THE STRICT ORDER PROPERTY AND GENERIC AUTOMORPHISMS

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Abstract. If T is a model complete theory with the strict order property, then the theory of the models of T with an automorphism has no model companion.

1. Introduction

Given a model complete theory T in a language $\mathcal{L}$, we consider the (incomplete) theory $T_\sigma = T \cup \{$"σ is an $\mathcal{L}$-automorphism"$\}$ in the language $\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}$. For M a model of T, and $\sigma \in \mathrm{Aut}_{\mathcal{L}}(M)$ we call σ a generic automorphism of M if (M, σ) is an existentially closed model of T_σ. A general problem is to find necessary and sufficient conditions on T for the class of existentially models of T_σ to be elementary, namely to be the class of models of some first order theory in $\mathcal{L}_\sigma$. This first order theory, if it exists, is denoted TA, and it is the model companion of T_σ. This problem seems to be a difficult problem even if we assume T to be stable [1], [5], [8]. Generic automorphisms in the sense of this paper were first studied by Lascar [7]. The work of Chatzidakis and Hrushovski [2] on the case where T is the theory ACF of algebraically closed fields renewed interest in the topic and Chatzidakis and Pillay studied general properties of TA for stable T [3].

Kudaibergenov proved that if TA exists then T eliminates the quantifier "there exists infinitely many". Therefore, if T is stable and TA exists then T does not have the fcp. Pillay conjectured that if T has the fcp then TA does not exist after the first author observed that the theory of random graphs does not have TA. The first author then proved that if TA exists and T does not have the independence property then T is stable, and if TA exists and T_σ has the amalgamation property then T is stable [4]. The latter fact covers the case of the random graphs. The present paper extends the former case.

So the theorem here shows that model complete theories with T_σ having a model companion are "low" in the hierarchy of *classification*

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theory; previous results have shown it cannot be in some intermediate positions.

For other examples, Hrushovski observed that there are no TA for $ACFA$ and for the theory of pseudo-finite fields Psf (unpublished). His argument depends heavily on field theory. $ACFA_\sigma$ does not have the amalgamation property but it is not known if Psf_σ has the amalgamation property.

In the rest of the paper, small letters a, b, c, etc. denote finite tuples and x, y, etc. denote finite tuples of variables. If a is a tuple of elements and A a set of elements, $a \in A$ means that each element of a belongs to A.

2. Main Theorem

Theorem 1. *Let T be a model complete theory in a language $\mathcal{L}$ and σ a new unary function symbol. If T has a model whose theory has the strict order property then T_σ has no model companion.*

Proof. Let M_0 be a model of T with the strict order property. So, there are $\mathcal{L}$-definable partial order $<$ on k-tuples in M_0 for some k and a sequence $\langle a_i : i < \omega \rangle$ of k-tuples in M_0 such that $a_i < a_j$ for $i < j < \omega$. By Ramsey's Theorem, we can assume that $\langle a_i : i < \omega \rangle$ is an $\mathcal{L}$-indiscernible sequence in M_0. Also, we can assume that there is an $\mathcal{L}$-automorphism σ_0 of M_0 such that $\sigma_0(a_i) = a_{i+1}$. So, (M_0, σ_0) is a model of T_σ.

Now by way of contradiction, suppose that T_σ has a model companion, say TA. Extend (M_0, σ_0) to a model (N, σ) of TA. N is an $\mathcal{L}$-elementary extension of M_0 since T is model complete. We can assume that (N, σ) is sufficiently saturated. In the rest of the proof, we work in (N, σ).

Consider the partial type $p(x) = \{a_i < x : i < \omega\}$ and let $\psi(x) \equiv \exists y(a_0 < \sigma(y) \wedge \sigma(y) < y \wedge y < x)$.

Claim. In (N, σ),

(1) $p(x) \vdash \psi(x)$, and
(2) if $q(x)$ is a finite subset of $p(x)$ then $q(x) \not\vdash \psi(x)$.

If this claim holds, then it contradicts the saturation of (N, σ).

We first show (2). Let n^* be such that $q(x) \subset \{a_i < x : i < n^*\}$. Then a_{n^*} satisfies $q(x)$. Suppose a_{n^*} satisfies $\psi(x)$. Let $b \in N$ be such that $a_0 < \sigma(b) < b < a_{n^*}$. By $a_0 < \sigma(b)$, we have $a_{n^*} = \sigma^{n^*}(a_0) < \sigma^{n^*+1}(b)$. By $\sigma(b) < b < a_{n^*}$, we have

$$\sigma^{n^*+1}(b) < \sigma^{n^*}(b) < \cdots < \sigma(b) < b < a_{n^*}.$$

By transitivity, we get $a_{n^*} < a_{n^*}$, which is a contradiction.

Now we turn to a proof of (1). Suppose $c \in p(N)$. Let M be such that $a_0, c \in M$, $|M| = |T|$, and $(M, \sigma|M)$ is an $\mathcal{L}(\sigma)$-elementary substructure of (N, σ).

For each $d \in p(N)$, let $\Psi(d)$ be the set of $\mathcal{L}(M)$-formulas $\varphi(x)$ satisfied in N by some tuple d' such that $d' \in p(N)$ and $d' < d$. Here, $\mathcal{L}(M)$-formulas are the formulas in $\mathcal{L}$ with parameters in M.

Note that if $d_1, d_2 \in p(N)$ and $d_2 < d_1$, then $\Psi(d_2) \subseteq \Psi(d_1)$, and by compactness, if $d_1, d_2 \in p(N)$ then there is $d_3 \in p(N)$ such that $d_3 < d_1$ and $d_3 < d_2$.

Let $\Psi = \bigcap_{d \in p(N)} \Psi(d)$. Let $\{\varphi_i(x) : i < |M|\}$ be an enumeration of all $L(M)$-formulas which do not belong to Ψ. By the definition of Ψ, for each $i < |M|$, we can choose $d_i \in p(N)$ such that $\varphi_i(x) \notin \Psi(d_i)$. By saturation of N and the remark above, we can find $c^* \in p(N)$ such that $c^* < d_i$ for every $i < |M|$. Each $\varphi_i(x)$ does not belong to $\Psi(c^*)$ since $\Psi(c^*) \subseteq \Psi(d_i)$. Hence, $\Psi(c^*) \subseteq \Psi$. Therefore, $\Psi(c^*) = \Psi$.

Now we have that if $d \in p(N)$ and $d < c^*$ then $\Psi(d) = \Psi(c^*)$. We can also assume that $c^* < c$. Since the sets $p(N)$ and M are invariant under σ, $\Psi(c^*)$ is also invariant under σ, which means, for any $\mathcal{L}$-formula $\varphi(x, y)$ and tuple $a \in M$, $\varphi(x, a) \in \Psi(c^*)$ if and only if $\varphi(x, \sigma(a)) \in \Psi(c^*)$.

Now choose $b_1 \in p(N)$ such that $b_1 < c^*$ and consider $q_1(x) = \mathrm{tp}_{\mathcal{L}}(b_1/M)$. Then $q_1(x) \subseteq \Psi(c^*)$. Let $\sigma(q_1(x))$ be the set of formulas $\varphi(x, \sigma(a))$ such that $\varphi(x, a) \in q_1(x)$, where $\varphi(x, y)$ is a formula in $\mathcal{L}$ and $a \in M$. Since $\Psi(c^*)$ is invariant under σ, we have $\sigma(q_1(x)) \subseteq \Psi(c^*)$. By the choice of c^*, $\Psi(c^*) = \Psi(b_1)$ and thus $\sigma(q_1(x)) \subseteq \Psi(b_1)$. By the definition of $\Psi(b_1)$ and by compactness, there is $b_2 \in p(N)$ such that $b_2 < b_1$ and b_2 realizes $\sigma(q_1(x))$.

Since $\sigma(q_1(x))$ is a complete $\mathcal{L}$-type over M, there are an $\mathcal{L}$-elementary substructure M' of N and an $\mathcal{L}$-automorphism τ of M' such that $Mb_1b_2 \subset M'$, $\tau(b_1) = b_2$ and $\tau|M = \sigma|M$. Now we have,

$$(M', \tau) \models a_0 < \tau(b_1) < b_1 < c.$$

Since $(M, \sigma|M)$ is a model of TA, it is an existentially closed model of T_σ. Note that the partial order $<$ is definable by an existential $\mathcal{L}$-formula modulo T. So, the formula $a_0 < \sigma(y) < y < c$ has a solution in $(M, \sigma|M)$. Hence, we have $(M, \sigma|M) \models \psi(c)$. This proves Claim (1) and we are done. $\square$

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