ON A PROBLEM OF STEVE KALIKOW

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ABSTRACT. The Kalikow problem for a pair (λ,κ) of cardinal numbers, $\lambda>\kappa$ (in particular $\kappa=2$) is whether we can map the family of ω–sequences from λ to the family of ω–sequences from κ in a very continuous manner. Namely, we demand that for $\eta,\nu\in{}^{\omega}\lambda$ we have: η,ν are almost equal if and only if their images are.

We show consistency of the negative answer, e.g., for $\aleph_\omega$ but we prove it for smaller cardinals. We indicate a close connection with the free subset property and its variants.

0. Introduction

In the present paper we are interested in the following property of pairs of cardinal numbers:

Definition 0.1. Let λ,κ be cardinals. We say that the pair (λ,κ) has the Kalikow property (and then we write $\mathcal{KL}(\lambda,\kappa)$) if

there is a sequence $\langle F_n : n<\omega\rangle$ of functions such that

$$F_n : {}^{n}\lambda \longrightarrow \kappa \qquad (\text{for } n<\omega)$$

and if $F:{}^{\omega}\lambda\longrightarrow{}^{\omega}\kappa$ is given by

$$(\forall\eta\in{}^{\omega}\lambda)(\forall n\in\omega)\big(F(\eta)(n)=F_n(\eta\restriction n)\big)$$

then for every $\eta,\nu\in{}^{\omega}\lambda$

$$(\forall^\infty n)(\eta(n)=\nu(n)) \ \ \underline{\text{iff}}\ \ (\forall^\infty n)(F(\eta)(n)=F(\nu)(n)).$$

In particular we answer the following question of Kalikow:

Kalikow Problem 0.2. Is $\mathcal{KL}(2^{\aleph_0},2)$ provable in ZFC?

The Kalikow property of pairs of cardinals was studied in [**?**]. Several results are known already. Let us mention some of them. First, one can easily notice that

$$\mathcal{KL}(\lambda,\kappa)\ \&\ \lambda'\le\lambda\ \&\ \kappa'\ge\kappa \quad\Rightarrow\quad \mathcal{KL}(\lambda',\kappa').$$

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Also ("transitivity")

$$\mathcal{KL}(\lambda_2,\lambda_1)\ \&\ \mathcal{KL}(\lambda_1,\lambda_0) \quad\Rightarrow\quad \mathcal{KL}(\lambda_2,\lambda_0)$$

and

$$\mathcal{KL}(\lambda,\kappa) \quad\Rightarrow\quad \lambda\leq\kappa^{\aleph_0}.$$

Kalikow proved that CH implies $\mathcal{KL}(2^{\aleph_0},2)$ (in fact that $\mathcal{KL}(\aleph_1,2)$ holds true) and he conjectured that CH is equivalent to $\mathcal{KL}(2^{\aleph_0},2)$.

The question 0.2 is formulated in [**?**, Problem 15.15, p.653].

We shall prove that $\mathcal{KL}(\lambda,2)$ is closely tied with some variants of the free subset property (both positively and negatively). First we present an answer to the problem 0.2 proving the consistency of $\neg\mathcal{KL}(2^{\aleph_0},2)$ in 1.1 (see 2.8 too). Later we discuss variants of the proof (concerning the cardinal and the forcing). Then we deal with positive answer, in particular $\mathcal{KL}(\aleph_n,2)$ and we show that the negation of a relative of the free subset property for λ implies $\mathcal{KL}(\lambda,2)$.

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Notation: We will use Greek letters κ,λ,χ to denote (infinite) cardinals and letters $\alpha,\beta,\gamma,\zeta,\xi$ to denote ordinals. Sequences of ordinals will be called $\bar\alpha$, $\bar\beta$, $\bar\zeta$ with the usual convention that $\bar\alpha=\langle\alpha_n:n<\ell g(\bar\alpha)\rangle$ etc. Sets of ordinals will be denoted by u, v, w (with possible indexes).

The quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ are abbreviations for "for all but finitely many $n\in\omega$" and "for infinitely many $n\in\omega$", respectively.

1. The negative result

For a cardinal χ, the forcing notion $\mathbb{C}_\chi$ for adding χ many Cohen reals consists of finite functions p such that for some $w\in[\chi]^{<\omega}$, $n<\omega$

$$\mathrm{dom}(p)=\{(\zeta,k):\zeta\in w\ \&\ k<n\}\quad\text{ and }\quad\mathrm{rang}(p)\subseteq 2$$

ordered by the inclusion.

Theorem 1.1. *Assume $\lambda\rightarrow(\omega_1\cdot\omega)^{<\omega}_{2^\kappa}$, $2^\kappa<\lambda\leq\chi$. Then*

$$\Vdash_{\mathbb{C}_\chi}\neg\mathcal{KL}(\lambda,\kappa)\quad\textit{ and hence }\quad\Vdash_{\mathbb{C}_\chi}\neg\mathcal{KL}(2^{\aleph_0},2).$$

Proof. Suppose that $\mathbb{C}_\chi$-names $\underset{\sim}{F}_n$ (for $n\in\omega$) and a condition $p\in\mathbb{C}_\chi$ are such that

$$p\Vdash_{\mathbb{C}_\chi}\text{“}\langle\underset{\sim}{F}_n:n<\omega\rangle\text{ exemplifies }\mathcal{KL}(\lambda,\kappa)\text{”}.$$

For $\bar\alpha\in{}^n\lambda$ choose a maximal antichain $\langle p^n_{\bar\alpha,\ell}:\ell<\omega\rangle$ of $\mathbb{C}_\chi$ deciding the values of $\underset{\sim}{F}_n(\bar\alpha)$. Thus we have a sequence $\langle\gamma^n_{\bar\alpha,\ell}:\ell<\omega\rangle\subseteq\kappa$ such that

$$p^n_{\bar\alpha,\ell}\Vdash_{\mathbb{C}_\chi}\underset{\sim}{F}_n(\bar\alpha)=\gamma^n_{\bar\alpha,\ell}.$$

Let χ^* be a sufficiently large regular cardinal. Take an elementary submodel M of $\big(\mathcal{H}(\chi^*), \in, <^*_{\chi^*}\big)$ such that
$\|M\| = \chi$, $\chi+1 \subseteq M$,
$\langle p^n_{\bar{\alpha},\ell} : \ell<\omega, n\in\omega, \bar{\alpha}\in {}^n\lambda\rangle, \langle \gamma^n_{\bar{\alpha},\ell} : \ell<\omega, n\in\omega, \bar{\alpha}\in {}^n\lambda\rangle \in M$.

By $\lambda \to (\omega_1\cdot\omega)^{<\omega}_{2^\kappa}$ (see [?, Claim 1.3]), we find a set $B\subseteq\lambda$ of indescernibles in M over

$$\kappa \cup \{\langle p^n_{\bar{\alpha},\ell} : \ell<\omega : n\in\omega, \bar{\alpha}\in {}^n\lambda\rangle, \langle \gamma^n_{\bar{\alpha},\ell} : \ell<\omega : n\in\omega, \bar{\alpha}\in {}^n\lambda\rangle, \chi, p\}$$

and a system $\langle N_u : u\in [B]^{<\omega}\rangle$ of elementary submodels of M such that

(a) B is of the order type $\omega_1\cdot\omega$ and for $u,v\in[B]^{<\omega}$:
(b) $\kappa+1\subseteq N_u$,
(c) $\chi, p, \langle p^n_{\bar{\alpha},\ell} : \ell<\omega, n<\omega, \bar{\alpha}\in {}^n\lambda\rangle, \langle \gamma^n_{\bar{\alpha},\ell} : \ell<\omega, n<\omega, \bar{\alpha}\in {}^n\lambda\rangle \in N_u$,
(d) $|N_u| = \kappa$, $N_u\cap B = u$,
(e) $N_u\cap N_v = N_{u\cap v}$,
(f) $|u|=|v| \quad\Rightarrow\quad N_u\cong N_v$, and let $\pi_{u,v} : N_v \longrightarrow N_u$ be this (unique) isomorphism,
(g) $\pi_{v,v} = \mathrm{id}_{N_v}$, $\pi_{u,v}(v) = u$, $\pi_{u_0,u_1}\circ\pi_{u_1,u_2} = \pi_{u_0,u_2}$,
(h) if $v'\subseteq v$, $|v|=|u|$ and $u' = \pi_{u,v}(v')$ then $\pi_{u',v'}\subseteq\pi_{u,v}$.

Note that if $u\subseteq B$ is of the order type ω then we may define

$$N_u = \bigcup\{N_v : v \text{ is a finite initial segment of } u\}.$$

Then the models N_u (for $u\subseteq B$ of the order type $\le\omega$) have the properties (b)–(h) too.

Let $\langle \beta_\zeta : \zeta<\omega_1\cdot\omega\rangle$ be the increasing enumeration of B. For a set $u\subseteq B$ of the order type $\le\omega$ let $\bar{\beta}^u$ be the increasing enumeration of u (so $\ell g(\bar{\beta}^u) = |u|$). Let $u^* = \{\beta_{\omega_1\cdot n} : n<\omega\}$. For $k\le\omega$ and a sequence $\bar{\xi} = \langle \xi_m : m<k\rangle \subseteq \omega_1$ we define

$$u[\bar{\xi}] = \{\beta_{\omega_1\cdot m+\xi_m} : m<k\} \cup \{\beta_{\omega_1\cdot n} : n\in\omega\setminus k\}.$$

Now, working in $\mathbf{V}^{\mathbb{C}_\chi}$, we say that a sequence $\underset{\sim}{\bar{\xi}}$ is k–*strange* if

(1) $\underset{\sim}{\bar{\xi}}$ is a sequence of countable ordinals greater than 0, $\ell g(\underset{\sim}{\bar{\xi}}) = k$
(2) $(\forall m<\omega)(\underset{\sim}{F}_m(\bar{\beta}^{u[\underset{\sim}{\bar{\xi}}]}{\restriction}m) = \underset{\sim}{F}_m(\bar{\beta}^{u^*}{\restriction}m))$.

Claim 1.1.1. *In* $\mathbf{V}^{\mathbb{C}_\chi}$*:*
if $\underset{\sim}{\bar{\xi}}^k$ *are* k*–strange sequences (for* $k<\omega$*) such that* $(\forall k<\omega)(\underset{\sim}{\bar{\xi}}^k \lhd \underset{\sim}{\bar{\xi}}^{k+1})$ then *the sequence* $\underset{\sim}{\bar{\xi}} \overset{\text{def}}{=} \bigcup\limits_{k<\omega} \underset{\sim}{\bar{\xi}}^k$ *is* ω*–strange.*

Proof of the claim. Should be clear (note that in this situation we have $\bar{\beta}^{u[\underset{\sim}{\bar{\xi}}]}{\restriction}m = \bar{\beta}^{u[\underset{\sim}{\bar{\xi}}^m]}{\restriction}m$). $\square$

Claim 1.1.2.

$$p \Vdash_{\mathbb{C}_\chi} \text{"\textit{there are no } } \omega\text{\textit{–strange sequences}"}.$$

Proof of the claim. Assume not. Then we find a name $\bar{\underset{\sim}{\xi}} = \langle \underset{\sim}{\xi}_m : m < \omega \rangle$ for an ω–sequence and a condition $q \geq p$ such that

$$q \Vdash_{\mathbb{C}_\chi} \text{"}(\forall m < \omega)(0 < \underset{\sim}{\xi}_m < \omega_1 \quad \& \quad \underset{\sim}{F}_m(\bar{\beta}^{u[\bar{\underset{\sim}{\xi}}]}{\restriction} m) = \underset{\sim}{F}_m(\bar{\beta}^{u^*}{\restriction} m))\text{"}.$$

By the choice of p and $\underset{\sim}{F}_m$ we conclude that

$$q \Vdash_{\mathbb{C}_\chi} \text{"}(\forall^\infty m)(\bar{\beta}^{u[\bar{\underset{\sim}{\xi}}]}(m) = \bar{\beta}^{u^*}(m))\text{"}$$

which contradicts the definition of $\bar{\beta}^{u[\bar{\underset{\sim}{\xi}}]}$, $\bar{\beta}^{u^*}$ and the fact that

$$q \Vdash_{\mathbb{C}_\chi} \text{"}(\forall m < \omega)(0 < \underset{\sim}{\xi}_m < \omega_1)\text{"}.$$

$\square$

By 1.1.1, 1.1.2, any inductive attempt to construct (in $\mathbf{V}^{\mathbb{C}_\chi}$) an ω–strange sequence $\bar{\underset{\sim}{\xi}}$ has to fail. Consequently we find a condition $p^* \geq p$, an integer $k < \omega$ and a sequence $\bar{\xi} = \langle \xi_\ell : \ell < k \rangle$ such that

$$p^* \Vdash_{\mathbb{C}_\chi} \text{"}\bar{\xi} \text{ is } k\text{–strange but } \neg(\exists \xi < \omega_1)(\bar{\xi}{}^\frown\langle\xi\rangle \text{ is } (k+1)\text{–strange)"}.$$

Then in particular

$$(\boxtimes) \qquad p^* \Vdash_{\mathbb{C}_\chi} \text{"}(\forall m < \omega)(\underset{\sim}{F}_m(\bar{\beta}^{u[\bar{\xi}]}{\restriction} m) = \underset{\sim}{F}_m(\bar{\beta}^{u^*}{\restriction} m))\text{"}.$$

[It may happen that $k = 0$, i.e., $\bar{\xi} = \langle\rangle$.]

For $\xi < \omega_1$ let $u_\xi = u[\bar{\xi}{}^\frown\langle\xi\rangle]$ and $w_\xi = u_\xi \cup (u^* \setminus \{\omega_1 \cdot k\})$. Thus $w_0 = u[\bar{\xi}] \cup u^*$ and all w_ξ have order type ω and $\pi_{w_{\xi_1}, w_{\xi_2}}$ is the identity on $N_{w_\xi \setminus \{\omega_1 \cdot k + \xi_2\}}$. Let $q \stackrel{\text{def}}{=} p^* {\restriction} N_{w_0}$ and $q_\xi = \pi_{w_\xi, w_0}(q) \in N_{w_\xi}$ (so $q_0 = q$). As the isomorphism π_{w_ξ, w_0} is the identity on $N_{w_0} \cap N_{w_\xi} = N_{w_0 \cap w_\xi}$ (and by the definition of Cohen forcing), we have that the conditions q, q_ξ are compatible. Moreover, as $p^* \geq p$ and $p \in N_\emptyset$, we have that both q and q_ξ are stronger than p.

Now fix $\xi_0 \in (0, \omega_1)$ (e.g. $\xi_0 = 1$) and look at the sequences $\bar{\beta}^{u_{\xi_0}}$ and $\bar{\beta}^{u^*}$. They are eventually equal and hence

$$p \Vdash_{\mathbb{C}_\chi} \text{"}(\forall^\infty m)(\underset{\sim}{F}_m(\bar{\beta}^{u_{\xi_0}}{\restriction} m) = \underset{\sim}{F}_m(\bar{\beta}^{u^*}{\restriction} m))\text{"}.$$

So we find $m^* < \omega$ and a condition $q'_{\xi_0} \geq q_{\xi_0}, q$ such that

$(\otimes^{\xi_0, m^*}_{q'_{\xi_0}})$ $q'_{\xi_0} \Vdash_{\mathbb{C}_\chi}$ "$(\forall m \geq m^*)(\underset{\sim}{F}_m(\bar{\beta}^{u_{\xi_0}}{\restriction} m) = \underset{\sim}{F}_m(\bar{\beta}^{u^*}{\restriction} m))$"

and (as we can increase q'_{ξ_0})

$(\oplus^{\xi_0, m^*}_{q'_{\xi_0}})$ the condition q'_{ξ_0} decides the values of $\underset{\sim}{F}_m(\bar{\beta}^{u_{\xi_0}}{\restriction} m)$ and $\underset{\sim}{F}_m(\bar{\beta}^{u^*}{\restriction} m)$ for all $m \leq m^*$.

Note that the condition $(\otimes^{\xi_0, m^*}_{q'_{\xi_0}})$ means that

there are NO $m \geq m^*$, $\ell_0, \ell_1 < \omega$ with
$\gamma^m_{\bar\beta^{u_{\xi_0}}\restriction m,\ell_0} \neq \gamma^m_{\bar\beta^{u^*}\restriction m,\ell_1}$ and the three conditions q'_{ξ_0}, $p^m_{\bar\beta^{u_{\xi_0}}\restriction m,\ell_0}$ and $p^m_{\bar\beta^{u^*}\restriction m,\ell_1}$ have a common upper bound in $\mathbb{C}_\chi$

(remember the choice of the $p^n_{\bar\alpha,\ell}$'s and $\gamma^n_{\bar\alpha,\ell}$'s). Similarly, the condition $(\oplus^{\xi_0,m^*}_{q'_{\xi_0}})$ means

there are NO $m \leq m^*$, $\ell_0, \ell_1 < \omega$ with
either $\gamma^m_{\bar\beta^{u_{\xi_0}}\restriction m,\ell_0} \neq \gamma^m_{\bar\beta^{u_{\xi_0}}\restriction m,\ell_1}$ and both q'_{ξ_0} and $p^m_{\bar\beta^{u_{\xi_0}}\restriction m,\ell_0}$, and q'_{ξ_0} and $p^m_{\bar\beta^{u_{\xi_0}}\restriction m,\ell_1}$ are compatible in $\mathbb{C}_\chi$
or $\gamma^m_{\bar\beta^{u^*}\restriction m,\ell_0} \neq \gamma^m_{\bar\beta^{u^*}\restriction m,\ell_1}$ and both q'_{ξ_0} and $p^m_{\bar\beta^{u^*}\restriction m,\ell_0}$, and q'_{ξ_0} and $p^m_{\bar\beta^{u^*}\restriction m,\ell_1}$ are compatible in $\mathbb{C}_\chi$.

Consequently the condition $q^*_{\xi_0} \stackrel{\rm def}{=} q'_{\xi_0}\restriction N_{w_0\cup w_{\xi_0}}$ has both properties $(\otimes^{\xi_0,m^*}_{q^*_{\xi_0}})$ and $(\oplus^{\xi_0,m^*}_{q^*_{\xi_0}})$ (and it is stronger than both q and q_{ξ_0}).

Now, for $0<\xi<\omega_1$ let

$$q^*_\xi \stackrel{\rm def}{=} \pi_{w_0\cup w_\xi, w_0\cup w_{\xi_0}}(q^*_{\xi_0}) \in N_{w_0\cup w_\xi}.$$

Then (for $\xi \in (0,\omega_1)$) the condition q^*_ξ is stronger than

$$\text{both}\ \ q = \pi_{w_0\cup w_\xi, w_0\cup w_{\xi_0}}(q)\ \ \text{and}\ \ q_\xi = \pi_{w_0\cup w_\xi, w_0\cup w_{\xi_0}}(q_{\xi_0})$$

and it has the properties $(\otimes^{\xi,m^*}_{q^*_\xi})$ and $(\oplus^{\xi,m^*}_{q^*_\xi})$. Moreover for all ξ_1,ξ_2 the conditions $q^*_{\xi_1}, q^*_{\xi_2}$ are compatible. [Why? By the definition of Cohen forcing, and $\pi_{w_0\cup w_{\xi_2}, w_0\cup w_{\xi_1}}(q^*_{\xi_1}) = q^*_{\xi_2}$ (chasing arrows) and $\pi_{w_0\cup w_{\xi_2}, w_0\cup w_{\xi_1}}$ is the identity on $N_{w_0\cup w_{\xi_2}} \cap N_{w_0\cup w_{\xi_1}} = N_{(w_0\cup w_{\xi_2})\cap(w_0\cup w_{\xi_1})}$ (see clauses (e), (f), (h) above).]

Claim 1.1.3. *For each $\xi_1,\xi_2 \in (0,\omega_1)$ the condition $q^*_{\xi_1} \cup q^*_{\xi_2}$ forces in $\mathbb{C}_\chi$ that*

$$(\forall m<\omega)(\underset{\sim}{F}_m(\bar\beta^{u_{\xi_1}}\restriction m) = \underset{\sim}{F}_m(\bar\beta^{u_{\xi_2}}\restriction m)).$$

Proof of the claim. If $m \geq m^*$ then, by $(\otimes^{\xi_1,m^*}_{q^*_{\xi_1}})$ and $(\otimes^{\xi_2,m^*}_{q^*_{\xi_2}})$ (passing through $\underset{\sim}{F}(\bar\beta^{u^*}\restriction m)$), we get

$$q^*_{\xi_1} \cup q^*_{\xi_2} \Vdash_{\mathbb{C}_\chi} \text{``}\underset{\sim}{F}_m(\bar\beta^{u_{\xi_1}}\restriction m) = \underset{\sim}{F}_m(\bar\beta^{u_{\xi_2}}\restriction m)\text{''}.$$

If $m<m^*$ then we use $(\oplus^{\xi_1,m^*}_{q^*_{\xi_1}})$ and $(\oplus^{\xi_1,m^*}_{q^*_{\xi_2}})$ and the isomorphism: the values assigned by $q^*_{\xi_1}$, $q^*_{\xi_2}$ to $\underset{\sim}{F}_m(\bar\beta^{u_{\xi_1}}\restriction m)$ and $\underset{\sim}{F}_m(\bar\beta^{u_{\xi_2}}\restriction m)$ have to be equal (remember $\kappa \subseteq N_\emptyset$, so the isomorphism is the identity on κ). $\square$

Look at the conditions

$$q_{\xi_1,\xi_2} \stackrel{\rm def}{=} q^*_{\xi_1}\restriction N_{w_{\xi_1}} \cup q^*_{\xi_2}\restriction N_{w_{\xi_2}} \in N_{w_{\xi_1}\cup w_{\xi_2}}.$$

It should be clear that for each $\xi_1, \xi_2 \in (0, \omega_1)$

$$q_{\xi_1,\xi_2} \Vdash_{\mathbb{C}_\chi} \text{“}(\forall m < \omega)(\underset{\sim}{F}_m(\bar\beta^{u_{\xi_1}}\restriction m) = \underset{\sim}{F}_m(\bar\beta^{u_{\xi_2}}\restriction m))\text{”}.$$

Now choose $\xi \in (0, \omega_1)$ so large that

$$\mathrm{dom}(p^*) \cap (N_{w_\xi} \setminus N_{w_0}) = \emptyset$$

(possible as $\mathrm{dom}(p^*)$ is finite, use (e)). Take any $0 < \xi_1 < \xi_2 < \omega_1$ and put

$$q^* \stackrel{\text{def}}{=} \pi_{w_0 \cup w_\xi, w_{\xi_1} \cup w_{\xi_2}}(q_{\xi_1,\xi_2}).$$

(Note: $\pi_{w_0,w_{\xi_1}} \subseteq \pi_{w_0\cup w_\xi, w_{\xi_1}\cup w_{\xi_2}}$ and $\pi_{w_\xi,w_{\xi_2}} \subseteq \pi_{w_0\cup w_\xi, w_{\xi_1}\cup w_{\xi_2}}$.) By the isomorphism we get that

$$q^* \Vdash_{\mathbb{C}_\chi} \text{“}(\forall m < \omega)(\underset{\sim}{F}_m(\bar\beta^{u_\xi}\restriction m) = \underset{\sim}{F}_m(\bar\beta^{u[\bar\xi]}\restriction m))\text{”}.$$

Now look back:

$$\begin{aligned} q^*_{\xi_1} \geq q_{\xi_1} = \ & \pi_{w_0\cup w_{\xi_1}, w_0\cup w_{\xi_0}}(q_{\xi_0}) = \pi_{w_{\xi_1},w_{\xi_0}}(q_{\xi_0}) = \\ & = \pi_{w_{\xi_1},w_{\xi_0}}(\pi_{w_{\xi_0},w_0}(q)) = \pi_{w_{\xi_1},w_0}(q) \end{aligned}$$

and hence

$$q^*_{\xi_1}\restriction N_{w_{\xi_1}} \geq \pi_{w_{\xi_1},w_0}(q)$$

and thus

$$q^*\restriction N_{w_0} \geq \pi_{w_0,w_{\xi_1}}(q^*_{\xi_1}\restriction N_{w_{\xi_1}}) \geq q = p^*\restriction N_{w_0}.$$

Consequently, by the choice of ξ, the conditions q^* and p^* are compatible (remember the definition of q_{ξ_1,ξ_2} and q^*). Now use ($\boxtimes$) to conclude that

$$q^* \cup p^* \Vdash_{\mathbb{C}_\chi} \text{“}(\forall m < \omega)(\underset{\sim}{F}_m(\bar\beta^{u^*}\restriction m) = \underset{\sim}{F}_m(\bar\beta^{u[\bar\xi]}\restriction m) = \underset{\sim}{F}_m(\bar\beta^{u_\xi}\restriction m))\text{”}$$

which implies that

$$q^* \cup p^* \Vdash_{\mathbb{C}_\chi} \text{“}\bar\xi^\frown\langle\xi\rangle \text{ is } (k+1)\text{–strange”},$$

a contradiction. □

Remark 1.2. About the proof of 1.1:

(1) No harm is done by forgetting 0 and replacing it by ξ_1, ξ_2.
(2) A small modification of the proof shows that in $\mathbf{V}^{\mathbb{C}_\chi}$:

If $F_n : {}^n\lambda \longrightarrow \kappa$ $(n \in \omega)$ are such that

$$(\forall \eta, \nu \in {}^\omega\lambda)[(\forall^\infty n)(\eta(n) = \nu(n)) \quad \Rightarrow \quad (\forall^\infty n)(F_n(\eta\restriction n) = F_n(\nu\restriction n))]$$

then there are infinite sets $X_n \subseteq \lambda$ (for $n < \omega$) such that

$$(\forall n < \omega)(\forall \nu, \eta \in \prod_{\ell<n} X_\ell)(F_n(\nu) = F_n(\eta)).$$

Say we shall have $X_n = \{\gamma_{n,i} : i < \omega\}$. Starting we have $\gamma^*_0, \dots, \gamma^*_n, \dots$. In the proof at stage n we have determined $\gamma_{\ell,i}$ $(\ell, i < n)$ and $p \in G$, $p \in N_{\{\gamma_{\ell,i}:\ell,i<\omega\}\cup\{\gamma^*_n,\gamma^*_{n+1},\dots\}}$. For $n = 0, 1, 2$ as before. For $n + 1 > 2$ first $\gamma_{0,n}, \dots, \gamma_{n-1,n}$ are easy by transitivity of equalities. Then find $\gamma_{n,0}, \gamma_{n,1}$ as before then again duplicate.

(3) In the proof it is enough to use $\{\beta_{\omega\cdot n+\ell} : n < \omega, \ell < \omega\}$. Hence, by 1.2 of [?] it is enough to assume $\lambda \to (\omega^3)^{<\omega}_{2^\kappa}$. This condition is compatible with $\mathbf{V} = \mathbf{L}$.

(4) We can use only $\lambda \to (\omega^2)^{<\omega}_{2^\kappa}$.

Definition 1.3. (1) For a sequence $\bar\lambda = \langle \lambda_n : n < \omega\rangle$ of cardinals we define the property $(\circledast)_{\bar\lambda}$:

$(\circledast)_{\bar\lambda}$ for every model M of a countable language, with universe $\sup\limits_{n\in\omega} \lambda_n$ and Skolem functions (for simplicity) there is a sequence $\langle X_n : n < \omega\rangle$ such that

(a) $X_n \in [\lambda_n]^{\lambda_n}$ (actually $X_n \in [\lambda_n]^{\omega_1}$ suffices)

(b) for every $n < \omega$ and $\bar\alpha = \langle \alpha_\ell : \ell \in [n+1, \omega)\rangle \in \prod\limits_{\ell \ge n+1} X_\ell$, letting (for $\xi \in X_n$)

$$M^\xi_{\bar\alpha} = \mathrm{Sk}(\bigcup_{\ell<n} X_\ell \cup \{\xi\} \cup \{\alpha_\ell : \ell \in [n+1, \omega)\})$$

we have:

$(\bigoplus)$ the sequence $\langle M^\xi_{\bar\alpha} : \xi \in X_n\rangle$ forms a Δ–system with the heart $N_{\bar\alpha}$ and its elements are pairwise isomorphic over the heart $N_{\bar\alpha}$.

(2) For a cardinal λ the condition $(\circledast)^\lambda$ is:

$(\circledast)^\lambda$ there exists a sequence $\bar\lambda = \langle \lambda_n : n < \omega\rangle$ such that $\sum\limits_{n<\omega} \lambda_n = \lambda$ and the condition $(\circledast)_{\bar\lambda}$ holds true.

In [?] a condition $(*)_\lambda$, weaker than $(\circledast)^\lambda$ was considered. Now, [?] continues [?] to get stronger indiscernibility. But by the same proof (using ω-measurable) one can show the consistency of $(\circledast)^{\aleph_\omega}$ + GCH.

Note that to carry out the proof of 1.1 we need even less then $(\circledast)^\lambda$: the $\bigcup\limits_{\ell<n} X_\ell$ (in (b) of 1.3) is much more then needed; it suffices to have $\bar\beta^0 \cup \bar\beta^1$ where $\bar\beta^0, \bar\beta^1 \in \prod\limits_{\ell<n} X_\ell$.

Conclusion 1.4. It is consistent that

$$2^{\aleph_0} = \aleph_{\omega+1} \quad \text{and} \quad \bigwedge_{n<\omega} \neg\mathcal{KL}(\aleph_\omega, \aleph_n) \quad \text{so } \neg\mathcal{KL}(2^{\aleph_0}, 2).$$

Remark 1.5. Koepke [?] continues [?] to get equiconsistency. His refinement of [?] (for the upper bound) works below too.

2. The positive result

For an algebra M on λ and a set $X \subseteq \lambda$ the closure of X under functions of M is denoted by $\mathrm{cl}_M(X)$. Before proving our result (2.6) we remind the reader of some definitions and propositions.

Proposition 2.1. *For an algebra M on λ the following conditions are equivalent*

$(\bigstar)^0_M$ *for each sequence $\langle \alpha_n : n \in \omega\rangle \subseteq \lambda$ we have*

$$(\forall^\infty n)(\alpha_n \in \mathrm{cl}_M(\{\alpha_k : n < k < \omega\})),$$

$(\bigstar)^1_M$ *there is no sequence $\langle A_n : n \in \omega\rangle \subseteq [\lambda]^{\aleph_0}$ such that*

$$(\forall n \in \omega)(\mathrm{cl}_M(A_{n+1}) \subsetneq \mathrm{cl}_M(A_n)),$$

$(\bigstar)^2_M$ $(\forall A \in [\lambda]^{\aleph_0})(\exists B \in [A]^{\aleph_0})(\forall C \in [B]^{\aleph_0})(\mathrm{cl}_M(B) = \mathrm{cl}_M(C))$.

Definition 2.2. We say that a cardinal λ has the $(\bigstar)$–property for κ (and then we write $\mathrm{Pr}^\bigstar(\lambda,\kappa)$) if there is an algebra M on λ with vocabulary of cardinality $\leq \kappa$ satisfying one (equivalently: all) of the conditions $(\bigstar)^i_M$ $(i < 3)$ of 2.1. If $\kappa = \aleph_0$ we may omit it.

Remember

Proposition 2.3. *If $\mathbf{V}_0 \subseteq \mathbf{V}_1$ are universes of set theory, $\mathbf{V}_1 \models \neg\mathrm{Pr}^\bigstar(\lambda)$ then $\mathbf{V}_0 \models \neg\mathrm{Pr}^\bigstar(\lambda)$.*

Proof. By absoluteness of the existence of an ω–branch to a tree. □

Remark 2.4. The property $\neg\mathrm{Pr}^\bigstar(\lambda)$ is a kind of a large cardinal property. It was clarified in $\mathbf{L}$ (remember that it is inherited from $\mathbf{V}$ to $\mathbf{L}$) by Silver [?] to be equiconsistent with "there is a beautiful cardinal" (terminology of 2.3 of [?]), another partition property inherited by $\mathbf{L}$.

Proposition 2.5. *For each $n \in \omega$, $\mathrm{Pr}^\bigstar(\aleph_n)$.*

Proof. This was done in [?, Chapter XIII], see [?, Chapter VII] too, and probably earlier by Silver. However, for the sake of completeness we will give the proof.

First note that clearly $\mathrm{Pr}^\bigstar(\aleph_0)$ and thus we have to deal with the case when $n > 0$. Let $f, g : \aleph_n \longrightarrow \aleph_n$ be two functions such that

> if $m < n$, $\alpha \in [\aleph_m, \aleph_{m+1})$
> then $f(\alpha,\cdot)\restriction\alpha : \alpha \xrightarrow{1-1} \aleph_m$, $g(\alpha,\cdot)\restriction\aleph_m : \aleph_m \xrightarrow{1-1} \alpha$ are functions inverse each to the other.

Let M be the following algebra on $\aleph_n$:

$$M = (\aleph_n, f, g, m)_{m\in\omega}.$$

We want to check the condition $(\bigstar)^1_M$:
assume that a sequence $\langle A_k : k<\omega\rangle \subseteq [\aleph_n]^{\aleph_0}$ is such that for each $k<\omega$

$$\mathrm{cl}_M(A_{k+1}) \subsetneq \mathrm{cl}_M(A_k).$$

For each $m<n$, the sequence $\langle \sup(\mathrm{cl}_M(A_k)\cap\aleph_{m+1}) : k<\omega\rangle$ is non-increasing and therefore it is eventually constant. Consequently we find k^* such that

$$(\forall m<n)(\sup(\mathrm{cl}_M(A_{k^*+1})\cap\aleph_{m+1}) = \sup(\mathrm{cl}_M(A_{k^*})\cap\aleph_{m+1})).$$

By the choice of $\langle A_k : k<\omega\rangle$ we have $\mathrm{cl}_M(A_{k^*+1}) \subsetneq \mathrm{cl}_M(A_{k^*})$. Let

$$\alpha_0 \stackrel{\mathrm{def}}{=} \min(\mathrm{cl}_M(A_{k^*}) \setminus \mathrm{cl}_M(A_{k^*+1})).$$

As the model M contains individual constants m (for $m\in\omega$) we know that $\aleph_0 \subseteq \mathrm{cl}_M(\emptyset)$ and hence $\aleph_0 \le \alpha_0$. Let $m<n$ be such that $\aleph_m \le \alpha_0 < \aleph_{m+1}$. By the choice of k^* we find $\beta \in \mathrm{cl}_M(A_{k^*+1})\cap\aleph_{m+1}$ such that $\alpha_0 \le \beta$. Then necessarily $\alpha_0<\beta$. Look at $f(\beta,\alpha_0)$: we know that $\alpha_0,\beta\in\mathrm{cl}_M(A_{k^*})$ and therefore $f(\beta,\alpha_0)\in\mathrm{cl}_M(A_{k^*})\cap\aleph_m$ and $f(\beta,\alpha_0)<\alpha_0$. The minimality of α_0 implies that $f(\beta,\alpha_0)\in\mathrm{cl}_M(A_{k^*+1})$ and hence

$$\alpha_0 = g(\beta, f(\beta,\alpha_0)) \in \mathrm{cl}_M(A_{k^*+1}),$$

a contradiction. $\square$

Explanation: Better think of the proof from the end. Let $\bar\alpha = \langle \alpha_n : n<\omega\rangle \in {}^{\omega}\lambda$. So for some $n(*)$, $n(*) \le n < \omega \;\Rightarrow\; \alpha_n \in \mathrm{cl}_M(\alpha_\ell : \ell > n)$. So for some $m_n > n$, $\{\alpha_{n(*)},\dots,\alpha_{n-1}\} \subseteq \mathrm{cl}_M(\alpha_n,\dots,\alpha_{m-1})$ and

$$(\forall \ell < n(*))(\alpha_\ell \in \mathrm{cl}_M(\alpha_\ell : \ell > n(*)) \;\Rightarrow\; \alpha_\ell \in \mathrm{cl}_M(\alpha_\ell : \ell \in [n, m_n))).$$

Let $W^* = \{\ell < n(*) : \alpha_\ell \in \mathrm{cl}_M(\alpha_n : n \ge n(*)\}$. It is natural to aim at:

- $(*)$ for n large enough (say $n > m_{n(*)}$), $F_n(\langle \alpha_\ell : \ell < n\rangle)$ depends just on $\{\alpha_\ell : \ell \in [n(*), n)$ or $\ell \in w\}$ and $\langle F_m(\bar\alpha{\restriction} m) : m \ge n\rangle$ codes $\bar\alpha{\restriction}(w \cup [n(*),\omega))$.

Of course, we are a given n and we do not know how to compute the real $n(*)$, but we can approximate. Then we look at a late enough end segment where we compute down.

Theorem 2.6. *Assume that $\lambda \le 2^{\aleph_0}$ is such that $\Pr^{\bigstar}(\lambda)$ holds. Then $\mathcal{KL}(\lambda,\omega)$ (and hence $\mathcal{KL}(\lambda,2)$).*

Proof. We have to construct functions $F_n : {}^{n}\lambda \longrightarrow \omega$ witnessing $\mathcal{KL}(\lambda,\omega)$. For this we will introduce functions $\mathbf{k}$ and $\mathbf{l}$ such that for $\bar\alpha \in {}^{n}\lambda$ the value of $\mathbf{k}(\bar\alpha)$ will say which initial segment of $\bar\alpha$ will be irrelevant for $F_n(\bar\alpha)$

and $\mathbf{l}(\bar\alpha)$ will be such that (under certain circumstances) elements α_i (for $\mathbf{k}(\bar\alpha) \le i < \mathbf{l}(\bar\alpha)$) will be encoded by $\langle \alpha_j : j \in [\mathbf{l}(\bar\alpha), n)\rangle$.

Fix a sequence $\langle \eta_\alpha : \alpha < \lambda\rangle \subseteq {}^\omega 2$ with no repetitions.

Let M be an algebra on λ such that $(\bigstar)^0_M$ holds true. We may assume that there are no individual constants in M (so $\mathrm{cl}_M(\emptyset) = \emptyset$).
Let $\langle \tau^n_\ell(x_0,\ldots,x_{n-1}) : \ell < \omega\rangle$ list all n-place terms of the language of the algebra M (and $\tau^1_0(x)$ is x). For $\bar\alpha \in {}^{\omega\ge}\lambda$ (with α_j the j-th element in $\bar\alpha$) let

$$u(\bar\alpha) = \{\ell < \ell g(\bar\alpha) : \alpha_\ell \notin \mathrm{cl}_M\big(\bar\alpha{\restriction}(\ell, \ell g(\bar\alpha))\big)\} \cup \{0\}$$

and for $\ell \notin u(\bar\alpha)$, $\ell < \ell g(\bar\alpha)$ let

$$\begin{aligned} f_\ell(\bar\alpha) &= \min\{j : \alpha_\ell \in \mathrm{cl}_M(\bar\alpha{\restriction}(\ell, j))\}\\ g_\ell(\bar\alpha) &= \min\{i : \alpha_\ell = \tau_i^{f_\ell(\bar\alpha)-\ell-1}(\bar\alpha{\restriction}(\ell, f_\ell(\bar\alpha)))\}. \end{aligned}$$

For $\bar\alpha \in {}^n\lambda$ $(1 < n < \omega)$ put

$$\begin{aligned} k_1(\bar\alpha) &= \min\big((u(\bar\alpha{\restriction}(n-1)) \setminus u(\bar\alpha)) \cup \{n-1\}\big)\\ k_0(\bar\alpha) &= \max\big(u(\bar\alpha) \cap k_1(\bar\alpha)\big). \end{aligned}$$

Note that if ($n > 1$ and) $\bar\alpha \in {}^n\lambda$ then $n-1 \in u(\bar\alpha)$ (as $\mathrm{cl}_M(\emptyset) = \emptyset$) and $k_1(\bar\alpha) > 0$ (as always $0 \in u(\bar\beta)$) and $k_0(\bar\alpha)$ is well defined (as $0 \in u(\bar\alpha) \cap k_1(\bar\alpha)$) and $k_0(\bar\alpha) < k_1(\bar\alpha) < n$. Moreover, for all $\ell \in (k_0(\bar\alpha), k_1(\bar\alpha))$ we have $\alpha_\ell \notin u(\bar\alpha{\restriction}(n-1))$ and thus $\alpha_\ell \in \mathrm{cl}_M(\bar\alpha{\restriction}(\ell, n-1))$. Now, for $\bar\alpha \in {}^{\omega>}\lambda$, $\ell g(\bar\alpha) > 1$ we define

$$\begin{aligned} \mathbf{l}(\bar\alpha) &= \max\{j \le k_1(\bar\alpha) : j > k_0(\bar\alpha) \;\Rightarrow\; (\forall i \in (k_0(\bar\alpha), j))(g_i(\bar\alpha) \le \ell g(\bar\alpha))\}\\ \mathbf{m}(\bar\alpha) &= \max\{j \le \mathbf{l}(\bar\alpha) : j > \max\{1, k_0(\bar\alpha)\} \;\Rightarrow\; k_0(\bar\alpha{\restriction}j) = k_0(\bar\alpha)\}\\ \mathbf{k}(\bar\alpha) &= \mathbf{l}(\bar\alpha{\restriction}\mathbf{m}(\bar\alpha)) \quad (\text{if } \mathbf{m}(\bar\alpha) \le 1 \text{ then put } \mathbf{k}(\bar\alpha) = -1). \end{aligned}$$

Clearly $\mathbf{k}(\bar\alpha) < \mathbf{m}(\bar\alpha) \le \mathbf{l}(\bar\alpha) \le k_1(\bar\alpha) < \ell g(\bar\alpha)$.

Claim 2.6.1. *For each $\bar\alpha \in {}^\omega\lambda$, the set $u(\bar\alpha)$ is finite and:*

(1) *The sequence $\langle k_1(\bar\alpha{\restriction}n) : n < \omega\rangle$ diverges to ∞.*
(2) *The sequence $\langle k_0(\bar\alpha{\restriction}n) : n < \omega \;\&\; k_0(\bar\alpha) \ne \max u(\bar\alpha)\rangle$, if infinite, diverges to ∞. There are infinitely many $n < \omega$ with $k_0(\bar\alpha{\restriction}n) = \max u(\bar\alpha)$.*
(3) *The sequence $\langle \mathbf{l}(\bar\alpha{\restriction}n) : n < \omega\rangle$ diverges to ∞.*
(4) *The sequences $\langle \mathbf{m}(\bar\alpha{\restriction}n) : n < \omega\rangle$ and $\langle \mathbf{k}(\bar\alpha{\restriction}n) : n < \omega\rangle$ diverge to ∞.*

Proof of the claim. Let $\bar\alpha = \langle \alpha_n : n < \omega\rangle \in {}^\omega\lambda$. By the property $(\bigstar)^0_M$ we find $n^* < \omega$ such that $u(\bar\alpha) \subseteq n^*$. Fix $n_0 > n^*$ and define

$$n_1 = \max\{f_n(\bar\alpha) + g_n(\bar\alpha) + 2 : n \in (n_0+1) \setminus u(\bar\alpha)\}$$

(so $n_1 \geq f_{n_0}(\bar{\alpha}) + 2 > n_0 + 3$ and for all $\ell \in (n_0+1) \setminus u(\bar{\alpha})$ we have: $\alpha_\ell \in \mathrm{cl}_M(\alpha_{\ell+1}, \ldots, \alpha_{n_1-1})$ is witnessed by $\tau^{f_\ell(\bar{\alpha})-\ell-1}_{g_\ell(\bar{\alpha})}(\alpha_{\ell+1}, \ldots, \alpha_{f_\ell(\bar{\alpha})-1})$ with $f_\ell(\bar{\alpha}), g_\ell(\bar{\alpha}) < n_1 - 1$).

1) Note that $u(\bar{\alpha}\restriction n) \cap (n_0+1) = u(\bar{\alpha})$ for all $n \geq n_1 - 1$ and hence for $n \geq n_1$

$$u(\bar{\alpha}\restriction n) \cap (n_0+1) = u(\bar{\alpha}\restriction(n-1)) \cap (n_0+1).$$

Consequently for all $n \geq n_1$ we have that $k_1(\bar{\alpha}\restriction n) > n_0$. As we could have chosen n_0 arbitrarily large we may conclude that $\lim\limits_{n\to\infty} k_1(\bar{\alpha}\restriction n) = \infty$.

2) Note that for all $n \geq n_1$

$$\text{either}\quad k_0(\bar{\alpha}\restriction n) = \max(u(\bar{\alpha}))\quad \text{or}\quad k_0(\bar{\alpha}\restriction n) > n_0.$$

Hence, by the arbitrarity of n_0, we get the first part of 2).
Let $\ell^* = \min(u(\bar{\alpha}\restriction n_1) \setminus u(\bar{\alpha}))$ (note that $n_1 - 1 \in u(\bar{\alpha}\restriction n_1) \setminus u(\bar{\alpha})$). Clearly $\ell^* > n_0$ and $\alpha_{\ell^*} \notin u(\bar{\alpha})$. Consider $n = f_{\ell^*}(\bar{\alpha})$ (so $\ell^* \leq n-2$, $n_1 \leq n-1$). Then $\ell^* \in u(\bar{\alpha}\restriction(n-1)) \setminus u(\bar{\alpha}\restriction n)$. As

$$\ell^* \cap u(\bar{\alpha}\restriction n_1) = \ell^* \cap u(\bar{\alpha}\restriction n-1) = u(\bar{\alpha})$$

(remember the choice of ℓ^*) we conclude that

$$\ell^* = k_1(\bar{\alpha}\restriction n) \quad \text{and} \quad k_0(\bar{\alpha}\restriction n) = \max u(\bar{\alpha}).$$

Now, since n_0 was arbitrarily large, we get that for infinitely many n, $k_0(\bar{\alpha}\restriction n) = \max u(\bar{\alpha})$.

3) Suppose that $n \geq n_1$. Then we know that $k_1(\bar{\alpha}\restriction n) > n_0$ and either $k_0(\bar{\alpha}\restriction n) = \max u(\bar{\alpha})$ or $k_0(\bar{\alpha}\restriction n) > n_0$ (see above). If the first possibility takes place then, as $n \geq n_1$, we may use $j = n_0 + 1$ to witness that $\mathbf{l}(\bar{\alpha}\restriction n) > n_0$ (remember the choice of n_1). If $k_0(\bar{\alpha}\restriction n) > n_0$ then clearly $\mathbf{l}(\bar{\alpha}\restriction n) > n_0$. As n_0 could be arbitrarily large we are done.

4) Suppose we are given $m_0 < \omega$. Take $m_1 > m_0$ such that for all $n \geq m_1$

$$\text{either } k_0(\bar{\alpha}\restriction n) = \max u(\bar{\alpha}) \quad \text{or} \quad k_0(\bar{\alpha}\restriction n) > m_0$$

(possible by 2)) and then choose $m_2 > m_1$ such that $k_0(\bar{\alpha}\restriction m_2) = \max u(\bar{\alpha})$ (by 2)). Due to 3) we find $m_3 > m_2$ such that for all $n \geq m_3$, $\mathbf{l}(\bar{\alpha}\restriction n) > m_2$. Now suppose that $n \geq m_3$. If $k_0(\bar{\alpha}\restriction n) = \max u(\bar{\alpha})$ then, as $\mathbf{l}(\bar{\alpha}\restriction n) > m_2$, we get $\mathbf{m}(\bar{\alpha}\restriction n) \geq m_2 > m_0$. Otherwise $k_0(\bar{\alpha}\restriction n) > m_0$ (as $n > m_1$) and hence $\mathbf{m}(\bar{\alpha}\restriction n) > m_0$. This shows that $\lim\limits_{n\to\infty} \mathbf{m}(\bar{\alpha}\restriction n) = \infty$. Now, immediately by the definition of $\mathbf{k}$ and 3) above we conclude that $\lim\limits_{n\to\infty} \mathbf{k}(\bar{\alpha}\restriction n) = \infty$. $\square$

Claim 2.6.2. *If $\bar{\alpha}^1, \bar{\alpha}^2 \in {}^\omega\lambda$ are such that $(\forall^\infty n)(\alpha^1_n = \alpha^2_n)$ then*

$$(\forall^\infty n)\Big(\mathbf{l}(\bar{\alpha}^1\restriction n) = \mathbf{l}(\bar{\alpha}^2\restriction n) \ \&\ \mathbf{m}(\bar{\alpha}^1\restriction n) = \mathbf{m}(\bar{\alpha}^2\restriction n) \ \&\ \mathbf{k}(\bar{\alpha}^1\restriction n) = \mathbf{k}(\bar{\alpha}^2\restriction n)\Big).$$

Proof of the claim. Let n_0 be greater than $\max(u(\bar{\alpha}^1)\cup u(\bar{\alpha}^2))$ and such that

$$\bar{\alpha}^1\restriction[n_0,\omega)=\bar{\alpha}^2\restriction[n_0,\omega).$$

For $k=1,2,3$ define n_k by

$$n_{k+1}=\max\{f_n(\bar{\alpha}^i)+g_n(\bar{\alpha}^i)+2: n\in(n_k+1)\setminus u(\bar{\alpha}^i),\ i<2\}.$$

As in the proof of 2.6.1 we have that then for $i=1,2$ and $j<3$:

($\otimes^1$) $(\forall n\geq n_{j+1})(k_0(\bar{\alpha}^i\restriction n)=\max u(\bar{\alpha}^i)\quad\text{or}\quad k_0(\bar{\alpha}^i\restriction n)>n_j)$

($\otimes^2$) $(\forall n\geq n_{j+1})(k_1(\bar{\alpha}^i\restriction n)>n_j\ \&\ \mathbf{l}(\bar{\alpha}^i\restriction n)>n_j)$

($\otimes^3$) $(\exists n'\in(n_1,n_2))(k_0(\bar{\alpha}^1\restriction n')=\max u(\bar{\alpha}^1)\ \&\ k_0(\bar{\alpha}^2\restriction n')=\max u(\bar{\alpha}^2))$

(for ($\otimes^3$) repeat arguments from 2.6.1.(2) and use the fact that $\bar{\alpha}^1\restriction[n_0,\omega)=\bar{\alpha}^2\restriction[n_0,\omega)$). Clearly

($\otimes^4$) $(\forall n>n_0)(u(\bar{\alpha}^1\restriction n)\setminus n_0=u(\bar{\alpha}^2\restriction n)\setminus n_0)$.

Hence, applying ($\otimes^1$), ($\otimes^2$), we conclude that:

($\otimes^5$) $(\forall n\geq n_1)(k_1(\bar{\alpha}^1\restriction n)=k_1(\bar{\alpha}^2\restriction n))$ and

($\otimes^6$) for all $n\geq n_1$:
either $k_0(\bar{\alpha}^1\restriction n)=\max u(\bar{\alpha}^1)$ and $k_0(\bar{\alpha}^2\restriction n)=\max u(\bar{\alpha}^2)$
or $k_0(\bar{\alpha}^1\restriction n)=k_0(\bar{\alpha}^2\restriction n)$.

Since

$$(\forall n\geq n_0)(f_n(\bar{\alpha}^1)=f_n(\bar{\alpha}^2)\ \&\ g_n(\bar{\alpha}^1)=g_n(\bar{\alpha}^2))$$

and by ($\otimes^2$) + ($\otimes^5$), we get (compare the proof of 2.6.1):

$$(\forall n\geq n_1)(\mathbf{l}(\bar{\alpha}^1\restriction n)=\mathbf{l}(\bar{\alpha}^2\restriction n))$$

and by ($\otimes^2$) + ($\otimes^3$) + ($\otimes^6$)

$$(\forall n\geq n_3)(\mathbf{m}(\bar{\alpha}^1\restriction n)=\mathbf{m}(\bar{\alpha}^2\restriction n)\geq n_1).$$

Moreover, now we easily get that

$$(\forall n\geq n_3)(\mathbf{k}(\bar{\alpha}^1\restriction n)=\mathbf{k}(\bar{\alpha}^2\restriction n)).$$

$\square$

For integers $n_0\leq n_1\leq n_2$ we define functions $F^0_{n_0,n_1,n_2}: {}^{n_2}\lambda\longrightarrow\mathcal{H}(\aleph_0)$ by letting $F^0_{n_0,n_1,n_2}(\alpha_0,\ldots,\alpha_{n_2-1})$ (for $\langle\alpha_0,\ldots,\alpha_{n_2-1}\rangle\in{}^{n_2}\lambda$) be the sequence consisting of:

(a) $\langle n_0,n_1,n_2\rangle$,

(b) the set T_{n_1,n_2} of all terms τ^n_ℓ such that $n\leq n_2-n_1$ and
either $\ell\leq n_2$ (we will call it *the simple case*)
or τ^n_ℓ is a composition of depth at most n_2 of such terms,

(c) $\langle\eta_\alpha\restriction n_2,n,\ell,\langle i_0,\ldots,i_{n-1}\rangle\rangle$ for $n\leq n_2-n_1$, $i_0,\ldots,i_{n-1}\in[n_1,n_2)$ and ℓ such that $\tau^n_\ell\in T_{n_1,n_2}$ and $\alpha=\tau^n_\ell(\alpha_{i_0},\ldots,\alpha_{i_{n-1}})$,

(d) $\langle n, \ell, \langle i_0, \ldots, i_{n-1}\rangle, i\rangle$ for $n \leq n_2 - n_1$, $i_0, \ldots, i_{n-1} \in [n_1, n_2)$, $i \in [n_0, n_1)$ and ℓ such that $\tau^n_\ell \in T_{n_1,n_2}$ and $\alpha_i = \tau^n_\ell(\alpha_{i_0}, \ldots, \alpha_{i_{n-1}})$,

(e) equalities among appropriate terms, i.e. all tuples

$$\langle n', \ell', n'', \ell'', \langle i'_0, \ldots, i'_{n'-1}\rangle, \langle i''_0, \ldots, i''_{n''-1}\rangle\rangle$$

such that $n_1 \leq i'_0 < \ldots < i'_{n'-1} < n_2$, $n_1 \leq i''_0 < \ldots < i''_{n''-1} < n_2$, $n', n'' \leq n_2 - n_1$, ℓ', ℓ'' are such that $\tau^{n'}_{\ell'}, \tau^{n''}_{\ell''} \in T_{n_1,n_2}$ and

$$\tau^{n'}_{\ell'}(\alpha_{i'_0}, \ldots, \alpha_{i'_{n'-1}}) = \tau^{n''}_{\ell''}(\alpha_{i''_0}, \ldots, \alpha_{i''_{n''-1}}).$$

(Note that the value of $F^0_{n_0,n_1,n_2}(\bar\alpha)$ does not depend on $\bar\alpha\restriction n_0$.)
Finally we define functions $F_n : {}^n\lambda \longrightarrow \mathcal{H}(\aleph_0)$ (for $1 < n < \omega$) by:

if $\bar\alpha \in {}^n\lambda$
then $F_n(\bar\alpha) = F^0_{\mathbf{k}(\bar\alpha),\mathbf{l}(\bar\alpha),n}(\bar\alpha)$.

As $\mathcal{H}(\aleph_0)$ is countable we may think that these functions are into ω. We are going to show that they witness $\mathcal{KL}(\lambda, \omega)$.

Claim 2.6.3. *If $\bar\alpha^1, \bar\alpha^2 \in {}^\omega\lambda$ are such that $(\forall^\infty n)(\alpha^1_n = \alpha^2_n)$ then $(\forall^\infty n)(F_n(\bar\alpha^1\restriction n) = F_n(\bar\alpha^2\restriction n))$.*

Proof of the claim. Take $m_0 < \omega$ such that for all $n \in [m_0, \omega)$ we have

$$\alpha^1_n = \alpha^2_n, \quad \mathbf{l}(\bar\alpha^1\restriction n) = \mathbf{l}(\bar\alpha^2\restriction n), \quad \text{and } \mathbf{k}(\bar\alpha^1\restriction n) = \mathbf{k}(\bar\alpha^2\restriction n)$$

(possible by 2.6.2). Let $m_1 > m_0$ be such that for all $n \geq m_1$:

$$\mathbf{k}(\bar\alpha^1\restriction n) = \mathbf{k}(\bar\alpha^2\restriction n) > m_0$$

(use 2.6.1). Then, for $n \geq m_1$, $i = 1, 2$ we have

$$F_n(\bar\alpha^i\restriction n) = F^0_{\mathbf{k}(\bar\alpha^i\restriction n),\mathbf{l}(\bar\alpha^i\restriction n),n}(\bar\alpha^i\restriction n) = F^0_{\mathbf{k}(\bar\alpha^1\restriction n),\mathbf{l}(\bar\alpha^1\restriction n),n}(\bar\alpha^i\restriction n).$$

Since the value of $F^0_{n_0,n_1,n_2}(\bar\beta)$ does not depend on $\bar\beta\restriction n_0$ and the sequences $\bar\alpha^1\restriction n$, $\bar\alpha^2\restriction n$ agree on $[m_0, \omega)$, we get

$$F^0_{\mathbf{k}(\bar\alpha^1\restriction n),\mathbf{l}(\bar\alpha^1\restriction n),n}(\bar\alpha^1\restriction n) = F^0_{\mathbf{k}(\bar\alpha^1\restriction n),\mathbf{l}(\bar\alpha^1\restriction n),n}(\bar\alpha^2\restriction n) = F^0_{\mathbf{k}(\bar\alpha^2\restriction n),\mathbf{l}(\bar\alpha^2\restriction n),n}(\bar\alpha^2\restriction n),$$

and hence

$$(\forall n \geq m_1)(F_n(\bar\alpha^1\restriction n) = F_n(\bar\alpha^2\restriction n)),$$

finishing the proof of the claim. $\square$

Claim 2.6.4. *If $\bar\alpha^1, \bar\alpha^2 \in {}^\omega\lambda$ and $(\forall^\infty n)(F_n(\bar\alpha^1\restriction n) = F_n(\bar\alpha^2\restriction n))$ then $(\forall^\infty n)(\alpha^1_n = \alpha^2_n)$*

Proof of the claim. Take $n_0 < \omega$ such that

$$u(\bar\alpha^1) \cup u(\bar\alpha^2) \subseteq n_0 \quad \text{and} \quad (\forall n \geq n_0)(F_n(\bar\alpha^1\restriction n) = F_n(\bar\alpha^2\restriction n)).$$

Then for all $n \geq n_0$ we have (by clause (a) of the definition of $F^0_{n_0,n_1,n_2}$):

$$\mathbf{l}(\bar\alpha^1\restriction n) = \mathbf{l}(\bar\alpha^2\restriction n) \quad \& \quad \mathbf{k}(\bar\alpha^1\restriction n) = \mathbf{k}(\bar\alpha^2\restriction n).$$

Further, let $n_1 > n_0$ be such that for all $n \geq n_1$, $\mathbf{k}(\bar{\alpha}^1{\restriction}n) > n_0$.

We are going to show that $\alpha^1_n = \alpha^2_n$ for all $n > n_1$. Assume not. Then we have $n > n_1$ with $\alpha^1_n \neq \alpha^2_n$ and thus $\eta_{\alpha^1_n} \neq \eta_{\alpha^2_n}$. Take $n' > n$ such that $\eta_{\alpha^1_n}{\restriction}n' \neq \eta_{\alpha^2_n}{\restriction}n'$. Applying 2.6.1 (2) and (4) choose $n'' > n'$ such that

$$\mathbf{m}(\bar{\alpha}^1{\restriction}n'') > n' \ \text{ and } \ k_0(\bar{\alpha}^1{\restriction}n'') = \max u(\bar{\alpha}^1).$$

Now define inductively: $m_0 = n''$, $m_{k+1} = \mathbf{m}(\bar{\alpha}^1{\restriction}m_k)$.
Thus

$$n'' = m_0 > \mathbf{l}(\bar{\alpha}^1{\restriction}m_0) \geq m_1 > \mathbf{l}(\bar{\alpha}^1{\restriction}m_1) \geq m_2 > \ldots$$

and

$$m_k > \max u(\bar{\alpha}^1) \quad \Rightarrow \quad k_0(\bar{\alpha}^1{\restriction}m_k) = \max u(\bar{\alpha}^1)$$

(see the definition of $\mathbf{m}$). Let k^* be the first such that $n \geq m_{k^*}$ (so $k^* \geq 2$). Note that by the choice of n_1 above we necessarily have

$$m_{k^*} > \mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*}) = \mathbf{k}(\bar{\alpha}^1{\restriction}m_{k^*-1}) > n_0.$$

Hence for all $k < k^*$:

$$\begin{array}{l} F_{m_k}(\bar{\alpha}^1{\restriction}m_k) = F_{m_k}(\bar{\alpha}^2{\restriction}m_k) \quad \text{ and} \\ \mathbf{l}(\bar{\alpha}^1{\restriction}m_{k+1}) = \mathbf{l}(\bar{\alpha}^2{\restriction}m_{k+1}) = \mathbf{k}(\bar{\alpha}^1{\restriction}m_k) = \mathbf{k}(\bar{\alpha}^2{\restriction}m_k). \end{array}$$

By the definition of the functions $\mathbf{l}, \mathbf{m}, \mathbf{k}$ and the choice of m_0 (remember $k_0(\bar{\alpha}^1{\restriction}m_0) = \max u(\bar{\alpha}^1)$) we know that for each $i \in [\mathbf{k}(\bar{\alpha}^1{\restriction}m_k), \mathbf{l}(\bar{\alpha}^1{\restriction}m_k))$, $k < k^*$ for some $\tau^m_\ell \in T_{\mathbf{l}(\bar{\alpha}^1{\restriction}m_k),m_k}$ and $i_0, \ldots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1{\restriction}m_k), m_k)$ we have $\alpha^1_i = \tau^m_\ell(\alpha^1_{i_0}, \ldots, \alpha^1_{i_{m-1}})$. Moreover we may demand that τ^m_ℓ is a composition of depth at most $\mathbf{l}(\bar{\alpha}^1{\restriction}m_k) - i$ of simple case terms. Since

$$F^0_{\mathbf{k}(\bar{\alpha}^1{\restriction}m_k),\mathbf{l}(\bar{\alpha}^1{\restriction}m_k),m_k}(\bar{\alpha}^1{\restriction}m_k) = F^0_{\mathbf{k}(\bar{\alpha}^2{\restriction}m_k),\mathbf{l}(\bar{\alpha}^2{\restriction}m_k),m_k}(\bar{\alpha}^2{\restriction}m_k)$$

we conclude that (by clause (d) of the definition of the functions $F^0_{n_0,n_1,n_2}$):

$$\alpha^2_i = \tau^m_\ell(\alpha^2_{i_0}, \ldots, \alpha^2_{i_{m-1}}).$$

Now look at our n.
If $\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1}) > n$ then $\mathbf{k}(\bar{\alpha}^1{\restriction}m_{k^*-1}) \leq n < \mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1})$ and thus we find $i_0, \ldots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1}), m_{k^*-1})$ and $\tau^m_\ell \in T_{\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1}),m_{k^*-1}}$ such that

$$\alpha^1_n = \tau^m_\ell(\alpha^1_{i_0}, \ldots, \alpha^1_{m-1}) \ \ \& \ \ \alpha^2_n = \tau^m_\ell(\alpha^2_{i_0}, \ldots, \alpha^2_{m-1}).$$

If $\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1}) \leq n$ then $n \in [\mathbf{k}(\bar{\alpha}^1{\restriction}m_{k^*-2}), \mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-2}))$ (as $\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1}) = \mathbf{k}(\bar{\alpha}^1{\restriction}m_{k^*-2})$ and $n < m_{k^*-1} \leq \mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-2})$). Hence, for some $i_0, \ldots, i_{m-1} \in [\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-2}), m_{k^*-2})$ and $\tau^m_\ell \in T_{\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-2}),m_{k^*-2}}$, we have

$$\alpha^1_n = \tau^m_\ell(\alpha^1_{i_0}, \ldots, \alpha^1_{m-1}) \ \ \& \ \ \alpha^2_n = \tau^m_\ell(\alpha^2_{i_0}, \ldots, \alpha^2_{m-1}).$$

In both cases we may additionally demand that the respective term τ^m_ℓ is a composition of depth $\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-1}) - n$ (or $\mathbf{l}(\bar{\alpha}^1{\restriction}m_{k^*-2}) - n$, respectively) of terms of the simple case. Now we proceed inductively (taking care of the depth of involved terms) and we find a term $\tau \in T_{\mathbf{l}(\bar{\alpha}^1{\restriction}m_0),m_0}$ (which is a composition of depth at most $\mathbf{l}(\bar{\alpha}^1{\restriction}m_0)-n$ of terms of the simple case) and $i_0,\ldots,i_{m-1}\in[\mathbf{l}(\bar{\alpha}^1{\restriction}m_0),m_0)$ such that

$$\alpha^1_n=\tau(\alpha^1_{i_0},\ldots,\alpha^1_{m-1})\ \ \&\ \ \alpha^2_n=\tau(\alpha^2_{i_0},\ldots,\alpha^2_{m-1}).$$

But now applying the clause (c) of the definition of the functions $F^0_{n_0,n_1,n_2}$ we conclude that $\eta_{\alpha^1_n}{\restriction}m_0=\eta_{\alpha^2_n}{\restriction}m_0$. Contradiction to the choice of n' and the fact that $m_0>n'$. □

The last two claims finish the proof of the theorem. □

Remark 2.7. If the models M have $\kappa<\lambda$ functions (so $\langle\tau^n_i(x_0,\ldots,x_{n-1}):i<\kappa\rangle$ lists the n–place terms) we can prove $\mathcal{KL}(\lambda,\kappa)$ and the proof is similar.

* * *

Final Remarks 2.8. 1) Now we phrase exactly what is needed to carry the proof of theorem 1.1 for $\lambda>\kappa$. It is:

($\boxtimes$) for every model M with universe λ and Skolem functions and with countable vocabulary, we can find pairwise distinct $\alpha_{n,\ell}<\lambda$ (for $n<\omega,\ell<\omega$) such that

($\otimes$) if $m_0<m_1<\omega$ and $\ell'_i<\ell''_i$ for $i<m_0$ and $\ell_i<\omega$ for $i\in[m_0,m_1)$ then the models

$$\begin{aligned}&(\mathrm{Sk}(\{\alpha_{i,\ell'_i},\alpha_{i,\ell''_i}:i<m_0\}\cup\{\alpha_{m_0,k_0},\alpha_{m_0,k_1}\}\cup\{\alpha_{i,\ell_i}:i\in(m_0,m_1)\}),\\&\quad\alpha_{0,\ell'_0},\alpha_{0,\ell''_0},\alpha_{1,\ell'_1},\alpha_{1,\ell''_1},\ldots,\alpha_{m_0-1,\ell'_{m_0-1}},\alpha_{m_0-1,\ell''_{m_0-1}},\alpha_{m_0,k_0},\\&\quad\alpha_{m_0,k_1},\alpha_{m_0+1,\ell_{m_0+1}},\ldots,\alpha_{m_1-1,\ell_{m_1-1}})\end{aligned}$$

and

$$\begin{aligned}&(\mathrm{Sk}(\{\alpha_{i,\ell'_i},\alpha_{i,\ell''_i}:i<m_0\}\cup\{\alpha_{m_0,k_0},\alpha_{m_0,k_2}\}\cup\{\alpha_{i,\ell_i}:i\in(m_0,m_1)\}),\\&\quad\alpha_{0,\ell'_0},\alpha_{0,\ell''_0},\alpha_{1,\ell'_1},\alpha_{1,\ell''_1},\ldots,\alpha_{m_0-1,\ell'_{m_0-1}},\alpha_{m_0-1,\ell''_{m_0-1}},\alpha_{m_0,k_0},\\&\quad\alpha_{m_0,k_2},\alpha_{m_0+1,\ell_{m_0+1}},\ldots,\alpha_{m_1-1,\ell_{m_1-1}})\end{aligned}$$

are isomorphic and the isomorphism is the identity on their intersection and they have the same intersection with κ.

For more details and more related results we refer the reader to [**?**].
2) Together with 1.5, 2.7 this gives a good bound to the consistency strength of $\neg\mathcal{KL}(\lambda,\kappa)$.
3) What if we ask $F_n:{}^n\lambda\longrightarrow{}^{\omega>}\kappa$ such that $F_n(\eta)\trianglelefteq F_{n+1}(\eta)$ and $\eta\in{}^\omega\lambda\ \Rightarrow\ F(\eta)=\bigcup F_n(\eta{\restriction}n)\in{}^\omega\kappa$? No real change.

References

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