RADICALS AND PLOTKIN'S PROBLEM CONCERNING GEOMETRICALLY EQUIVALENT GROUPS

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ABSTRACT. If G and X are groups and N is a normal subgroup of X, then the $G-$closure of N in X is the normal subgroup $\overline{X}^G = \bigcap\{\ker\varphi | \varphi : X \to G, \text{ with } N \subseteq \ker\varphi\}$ of X. In particular, $\overline{1}^G = R_G X$ is the $G-$radical of X. Plotkin [2, 6, 3] calls two groups G and H geometrically equivalent, written $G \sim H$, if for any free group F of finite rank and any normal subgroup N of F the $G-$closure and the $H-$closure of N in F are the same. Quasiidentities are formulas of the form $(\bigwedge_{i\leq n} w_i = 1 \to w = 1)$ for any words w, w_i $(i \leq n)$ in a free group. Generally geometrically equivalent groups satisfy the same quasiidentiies. Plotkin showed that nilpotent groups G and H satisfy the same quasiidenties if and only if G and H are geometrically equivalent. Hence he conjectured that this might hold for any pair of groups; see the Kourovka Notebook [2]. We provide a counterexample.

In a series of paper, B. I. Plotkin and his collaborators [6, 3, 4, 5] investigated radicals of groups and their relation to quasiidentities. If G is a group, then the *G-radical R_GX of a group X* is defined by

$$R_GX = \bigcap\{\ker\varphi; \varphi : X \to G \text{ any homomorphism }\}.$$

Clearly, R_GX is a characteristic, hence a normal subgroup of X. The radical R_G can also be used to define the *G-closure $\overline{U}^G = \overline{U}$ of a normal subgroup U of X*, by saying that $\overline{U}/U = R_G(X/U)$. This immediately leads to Plotkin's definition of geometrically equivalent groups, see [6, 3, 4, 5] and [2, p. 113].

Definition 0.1. *Let G and H be two groups. Then G and H are geometrically equivalent, written $G \sim H$, if for any free group F of finite rank and any normal subgroup U of F the G- and H-closure of U in F are the same, i.e. for any normal subgroup U we have $\overline{U}^G = \overline{U}^H$.*

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It is easy to see that $G \sim H$ if and only if $R_G K = R_H K$ for all finitely generated groups K. Plotkin notes that geometrically equivalent groups satisfy the same quasi-identities. The well-known notion of quasiidentities relates to quasivarieties of groups. A *quasiidentity* is an expression of the form

$$w_1 = 1 \wedge \cdots \wedge w_n = 1 \to w = 1 \text{ where } w_i, w \in F \ (i \leq n) \text{ are words.}$$

Moreover the following was shown in [6], see [2, p.113].

Theorem 0.2. *(a) If $G \sim H$, and G is torsion-free, then H is torsion-free.*

(b) If G, H are nilpotent, then $G \sim H$ if and only if G and H satisfy the same quasiidentities.

This lead Plotkin to conjecture that two groups might be geometrically equivalent if and only if they satisfy the same quasiidentities, see the Kourovka Notebook [2, p.113, problem 14.71]. In this note we refute this conjecture. Clearly there are only countably many finitely presented groups which we enumerate as the set $\mathfrak{K} = \{K_n : n \in w\}$ and let $G = \prod_{n \in w} K_n$ be the restricted direct product. Then G satisfies only those quasiidentities satisfied by all groups and so if H is any group with $G \leq H$, G satisfies the same quasiidenties as H.

R. Camm [1, p. 68, p. 75 Corollary] proved there are $2^{\aleph_0}$ non-isomorphic, two-generator, simple groups, see also Lyndon, Schupp [7, p. 188, Theorem 3.2]. So there exists a 2-generated simple group L which cannot be mapped nontrivially into G. We consider the pair G, $H = L \times G$ and show the following:

Theorem 0.3. *If G, H and L are as above, $R_G L = L$ and $R_H L = 1$. In particular G and H are not geometrically equivalent. Since $G \leq H$ satisfy the same quasiidentities, this is the required counterexample.*

Proof. Since L is a two-generated simple group, L is an epimorphic image of a free group of rank 2. So it is enough to prove that $R_G L = L$ and $R_H L = 1$. The first equality follows since there is no nontrivial homomorphism of L into G. On the other hand, there is a canonical embedding $L \to H = L \times G$, so $R_H L = 1$. $\square$

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