STRONGLY MEAGER AND STRONG MEASURE ZERO SETS

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ABSTRACT. In this paper we present two consistency results concerning the existence of large strong measure zero and strongly meager sets.

1. INTRODUCTION

Let $\mathcal{M}$ denote the collection of all meager subsets of 2^ω and let $\mathcal{N}$ be the collection of all subsets of 2^ω that have measure zero with respect to the standard product measure on 2^ω.

Definition 1.1. *Suppose that $X \subseteq 2^\omega$ and let $+$ denote the componentwise addition modulo 2. We say that X is strongly meager if for every $H \in \mathcal{N}$, $X + H = \{x + h : x \in X, h \in H\} \neq 2^\omega$.*

We say that X is a strong measure zero set if for every $F \in \mathcal{M}$, $X + F \neq 2^\omega$. Let $\mathcal{SM}$ denote the collection of strongly meager sets and let $\mathcal{SN}$ denote the collection of strong measure zero sets.

For a family of sets $\mathcal{J} \subseteq P(\mathbb{R})$ let
$\mathsf{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ and } \bigcup \mathcal{A} = 2^\omega\}$.
$\mathsf{non}(\mathcal{J}) = \min\{|X| : X \notin \mathcal{J}\}$.

Strong measure zero sets are usually defined as those subsets X of 2^ω such that for every sequence of positive reals $\{\varepsilon_n : n \in \omega\}$ there exists a sequence of basic open sets $\{I_n : n \in \omega\}$ with diameter of I_n smaller than ε_n and $X \subseteq \bigcup_n I_n$. The Galvin-Mycielski-Solovay theorem ([4]) guarantees that both definitions are yield the same families of sets.

Recall the following well–known facts. Any of the following sentences is consistent with ZFC,

(1) $\mathcal{SN} = [2^\omega]^{\leq\aleph_0}$, (Laver [7])
(2) $\mathcal{SN} = [2^\omega]^{\leq\aleph_1}$, (Corazza [3], Goldstern-Judah-Shelah [5])
(3) $\mathcal{SM} = [2^\omega]^{\leq\aleph_0}$. (Carlson, [2])
(4) $\mathsf{non}(\mathcal{SN}) = \mathfrak{d} = 2^{\aleph_0} > \aleph_1$, $\mathsf{cov}(\mathcal{M}) = \aleph_1$ and there exists a strong measure zero set of size $2^{\aleph_0}$. (Goldstern-Judah-Shelah [5])

The proofs of the above results as well as all other results quoted in this paper can be also found in [1].

In this paper we will show that the following statements are consistent with ZFC:

- for any regular $\kappa > \aleph_0$, $\mathcal{SM} = [2^\omega]^{<\kappa}$,
- $\mathcal{SM}$ is an ideal and $\mathsf{add}(\mathcal{SM}) \geq \mathsf{add}(\mathcal{M})$,

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- $\mathsf{non}(\mathcal{SN}) = 2^{\aleph_0} > \aleph_1$, $\mathfrak{d} = \aleph_1$ and there is a strong measure zero set of size $2^{\aleph_0}$.

2. $\mathcal{SM}$ may have large additivity

In this section we will show that $\mathcal{SM}$ can be an ideal with large additivity. Let

$$\mathfrak{m} = \min\{\gamma : \mathbf{MA}_\gamma \text{ fails}\}.$$

We will show that $\mathcal{SM} = [2^\omega]^{<\mathfrak{m}}$ is consistent with ZFC, provided $\mathfrak{m}$ is regular. In particular, the model that we construct will satisfy $\mathsf{add}(\mathcal{SM}) = \mathsf{add}(\mathcal{M})$.

Note that if $\mathcal{SM} = [2^\omega]^{<\mathfrak{m}}$ then $2^{\aleph_0} > \mathfrak{m}$, since Martin's Axiom implies the existence of a strongly meager set of size $2^{\aleph_0}$. Our construction is a generalization of the construction from [2].

To witness that a set is not strongly meager we need a measure zero set. The following theorem is crucial.

Theorem 2.1 (Lorentz)**.** *There exists a function $K \in \omega^{\mathbb{R}}$ such that for every $\varepsilon > 0$, if $A \in [2^\omega]^{\geq K(\varepsilon)}$ then for all except finitely many $k \in \omega$ there exists $C \subseteq 2^k$ such that*

1. $|C| \cdot 2^{-k} \leq \varepsilon$,
2. $(A{\restriction}k) + C = 2^k$.

Proof Proof of this lemma can be found in [8] or [1]. $\square$

Definition 2.2. *For each $n \in \omega$ let $\{C^n_m : n, m \in \omega\}$ be an enumeration of all clopen sets in 2^ω of measure $\leq 2^{-n}$. For a real $r \in \omega^\omega$ and $n \in \omega$ define an open set*

$$H^r_n = \bigcup_{m>n} C^m_{r(m)}.$$

It is clear that H^r_n is an open set of measure not exceeding 2^{-n}. In particular, $H^r = \bigcap_{n\in\omega} H^r_n$ is a Borel measure zero set of type G_δ.

Theorem 2.3. *Let $\kappa > \aleph_0$ be a regular cardinal. It is consistent with ZFC that $\mathbf{MA}_{<\kappa} + \mathcal{SM} = [2^\omega]^{<\kappa}$ holds. In particular, it is consistent that $\mathcal{SM}$ is an ideal and $\mathsf{add}(\mathcal{SM}) = \mathsf{add}(\mathcal{M}) > \aleph_1$.*

Proof Fix κ such that $\mathsf{cf}(\kappa) = \kappa > \aleph_0$. Let $\lambda > \kappa$ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$. Start with a model $\mathbf{V} \models \mathsf{ZFC} + 2^{\aleph_0} = \lambda$.

Suppose that $\mathcal{P}$ is a forcing notion of size $< \kappa$. We can assume that there is $\gamma < \kappa$ such that $\mathcal{P} = \gamma$ and $\leq, \perp \subseteq \gamma \times \gamma$.

Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \lambda\}$ be a finite support iteration such that for each $\alpha < \lambda$,

1. $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbf{C}$, if α is limit,
2. there is $\gamma = \gamma_\alpha$ such that $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq (\gamma, \leq, \perp)$ is a ccc forcing notion.

By passing to a dense subset we can assume that if $p \in \mathcal{P}_\lambda$ then $p : \mathsf{dom}(p) \longrightarrow \kappa$, where $\mathsf{dom}(p)$ is a finite subset of λ.

By bookkeeping we can guarantee that $\mathbf{V}^{\mathcal{P}_\lambda} \models \mathbf{MA}_{<\kappa}$. In particular, $\mathbf{V}^{\mathcal{P}_\lambda} \models [2^\omega]^{<\kappa} \subseteq \mathcal{SM}$.

It remains to show that no set of size κ is strongly meager.

Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}_\lambda} \cap 2^\omega$ is a set of size κ. Find limit ordinal $\alpha < \lambda$ such that $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}_\alpha}$. As usual we can assume that $\alpha = 0$. Let c be the Cohen real added at the step $\alpha = 0$. We will show that $\mathbf{V}^{\mathcal{P}_\lambda} \models X + H^c = 2^\omega$, which will end the proof.

Suppose that the above assertion is false. Let $p \in \mathcal{P}_\lambda$ and let $\dot{z}$ be a $\mathcal{P}_\lambda$-name for a real such that

$$p \Vdash_\lambda \dot{z} \notin X + H^{\dot{c}}.$$

Let $X = \{x_\xi : \xi < \kappa\}$ and for each ξ find $p_\xi \geq p$ and $n_\xi \in \omega$ such that

$$p_\xi \Vdash_\lambda \dot{z} \notin x_\xi + H^{\dot{c}}_{n_\xi}.$$

Let $Y \subseteq \kappa$ be a set of size κ such that

(1) $n_\xi = \widetilde{n}$ for $\xi \in Y$,
(2) $\{\mathsf{dom}(p_\xi) : \xi \in Y\}$ form a Δ-system with root $\widetilde{\Delta}$,
(3) $p_\xi \restriction \widetilde{\Delta} = \widetilde{p}$, for $\xi \in Y$,
(4) $p_\xi(0) = \widetilde{s}$, with $|\widetilde{s}| = \ell > \widetilde{n}$, for $\xi \in Y$.

Fix a subset $X' = \{x_{\xi_j} : j < K(2^{-\ell})\} \subseteq Y$ and let $\widetilde{m} \in \omega$ be such that $C^\ell_{\widetilde{m}} + X' = 2^\omega$.

Define condition $p^\star$ as

$$p^\star(\beta) = \begin{cases} p_{\xi_j} & \text{if } \alpha \neq \beta \ \&\ \beta \in \mathsf{dom}(p_{\xi_j}),\ j < K(2^{-\ell}) \\ \widetilde{s}^\frown \widetilde{m} & \text{if } \alpha = \beta \end{cases} \quad \text{for } \beta < \lambda.$$

On one hand $p^\star \Vdash_\lambda C^\ell_{\widetilde{m}} \subseteq H^{\dot{c}}_{\widetilde{n}}$, so $p^\star \Vdash_\lambda X' + H^{\dot{c}}_{\widetilde{n}} = 2^\omega$. On the other hand, $p^\star \geq p_{\xi_j}$, $j \leq K(2^{-\ell})$, so $p^\star \Vdash_\lambda \dot{z} \notin X' + H^{\dot{c}}_{\widetilde{n}}$. Contradiction.

To finish the proof we show that $\mathbf{V}^{\mathcal{P}_\lambda} \models \mathsf{add}(\mathcal{M}) = \kappa$. First note that $\mathbf{MA}_{<\kappa}$ implies that $\mathsf{add}(\mathcal{M}) \geq \kappa$ in $\mathbf{V}^{\mathcal{P}_\lambda}$. The other inequality is a consequence of the general theory. Recall that (see [1])

(1) $\mathsf{add}(\mathcal{M}) = \min\{\mathsf{cov}(\mathcal{M}), \mathfrak{b}\}$

Suppose that $F \subset \omega^\omega$ is an unbounded family of size $\geq \kappa$.

2. if $\mathcal{P}$ is a forcing notion of cardinality $< \kappa$ then F remains unbounded in $\mathbf{V}^{\mathcal{P}}$.
3. if $\{\mathcal{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \lambda\}$ is a finite support iteration such that $\Vdash_\alpha |\mathcal{Q}_\alpha| < \kappa$ then $\mathbf{V}^{\mathcal{P}_\lambda} \models F$ is unbounded..

From the results quoted above follows that $\mathsf{add}(\mathcal{M}) \leq \mathfrak{b} \leq \kappa$ in $\mathbf{V}^{\mathcal{P}_\lambda}$, which ends the proof. $\square$

3. Strong measure zero sets

In this section we will discuss models with strong measure zero sets of size $2^{\aleph_0}$.
We start with the definition of forcing that will be used in our construction.

Definition 3.1. *The infinitely equal forcing notion* $\mathbf{EE}$ *is defined as follows:* $p \in \mathbf{EE}$ *if the following conditions are satisfied:*

(1) $p : \mathsf{dom}(p) \longrightarrow 2^{<\omega}$,
(2) $\mathsf{dom}(p) \subseteq \omega$, $|\omega \setminus \mathsf{dom}(p)| = \aleph_0$,
(3) $p(n) \in 2^n$ *for all* $n \in \mathsf{dom}(p)$.

For $p, q \in \mathbf{EE}$ *and* $n \in \omega$ *we define:*

(1) $p \geq q \iff p \supseteq q$, *and*
(2) $p \geq_n q \iff p \geq q$ *and the first* n *elements of* $\omega \setminus \mathsf{dom}(p)$ *and* $\omega \setminus \mathsf{dom}(q)$ *are the same.*

It is easy to see (see [1]) that **EE** is proper (satisfies axiom A), and strongly ω^ω bounding, that is if $p \Vdash \tau \in \omega$ and $n \in \omega$ then there is $q \geq_n p$ and a finite set $F \subseteq \omega$ such that $q \Vdash \tau \in F$.

In [5] it is shown that a countable support iteration of **EE** and rational perfect set forcing produces a model where there is a strong measure zero set of size $2^{\aleph_0}$. In particular, one can construct (consistently) a strong measure zero of size $2^{\aleph_0}$ without Cohen reals. The remaining question is whether such a construction can be carried out without unbounded reals.

Theorem 3.2 ([5]). *Suppose that $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ is a countable support iteration of proper, strongly ω^ω-bounding forcing notions. Then*

$$\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathcal{SN} \subseteq [\mathbb{R}]^{\leq \aleph_1}. \quad \square$$

The theorem above shows that using countable support iteration we cannot build a model with a strong measure zero set of size $> \mathfrak{d}$. Since countable support iteration seems to be the universal method for constructing models with $2^{\aleph_0} = \aleph_2$ the above result seems to indicate that a strong measure zero set of size $> \mathfrak{d}$ cannot be constructed at all. Strangely it is not the case.

Theorem 3.3. *It is consistent that $\mathsf{non}(\mathcal{SN}) = 2^{\aleph_0} > \mathfrak{d} = \aleph_1$ and there are strong measure zero sets of size $2^{\aleph_0}$.*

Proof Suppose that $\mathbf{V} \models \mathsf{CH}$ and $\kappa = \kappa^{\aleph_0} > \aleph_1$. Let $\mathcal{P}$ be a countable support product of κ copies of **EE**. The following facts are well-known (see [6])

(1) $\mathcal{P}$ is proper,
(2) $\mathcal{P}$ satisfies $\aleph_2$-cc,
(3) $\mathcal{P}$ is ω^ω-bounding,
(4) for $f \in \mathbf{V}[G] \cap \omega^\omega$ there exists a countable set $A \subseteq \kappa$, $A \in \mathbf{V}$ such that $f \in \mathbf{V}[G{\restriction}A]$.

It follows from (3) that $\mathbf{V}^{\mathcal{P}} \models \mathfrak{d} = \aleph_1$. Moreover, (1) and (2) imply that $2^{\aleph_0} = \kappa$ in $\mathbf{V}^{\mathcal{P}}$.

For a set $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}}$ let $\mathsf{supp}(X) \subseteq \kappa$ be a set such that $X \in \mathbf{V}[G{\restriction}\mathsf{supp}(X)]$. Note that $\mathsf{supp}(X)$ is not determined uniquely, but we can always choose it so that $|\mathsf{supp}(X)| = |X| + \aleph_0$.

Lemma 3.4. *Suppose that $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}}$ and $\mathsf{supp}(X) \neq \kappa$. Then $\mathbf{V}^{\mathcal{P}} \models X \in \mathcal{SN}$*

Note that this lemma finishes the proof. Clearly the assumptions of the lemma are met for all sets of size $< \kappa$ and also for many sets of size κ.

Proof We will use the following characterization (see [1]):

Lemma 3.5. *The following conditions are equivalent.*

(1) *$X \subseteq 2^\omega$ has strong measure zero.*
(2) *For every $f \in \omega^\omega$ there exists $g \in (2^{<\omega})^\omega$ such that $g(n) \in 2^{f(n)}$ for all n and*

$$\forall x \in X \ \exists n \ x{\restriction}f(n) = g(n). \quad \square$$

Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}} \cap 2^\omega$ is given and $\mathsf{supp}(X) \neq \kappa$. Let $\alpha^\star \in \kappa \setminus \mathsf{supp}(X)$. We will check condition (2) of the previous lemma.

Fix $f \in \mathbf{V}^{\mathcal{P}} \cap \omega^\omega$. Since $\mathcal{P}$ is ω^ω-bounding we can assume that $f \in \mathbf{V}$. Consider a condition $p \in \mathcal{P}$. Fix $\{k_n : n \in \omega\}$ such that $k_n \geq f(n)$ and $k_n \notin \mathsf{dom}(p(\alpha^\star))$ for

$n \in \omega$. Let $p_f \geq p$ be any condition such that $\omega \setminus \{k_n : n \in \omega\} \subseteq \mathsf{dom}(p_f(\alpha^\star))$. We will check that

$$p_f \Vdash_{\mathcal{P}} \forall x \in X \; \exists n \; x{\restriction}f(n) = \dot{G}(\alpha^\star)(k_n){\restriction}f(n),$$

where $\dot{G}$ is the canonical name for the generic object. Take $x \in X$ and $r \geq p_f$. Find n such that $k_n \notin \mathsf{dom}\big(r(\alpha^\star)\big)$. Let $r' \geq r$ and s be such that

(1) $\mathsf{supp}(r') \subseteq \mathsf{supp}(X)$
(2) $r' \geq r{\restriction}\mathsf{supp}(X)$,
(3) $r' \Vdash_{\mathcal{P}} x{\restriction}k_n = s$.

Let

$$r''(\beta) = \begin{cases} r'(\beta) & \text{if } \beta \neq \alpha^\star \\ r'(\alpha^\star) \cup \{(k_n, s)\} & \text{if } \beta = \alpha^\star \end{cases} .$$

It is easy to see that $r'' \Vdash x{\restriction}f(n) = \dot{G}(\alpha^\star)(k_n){\restriction}f(n)$. Since f and x were arbitrary we are done. $\square$

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