

# CONSISTENTLY THERE IS NO NON TRIVIAL CCC FORCING NOTION WITH THE SACKS OR LAVER PROPERTY

SAHARON SHELAH

*Dedicated to the memory of Paul Erdős*

## 1. INTRODUCTION

At the recent set theory conference, Boban Velickovic asked the following question:

1.1. **Question.** *Is there a nontrivial forcing notion with the Sacks property which is also ccc?*

(See below for a definition of the Sacks property.)

A “definable” variant of this question has been answered in [Sh 480]:

Every nontrivial Souslin forcing notion which has the Sacks property has an uncountable antichain.

(A Souslin forcing notion is a forcing notion for which the set of conditions, the comparability relation and the incompatibility relation are all analytic subsets of the reals. See [JdSh 292] and [Sh 480] for details).

We show here

1.2. **Theorem.** *The following statement is equiconsistent with ZFC:*

(\*) *Every nontrivial forcing notion which has the Sacks property has an uncountable antichain.*

Our proof follows the ideas from [Sh 480].

Independently, Velickovic has also proved the consistency of (\*), following [Sh 480] and some of his works. In fact, he shows that the proper forcing axiom (PFA), and even the open coloring axiom implies (\*).

Our proof shows that also the following strengthening of (\*):

(\*\*) *Every nontrivial forcing notion which has the Laver property has an uncountable antichain.*

is equiconsistent with ZFC.

Note that if  $\text{cov}(\text{meagre}) = \text{continuum}$  (which follows e.g. from PFA) then there is a (non principal) Ramsey ultrafilter on  $\omega$ . The “Mathias” forcing notion for shooting making this ultrafilter principal has the Laver property and is ccc, so (\*\*) does not follow from PFA.

So our result and Velickovic’ result are incomparable.

---

The research supported by the Israel Science Foundation (founded by the Israel Academy of Sciences). Publication number 723.

We would like to thank Martin Goldstern for many improvements in this paper.

**1.3. Definition.** Let  $g \in \omega^\omega$  be increasing. A  $g$ -slalom is a sequence  $\bar{A} = (A_n : n \in \omega)$ ,  $A_n \subseteq \omega$ ,  $|A_n| \leq g(n)$ . We say that  $\bar{A}$  covers  $\eta \in \omega^\omega$  iff  $\forall n \eta(n) \in A_n$ .

Let  $V_1 \subseteq V_2$  be models of set theory. We say that  $(V_1, V_2)$  is  $(f, g)$ -bounding iff:

For all  $\eta \in \prod_n f(n) \cap V_2$  there is a  $g$ -slalom in  $V_1$  covering  $f$ .

We say that a forcing notion  $\mathbb{R}$  is  $(f, g)$ -bounding if the pair  $(V, V^{\mathbb{R}})$  is  $(f, g)$ -bounding

A pair  $(V_1, V_2)$  has the *Laver property* iff  $(V_1, V_2)$  is  $(f, g)$ -bounding for all increasing  $f, g \in V_1$ , or in other words: For all  $g \in V_1$ , every function in  $V_2$  which is bounded by a function in  $V_1$  is covered by a  $g$ -slalom from  $V_1$ .

Similarly,  $\mathbb{R}$  has the Laver property iff  $(V, V^{\mathbb{R}})$  has the Laver property.

$(V_1, V_2)$  has the Sacks property if it has the Laver property, and every function in  $V_2$  is bounded by a function in  $V_1$ .

## 2. A LEMMA ON MATHIAS FORCING

The following lemma is a theorem of ZFC.

**2.1. Lemma.** Let  $\mathbb{M}$  be the Mathias forcing,  $\eta$  the increasing enumeration of the  $\mathbb{M}$ -generic subset of  $\omega$ , and assume that  $\underline{T}$  is a name of a tree  $\subseteq \omega^{>2}$  such that

$$(!) \quad \Vdash_{\mathbb{M}} |\underline{T} \cap \eta^{(n)} 2| \leq 2^{\eta^{(n-1)}}$$

Then there is a family  $(T_i, q_i : i < 2^{\aleph_0})$  such that for all  $i, j$ :

- (1)  $q_i \in \mathbb{M}$
- (2)  $T_i \subseteq \omega^{>2}$  is a tree
- (3)  $q_i \Vdash \underline{T} \subseteq T_i$
- (4) Whenever  $i \neq j$ , then  $\lim T_i \cap \lim T_j$  is finite.

(We write  $\lim T$  for the set of branches of a tree  $T$ .)

*Proof.* Conditions in the Mathias forcing are of the form  $(w, A)$ , where  $w$  is a finite set of natural numbers and  $A$  is an infinite subset of  $\omega$  such that  $\sup(w) < \min A$ . If  $(w, A) \in \mathbb{M}$ , and  $w = \{w_0 < \dots < w_{\ell-1}\}$ , then

$$(w, A) \Vdash \underline{\eta}(i) = w_i \ (i = 0, \dots, \ell - 1), \quad \underline{\eta}(i) \in A \ (i = \ell, \ell + 1, \dots)$$

Using the fact that truth values in  $V^{\mathbb{M}}$  can be decided by pure extensions, we can find a condition  $(w^*, A^*)$  which forces that  $\underline{T}$  is “decided continuously” by  $\underline{\eta}$ , more specifically:

(\*)<sub>1</sub> there is a function  $t$  with domain  $[A^*]^{<\omega}$ ,  $\forall w : t(w) \subseteq \omega^{>2}$ , such that

For all finite  $u \subseteq A^*$  and all  $n$  with  $\sup(u) < n \in A^*$ :

$$(w^* \cup u, A^* \setminus (n + 1)) \Vdash \underline{T}_\delta \cap^n 2 = t(w) \cap^n 2.$$

For any finite  $u \subseteq A^*$  let

$$T_u =^{df} \{\eta \in \omega^{>2} : \text{for any large enough } k < \omega \text{ we have } (w^*, A^* \setminus k) \Vdash_{\mathbb{Q}} \text{“}\eta \in \underline{T}\text{”}\}$$

So  $T_u$  is a subtree of  $\omega^{>2}$ .

We have

(\*)<sub>2</sub> If  $w^* \cup u = \{w_0 < \dots < w_{\ell-1}\}$ , then  $(w^* \cup u, A \setminus k) \Vdash \underline{\eta}(\ell - 1) = \max u$ ,  $\underline{\eta}(\ell) \geq k$  hence (recall condition (!)):  $(w^* \cup u, A \setminus k) \Vdash |\underline{T} \cap^k 2| \leq 2^{\eta^{(\ell-1)}} = 2^{\max u}$ .

It follows that:

(\*)<sub>3</sub> for every finite  $u \subseteq A^*$ , non empty for simplicity, we have

$$k < \omega \rightarrow |T_u \cap {}^k 2| \leq 2^{\max(u)},$$

hence  $\lim(T_u)$  is a finite subset of  ${}^\omega 2$ .

We also get:

(\*)<sub>4</sub> if  $u \cup \{m, k\} \subseteq A^*$  and  $\sup(u) < m < k$ , then  $T_u \cap {}^m 2 = T_{u \cup \{k\}} \cap {}^m 2$ .

[Proof: We know that already  $(w^* \cup u, A \setminus m)$  decides  $\underline{T} \cap {}^m 2$ , and the conditions  $(w^* \cup u \cup \{m\}, A \setminus k)$  and  $(w^* \cup u, A \setminus k)$  (for  $k > m$ ) are both stronger than  $(w^* \cup u, A \setminus m)$ .]

In particular, we get:

(\*)<sub>5</sub> for all finite  $u \subseteq A^*$  the sequence  $\langle T_{u \cup \{k\}} : k \in A^* \rangle$  converges to  $T_u$ .

This means that for every  $m < \omega$  for every large enough  $k \in A^*$  we have  $T_{u \cup \{k\}} \cap {}^{m>} 2 = T_u \cap {}^{m>} 2$ .

(\*)<sub>6</sub> ( $\alpha$ ) for  $A$  an infinite subset of  $A^*$  we let

$$T_A = \bigcup \{T_{A \cap n} : n < \omega\} \quad T[A] = \bigcup \{T_u : u \subseteq A \text{ finite}\}$$

( $\beta$ ) for  $u, v$  finite subsets of  $A^*$  we let  $\mathbf{n}(u, v)$  the smallest  $m$  such that

- whenever  $\eta, \nu$  are distinct members of  $\lim(T_u) \cup \lim(T_v)$  then the length of  $\eta \cap \nu$  is  $< m$
- $\sup(w^* \cup u \cup v) < m$

Note that  $\mathbf{n}(u, v)$  is well defined, as both  $\lim(T_u)$  and  $\lim(T_v)$  are finite.

( $\gamma$ ) for  $m < \omega$  let

$$\mathbf{n}(m) =^{df} \max\{\mathbf{n}(u, v) : u, v \subseteq A^* \cap (m+1)\},$$

so  $\mathbf{n}(m) < \omega$  is well defined being the maximum of a finite set of natural numbers.

(\*)<sub>7</sub> Note that  $(w^*, A) \Vdash \underline{T} \subseteq T[A]$ .

So without loss of generality (as we can replace  $A^*$  by any infinite subset):

(\*)<sub>8</sub> if  $n \in A^*$  then  $\mathbf{n}(n) < \min(A^* \setminus (n+1))$

hence

(\*)<sub>9</sub> if  $n \in A^*$ ,  $u \subseteq A^* \cap (n+1)$ ,  $k \in A^* \setminus (n+1)$   
then  $T_u \cap {}^{\mathbf{n}(n)} 2 = T_{u \cup \{k\}} \cap {}^{\mathbf{n}(n)} 2$ .

(\*)<sub>10</sub> if  $u, v$  are finite subsets of  $A^*$  and  $\sup(u \cup v) < m \in A^*$   
then  $T_{u \cup \{m\}} \cap T_v \subseteq T_u \cap T_v$

Hence

(\*)<sub>11</sub> if  $u, v$  are finite subsets of  $A^*$ , not disjoint for notational simplicity, and  $m = \sup(u \cap v)$  then

$$T_u \cap T_v \subseteq T_{u \cap (m+1)} \cap T_{v \cap (m+1)}$$

[ why? we can prove this by induction on  $\max(u \cup v)$  using (\*)<sub>8</sub>]

Hence, letting  $Y = \bigcup \{\lim(T_u) : u \text{ a finite subset of } A^*\}$  (=a countable set), we have

(\*)<sub>12</sub> if  $A, B$  are infinite subsets of  $A^*$ , with intersection finite non empty then  $\lim T[A] \cap \lim T[B]$  is included in

$$\bigcup \{ \lim T_u \cap \lim T_v : u \subseteq A \cap (\max(A \cap B) + 1), v \subseteq B \cap (\max(A \cap B) + 1) \}$$

so  $\lim T[A] \cap \lim T[B]$  is a finite subset of  $Y$ .

[ why ? just use (\*)<sub>9</sub> and the definition of  $T[A]$ ].

Now fix an uncountable family  $(A_i : i < 2^{\aleph_0})$  of almost disjoint subsets of  $A^*$ . Let  $q_i := (w^*, A_i)$ . Then by (\*)<sub>7</sub>, we have  $q_i \Vdash \underline{T} \subseteq T[A_i]$ . Hence  $(q_i, A_i : i < 2^{\aleph_0})$  satisfies the conditions 1.,2.,3.,4. of lemma 2.1.  $\square$

### 3. AN ITERATION ARGUMENT

3.1. **Notation.** For any  $f : \omega \rightarrow \omega$  we define

- $f^- : \omega \rightarrow \omega$  by:  $f^-(n) = f(n-1)$  for  $n > 0$ ,  $f^-(0) = 1$ .
- $\hat{f} : \omega \rightarrow \omega$  by:  $\hat{f}(n) = 2^{f(n)}$ .

3.2. **Framework.** We will start with a universe where  $2^{\aleph_0} = \aleph_1$ ,  $2^{\aleph_1} = \aleph_2$  and use an iteration of length  $\kappa = \aleph_2$ ,  $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \aleph_2 \rangle$  satisfying the following:

- (A)  $\bar{\mathbb{Q}}$  is a countable support iteration of proper forcing notions.
- (B)  $\mathbb{P}_\kappa$ , the union of  $\mathbb{P}_i$  for  $i < \kappa$ , satisfies the  $\kappa$ -cc
- (C) The set

$$S := \{ \delta < \kappa : cf(\delta) > \aleph_0, \Vdash_{\mathbb{P}_i} \text{“}\mathbb{Q}_\delta = \text{Mathias forcing, with generic real } \eta_\delta \text{”} \}$$

is stationary.

- (D) Each forcing notion  $\mathbb{Q}_i$  has the Laver property.
- (E) Whenever  $\alpha < \kappa$  and  $\underline{T}$  is a  $\mathbb{P}_\alpha$ -name of an Aronszajn tree, then for some  $i \in [\alpha, \kappa)$ ,  $\Vdash_{\mathbb{P}_i}$  “if  $\underline{T}$  is a Souslin tree then  $\mathbb{Q}_i$  is forcing by  $\underline{T}$  (or an isomorphic forcing notion)”

(See the remarks 4.2 and 4.1 for weaker assumptions)

3.3. **Fact.** Let  $\bar{\mathbb{Q}}$  satisfy properties (A)–(E) above, and let  $\mathbb{P}_\kappa$  be the CS limit of this iteration. Then:

- (1)  $\mathbb{P}_\kappa$  is proper, making  $\kappa$  to  $\aleph_2$ .
- (2) If  $\delta \in S$  then the forcing notion  $\mathbb{P}_\kappa / \mathbb{P}_{\delta+1}$  has the Laver property.

*Proof.* (1) By [Sh:f]

(2) By [Sh:f, ch VI, section 3]  $\square$

3.4. **Remark.** Assume (say) GCH, then there is a forcing iteration as above. Define  $\mathbb{Q}_i$  as follows: If  $i$  is even, then let  $\mathbb{Q}_i$  be the Mathias forcing, and if  $i$  is odd, then let  $\mathbb{Q}_i$  be either trivial or a Souslin tree.

Note that in all intermediate universes we will have GCH, all forcing notions  $\mathbb{P}_i$  and  $\mathbb{Q}_i$  (for  $i < \aleph_2$ ) will have a dense subset of size  $\aleph_1$ ; this will be sufficient for  $\kappa$ -cc. All the forcing notions  $\mathbb{Q}_i$  will have the Laver property (recall that a Souslin tree does not add reals), and the usual bookkeeping argument can take care of killing all Souslin trees on  $\omega_1$ .

**3.5. Theorem.** *Let  $\bar{\mathbb{Q}}$  satisfy conditions (A)–(E) above.*

*Then in the universe  $\mathbf{V}^{\mathbb{P}^\kappa}$  the following holds:*

- (1) *Souslin's hypothesis*
- (2) *Any nontrivial ccc forcing notion adds a real. (See 3.6 below)*
- (3) *Whenever  $\mathbb{T}$  is a function satisfying the following conditions (α)–(γ):*
  - (α)  $\text{Dom}(\mathbb{T}) = \{f : f \text{ is a function from } \omega \text{ to } \omega, \text{ (strictly) increasing}\}$
  - (β)  $\mathbb{T}(f)$  *is a subtree of  ${}^\omega 2$*
  - (γ) *for every  $f \in \text{Dom}(\mathbb{T})$  and  $n < \omega$  we have*

$$1 \leq |\mathbb{T}(f) \cap f^{(n)} 2| \leq f^-(n)$$

*then  $\mathbb{T}$  also satisfies condition (δ):*

- (δ) *there is a countable subset  $Y \subseteq {}^\omega 2$  and an uncountable subset  $\mathbf{A} \subseteq \text{Dom}(\mathbb{T})$  with:*

*whenever  $f \neq g$  are from  $\mathbf{A}$  then  $\text{lim}(\mathbb{T}(f)) \cap \text{lim}(\mathbb{T}(g))$  is a finite subset of  $Y$ .*

**3.6. Fact.** If there is a nontrivial ccc forcing which does not add reals, then there is a Souslin tree on  $\omega_1$ .

In other words: If Souslin Hypothesis holds then

for every ccc forcing  $\mathbb{R}$  which is not trivial, there are  $p \in \mathbb{R}$  and an  $\mathbb{R}$ -name

$\eta$  such that:

$p \Vdash_{\mathbb{R}} \text{“}\eta \in {}^\omega 2 \text{ is new, that is does not belong to } \mathbf{V}\text{”}$

*Proof.* Let  $\mathbb{R}$  be a nontrivial ccc forcing. So for some  $q \in \mathbb{R}$  we have

$q \Vdash_{\mathbb{R}} \text{“}\mathcal{G}_{\mathbb{R}} \text{ does not belong to } \mathbf{V}\text{”}$

Hence for some quadruple  $(p, \alpha, \beta, \eta)$  we have:

(\*)<sub>13</sub>  $p \in \mathbb{R}$ ,  $\alpha, \beta$  are ordinals,  $\eta$  is a  $\mathbb{R}$ -name and  $p \Vdash_{\mathbb{R}} \text{“}\eta \in {}^\alpha \beta \text{ is not from } \mathbf{V}\text{”}$

We can choose such quadruple with the ordinal  $\alpha$  minimal. Necessarily  $\alpha$  is a limit ordinal and for  $\gamma < \alpha$ ,  $\eta \upharpoonright \gamma$  is forced by  $p$  to belong to  $\mathbf{V}$ .

For  $\gamma < \alpha$ , let

$$T_\gamma = \{\nu : \nu \text{ is a function from } \gamma \text{ to } \beta \text{ so } p \text{ does not force that } \nu \neq \eta \upharpoonright \gamma\}$$

and let  $T = \cup\{T_\gamma : \gamma < \alpha\}$ . Clearly  $T$  is a tree with  $\alpha$  levels, and  $p$  forces that  $\eta$  is a new  $\alpha$ -branch of it.

Now  $T$  cannot have  $\aleph_1$  pairwise incomparable elements, as if  $\nu_\zeta$  for  $\zeta < \omega_1$  are like that, we can find  $p_\zeta$  such that:  $p \leq p_\zeta \in \mathbb{R}$  and  $p_\zeta \Vdash_{\mathbb{R}} \text{“}\nu_\zeta \text{ is an initial segment of } \eta\text{”}$ ; now if  $p_\zeta, p_\xi$  are compatible in  $\mathbb{R}$  then  $\nu_\zeta, \nu_\xi$  are comparable in  $T$  (being, both, the initial segment of some possible  $\eta$ ). So  $\{p_\zeta : \zeta < \omega_1\}$  are pairwise incompatible contradiction to  $\mathbb{R}$  satisfies the ccc”

Also in  $T$ , by its choice, every member has above it elements of every higher level and there is no node above which the tree has no two distinct members of the same level (as then  $\eta$  will be forced to belong to  $\mathbf{V}$  by some condition above  $p$ ).

Also as  $\mathbb{R}$  satisfies the ccc, every level is countable, and by the minimality of  $\alpha$  (as we are allowed to change  $\beta$ ) clearly  $\alpha$  is a regular cardinal. Now  $\alpha > \omega_1$  is impossible by “ $\mathbb{R}$  satisfies the  $\kappa$ -cc”. As there are no Souslin tree also  $\alpha = \omega_1$  is impossible. So clearly  $p$  forces that  $\mathbb{R}$  add reals so there is  $\underline{\eta}$  as required.  $\square$

**3.7. Observation.** Theorem 3.5 suffices to prove 1.2 and its strengthening (\*\*) mentioned in the introduction. That is, conditions (2)&(3) of 3.5 imply:

Any nontrivial forcing with the Laver property has an uncountable antichain.

*Proof.* Let  $\mathbb{R}$  be a forcing notion with the Laver property which adds a real, say  $p \Vdash \underline{\eta} \in {}^\omega 2, \underline{\eta} \notin V$ .

Consider any increasing function  $f$ . The function  $n \mapsto \eta \upharpoonright f(n+1)$  has only  $2^{f(n+1)}$  many possible values, i.e., is bounded. So, by the Laver property there is a tree  $T_f \subseteq {}^{\omega > 2}$  and a condition  $q_f$  stronger than  $q$  with

$$\forall n |T_f \cap 2^{f(n+1)}| \leq f(n), \quad q_f \Vdash \underline{\eta} \in \lim T_f$$

We have thus defined a family  $\mathbb{T} = (T_f : f \text{ increasing})$ . By theorem 3.5, there is a family  $(f_i : i \in \omega_1)$  such that

$$\forall i \neq j : \lim T_{f_i} \cap \lim T_{f_j} \text{ is finite}$$

Clearly, for  $i \neq j$  the conditions  $q_{f_i}$  and  $q_{f_j}$  must be incompatible, since any condition  $r$  stronger than both would force

$$r \Vdash_{\mathbb{R}} \underline{\eta} \in \lim T_{f_i} \cap \lim T_{f_j}$$

which implies  $r \Vdash \underline{\eta} \in V$ , a contradiction.

(Remark: While  $\lim T_{f_i}$  and  $\lim T_{f_j}$  can of course contain branches in  $V^{\mathbb{R}}$  which did not exist in  $V$ , the fact that their intersection is a certain finite set is absolute between  $V$  and  $V^{\mathbb{R}}$ .)  $\square$

**Proof of theorem 3.5, part 3.** Assume that  $\underline{\mathbb{T}}$  is a  $\mathbb{P}_\kappa$ -name such that  $p^* \in \mathbb{P}_\kappa$  forces “ $\underline{\mathbb{T}}$  satisfies  $(\alpha), (\beta), (\gamma)$ ”.

Without loss of generality (replacing the ground model by an intermediate model  $V[G_\alpha]$ ,  $G_\alpha \subseteq \mathbb{P}_\alpha$ ,  $p^* \in G_\alpha$ , if necessary) we can assume that  $p^*$  is really the empty condition.

Let  $S \subseteq \kappa$  be unbounded,  $\delta \in S \Rightarrow \mathbb{Q}_\delta$  is Mathias forcing, with generic real  $\underline{\eta}_\delta$ .

For every  $\delta \in S$  let  $\underline{\mathbb{T}}_\delta^0 = \underline{\mathbb{T}}(\underline{\eta}_\delta^0)$ ; clearly :

$$(*)_{14} \Vdash_{\mathbb{P}_\kappa} \text{“}\underline{\mathbb{T}}_\delta^0 \text{ is a subtree of } {}^{\omega > 2} \text{ such that } (\forall n) |\underline{\mathbb{T}}_\delta^0 \cap \underline{\eta}_\delta^{(n)} 2| \leq \underline{\eta}_\delta^-(n)\text{.”}$$

There is only a bounded number of possibilities for  $\underline{\mathbb{T}}_\delta^0 \cap \underline{\eta}_\delta^{(n)} 2$ , so since  $\mathbb{P}_\kappa / \mathbb{P}_{\delta+1}$  has the Laver property, we can find a pair  $(p_\delta, \underline{\mathbb{T}}_\delta)$  satisfying

$$(*)_{15} \begin{array}{l} (1) p_\delta \in \mathbb{P}_\kappa \\ (2) \underline{\mathbb{T}}_\delta \text{ is a } \mathbb{P}_{\delta+1}\text{-name} \\ (3) \Vdash_{\mathbb{P}_{\delta+1}} \text{“}\underline{\mathbb{T}}_\delta \text{ is a subtree of } {}^{\omega > 2} \text{ and } n < \omega \rightarrow |\underline{\mathbb{T}}_\delta \cap \underline{\eta}_\delta^{(n)} 2| \leq \hat{\eta}_\delta^-(n)\text{”} \end{array}$$

$$(4) \Vdash_{\mathbb{P}} \text{“}\underline{T}_\delta^0 \subseteq \underline{T}_\delta\text{”}$$

So there is stationary subset  $S_1 \subseteq S$  and a condition  $q_1 \in \mathbb{P}_\kappa$  such that  $\delta \in S_1 \rightarrow p_\delta \restriction \delta = q_1$ . (Again we may assume that  $q_1$  is the trivial condition.)

Possibly increasing  $p_\delta(\delta)$  we can find a  $\mathbb{P}_\delta$ -name such that  $p_\delta \restriction \delta$  forces:

(\*)<sub>16</sub> Above  $p_\delta(\delta)$ , the  $\mathbb{Q}_\delta$ -name  $\underline{T}_\delta$  can be read continuously from  $\eta_\delta$  as in (\*)<sub>1</sub>, through the function  $\underline{t}_\delta$ .

For  $\delta \in S_1, p_\delta(\delta)$ ,  $\underline{t}_\delta \restriction \underline{T}_\delta$  are members of  $\mathcal{H}(\aleph_1)^{\mathbf{V}[G_\delta]}$ . So we can find  $q_\delta \geq p_\delta \restriction \delta$  forcing  $p_\delta(\delta)$ ,  $\underline{t}_\delta$  and  $\underline{T}_\delta$  to be equal to hereditarily countable  $\mathbb{P}_\delta$ -names  $p'_\delta(\delta)$ ,  $\underline{t}'_\delta$  and  $\underline{T}'_\delta$ . [Here, “hereditarily countable” is taken in the sense of [Sh:f, III 4.1A].]

Since  $cf(\delta) > \aleph_0$  for  $\delta \in S_1$  we can find a stationary subset  $S_2 \subseteq S_1$  on which  $p'_\delta(\delta)$ ,  $\underline{t}'_\delta$  and  $\underline{T}'_\delta$  are all constant, say with values  $p^*$ ,  $\underline{t}^*$  and  $\underline{T}^*$ . Again we change our base universe to some intermediate universe so that  $\underline{T}^*$  is now a  $\mathbb{M}$ -name, and  $t^*$  is an actual function, and  $p^* = (w^*, A^*) \in \mathbb{M}$ .

We now use our main lemma 2.1 to find an almost disjoint family  $(A_i : i \in \omega_1)$  and  $(T_i : i \in \omega_1)$  such that  $A_i \subseteq A^*$ ,  $(w^*, A_i) \Vdash_{\mathbb{M}} \underline{T}^* \subseteq T[A_i]$ .

Note because of (\*)<sub>1</sub> this relation can already be computed from  $t^*$ , so we also have:

(\*)<sub>17</sub>  $\forall \delta \in S_2 \Vdash_{\mathbb{P}_\delta} \text{“}q_i \Vdash_{\mathbb{Q}_\delta} \underline{T}_\delta \subseteq T[A_i]\text{”}$ .

Now consider the model  $V[G_\kappa]$ . A density argument shows that

(\*)<sub>18</sub>  $\forall i : \{\delta \in S_1 : q_i \in G_{\mathbb{Q}_\delta}\} \neq \emptyset$

So for all  $i$  there is  $\delta = \delta(i)$  with  $\mathbb{T}(\eta_{\delta(i)}) \subseteq T[A_i]$ . Letting

$$\mathbf{A} := \{\delta_i : i < \omega_1\}$$

we have found an uncountable family as required.

#### 4. REFINEMENTS

Theorem 3.5 answers the original question, but essentially the same proof gives a somewhat stronger theorem. The following remarks point a few places where assumptions can be weakened or conclusions strengthened. We leave the details to the reader.

**4.1. Remark.**  $2^{2^{\aleph_0}} = \aleph_2$  in the ground model is not necessary. The length of our iteration can be any regular cardinal  $\kappa$  satisfying  $\mu^{\aleph_0} < \kappa$  for all  $\mu < \kappa$ . In the final model we will have  $2^{\aleph_0} = \aleph_2 = \kappa$ .

**4.2. Remark.** It is not necessary that all forcing notions have the Laver property. All we need is that  $\mathbb{P}_\kappa/P_{\delta+1}$  is  $(\hat{\eta}_\delta, \hat{\eta}_\delta^-/\eta_\delta^-)$ -bounding, which can be ensured by a slightly stronger condition on all the  $\mathbb{Q}_i$ ,  $\delta < i < \kappa$ .

**4.3. Remark.** We showed that in our model every forcing notion with the Laver property which adds reals will have an uncountable antichain. We can strengthen this conclusion by remarking that such forcing notions will actually have an antichain of size  $\kappa = \aleph_2$ .

*Proof.* Recall the construction of the almost disjoint family after condition (\*<sub>12</sub>), which was used in (\*<sub>17</sub>). Instead of using an almost disjoint family of size  $\aleph_1$  in the intermediate

model we can use a  $\mathbb{P}_\kappa$ -name of an almost disjoint family of size continuum: Identify the set  $A^*$  there with  ${}^\omega 2$ , then every  $\mathbb{P}_\alpha$ -name  $\rho$  of an element of  ${}^\omega 2$  will induce a set  $A_\rho \subseteq A^*$ . Clearly,

$$\Vdash_{\mathbb{P}_\kappa} \rho_1 \neq \rho_2 \Rightarrow A_{\rho_1} \cap A_{\rho_2} \text{ finite.}$$

As before, a density argument ensures that there will be  $\kappa$  many different functions  $\rho$  such that  $(w^*, A_\rho)$  appears in one of the generic Mathias filters for some  $\mathbb{Q}_\delta = \mathbb{M}$ , so we can strengthen the conclusion in 3.5, 3( $\delta$ ) to get a  $\kappa$ -size set  $\mathbf{A}$  rather than just an uncountable one.  $\square$

**4.4. Remark.** We do not need that all forcing notions  $\mathbb{Q}_i$  have size at most  $\aleph_1$ , there are also weaker conditions (e.g.  $\kappa$ -pic, see [Sh:f, Ch VIII]) that will ensure  $\kappa$ -cc of  $\mathbb{P}_\kappa$ .

For example, instead of forcing only with Souslin trees in the odd stages we can use the forcing from [Sh:f, Ch V, Section 6], it specializes the tree (so we can specialize all Aronszajn trees). Here we can prove the  $\kappa$ -cc using the  $\kappa$ -pic condition. ,

**4.5. Remark.** Finally, in  $V^{\mathbb{P}_\kappa}$  we can strengthen the conclusion

Every ccc nontrivial forcing fails the Laver property

as follows:

For every ccc forcing notion  $\mathbb{R}$ , whenever  $\eta$  is an  $\mathbb{R}$ -name of a new member of  ${}^\omega 2$  and  $h$  is a strictly increasing function from  $\omega$  to  $\omega$ , then we can find an increasing sequence  $\langle n_i : i < \omega \rangle$  of natural numbers such that  $h(n_i) < n_{i+1}$  and for no  $p, T$  do we have:

$$p \in \mathbb{R}, T \text{ a subtree of } {}^\omega 2, p \Vdash_{\mathbb{R}} \eta \in \lim(T) \text{ and for every } i < \omega \\ \text{we have } |T \cap {}^{n_{2i+1}} 2| \leq h(n_{2i})$$

The proof is similar to the proof above.

## REFERENCES

- [Sh 480] Saharon Shelah. How special are Cohen and random forcings i.e. Boolean algebras of the family of subsets of reals modulo meagre or null. *Israel Journal of Mathematics*, **88**:159–174, 1994.
- [JdSh 292] Jaime Ihoda (Haim Judah) and Saharon Shelah. Souslin forcing. *The Journal of Symbolic Logic*, **53**:1188–1207, 1988.
- [Sh:f] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer, 1998.