpha\varphi \in G_\gamma$ and the repetition (c2) (Section 2) of the enumeration of c_α 's provides $\gamma < \beta < \lambda$ such that $c_\beta = c_\alpha$, hence
(i) $c_\beta\varphi \notin c_\beta R$ and $c_\beta\varphi \in G_\beta$.
However, $G_\beta \otimes R_\beta$ is a free R_β–module by Proposition 3.2 and c_β is a basic element of the R_β–module $G_\beta \otimes R_\beta$; we find a free decomposition $G_\beta \otimes R_\beta = c_\beta R_\beta \oplus C$. The pushout $G_{\beta+1} = G_\beta + A_\beta$ gives $G_{\beta+1} \otimes R_\beta = (A_\beta \otimes R_\beta) \oplus C$ and Proposition 3.3(b) provides an R_β–module D such that $L = C \oplus D$ satisfies

(ii) $(A_\beta \otimes R_\beta) \oplus L = G \otimes R_\beta$, $G_\beta \otimes R_\beta = c_\beta R_\beta \oplus C$
where $C = L \cap (G_\beta \otimes R_\beta)$ by the modular law.
The element $c_\beta \varphi \in G_\beta \subseteq G_\beta \otimes R_\beta$ has a unique decomposition $c_\beta \varphi = c_\beta r + c$ with $r \in R_\beta$ and $c \in C$. If $c = 0$, then $c_\beta \varphi \in c_\beta R_\beta \cap G_\beta = c_\beta R$ by purity of c_β is a contradiction. Hence $0 \neq c \in C$ which is a free R_β–module with a basis B. The element $c = \sum_{b \in [c]} bc_b$ has a unique decomposition and a B–support $[c] = \{b \in B : c_b \in R_\beta \backslash \{0\}\} \neq \emptyset$.
On the other hand $c \in C \subseteq G_\beta \otimes R_\beta$ and $cm = \sum_{[c]} bc_b m \in G_\beta \cap C$ for some $m \neq 0$. However $G_\beta \cap C \subset G_\alpha$ for some $\alpha < \beta$, which is contained in the free R_α–module $G_\alpha \otimes R_\alpha$. Since $\alpha \neq \beta$, our choice of R_α, R_β provides an $h < \omega$ such that
(iii) p_j does not divide $c \in C$ for all $j > h$,
where the enumeration of primes is taken in $X_\beta = \{p_n : n < \omega\}$.
If $\pi : G_{\beta+1} \otimes R_\beta \to C$ denotes the canonical projection induced by (ii), then
(iv) $0 \neq c = c_\beta \varphi \pi$.
Moreover, the image $\eta \varphi \pi$ of any $\eta \in Y_\beta$ viewed as $\eta \in A_\beta \otimes R_\beta \subseteq G_{\beta+1} \otimes R_\beta$ can be expressed by

$$\eta \varphi \pi = \sum_{b \in [\eta]} b r_b^\eta \text{ with } \quad r_b^\eta \in R_\beta \backslash \{0\}$$

with a finite subset $[\eta]$ of B. Abusing notation we shall call $[\eta]$ the B–support of η as well. Recall that $|Y_\beta| = \lambda > |R_\beta| \geq \aleph_0$, and it is easy to find $Y' \subseteq Y_\beta$, $n \in \mathbb{N}$ and $r_b \in R_\beta$ for all $b \in B$ such that $|Y'| = \lambda$ and $|[\eta]| = n$, $r_b^\eta = r_b$ for all $\eta \in Y'$ and $b \in B$. Next we apply the Δ–Lemma to $\{[\eta] : \eta \in Y'\}$ (cf. Jech [22, p.225]) and find $Y'' \subseteq Y'$, $E \subset B$ such that $|Y''| = \lambda$ and $[\eta] \cap [\eta'] = E$ for all $\eta \neq \eta' \in Y''$. Since $[c] \subset B$ is finite, we also find $Y \subset Y''$ such that $|Y| = \lambda$ and $[\eta] \cap [c] \subseteq E$ for all $\eta \in Y$.
From $|Y| = \lambda > \aleph_0$ we find two distinct branches $\eta, \eta' \in Y$ with $\eta \restriction h = \eta' \restriction h$. The branch point $j > h$ of η, η' belongs to L_β by (b3), hence $p_j \in X_\beta$, where j is from the enumeration along branches. The definition branch point gives $\eta \restriction j = \eta' \restriction j$ and $\eta(j) = 1$, $\eta'(j) = 0$ without loss of generality. From the relations (X_β) in A_β (Section 2) we have $p_j | (\eta'_j + \eta' \restriction j + \eta'(j) \perp)$ in A_β and $p_j | (\eta'_j + \eta' \restriction j + \eta'(j) \perp)$ in A_β, hence $p_j | \eta_j - \eta'_j + \eta(j) \perp = \eta_j - \eta'_j + c_\beta$ in G_β and therefore $p_j | (\eta_j \varphi \pi - \eta'_j \varphi \pi) + c_\beta \varphi \pi$. However $[c] = [c_\beta \varphi \pi]$ and if $d = \eta_j \varphi \pi - \eta'_j \varphi \pi$, then $d \restriction E = 0$ by our choice of $\eta, \eta' \in Y$ with $\eta \neq \eta'$, hence d and c are linearly independent. We conclude $p_j | c$ in C which contradicts (iii) and Theorem 4.1 follows.

5. A counterexample

The reader might suspect that $\aleph_1$ in Theorem 4.1 can be replaced by $\aleph_2$ for instance. This is the case if we assume prediction principles as $\diamondsuit$ (which imply CH), see Dugas, Göbel [7]. However, in general it is no longer true as follows from

Theorem 5.1. *Assuming Martin's axiom, any $\aleph_2$–free group of cardinality $< 2^{\aleph_0}$ is separable.*

Recall that an $\aleph_1$–free group is separable if any pure cyclic subgroup is a summand. Preliminaries on (MA) can be seen in Jech [22] or Eklof, Mekler [13].
Proof If G is an $\aleph_2$–free group of cardinality $|G| < 2^{\aleph_0}$, $0 \neq e \in G$ pure in G and $\sigma : e\mathbb{Z} \to \mathbb{Z}$ taking $e\sigma = 1$, then we must extend σ to an homomorphism $\Phi : G \to \mathbb{Z}$.

Let $P = \{\varphi; \varphi : D_\varphi \to \mathbb{Z}, e \in D_\varphi, e\varphi = 1\}$ where D_φ is a pure and finitely generated subgroup of G. Obviously $|P| < 2^{\aleph_0}$ from $|G| < 2^{\aleph_0}$ and $(P, \subseteq)$ is partially ordered by extensions of maps. Suppose for a moment that P satisfies the hypothesis for MA and $P_g = \{\varphi \in P : g \in D_\varphi\}$ is dense for all $g \in G$. Then by MA there is a compatible set $F \subseteq P$ such that $F \cap P_g \neq \emptyset$ for all $g \in G$. So $\bigcup F = \Phi$ is a partial homomorphism from G to $\mathbb{Z}$. Since $F \cap P_g \neq \emptyset$, also $g \in \text{dom } \ \Phi$ for all $g \in G$, hence $\Phi \in \text{Hom }(G, \mathbb{Z})$ and Φ extends σ by definition of P. Thus it remains to show that $(P, \subseteq)$ satisfies the hypothesis of MA:
In order to show that P_g is dense in P, we consider any $\varphi \in P$ and find $\varphi \subset \varphi' \in P$ such that $g \in dom\ \varphi'$. Since G is $\aleph_2$–free, there is $D' \supseteq dom\ \varphi$ such that $g \in D'$ and D' is pure and finitely generated in G by Pontrjagin's theorem. Recall that dom φ is pure in G, hence pure in D' and $D'/\text{dom } \ \varphi$ must be finitely generated and torsion–free. We apply Gauß' theorem to see that $D'/\text{dom } \ \varphi$ is free, hence $D' = \text{dom } \ \varphi \oplus C$ for some $C \subseteq D'$ with $C \cong D'/\text{dom } \ \varphi$. Now it is easy to extend φ to a homomorphism $\varphi' : D' \to \mathbb{Z}$. Finally, we must show that $(P, \subseteq)$ satisfies ccc, the countable antichain condition. Let $F \subseteq P$ be an uncountable subset of P. We must find two distinct elements $\varphi_i \in F$ and $\Phi \in P$ such that $\varphi_i \subseteq \Phi$ for $i = 1, 2$. We may assume $|F| = \aleph_1$, hence $(\sum_{\varphi \in F} \text{dom } \ \varphi)_* = U$, the pure subgroup of G generated (purely) by all dom φ has cardinality $\aleph_1$ and must be free by hypothesis on G. We select a basis B of U and replace any $\varphi \in F$ by φ' with dom $\varphi' = \langle B_\varphi \rangle \supseteq \text{dom } \ \varphi$ with a finite subset of B_φ of B. The argument given above allows to extend φ to a homomorphism φ'.
Clearly, it is enough to find two compatible elements φ_i in the new F. By the Δ–Lemma (Jech [22, p.225]) we also find $E \subset B$ and $F' \subseteq F$ such that $|F'| = \aleph_1$ and dom $\varphi \cap \text{dom } \ \varphi' = E$ for all $\varphi \neq \varphi' \in F'$. By a pigeon–hole argument we can also find $F'' \subseteq F'$ such that $|F''| = \aleph_1$ and $\varphi \restriction E = \varphi' \restriction E$ for all $\varphi, \varphi' \in F''$. Now it is clear that we can extend two of these maps φ, φ' to dom $\varphi +$ dom φ' as required.

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