ON ABSOLUTELY DIVERGENT SERIES

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ABSTRACT. We show that in the $\aleph_2$-stage countable support iteration of Mathias forcing over a model of CH the complete Boolean algebra generated by absolutely divergent series under eventual dominance is not isomorphic to the completion of $P(\omega)/\text{fin}$.

This complements Vojtáš' result, that under $\text{cf}(\mathfrak{c}) = \mathfrak{p}$ the two algebras are isomorphic [15].

1. INTRODUCTION

One of the traditional fields of real analysis is the study of asymptotic behaviour of series and sequences; see e.g. the monographs of G. H. Hardy [8] and G. M. Fikhtengolz [7]. Among these topics is the classical problem of tests of absolute convergence and/or divergence of series of real numbers. Of specific importance is the comparison test, because many other tests, like Cauchy's (root) test, d'Alembert's (ratio) test, and Raabe's test, are special instances of it.

We employ here a global point of view (implicit) of set theory, rather than looking at explicit series and tests (because these are only countably many explicit ones, as our language is countable, and hence from a global point of view not very interesting). From this global — set theoretic — point of view the study of comparison tests is nothing else than the study of the

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ordering of eventual dominance on absolute values of the sequences, which describe the entries that have to be summed up in a series, or on sequences with nonnegative entries, to which we restrict ourselves. A sequence $\bar{b}$ is eventually smaller than a sequence $\bar{a}$, denoted as $\bar{b} \leqslant^* \bar{a}$, if we have that $b_n \leqslant a_n$ for all but finitely many n.

Note that the stronger information in the sense of convergence is carried by the eventually greater sequences, in contrast to divergence where it is carried by the smaller ones. Hence we are interested in $\leqslant^*$ on ℓ^1 upwards, whereas on the set of divergent series $c_0 \setminus \ell^1$ the relation $\leqslant^*$ is interesting downwards.

There is a substantial difference between $(\ell^1, \geqslant^*)$ and $(c_0 \setminus \ell^1, \leqslant^*)$, namely the first is directed and the second is not. For a directed ordering, questions about unbounded and dominating families are interesting. T. Bartoszyński [4] has shown that the minimum size of an unbounded family of absolutely convergent series $\mathfrak{b}(\ell^1, \geqslant^*)$ is equal to $\mathrm{add}(\mathcal{N})$, the additivity of the ideal of sets of Lebesgue measure zero. Dually, the minimal size of a dominating family $\mathfrak{d}(\ell^1, \geqslant^*)$ is $\mathrm{cof}(\mathcal{N})$, the minimal size of a base of the ideal of sets of measure zero. This result says that in order to decide the absolute convergence of all series we need $\mathrm{cof}(\mathcal{N})$ many series as parameters in the comparison test. This number is known to be consistently smaller than the size of the continuum $2^{\aleph_0} = \mathfrak{c}$.

On the opposite side, with divergence we need always continuum many divergent series as parameters for a comparison test in order to decide the divergence of all series. That is because there are continuum many incompatible divergent series below each divergent series. This observation together with the σ-closedness of $(c_0 \setminus \ell^1, \leqslant^*)$ raises the question what $(c_0 \setminus \ell^1, \leqslant^*)$ looks like from the Boolean theoretic point of view. In [15] P. Vojtáš has proved that the complete Boolean algebra generated by $(c_0 \setminus \ell^1, \leqslant^*)$ is isomorphic to the completion of the algebra $\mathcal{P}(\omega)/\mathrm{fin}$ of subsets of natural numbers equipped with eventual inclusion, assuming $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$ (e.g. under CH or MA). Moreover, T. Bartoszyński and M. Scheepers [3] have shown that the $\mathfrak{t}$-numbers of both orderings are the same without additional hypotheses. This leads to the formulation of the problem whether these two algebras are always isomorphic, in all models of axiomatic set theory.

There is yet another striking phenomenon: F. Hausdorff has shown (in [9]) that there is in ZFC an (ω_1, ω_1^*) gap in $(c_0, \leqslant^*)$, such that the lower

part of the gap consists of convergent series and the upper part consists of divergent series. This is especially interesting when both $\mathrm{add}(\mathcal{N})$ and $\mathfrak{t}$ are greater than ω_1. In this case we cannot approach the "border between convergence and divergence" from either single side in ω_1 steps, but we can do it in ω_1 steps if we do it simultaneously from both sides by a Hausdorff gap.

To finish this introductory motivation, let us state that we can consider the classical study of asymptotic behaviour in the real analysis as a sort of study of forcing notions, because a better estimate and/or a stronger result really corresponds to a stronger forcing condition (in the case of non-directed orderings). Although it is historically a part of real analysis, it has gained new interest, because of numerous applications in complexity theory in computer science.

We consider the following complete Boolean algebras:

1. The **algebra of regular open sets in the partial order** $(\mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}, \subseteq^*)$, called $\mathrm{RO}(\mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}, \subseteq^*)$, where fin is the ideal of finite subsets of ω and $\mathcal{P}(\omega)/\mathrm{fin}$ is the set of all equivalence classes $a/\mathrm{fin} = \{b \in \mathcal{P}(\omega) \mid b \triangle a \text{ is finite }\}$. ($a \triangle b = (a \setminus b) \cup (b \setminus a)$ is the symmetric difference of a and b.)

We have that $a/\mathrm{fin} \subseteq^* b/\mathrm{fin}$ iff $a \subseteq^* b$, i.e. iff $a \setminus b$ is finite. The element 0 is the class $\emptyset/\mathrm{fin} = \mathrm{fin}$.

The partial order $P = (\mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}, \subseteq^*)$ is separative, i.e.

$$\forall p, q \in P \, (p \nleqslant q \longrightarrow \exists r \in P \, (r \leqslant p \, \wedge \, r \perp q)),$$

(where $r \perp q$ iff $\neg \exists s \, (s \leqslant r \, \wedge \, s \leqslant q)$) or, in topological terms, for $p \neq q \in P$ we have that

$$\mathrm{int}(\mathrm{cl}(\{p' \mid p' \leqslant p\})) \neq \mathrm{int}(\mathrm{cl}(\{q' \mid q' \leqslant q\})),$$

where the interiors and closures are taken in the so-called cut topology on $(P, \leqslant)$, which is generated by the basic open sets $\{\{p' \mid p' \leqslant p\} \mid p \in P\}$. Hence the map $p \mapsto \mathrm{int}(\mathrm{cl}(\{p' \mid p' \leqslant p\}))$ is an embedding into the algebra of regular open subsets of P, called $\mathrm{RO}(P)$.

In general, for a partial order $(P, \leqslant)$, $A \subseteq P$ is called regular open iff

$$\mathrm{int}(\mathrm{cl}(A)) = A.$$

As shown in [10, page 152], for any separative $(P, \leqslant)$ there is a unique complete Boolean algebra $\mathrm{RO}(P)$ into which — leaving out the Boolean algebra's zero element, of course — it can be densely embedded.

2. The algebra of regular open sets $\mathrm{RO}((c_0 \setminus \ell^1, \leqslant^*)/\approx)$, where $c_0 \setminus \ell^1 = \{\bar{c} = \langle c_n \mid n \in \omega \rangle \mid c_n \in \mathbb{R}_+ \wedge \lim c_n = 0 \wedge \sum c_n = \infty\}$, $\bar{d} \leqslant^* \bar{c}$ iff for all but finitely many n we have that $d_n \leqslant c_n$. This partial order $(c_0 \setminus \ell^1, \leqslant^*)$ is not separative, see [16]. Hence we take the separative quotient (see [10, page 154]): We set $\bar{d} \approx \bar{c}$ iff $\forall \bar{e}(\bar{e} \perp \bar{d} \leftrightarrow \bar{e} \perp \bar{c})$. Then we have that

$$(\bar{d}/\approx)\ (\leqslant /\approx)\ (\bar{c}/\approx) \text{ iff } \forall \bar{e} \leqslant^* \bar{d}\ \bar{e} \not\perp \bar{c}.$$

We write $(c_0 \setminus \ell^1, \leqslant^*)/\approx$ for $(c_0 \setminus \ell^1/\approx, \leqslant^* /\approx)$, the separative quotient, which is densely embedded into $\mathrm{RO}((c_0 \setminus \ell^1, \leqslant^*)/\approx)$, the second object of our investigation.

The purpose of this paper is to prove the following

Main Theorem. *In any extension got by the $\aleph_2$-stage countable support iteration of Mathias forcing over a model of* CH, *the complete Boolean algebra generated by the separative quotient of absolutely divergent series under eventual dominance is not isomorphic to the completion of $P(\omega)/\mathrm{fin}$.*

Notation and precaution: We shall be using some partial orders as notions of forcing as well. Then the stronger condition is the *smaller* condition. Thus $\leqslant$ in forcing will often coincide with $\subseteq^*$ or $\leqslant^*$. For functions $f, g \colon \omega \to \mathbb{R}$ we say $f \leqslant^* g$ iff for all but finitely many n, $f(n) \leqslant g(n)$. For subsets $A, B \subseteq \omega$ we write $A \subseteq^* B$ iff $A \setminus B$ is finite. The quantifier $\forall^\infty$ means "for all but finitely many", and the quantifier $\exists^\infty$ means "there are infinitely many". Names for elements in forcing extensions are written with tildes under the object, like x, and names for elements of the ground model are written with checks above the objects, like $\check{x}$.

Our notation follows Jech [10] and Kunen [12]. Recall that a subset A of a partial order $(P, \leqslant_P)$ is called open iff it contains with any of its elements also all stronger (i.e. $\leqslant_P$ than the given element) conditions.

If the ordering is clear, we shall often write only P instead of $(P, \leqslant_P)$ and $\leqslant$ instead of $\leqslant_P$.

2. $\mathfrak{h}$-NUMBERS

The means to distinguish the two algebras are the $\mathfrak{h}$-numbers. Therefore this section collects the facts we need about this cardinal characteristic.

Note that by a result of Bartoszyński and Scheepers [3] our two partial orders have the same $\mathfrak{t}$-numbers. For information on $\mathfrak{t}$ and other cardinal characteristics we refer the reader to [6].

Definition 2.1. *(a) A complete Boolean algebra B is called κ-distributive iff for every sequence of sets $\langle I_\alpha \mid \alpha \in \kappa \rangle$ and every set $\{u_{\alpha,i} \mid i \in I_\alpha, \alpha \in \kappa\}$ of members of B the equation*

$$\prod_{\alpha \in \kappa} \sum_{i \in I_\alpha} u_{\alpha,i} = \sum_{f \in \prod_{\alpha \in \kappa} I_\alpha} \prod_{\alpha \in \kappa} u_{\alpha, f(\alpha)}$$

holds.

(b) For a partial order $(P, \leqslant)$, $\mathfrak{h}(P, \leqslant)$ is the minimal cardinal κ such that $\mathrm{RO}((P, \leqslant)/\approx)$ is not κ-distributive. If there is no such κ, let $\mathfrak{h}(P, \leqslant)$ be undefined.

(c) $\mathfrak{h} = \mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}, \subseteq^*)$ is the well-known $\mathfrak{h}$-number which was introduced by Balcar, Pelant and Simon in [2]. In fact, it could also be written $\mathfrak{h} = \mathfrak{h}(\mathcal{P}(\omega) \setminus \mathrm{fin}, \subseteq)$, since the separative quotient of $(\mathcal{P}(\omega) \setminus \mathrm{fin}, \subseteq)$ is $(\mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}, \subseteq^*)$.

The separative quotient of a separative order is (isomorphic to) the order itself, and the set of regular open sets of a complete Boolean algebra (minus its zero) is (isomorphic to) the algebra itself. Hence

(2.1) $$\mathfrak{h}(P) = \mathfrak{h}(P/\approx) = \mathfrak{h}(\mathrm{RO}(P/\approx)).$$

The following fact allows us to work with various equivalent definitions of $\mathfrak{h}(P, \leqslant)$.

Fact 2.2. *For any partial order $(P, \leqslant)$ and cardinal κ the following are equivalent:*

(1) $\mathrm{RO}((P, \leqslant)/\approx)$ is κ-distributive.

(2) The intersection of κ open dense subsets of $(P, \leqslant)$ that are closed under $\approx$ is dense in $(P, \leqslant)$.

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(3) Every family of κ maximal antichains in P has a refinement.

(3') Every family of κ maximal antichains in $P/\approx$ has a refinement.

(4) Forcing with $(P, \leqslant)/\approx$ does not add a new function from κ to ordinals.

(5)   *In the following game $G(P, \kappa)$ the player* INC *does not have a winning strategy. The game $G(P, \kappa)$ is played in κ rounds, and the two players* INC *and* COM *choose $p_\alpha^{\text{INC}}, p_\alpha^{\text{COM}}$ in the α-th round such that for all $\alpha < \beta < \kappa$,*

$$p_\alpha^{\text{INC}} \geqslant p_\alpha^{\text{COM}} \geqslant p_\beta^{\text{INC}} \geqslant p_\beta^{\text{COM}}.$$

In the end, player INC *wins iff the sequence of moves does not have a lower bound in P or if at some round he/she does not have a legal move. Of course,* INC *stands for "incomplete" and* COM *stands for "complete".*

Proof. The equivalence of (1) to (4) is well-known (even for not necessarily separative partial orders!). We show: a) that $\neg(2)$ implies $\neg(5)$ and b) $\neg(5)$ implies $\neg(3)$. This is also proved, for a different game, where COM begins, and for a special Boolean algebra in [14]. For $G(P, \omega)$, the equivalence of (2) and (5) is also proved in [11].

a) Suppose that we are given open dense sets $\langle D_\alpha \mid \alpha \in \kappa \rangle$ that are closed under $\approx$ and such that $A = \bigcap \{ D_\alpha \mid \alpha \in \kappa \}$ is not dense. Define a winning strategy for INC in $G(P, \kappa)$ as follows: For $\alpha \geqslant 0$, INC plays $p_\alpha^{\text{INC}} \in D_\alpha$ such that $p_\alpha^{\text{INC}} \leqslant p_\beta^{\text{COM}}$ for all $\beta < \alpha$ and such that A contains no element $\leqslant p_\alpha^{\text{INC}}$. The first move is possible because A is not dense. This is clearly a winning strategy for INC.

b) Let σ be a winning strategy for INC in the game $G(P, \kappa)$. We define maximal antichains $\langle A_\alpha \mid \alpha \in \gamma \leqslant \kappa \rangle$ in P such that if $\alpha < \beta < \gamma$ then A_β is a refinement of A_α and if $p_\beta \in A_\beta$ and $p_\alpha \in A_\alpha$ is the unique member of A_α such that $p_\alpha \geqslant p_\beta$ then $\langle p_\alpha \mid \alpha \in \beta \rangle$ are responses by σ in an initial segment of a play, i.e.,

$$\forall \alpha \leqslant \beta \quad \text{for some } \langle p_\gamma^{\text{COM}} \mid \gamma \in \alpha \rangle \ \ p_\alpha = p_\alpha^{\text{INC}} = \sigma(\langle p_\gamma^{\text{INC}}, p_\gamma^{\text{COM}} \mid \gamma < \alpha \rangle).$$

Suppose first that $\langle A_\alpha \mid \alpha \in \delta \rangle$ has been constructed. If the sequence does not have a refinement, then $\neg(3)$ is proved. Otherwise suppose that there is some refinement B (which is of course, an antichain). Suppose that $\delta = \delta' + 1$. Then set

$$A_\delta' = \{ \sigma(\langle p_\alpha^{\text{INC}}, p_\alpha^{\text{COM}} \mid \alpha \leqslant \delta' \rangle) \mid \langle p_\alpha \mid \alpha^{\text{INC}} \leqslant \delta' \rangle \text{ is decreasing through}$$
$$\text{all the } A_\alpha, \text{ and } p_{\delta'}^{\text{COM}} \in B, \text{ and for } \alpha < \delta',$$
$$p_\alpha^{\text{COM}} \text{ is such that } p_\alpha^{\text{INC}} \geqslant p_\alpha^{\text{COM}} \geqslant p_{\alpha+1}^{\text{INC}} \},$$

and take $A_\delta \supseteq A'_\delta$ such that A_δ is a maximal antichain If δ is a limit, then

$$A'_\delta = \{\sigma(\langle p_\alpha^{\mathrm{INC}}, p_\alpha^{\mathrm{COM}} \,|\, \alpha < \delta\rangle) \,|\, \langle p_\alpha \,|\, \alpha^{\mathrm{INC}} \leqslant \delta\rangle \text{ is decreasing through}$$

$$\text{all the } A_\alpha, \text{ and for } \alpha < \delta, \, p_\alpha^{\mathrm{COM}} \text{ is}$$

$$\text{such that } p_\alpha^{\mathrm{INC}} \geqslant p_\alpha^{\mathrm{COM}} \geqslant p_{\alpha+1}^{\mathrm{INC}}\},$$

and again take for A_δ a maximal antichain containing A'_δ.

If the construction did not stop before κ, then we would have found a $\leqslant$-cofinal part $\langle p_\alpha \,|\, \alpha \in \kappa\rangle$ of a play $\langle p_\alpha^{\mathrm{INC}}, p_\alpha^{\mathrm{COM}} \,|\, \alpha \in \kappa\rangle$ according to σ in which INC loses, which would be a contradiction. $\square_{2.2}$

From Fact 2.2 we also get

Corollary 2.3. *The following are equivalent:*

(a) INC *has a winning strategy in* $G(P, \kappa)$.

(b) INC *has a winning strategy in* $G(P/\approx, \kappa)$.

(c) INC *has a winning strategy in* $G(\mathrm{RO}(P/\approx), \kappa)$.

3. Distinguishing $\mathfrak{h}$-numbers; $\mathcal{P}(\omega)/\mathrm{fin}$

Complete Boolean algebras that are isomorphic have the same $\mathfrak{h}$-numbers. We use this obvious fact in order to derive our main theorem from

Theorem 3.1. *Let G be generic for the $\aleph_2$-stage countable support iteration of Mathias forcing over a model of* CH. *Then we have that in $V[G]$,*

(a) $\mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin}, \subseteq^*) = \aleph_2$, *and*

(b) $\mathfrak{h}((c_0 \setminus \ell^1, \leqslant^*)/\approx) = \aleph_1$.

Beginning of proof. We start with a ground model $V \models$ CH and take an ω_2-stage countable support iteration $P = \langle P_\alpha, Q_\beta \,|\, \beta \in \omega_2, \alpha \leqslant \omega_2\rangle$ of Mathias forcing, i.e. $\forall \alpha \in \omega_2$, $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is Mathias forcing".

Remember that the conditions of Mathias forcing are pairs $\langle u, A\rangle \in [\omega]^{<\omega} \times [\omega]^\omega$ such that $\max u < \min A$, ordered by $\langle v, B\rangle \leqslant \langle u, A\rangle$ iff $u \subseteq v \subseteq u \cup A$ and $B \subseteq A$. Mathias forcing will also (outside the iteration) be denoted by Q_M.

It is well-known (see [14]) that Mathias forcing can be decomposed as $Q_M = Q'_M * Q''_M$, where Q'_M is $(\mathcal{P}(\omega)/\mathrm{fin} \setminus \{0\}, \subseteq^*)$, which is σ-closed and adds as a generic a Ramsey ultrafilter $\underset{\sim}{G}'_M$, and $\underset{\sim}{Q}''_M$ denotes a name for Mathias forcing with conditions with second component in $\underset{\sim}{G}'_M$ (also know

in the literature as $\mathbb{M}_{G'_M}$). The (Q'_M-name for the) generic filter for Q''_M (which determines the Mathias real) will be denoted by $\underset{\sim}{G''_M}$. The map sending $\langle u, A \rangle$ to $\langle A, \langle u, A \rangle \rangle$ is a dense embedding from Q_M into $Q'_M * \underset{\sim}{Q''_M}$.

Since the first component is σ-closed and the second component is σ-centred (hence c.c.c.) the whole forcing is proper [13] and any iteration with countable support will not collapse $\aleph_1$. Since for $\alpha < \omega_2$, $\Vdash_{P_\alpha} |Q_\alpha| \leq \omega_1$ and since the iteration length is $\leqslant \omega_2$, by [13, III,4.1], P has the $\aleph_2$-c.c. and hence does not collapse any cardinals.

The next lemma is folklore. A proof of it with a slightly more complicated argument can be found in [14].

Lemma 3.2. *In the model $V[G]$ from above we have that $\mathfrak{h} = \aleph_2$.*

Proof. Since in $V[G]$ we have that $2^\omega = \aleph_2$, we clearly have $\mathfrak{h} \leqslant \aleph_2$. We are now going to show that $\mathfrak{h} \geqslant \aleph_2$. We verify Fact 2.2(2) for $\kappa = \aleph_1$. In $V[G]$, let $\langle D_\nu \mid \nu < \omega_1 \rangle$ be a family of open dense sets of $\mathcal{P}(\omega)/\text{fin} \setminus \{0\}$.

By a Löwenheim-Skolem argument, there is some ω_1-club (this is an unbounded set which is closed under suprema of strictly increasing ω_1-sequences) $C \subseteq \omega_2$, $C \in V$, such that for every $\alpha \in C$ $\forall \nu \in \omega_1$, $D_\nu \cap V[G_\alpha]$ is in $V[G_\alpha]$ and is open dense in $(\mathcal{P}(\omega)/\text{fin})^{V[G_\alpha]} \setminus \{0\}$. We want to prove that $\bigcap_{\nu \in \omega_1} D_\nu$ is not empty below a given $B \in (\mathcal{P}(\omega)/\text{fin})^{V[G]} \setminus \{0\}$. By [13], there is some $\delta < \aleph_2, \delta \in C$ such that $B \in V[G_\delta]$. By mapping B bijectively, say via f, onto ω and changing the D_ν by mapping each of their members pointwise with the same map f we get D'_ν, $\nu \in \omega_1$. We claim the next Mathias real hits all the D_ν below B. Now it is easy to see that for $\nu \in \aleph_1$, that

$$D_M(\nu) := \{(u, A) \in Q_\delta \mid A \in D'_\nu \cap V[G_\delta]\}$$

is dense in Q_δ. So the Mathias real $r \in [\omega]^\omega$ will be in all the D'_ν. Now $f^{-1\prime\prime}r$ is below B and is in all the D_ν. $\qquad\qquad\square_{3.2}$

4. DISTINGUISHING $\mathfrak{h}$-NUMBERS; $c_0 \setminus \ell^1$

In this section, we are going to prove $\mathfrak{h}((c_0 \setminus \ell^1, \leqslant^*)/\approx) = \aleph_1$ in $V[G]$. We work with the formulation 2.2(2) and shall show something slightly stronger:

> For any given $\bar{b} \in (c_0 \setminus \ell^1)^{V[G]}$, there are $\langle D_\nu \mid \nu \in \omega_1 \rangle \in V[G]$ such that D_ν is open and dense in $(c_0 \setminus \ell^1, \leqslant^*)^{V[G]}$ and closed under $\approx$ and such that their intersection is not dense below $\bar{b}$.

Suppose that $\bar{b} \in (c_0 \setminus \ell^1)^{V[G]}$. There is some $\delta < \omega_2$ such that $\bar{b} \in V[G_\delta]$. We choose a family $\langle D_\nu \mid \nu \in \omega_1 \rangle \in V[G]$ such that $\langle D_\nu \mid \nu \in \omega_1 \rangle$ is an enumeration of

(4.1)
$$\left\{ \left\{ \bar{a} \in (c_0 \setminus \ell^1)^{V[G]} \;\middle|\; \sum_{\ell \in H} a_\ell < \infty \text{ or } \sum_{\ell \in \omega \setminus H} a_\ell < \infty \right\} \;\middle|\; H \in ([\omega]^\omega)^{V[G_\delta]} \right\}.$$

This is possible, because in $V[G_\delta]$ the continuum has still cardinality $\aleph_1$. All the sets in the set above are closed under $\approx$ and open and dense in $(c_0 \setminus \ell^1)^{V[G]}$; the latter is shown as in Lemma 4.3.

First let $\bar{m}(\bar{b}) = \bar{m} = \langle m_i \mid i \in \omega \rangle \in (\omega^\omega)^{V[G_\delta]}$ be a sequence of natural numbers such that for every $i \in \omega$,

(4.2)
$$m_0 = 0 \text{ and } m_{i+1} > 2^{m_i} \text{ and}$$

(4.3)
$$\frac{2^{m_i - 2}}{36 \cdot (i+1)^2} \geqslant 2^{(i+1)^2} \text{ and}$$

(4.4)
$$\forall \ell \geqslant m_{i+1} \; b_\ell \leqslant 2^{-m_i}.$$

Now we begin an indirect proof. We assume

(4.5)
$$\bigcap D_\nu \text{ is dense } (\leqslant_{c_0 \setminus \ell^1}) \text{ below } \bar{b}.$$

The following chain of conclusions, including three lemmata, serves to derive a contradiction from our assumption. Following [5], we factorise $P = P_\delta * P_{\delta,\omega_2}$. We consider $V[G_\delta]$ as the ground model. So there is a condition $p \in P_{\delta,\omega_2} \cap G$ and

(4.6)
$$p \Vdash_{P_{\delta,\omega_2}} \text{``} \bigcap_{\nu} \underset{\sim}{D_\nu} \text{ is dense below } \check{\bar{b}} \text{''}.$$

For technical reasons we have to "discretize" the partial order $(c_0 \setminus \ell^1)^{V[G]}$ a bit. We set

$$(c_0 \setminus \ell^1)^{\bar{m}}_{\text{discr}} = \left\{ \bar{e} = \langle e_\ell \mid \ell \in \omega \rangle \in c_0 \setminus \ell^1 \;\middle|\; \forall i \in \omega \setminus \{0\} \; \forall \ell \in [m_i, m_{i+1}) \right.$$

$$\left. e_\ell \in \left\{ \frac{j}{2^{m_{i+1}}} \;\middle|\; j = 0, 1, \ldots, 2^{m_{i+1} - m_{i-1}} \right\} \right\}.$$

It is easy to see that $((c_0 \setminus \ell^1)^{\bar{m}}_{\text{discr}})^{V[G]}$ (— we interpret $c_0 \setminus \ell^1$ as a defining formula, which has to be evaluated according to the model of set theory —) is dense in $(c_0 \setminus \ell^1)^{V[G]}$ below $\bar{b}$, the calculation that

$$\sum_i \frac{m_{i+1} - m_i}{2^{m_{i+1}}} < \infty$$

together with the formula (4.4) helps to see it.

Because of $(c_0 \setminus \ell^1)^{\bar{m}}_{\mathrm{discr}}$'s density below $\bar{b}$ and of (4.6) we may assume that

$$(4.7) \qquad p \Vdash_{P_{\delta,\omega_2}} \exists \bar{c} \leqslant^* \check{\bar{b}} \;\; \bar{c} \in (c_0 \setminus \ell^1)^{\bar{m}}_{\mathrm{discr}} \cap \bigcap_{\nu \in \omega_1} D_\nu,$$

and we do so.

By the maximum principle, there is a name $\undertilde{\bar{c}}$ such that

$$(4.8) \qquad p \Vdash_{P_{\delta,\omega_2}} \undertilde{\bar{c}} \in (c_0 \setminus \ell^1)^{\bar{m}}_{\mathrm{discr}} \cap \bigcap_{\nu \in \omega_1} \undertilde{D}_\nu \wedge \undertilde{\bar{c}} \leqslant^* \check{\bar{b}}.$$

We set for $i \in \omega \setminus \{0\}$

$$x_i = \left\{ s \;\middle|\; s \colon [m_i, m_{i+1}) \to \left\{ \frac{j}{2^{m_{i+1}}} \;\middle|\; j = 0, 1, \ldots, 2^{m_{i+1}-m_{i-1}} \right\} \right\}.$$

Then we use

Lemma 4.1. *(The Laver property for P_{δ,ω_2}.)* *Suppose that $\langle x_i \,|\, i \in \omega \setminus \{0\} \rangle \in V[G_\delta]$ is a family of finite sets and that*

$$p \Vdash_{P_{\delta,\omega_2}} \forall i \in \omega \;\; \undertilde{\bar{c}} \restriction [m_i, m_{i+1}) \in \check{x}_i.$$

Then there are some $q \leqslant_{P_{\delta,\omega_2}} p$ and some $\langle y_i \,|\, i \in \omega \setminus \{0\} \rangle \in V[G_\delta]$ such that in $V[G_\delta]$

(1) $\forall i \in \omega \setminus \{0\} \;\; |y_i| \leqslant 2^{i^2}$, *and*
(2) $\forall i \in \omega \setminus \{0\} \;\; y_i \subseteq x_i$, *and*
(3) $q \Vdash_{P_{\delta,\omega_2}} \forall i \in \omega \setminus \{0\} \;\; \undertilde{\bar{c}} \restriction [m_i, m_{i+1}) \in \check{y}_i$.

Proof. See Lemma 9.6. in [5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Now we apply Lemma 4.1 to our given x_i and $\bar{c}$ and get $\langle y_i \,|\, i \in \omega \setminus \{0\} \rangle \in V[G_\delta]$ as in the lemma. We also fix some q as in the lemma. Since there are densely many such q below p and since $p \in G$ we may assume that

$$(4.9) \qquad\qquad\qquad\qquad q \in G.$$

For $i > 0$, we set

$$w_i = \left\{ s \in y_i \;\middle|\; \sum_{\ell \in [m_i, m_{i+1})} s_\ell > \frac{1}{i^2} \right\}.$$

Since $\sum_{i \in \omega \setminus \{0\}} \frac{1}{i^2} < \infty$, we have that for any $\bar{e} \in (c_0 \setminus \ell^1)^{V[G]}$,

(4.10) $\quad \forall^\infty i \ \ \bar{e} \restriction [m_i, m_{i+1}) \in y_i$

$\longrightarrow \exists A \in [\omega]^\omega \ (\forall i \in A \setminus \{0\} \ \ \bar{e} \restriction [m_i, m_{i+1}) \in w_i \wedge \sum_{i \in A} \sum_{\ell \in [m_i, m_{i+1})} e_\ell = \infty).$

Note that by our choice of $\bar{m}$ we have for $i > 0$,

(4.11) $\qquad\qquad\qquad |w_i| \leq |y_i| \leq 2^{i^2} \leq \dfrac{2^{m_{i-1}-2}}{36 \cdot i^2}.$

Before continuing in the main stream of conclusions, we now record a useful lemma from probability theory. The methods presented in [1] led us to prove this lemma.

Lemma 4.2. *Assume that $\beta > 0$, and*

(a) $\quad m < m' < m''$ *are natural numbers, $m' > 2^m$ and $m'' > 2^{m'}$.*

(b) $\quad w \subseteq \left\{ \bar{d} \mid \bar{d} = \langle d_\ell \mid m' \leq \ell < m'' \rangle, d_\ell \in \left\{ \frac{j}{2^{m''}} \mid 0 \leq j \leq 2^{m''-m} \right\} \right\}.$

(c) $\quad$ *If $\bar{d} \in w$, then $\sum \{ d_\ell \mid \ell \in [m', m'') \} \geq \frac{1}{\beta}$.*

(d) $\quad |w| \leq \frac{2^{m-2}}{36\beta}.$

Then we can find a partition (u_0, u_1) of $[m', m'')$ such that

(4.12) $\qquad$ *If $\bar{d} \in w$ and $h \in \{0, 1\}$, then* $\dfrac{1}{3} \leq \dfrac{\sum \{ d_\ell \mid \ell \in u_h \}}{\sum \{ d_\ell \mid \ell \in [m', m'') \}} \leq \dfrac{2}{3}.$

Proof. We flip a fair coin for every $\ell \in [m', m'')$ to decide whether ℓ is in u_0 or in u_1 (so probabilities are $\frac{1}{2}$ and $\frac{1}{2}$).

We use $d = \sum_{\ell \in [m', m'')} d_\ell$ as an abbreviation. Given $\bar{d} \in w$ and $h \in \{0, 1\}$, we shall estimate the probability

$$\text{Prob} \left(\frac{\sum \{ d_\ell \mid \ell \in u_h \}}{d} < \frac{1}{3} \right).$$

The expected value of

$$\frac{\sum \{ d_\ell \mid \ell \in u_h \}}{d}$$

is $\frac{1}{2}$.

TV denotes the truth value of an event φ: $\text{TV}(\varphi) = 1$ if φ is true, and $\text{TV}(\varphi) = 0$ if φ is not true. We compute the variance

$$\text{Var} = \text{Exp}\left(\left(\frac{\sum\{d_\ell \mid \ell \in u_h\}}{d} - \text{Exp}\left(\frac{\sum\{d_\ell \mid \ell \in u_h\}}{d}\right)\right)^2\right)$$

which equals, as the coins are thrown independently,

$$= \frac{1}{d^2} \cdot \sum_\ell \left(\text{Exp}(d_\ell^2 \cdot \text{TV}(\ell \in u_h)) - (\text{Exp}(d_\ell \cdot \text{TV}(\ell \in u_h)))^2\right)$$

$$\leqslant \frac{1}{d^2} \cdot \sum_\ell d_\ell^2 \cdot \frac{1}{2}.$$

For the next argument, we allow, in contrast to our assumption (b) of Lemma 4.2, that the d_ℓ be reals such that

$$0 \leqslant d_\ell \leqslant \frac{1}{2^m}.$$

We maximize

$$\frac{1}{d^2} \cdot \sum_\ell d_\ell^2 \cdot \frac{1}{2}$$

under the given requirements. The maximum of any variation is attained if the d_ℓ, $\ell \in [m', m'')$ are most unevenly distributed, i.e. if some of them are $\frac{1}{2^m}$, one is possibly between 0 and $\frac{1}{2^m}$ and the others are 0. In order to have them summed up to d, $v := \left\lfloor \frac{d}{\frac{1}{2^m}} \right\rfloor = \lfloor 2^m \cdot d \rfloor$ of them are $\frac{1}{2^m}$ (where $\lfloor x \rfloor$ denotes the largest $n \in \omega$ such that $n \leqslant x$).

Hence we get that

$$\begin{aligned}
\text{Var} \quad &\leqslant \quad \frac{1}{d^2} \cdot \sum_\ell d_\ell^2 \cdot \frac{1}{2} \\
&\leqslant \quad \frac{1}{2 \cdot d^2} \cdot \left(\left(\frac{1}{2^m}\right)^2 \cdot v + \left((2^m \cdot d - v) \cdot \frac{1}{2^m}\right)^2\right) \\
&\leqslant \quad \frac{1}{2 \cdot d^2} \cdot \left(\frac{1}{2^m}\right)^2 \cdot d \cdot 2^m \\
&= \quad \frac{1}{2d} \cdot \frac{1}{2^m} \leqslant \frac{\beta}{2^{m+1}} \quad \text{(see premise (c) of Lemma 4.2 for the last } \leqslant \text{).}
\end{aligned}$$

We set

$$
\begin{aligned}
\alpha \;&=\; \mathrm{Prob}\left(\frac{\sum\{d_\ell \mid \ell \in u_h\}}{d} < \frac{1}{3}\right) \\
&=\; \mathrm{Prob}\left(\frac{\sum\{d_\ell \mid \ell \in u_h\}}{d} > \frac{2}{3}\right).
\end{aligned}
$$

So we get another estimate

$$
\begin{aligned}
\frac{\beta}{2^{m+1}} \;&\geqslant\; \mathrm{Var} = \mathrm{Exp}\left(\frac{\sum\{d_\ell \mid \ell \in u_h\}}{d} - \frac{1}{2}\right)^2 \\
&\geqslant\; \alpha \cdot \left(\frac{-1}{6}\right)^2 + \alpha \cdot \left(\frac{1}{6}\right)^2 = \frac{\alpha}{18}.
\end{aligned}
$$

Hence we have that

$$
(4.13) \qquad\qquad \alpha \leqslant 18 \cdot \frac{\beta}{2^{m+1}}.
$$

The number of cases for a possible failure, which means $\bar{d} \in w$ such that

$$
\frac{\sum\{d_\ell \mid \ell \in u_h\}}{d} \notin \left[\frac{1}{3}, \frac{2}{3}\right],
$$

is $|w|$, and the probability of any one failure is 2α.

Hence we have at least one chance of success if

$$
(4.14) \qquad\qquad |w| \cdot 2\alpha < 1,
$$

because then

$$
\mathrm{Prob}(\text{no failure in } |w| \text{ cases}) \geqslant 1 - |w| \cdot 2\alpha > 0.
$$

However, since by (4.13) we have that $\alpha \leqslant 18 \cdot \frac{\beta}{2^{m+1}}$, and since by our premises we have that $|w| \leqslant \frac{2^{m-2}}{36\beta}$, our sufficient condition (4.14) for success is fulfilled. $\qquad\square_{4.2}$

Now in $V[G_\delta]$ we apply Lemma 4.2 for every $i \in \omega$, with $w = w_{i+1}$, $m = m_i$, $m' = m_{i+1}$, $m'' = m_{i+2}$, $\beta = (i+1)^2$, and we get for $h = 0, 1$ for all $i \in \omega$ some $u_{h,i+1} \subseteq [m_{i+1}, m_{i+2})$ as in Lemma 4.2.

With a real parameter in $V[G_\delta]$ (namely $\langle u_{0,i} \mid i \in \omega \setminus \{0\}\rangle$) we define the set

$$
J = \{\bar{d} \in (c_0 \setminus \ell^1)^{V[G]} \mid \exists h \in \{0,1\} \;\; \forall^\infty i \in \omega \setminus \{0\} \;\; \bar{d}{\restriction}u_{h,i} \equiv 0\}.
$$

J is obviously open in $(c_0 \setminus \ell^1, \leqslant^*)$.

The closure of J under $\approx$ is

$$(4.15) \quad J' = \{\bar{d} \mid \exists \bar{d}' \in J \ \ \forall \bar{e} \leqslant^* \bar{d} \ \ \bar{e} \not\perp \bar{d}'\} = \{\bar{d} \mid \exists h \ \sum_{i \in \omega \setminus \{0\}} \sum_{\ell \in u_{h,i}} d_\ell < \infty\}.$$

Note that we have

$$(4.16) \qquad\qquad (\bar{c}/\approx) \in \{\bar{d}/\approx \mid \bar{d} \in J\} \quad \text{iff} \quad \bar{c} \in J'.$$

In the end, J' will be the bad guy among the D_ν from (4.8).

Lemma 4.3. *J is dense in $(c_0 \setminus \ell^1)^{V[G]}$ under $\leqslant^*$.*

Proof. Let $\bar{d}$ be an arbitrary element of $(c_0 \setminus \ell^1)^{V[G]}$. For $h \in \{0, 1\}$ define $\bar{d}^h = \langle d_\ell^h \mid \ell \in \omega \rangle$ below $\bar{d}$ as follows

$$d_\ell^h = \begin{cases} d_\ell, & \text{if } \exists i \in \omega \setminus \{0\} \ (m_i \leqslant \ell < m_{i+1}, \text{ and } \ell \in u_{h,i}); \\ 0, & \text{else.} \end{cases}$$

At least one of the $\bar{d}^h$ is divergent, because

$$\sum d_\ell = \sum (d_\ell^0 + d_\ell^1).$$

The divergent ones among the $\bar{d}^h$'s are in J. $\qquad\qquad\qquad \Box_{4.3}$

Hence also J' is dense. So J' is one of the D_ν, namely with H from (4.1) being $\bigcup_{i \in \omega \setminus \{0\}} u_{0,i}$. Now we can finally reach a contradiction by showing that

$$(4.17) \qquad\qquad q \Vdash_{P_{\delta,\omega_2}} \tilde{c} \in \tilde{J}'$$

This will contradict (4.8).

In order to prove (4.17), we consider formula (4.10), which yields

(4.18)

$$q \Vdash_{P_{\delta,\omega_2}} \exists A \in [\omega]^\omega \ \ (\forall i \in A \setminus \{0\} \ \ \tilde{c} \restriction [m_i, m_{i+1}) \in w_i \wedge \sum_{i \in A} \sum_{\ell \in [m_i, m_{i+1})} \tilde{c}_\ell = \infty).$$

Hence, by (4.9), in $V[G]$ there is an infinite A such that for $h = 0, 1$ we have by (4.12) that

$$(4.19) \qquad V[G] \models \sum_{i \in A \setminus \{0\}} \sum_{\ell \in u_{h,i}} c_\ell \geqslant \frac{1}{3} \cdot \sum_{i \in A \setminus \{0\}} \sum_{\ell \in [m_i, m_{i+1})} c_\ell = \infty.$$

Hence for either choice of $h \in \{0, 1\}$ we have that $\bar{c}_h = \langle c_\ell^h \mid \ell \in \omega \rangle$, where

$$c_\ell^h = \begin{cases} c_\ell, & \text{if } \exists i \in A \setminus \{0\} \ (m_i \leqslant \ell < m_{i+1}, \text{ and } \ell \in u_{h,i}); \\ 0, & \text{else,} \end{cases}$$

is divergent.

We shall show that $\bar{c} \notin J'$ (though $q \in G$), that is according to the definition (4.15) of J':

$$(4.20) \qquad \forall \bar{d} \in J \ \exists \bar{c'} \leqslant^* \bar{c} \ \ \bar{c'} \perp \bar{d}.$$

(Remark: Of course, we could have worked with $\bar{c}/\approx$ and formulation 2.2(2′) all the time and could have shown that there is no $\bar{d} \in J$ that is $\approx \bar{c}$. But we just did not like to handle equivalence classes all the time.)

Suppose we are given $\bar{d} \in J$. Then we have that

$$(4.21) \qquad V[G] \models \exists h' \in \{0,1\} \ \forall^\infty i \in \omega \setminus \{0\} \ \ \bar{d} \!\restriction\! u_{h',i} \equiv 0.$$

We fix such a number h'. But now we take $h = 1 - h'$! Then we have that $\bar{c}_h \leqslant^* \bar{c}$, and $\bar{c}_h$ is divergent, and for every sequence $\bar{e}$ with $(\bar{e} \leqslant^* \bar{d} \wedge \bar{e} \leqslant^* \bar{c}_h)$ we have that

$$(4.22) \qquad V[G] \models \forall^\infty \ell \ \ e_\ell = 0.$$

Hence such an $\bar{e}$ cannot be a divergent series, and we proved that $\bar{c}_h \perp \bar{d}$ and hence $\bar{c} \notin J'$ (and, by (4.16), $\bar{c} \not\approx \bar{d}$ for any $\bar{d} \in J$). This proves (4.17). So finally we derived a contradiction from (4.5). $\qquad \square_{\text{MainTheorem}}$

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