COVERING OF THE NULL IDEAL MAY HAVE COUNTABLE COFINALITY

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Abstract. We prove that it is consistent that the covering number of the ideal of measure zero sets has countable cofinality.

0. Introduction

In the present paper we show that it is consistent that the covering of the null ideal has countable cofinality. Recall that the covering number of the null ideal (i.e. the ideal of measure zero sets) is defined as

$$\mathrm{cov}(\mathrm{null}) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathrm{null} \text{ and } \bigcup_{A\in\mathcal{P}} A = \mathbb{R}(= {}^{\omega}2)\}.$$

The question whether the cofinality of cov(null) is uncountable has been raised by D. Fremlin and has been around since the late seventies. It appears in the current Fremlin's list of problems, [**?**], as problem CO. Recall that for the ideal of meagre sets the answer is positive, i.e. A. Miller [**?**] proved that the cofinality of the covering of category is uncountable. T. Bartoszyński [**?**] saw that $\mathfrak{b} < \aleph_\omega$ is necessary (see [**?**, ch 5] for more results related to this problem). It should be noted that most people thought $\mathrm{cf}(\mathrm{cov}(\mathrm{null})) = \aleph_0$ is impossible

The main result of this paper is the following:

Theorem 0.1. $\mathrm{Con}(\mathrm{cov}(\mathrm{null}) = \aleph_\omega + \mathrm{MA}_{\aleph_n})$ *for each* $n < \omega$.

The presentation of the proof of 0.1 sacrifices generality for hopeful transparency. We finish by some further remarks, e.g. the exact cardinal assumption for 0.1. We try to make the paper self contained for readers with basic knowledge of forcing.

In a subsequent paper, [**?**], we deal with the question: "can every non-null set be partitioned to uncountably many non-null sets", equivalently: "can the ideal of null sets which are subsets of a non-null subset of $\mathbb{R}$ be $\aleph_1$-saturated". P. Komjath [**?**] proved that it is consistent that there is a non-meagre set A such that the ideal of meagre subsets of A is $\aleph_1$-saturated. The

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question whether a similar fact may hold for measure dates back to Ulam, see also Prikry's thesis. It appears as question $EL(a)$ on the Fremlin's list. In [**?**] we prove the following:

Theorem 1. *It is consistent that there is a non-null set $A \subseteq \mathbb{R}$ such that the ideal of null subsets of A is $\aleph_1$-saturated (of course, provided that "ZFC $+\exists$ measurable" is consistent).*

In [**?**] we also prove the following.

Theorem 2. *It is consistent that:*

($\oplus$) *there is a non null $A \subseteq \mathbb{R}$ such that: for every $f : A \to \mathbb{R}$, the function f as a subset of the plane $\mathbb{R} \times \mathbb{R}$ is null*

provided that "ZFC + there is a measurable cardinal" is consistent.

Notation 0.2. We denote:

- natural numbers by k, l, m, n and also i, j
- ordinals by α, β, γ, δ, ζ, ξ (δ always limit)
- cardinals by λ, κ, χ, μ
- reals by a, b and positive real (normally small) by ε.
- subsets of ω or ${}^{\omega\geq}2$ or Ord by A, B, C, X, Y, Z but
- $\mathcal{B}$ is a Borel function
- finitely additive measures by Ξ
- sequences of natural numbers or ordinals by η, ν, ρ,
- s is used for various things

$\mathcal{T}$ is as in definition 2.9, t is a member of $\mathcal{T}$.
We denote

- forcing notions by P, Q,
- forcing conditions by p, q

and use r to denote members of Random (see below) except in definition 2.2.

- Leb is the Lebesgue measure (on ${}^{\omega}2$),
- Random will be the family

$$\big\{r \subseteq {}^{\omega>}2 : \begin{array}{l} r \text{ is a subtree of } ({}^{\omega>}2, \lhd) \\ \text{(i.e. non-empty subset of } {}^{\omega>}2 \text{ closed under initial segments)} \\ \text{with no } \lhd\text{–maximal element} \\ \text{(so } \lim(r) =: \{\eta \in {}^{\omega}2 : (\forall n \in \omega)(\eta{\restriction}n \in t)\} \\ \text{ is a closed subset of } {}^{\omega}2) \\ \text{and } \mathrm{Leb}(\lim(r)) > 0\big\} \end{array}$$

ordered by inverse inclusion. We may sometimes use instead

$$\{B : B \text{ is a Borel non-null subset of } {}^{\omega}2\}.$$

For $\eta \in {}^{\omega>}2$, $A \subseteq {}^{\omega\geq}2$ let

$$A^{[\eta]} = \{\nu \in A : \nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu\}.$$

Let $\mathcal{H}(\chi)$ denote the family of sets with transitive closure of cardinality $< \chi$, and let $<^*_\chi$ denote a well ordering of $\mathcal{H}(\chi)$.

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1. Preliminaries

We review various facts on finitely additive measures.

Definition 1.1. (1) $\mathcal{M}$ is the set of functions Ξ from some Boolean subalgebra P of $\mathcal{P}(\omega)$ including the finite sets, to $[0,1]_{\mathbb{R}}$ such that:

- $\Xi(\emptyset) = 0$, $\Xi(\omega) = 1$,
- if $Y, Z \in P$ are disjoint, then $\Xi(Y \cup Z) = \Xi(Y) + \Xi(Z)$,
- $\Xi(\{n\}) = 0$ for $n \in \omega$.

(2) $\mathcal{M}^{\rm full}$ is the set of $\Xi \in \mathcal{M}$ with domain $\mathcal{P}(\omega)$ and members are called "finitely additive measures (on ω)".
(3) We say "A has Ξ-measure a (or $> a$, or whatever) if $A \in {\rm dom}(\Xi)$ and $\Xi(A)$ is a (or $> a$ or whatever).

Proposition 1.2. *Let a_α, b_α $(\alpha < \alpha^*)$ be reals, $0 \le a_\alpha \le b_\alpha \le 1$, and let $A_\alpha \subseteq \omega$ $(\alpha < \alpha^*)$ be given. The following conditions are equivalent:*

(A) *There exists $\Xi \in \mathcal{M}$ which satisfies $\Xi(A_\alpha) \in [a_\alpha, b_\alpha]$ for $\alpha < \alpha^*$.*
(B) *For every $\varepsilon > 0, m < \omega$ and $n < \omega$, and $\alpha_0 < \alpha_1, \ldots < \alpha_{n-1} < \alpha^*$ we can find a finite, non-empty $u \subseteq [m, \omega)$ such that for $l < n$*

$$a_{\alpha_l} - \varepsilon \le |A_{\alpha_l} \cap u|/|u| \le b_{\alpha_l} + \varepsilon.$$

(C) *For every real $\varepsilon > 0$, $n < \omega$ and $\alpha_0 < \alpha_1, \ldots < \alpha_{n-1} < \alpha^*$ there are $c_l \in [a_{\alpha_l} - \varepsilon, b_{\alpha_l} + \varepsilon]$ such that in the vector space $\mathbb{R}^n$, $\langle c_0, \ldots c_{n-1}\rangle$ is in the convex hull of $\{\rho \in {}^n\{0,1\} :$ for infinitely many $m \in \omega$ we have $\forall_{l<n}[\rho(l) = 1 \Leftrightarrow m \in A_{\alpha_l}]\}$.*
(D) *Like part (A) with $\Xi \in \mathcal{M}^{\rm full}$.*
(E) *like part (B) demanding $u \subseteq \omega$, $|u| \ge m$.*

Proof. Straightforward. On (C) see the 2.17. □

Proposition 1.3. *1) Assume that $\Xi_0 \in \mathcal{M}$ and for $\alpha < \alpha^*$, $A_\alpha \subseteq \omega$ and $0 \le a_\alpha \le b_\alpha \le 1$, a_α, b_α reals. The following are equivalent:*

(A) *There is $\Xi \in \mathcal{M}^{full}$ extending Ξ_0 such that $\alpha < \alpha^* \Rightarrow \Xi(A_\alpha) \in [a_\alpha, b_\alpha]$.*
(B) *For every partition $\langle B_0, \ldots, B_{m-1}\rangle$ of ω with $B_i \in {\rm dom}(\Xi_0)$ and $\varepsilon > 0$, $n < \omega$ and $\alpha_0 < \ldots < \alpha_{n-1} < \alpha^*$ we can find a finite set $u \subseteq \omega$ such that $\Xi(B_\alpha) - \varepsilon \le |u \cap B_\alpha|/|u| \le \Xi(B_\alpha) + \varepsilon$ and $a_{\alpha_l} - \varepsilon \le |u \cap A_{\alpha_l}|/|u| \le b_{\alpha_l} + \varepsilon$.*
(C) *For every partition $\langle B_0, \ldots, B_{m-1}\rangle$ of ω with $B_i \in {\rm dom}(\Xi_0)$ and $\varepsilon > 0$, $n < \omega$ and $\alpha_0 < \ldots < \alpha_{n-1} < \alpha^*$ we can find $c_{l,k} \in [0,1]_{\mathbb{R}}$ for $l < n, k < m$ such that*

(a) $\sum\limits_{k<m} c_{l,k} \in (a_{\alpha_l} - \varepsilon, b_{\alpha_l} + \varepsilon)$

(b) *for each* $k < m$ *and* $s < \omega$ *we can find* $u \subseteq B_k$ *with* $\geq s$ *members such that*

$$l < n \Rightarrow c_{l,k} - \varepsilon < (|u \cap A_l \cap B_k|/|u|) \times \Xi(B_k) < c_{l,k} + \varepsilon.$$

(D) *for every partition* $\langle B_0, \ldots, B_{m-1} \rangle$ *of* ω, *with* $B_i \in \mathrm{dom}(\Xi_0)$, $\varepsilon > 0$, $n < \omega$, *and* $\alpha_0, \ldots, \alpha_{n-1} < \alpha^*$ *we can find* $c_{l,k} \in [0,1]_{\mathbb{R}}$ *for* $l < n$, $k < m$ *such that*

(a) $\sum_{k<m} c_{l,k} \in [a_{\alpha_l} - \varepsilon, b_{\alpha_l} + \varepsilon]$,

(b) $\langle c_{l,k} : l < n \rangle$ *is in the convex hull of*

$$\{\rho \in {}^n\{0,1\} : \text{ for infinitely many } i \in B_k, \text{ we have: } (\forall l < n)[\rho(l) = 1 \Leftrightarrow i \in A_{\alpha_l}]\}.$$

2) The following are sufficient conditions for (A), (B), (C), (D) above:

(E) *For every* $\varepsilon > 0$, $A^* \in \mathrm{dom}(\Xi_0)$ *such that* $\Xi_0(A^*) > 0$, $n < \omega$, $\alpha_0 < \ldots < \alpha_{n-1} < \alpha^*$, *we can find a finite, non-empty* $u \subseteq A^*$ *such that* $a_{\alpha_l} - \varepsilon \leq |A_{\alpha_l} \cap u|/|u| \leq b_{\alpha_l} + \varepsilon$ *for* $l < n$.

(F) *For every* $\varepsilon > 0$, $n < \omega$, $\alpha_0 < \alpha_1, \ldots < \alpha_{n-1} < \alpha^*$ *and* $A^* \in \mathrm{dom}(\Xi_0)$ *such that* $\Xi_0(A^*) > 0$, *the set* $\prod_{l<n}[a_{\alpha_l} - \varepsilon, b_{\alpha_l} + \varepsilon] \subseteq \mathbb{R}^n$ *is not disjoint to the convex hull of*

$$\{\rho \in {}^n\{0,1\} : \text{ for infinitely many } m \in A^* \text{ we have: } \forall_{l<n}[\rho(l) = 1 \Leftrightarrow m \in A_{\alpha_l}]\}.$$

3) If in addition $b_\alpha = 1$ *for* $\alpha < \alpha^*$ then *a sufficient condition for (A) — (E) above is*

(G) *if* $A^* \in \mathrm{dom}(\Xi_0)$ *and* $\Xi(A^*) > 0$ *and* $n < \omega$ *and* $\alpha_0 < \ldots < \alpha_{n-1} < \alpha^*$ *then* $A^* \cap \bigcap\limits_{l<n} A_{\alpha_l} \neq \emptyset$.

Proof. Straightforward. □

Definition 1.4. (1) For $\Xi \in \mathcal{M}^{\rm full}$ and sequence $\bar{a} = \langle a_l : l < \omega \rangle$ of reals in $[0,1]_{\mathbb{R}}$ (or just $\sup\limits_{l<\omega}(|a_l|) < \infty$), let

$$Av_\Xi(\bar{a}) =$$
$$\sup\{\sum_{k<k^*} \Xi(A_k) \inf(\{a_l : l \in A_k\}) : \langle A_k : k < k^* \rangle \text{ is a partition of } \omega\} =$$
$$\inf\{\sum_{k<k^*} \Xi(A_k) \sup(\{a_l : l \in A_k\}) : \langle A_k : k < k^* \rangle \text{ is a partition of } \omega\}.$$

(Easily proved that they are equal.)

(2) For $\Xi \in \mathcal{M}$, $A \subseteq \omega$ such that $\Xi(A) > 0$ define $\Xi_A(B) = \Xi(A \cap B)/\Xi(A)$. Clearly $\Xi_A \in \mathcal{M}$ with the same domain, $\Xi_A(A) = 1$. If $B \subseteq \omega$ and $\Xi(B) > 0$ then we let

$$Av_{\Xi|B}(\langle a_k : k \in B \rangle) = Av_\Xi(\langle a'_k : k < \omega \rangle)/\Xi(B)$$

where $a'_k = \begin{cases} a_k & \text{if } k \in B \\ 0 & \text{if } k \notin B \end{cases}$

Proposition 1.5. *Assume that* $\Xi \in \mathcal{M}^{\rm full}$ *and* $a^i_l \in [0,1]_{\mathbb{R}}$ *for* $i < i^* < \omega$, $l < \omega$, $B \subseteq \omega$, $\Xi(B) > 0$ *and* $Av_{\Xi_B}(\langle a^i_l : l < \omega\rangle) = b_i$ *for* $i < i^*$, $m^* < \omega$ *and lastly* $\varepsilon > 0$. Then *for some finite* $u \subseteq B \setminus m^*$ *we have: if* $i < i^*$, *then* $b_i - \varepsilon < (\sum\{a^i_l : l \in u\})/|u| < b_i + \varepsilon$.

Proof. Let $B = \bigcup\limits_{j<j^*} B_j$ be a partition of B with $j^* < \omega$ such that for every $i < i^*$ we have

$$\sum_{j<j^*} \sup\{a^i_l : l \in B_j\}\Xi(B_j) - \sum_{j<j^*} \inf\{a^i_l : l \in B_j\}\Xi(B_j) < \varepsilon/2.$$

Now choose k^* large enough such that there are k_j satisfying $k^* = \sum\limits_{j<j^*} k_j$ and $|k_j/k^* - \Xi(B_j)/\Xi(B)| < \varepsilon/(2j^*)$ for $j < j^*$. Let $u_j \subseteq B_j \setminus m^*$, $|u_j| = k_j$ for $j < j^*$. Now let $u = \bigcup\limits_{j<j^*} u_j$. Now calculate:

$$\begin{aligned}
\sum_{l\in u} a^i_l/|u| &= \sum_{j<j^*} \sum\{a^i_l : l \in u_j\}/|u| \le \sum_{j<j^*} \sup\{a^i_l : l \in B_j\}k_j/k^* \\
&\le \sum_{j<j^*} \sup\{a^i_l : l \in B_j\}(\Xi(B_j)/\Xi(B) + \varepsilon/2j^*) \\
&\le b_i + \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = b_i + \varepsilon,
\end{aligned}$$

$$\begin{aligned}
\sum_{l\in u} a^i_l/|u| &= \sum_{j<j^*} \sum\{a^i_l : l \in u_j\}/|u| \ge \sum_{j<j^*} \inf\{a^i_l : l \in B_j\}k_j/k^* \\
&\ge \sum_{j<j^*} \inf\{a^i_l : l \in B_j\}(\Xi(B_j)/\Xi(B) - \varepsilon/(2j^*)) > b_i - \varepsilon
\end{aligned}$$

□

Claim 1.6. *Suppose* Q_1, Q_2 *are forcing notions,* $\Xi_0 \in \mathcal{M}^{\rm full}$ *in* V, *and for* $\ell = 1, 2$

$$\Vdash_{Q_\ell} \text{“ } \underset{\sim}{\Xi}_\ell \text{ is a finitely additive measure extending } \Xi_0 \text{ ”.}$$

Then

$$\Vdash_{Q_1\times Q_2} \text{“there is a finitely additive measure extending } \underset{\sim}{\Xi}_1 \text{ and } \underset{\sim}{\Xi}_2 \text{ (hence } \Xi_0\text{)”.}$$

Proof. Straightforward by 1.2 as:

($*$) if $\Vdash_{Q_l}$ “$\underset{\sim}{A}_l \subseteq \omega$” and $\Vdash_{Q_1\times Q_2}$ “$\underset{\sim}{A}_1 \cap \underset{\sim}{A}_2$ is finite” then

$$\Vdash_{Q_1\times Q_2} \text{“ for some } m \text{ and } A \subseteq \omega,\ A \in V \text{ we have: } \underset{\sim}{A}_1 \setminus m \subseteq A,\ \ (\underset{\sim}{A}_2 \setminus m) \cap A = \emptyset \text{ ”.}$$

□

Fact 1.7. *Assume* Ξ *is a partial finitely additive measure,* $\bar{a}^\alpha = \langle a^\alpha_k : k < \omega\rangle$ *sequence of reals for* $\alpha < \alpha^*$ *such that* $\limsup\limits_k |a^\alpha_k| < \infty$ *for each* α. Then *(B)* $\Rightarrow$ *(A) where*

(A) *there is* Ξ^*, $\Xi \subseteq \Xi^* \in \mathcal{M}^{\rm full}$ *such that* $Av_{\Xi^*}(\bar{a}^\alpha) \ge b_\alpha$ *for* $\alpha < \alpha^*$.

(B) *for every partition* $\langle B_0,\dots,B_{m^*-1}\rangle$ *of* ω *with* $B_m\in\mathrm{dom}(\Xi)$ *and* $\varepsilon>0$, $k^*>0$ *and* $\alpha_0<\ldots<\alpha_{n^*-1}<\alpha^*$, *there is a finite* $u\subseteq\omega\setminus k^*$ *such that:*
  (i) $\Xi(B_m)-\varepsilon<|u\cap B_m|/|u|<\Xi(B_m)+\varepsilon$,
  (ii) $\frac{1}{|u|}\sum\limits_{k\in u}a_k^{\alpha_l}>b_{\alpha_l}-\varepsilon$ *for* $l<n^*$.

Remark 1.8. If in (A) we demand $Av_\Xi(\bar a^\alpha)=b_\alpha$, *then* in (B)(ii) add $\frac{1}{|u|}\sum\limits_{k\in u}a_k^{\alpha_l}\le b_{\alpha_l}+\varepsilon$.

2. The iteration

Ignoring $\mathrm{MA}_{<\kappa}$ (which anyhow was a side issue) a quite natural approach in order to get 0.1 (i.e. $\mathrm{cov}(\mathrm{null})=\lambda$, say $\lambda=\aleph_\omega$) is to use finite support iteration, $\bar Q=\langle P_\alpha,Q_\alpha:\alpha<\alpha^*\rangle$, add in the first λ steps null sets N_α (the intention is that $\bigcup\limits_{\alpha<\lambda}N_\alpha={}^\omega 2$ in the final model), and then iterate with Q_α being $\mathrm{Random}^{\mathbf{V}^{P'_\alpha}}$ where $P'_\alpha\lessdot P_\alpha$ and $|P'_\alpha|<\lambda$. Say, for some $A_\alpha\subseteq\alpha$

$$P'_\alpha=\{p\in P_\alpha:\ \begin{array}{l}\mathrm{dom}(p)\subseteq A_\alpha\text{ and this holds for the conditions}\\ \text{involved in the }P_\gamma\text{-name for }\gamma\in\mathrm{dom}(p)\text{ etc}\}\end{array}$$

(so each Q_α is a partial random; see Definition 2.2). If every set of $<\lambda$ null sets from $\mathbf{V}^{P_{\alpha^*}}$ is included in some $\mathbf{V}^{P'_\alpha}$, clearly $\mathbf{V}^{P_{\alpha^*}}\models\mathrm{cov}(\mathrm{null})\ge\lambda$; but we need the other inequality too.

The problem is why does $\langle N_\alpha:\alpha<\lambda\rangle$ continue to cover? For $P_{\lambda+n}$ such that $\alpha\in[\lambda,\lambda+n)\Rightarrow A_\alpha=\alpha$ this is very clear (we get iteration of Random forcing) and if $\alpha\in[\lambda,\lambda+n)\Rightarrow A_\alpha\subseteq\lambda$ this is clear (we get product). But necessarily we get a quite chaotic sequence $\langle A_{\alpha_m}\cap\{\alpha_\ell:\ell<m\}:m<m^*\rangle$ for some $\alpha_0<\ldots<\alpha_{m^*-1}$. More concretely this is the problem of why there are no perfect sets of random reals (see 2.7) or even just no dominating reals. We need to "let the partial randoms whisper secrets one to another", in other words to pass information in some way. This is done by the finitely additive measures $\Xi^t_{\bar\alpha}$. We had tried with thinking of using $\aleph_\varepsilon$-support (see [**?**]), the idea is still clear in the proof of 3.3. In this proof we start with "no dominating reals" for which we can just use ultrafilters (rather than finitely additive measures).

Let us start with a ground model $\mathbf{V}$ satisfying the following hypothesis:

Hypothesis 2.1. (a) $\lambda=\sum\limits_{\zeta<\delta(*)}\lambda_\zeta$, $\aleph_0<\kappa=\mathrm{cf}(\kappa)$, $\kappa<\lambda_\zeta<\lambda_\gamma$ for $\zeta<\gamma<\delta(*)$, $\chi^\lambda=\chi$, $2^\kappa=\chi$ and $\zeta<\delta(*)\Rightarrow(\lambda_\zeta)^{\aleph_0}<\lambda$,
(b) we have one of the following[1]:
  (α) $\mathrm{cf}(\chi)>\lambda$, the length of the final iteration is χ,
  (β) length of the final iteration is $\chi\times\chi\times\lambda^+$.

[1] actually, any ordinal α^* of cardinality χ, divisible by χ and of cofinality $>\lambda$ is O.K.

We speak mainly on (α). In case (β) we should be careful to have no repetitions in $\bar\eta = \langle \eta_\alpha : \alpha < \delta^*\rangle$ (see below) or $\langle \eta_\alpha/\approx_\kappa : \alpha < \delta^*\rangle$ with no repetitions, where $\eta \approx_\kappa \nu$ iff $\eta, \nu \in {}^\kappa 2$ and $|\{i < \kappa : \eta(i) \neq \nu(i)\}| < \kappa$.

The reader may choose to restrict himself and start with $\mathbf{V}$ satisfying: GCH, $\lambda = \aleph_\omega$, $\delta(*) = \omega$, $\lambda_n = \aleph_{n(*)+n}$, $\kappa = \aleph_{n(*)} > \aleph_1$ and $\chi = \aleph_{\omega+1}$. Now add $\aleph_{\omega+1}$ generic subsets of κ, i.e., force with a product of χ copies of $({}^{\kappa>}2, \lhd)$ with support $< \kappa$. This model satisfies the hypothesis.

We intend to define a forcing P such that

$$\mathbf{V}^P \models 2^{\aleph_0} = \chi \ + \ \mathrm{cov}(\mathrm{null}) = \lambda \ + \ \mathrm{MA}_{<\kappa}.$$

Definition 2.2. 1) $\mathcal{K}$ is the family of sequences

$$\bar Q = (P_\alpha, \underset{\sim}{Q}_\alpha, A_\alpha, \mu_\alpha, \underset{\sim}{\tau}_\alpha : \alpha < \alpha^*)$$

satisfying:

(A) $(P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \alpha^*)$ is a finite support iteration of c.c.c. forcing notions, we call $\alpha^* = \ell g(\bar Q)$ (the length of $\bar Q$), P_{α^*} is the limit,

(B) $\underset{\sim}{\tau}_\alpha \subseteq \mu_\alpha < \kappa$ is the generic of $\underset{\sim}{Q}_\alpha$, (i.e. over $\mathbf{V}^{P_\alpha}$ from $G_{\underset{\sim}{Q}_\alpha}$ we can compute $\underset{\sim}{\tau}_\alpha$ and vice versa),

(C) $A_\alpha \subseteq \alpha$ (for proving theorem 0.1 we use $|A_\alpha| < \lambda$),

(D) $\underset{\sim}{Q}_\alpha$ is a P_α-name of a c.c.c. forcing notion but computable from $\langle \underset{\sim}{\tau}_\gamma[\underset{\sim}{G}_{P_\alpha}] : \gamma \in A_\alpha\rangle$; in particular it belongs to $\mathbf{V}_\alpha = \mathbf{V}[\langle \underset{\sim}{\tau}_\gamma[G_{P_\alpha}] : \gamma \in A_\alpha\rangle]$.

(E) $\alpha^* \geq \lambda$ and for $\alpha < \lambda$ we have $Q_\alpha = ({}^{\omega>}2, \lhd)$ (the Cohen forcing) and $\mu_\alpha = \aleph_0$ (well, identifies ${}^{\omega>}2$ with ω).

(F) For each $\alpha < \alpha^*$ one of the following holds, (and the case is determined in $\mathbf{V}$):

(α) $|\underset{\sim}{Q}_\alpha| < \kappa$, $|A_\alpha| < \kappa$ and (just for notational simplicity) the set of elements of $\underset{\sim}{Q}_\alpha$ is $\mu_\alpha < \kappa$ (but the order not necessarily the order of the ordinals) and $\underset{\sim}{Q}_\alpha$ is seperative (i.e. $\zeta \Vdash \xi \in G_{\underset{\sim}{Q}_\alpha} \Leftrightarrow \underset{\sim}{Q}_\alpha \models \xi \leq \zeta$),

(β) essentially $\underset{\sim}{Q}_\alpha = \mathrm{Random}^{\mathbf{V}_\alpha}$
$(= \{r \in \mathbf{V}_\alpha : r \subseteq {}^{\omega>}2,$ perfect tree, $\mathrm{Leb}(\lim(r)) > 0\})$
and $|A_\alpha| \geq \kappa$;

but for simplicity $\underset{\sim}{Q}_\alpha = \mathrm{Random}^{A_\alpha, \bar Q{\restriction}\alpha}$ where for $A \subseteq \ell g(\bar Q)$,

$\mathrm{Random}^{A,\bar Q} = \{p :$ there is (in V) a Borel function $\mathcal{B} = \mathcal{B}(x_0, x_1, \ldots)$,
with variables ranging on {true, false} and
range perfect subtrees r of ${}^{\omega>}2$ with $\mathrm{Leb}(\lim r) > 0$,
such that $(\forall \eta \in r)[\mathrm{Leb} \lim r^{[\eta]} > 0]$,
recalling $r^{[\eta]} = \{\nu \in r : \nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu\}$, and
there are pairs $(\gamma_\ell, \zeta_\ell)$ for $\ell < \omega, \gamma_\ell \in A$ and $\zeta_\ell < \mu_{\gamma_\ell}$,
such that $p = \mathcal{B}(\ldots, \text{truth value}(\zeta_\ell \in \underset{\sim}{\tau}_{\gamma_\ell}), \ldots)_{\ell<\omega}\}$

(in other notation, $p = \mathcal{B}(\text{truth value}(\zeta_\ell \in \underset{\sim}{\tau}_{\gamma_\ell}) : \ell < \omega)$);
in this case we let $\mathrm{supp}(p) = \{\gamma_\ell : \ell < \omega\}$.

In this case $\mu_\alpha = \omega$ and $\underset{\sim}{\tau}_\alpha$ is the random real, i.e.

$$\underset{\sim}{\tau}_\alpha(n) = \ell \Leftrightarrow (\exists \eta \in {}^n 2)({}^{\omega >}2)^{[\eta^\frown\langle \ell \rangle]} \in \underset{\sim}{G}_{Q_\alpha}.$$

2) Let

$$\begin{aligned} P'_\alpha = \{p \in P_\alpha : \ & \text{for every } \gamma \in \mathrm{dom}(p), \text{if } |A_\gamma| < \kappa \text{ then} \\ & p(\gamma) \text{ is an ordinal } < \mu_\gamma \text{ (not just a } P_\gamma\text{–name) and} \\ & \text{if } |A_\gamma| \geq \kappa \text{ then } p(\gamma) \text{ has the form mentioned in clause} \\ & (\mathrm{F})(\beta) \text{ above (and not just a } P_\gamma\text{–name of such object) } \} \end{aligned}$$

(this is a dense subset of P_α).

3) For $A \subseteq \alpha$ let

$$P'_A = \{p \in P'_\alpha : \mathrm{dom}(p) \subseteq A \text{ and } \gamma \in \mathrm{dom}(p) \Rightarrow \mathrm{supp}(p(\gamma)) \subseteq A\}$$

Fact 2.3. *Suppose $\bar{Q} \in \mathcal{K}$ with $\ell g(\bar{Q}) = \alpha^*$.*

(1) *For $\alpha \leq \alpha^*$, P'_α is a dense subset of P_α and P_α satisfies the c.c.c.*

(2) *If*
   (a) $\mathrm{cf}(\alpha^*) > \lambda$,
   (b) *for every $A \subseteq \alpha^*$, if $|A| < \lambda$,* then *there is $\beta < \alpha^*$ such that $A \subseteq A_\beta$ (and $|A_\beta| \geq \kappa$).*

   Then, *in the extension, ${}^\omega 2$ is not the union of $< \lambda$ null sets.*

(3) *In $\mathbf{V}^{P_\alpha}$, from $\underset{\sim}{\tau}_\alpha[G_{Q_\alpha}]$ we can reconstruct G_{Q_α} and vice versa. From $\langle \underset{\sim}{\tau}_\gamma : \gamma < \alpha \rangle[G_{P_\alpha}]$ we can reconstruct G_{P_α} and vice versa. So $\mathbf{V}^{P_\alpha} = \mathbf{V}[\langle \underset{\sim}{\tau}_\beta : \beta < \alpha \rangle]$.*

(4) *If $\mu < \lambda$, and $\underset{\sim}{X}$ is a P_{α^*}-name of a subset of μ,* then *there is a set $A \subseteq \alpha^*$ such that $|A| \leq \mu$ and $\Vdash_{P_{\alpha^*}}$ "$\underset{\sim}{X} \in \mathbf{V}[\langle \underset{\sim}{\tau}_\gamma : \gamma \in A \rangle]$". Moreover for each $\zeta < \mu$ there is in $\mathbf{V}$ a Borel function $\mathcal{B}(x_0, \ldots, x_n, \ldots)_{n<\omega}$ with domain and range the set $\{\mathrm{true}, \mathrm{false}\}$ and $\gamma_\ell \in A$, $\zeta_\ell < \mu_{\gamma_\ell}$ for $\ell < \omega$ such that*

$$\Vdash_{P_{\alpha^*}} \text{"}\zeta \in \underset{\sim}{X} \text{ iff } \mathrm{true} = \mathcal{B}(\ldots, \text{"truth value of } \zeta_\ell \in \underset{\sim}{\tau}_{\gamma_\ell}[G_{Q_{\gamma_\ell}}]\text{"}, \ldots)_{\ell<\omega}\text{"}$$

(5) *For $\bar{Q} \in \mathcal{K}$ and $A \subseteq \alpha^*$, every real in $\mathbf{V}[\langle \underset{\sim}{\tau}_\gamma : \gamma \in A \rangle]$ has the form mentioned in clause (F)(β) of2.2(1).*

(6) *If condition (c) below holds* then $\mathbf{V}^{P_{\alpha^*}} \models \mathrm{MA}_{<\kappa}$, *where*
   (c) *if $\underset{\sim}{Q}$ is a P_{α^*}-name of a c.c.c. forcing notion with set of elements $\mu < \kappa$ then for some $\alpha < \alpha^*$, $\underset{\sim}{Q}$ is a P_α-name $\mu_\alpha = \mu$ and $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q} = \underset{\sim}{Q}_\alpha$".*

(7) *if $|A_\beta| \geq \kappa$ then Q_β is actually* $\mathrm{Random}^{\mathbf{V}^{P_{A_\beta}}}$.

Proof. 2) Easy using parts (3) — (7). Note that for any $\beta < \alpha^*$ satisfying $|A_\beta| \geq \kappa$ the null sets from $\mathbf{V}^{[\langle \underset{\sim}{\tau}_\gamma : \gamma \in A_\beta \rangle]}$ do not cover ${}^\omega 2$ in $\mathbf{V}^{P_{\alpha^*}}$ as we have random reals over $\mathbf{V}^{[\langle \underset{\sim}{\tau}_\gamma : \gamma \in A_\beta \rangle]}$. So, by clause (b) of the assumption, it is enough to note that if $\underset{\sim}{y}$ is a P_{α^*}–name of a member of ${}^\omega 2$, then there is a countable $A \subseteq \alpha^*$, such that $\underset{\sim}{y}[\underset{\sim}{G}] \in \mathbf{V}[\langle \underset{\sim}{\tau}_\beta : \beta \in A \rangle]$. This follows by part (4).

3) By induction on α.

4) Let χ^* be such that $\{\bar{Q}, \lambda\} \in \mathcal{H}(\chi^*)$, and let $\zeta < \mu$; let M be a countable elementary submodel of $(\mathcal{H}(\chi^*), \in, <^*_{\chi^*})$ to which $\{\bar{Q}, \lambda, \kappa, \mu, \underset{\sim}{X}, \zeta\}$ belongs, so $\Vdash_{P_{\alpha^*}}$ "$M[\underset{\sim}{G}_{P_{\alpha^*}}] \cap \mathcal{H}(\chi^*) = M$". Hence by 2.3(3) (i.e. as $\mathbf{V}^{P_\alpha} = \mathbf{V}[\langle \underset{\sim}{\mathcal{I}}_\beta : \beta < \alpha\rangle]$) we have $M[\underset{\sim}{G}_{P_{\alpha^*}}] = M[\langle \underset{\sim}{\mathcal{I}}_i : i \in \alpha^* \cap M\rangle]$ and the conclusion should be clear.
5) By 2.3(4).
6) Straight.
7) Check. □

Definition 2.4. (1) Suppose that $\bar{a} = \langle a_l : l < \omega\rangle$ and $\langle n_l : l < \omega\rangle$ are such that:

(a) $a_l \subseteq {}^{n_l}2$,
(b) $n_l < n_{l+1} < \omega$ for $l < \omega$,
(c) $|a_l|/2^{n_l} > 1 - 1/10^l$.

Let $N[\bar{a}] =: \{\eta \in {}^\omega 2 : (\exists^\infty l)(\forall \nu \in a_l)\nu \not\lhd \eta\}$.
(2) For $\bar{a}$ as above and $n \in \omega$, let

$$\mathrm{tree}_n(\bar{a}) = \{\nu \in {}^{\omega >}2 : n_l > n \Rightarrow \nu \restriction n_l \in a_l\}.$$

It is well known that for $\bar{a}$ as above the set $N[\bar{a}]$ is null (and $N[\bar{a}] = {}^\omega 2 \setminus \bigcup\limits_{n<\omega} \lim \mathrm{tree}_n(\bar{a})$).

Definition 2.5. For $\alpha < \lambda$ we identify Q_α (the Cohen forcing) with:

$$\{\langle (n_l, a_l) : l < k\rangle : k < \omega, n_l < n_{l+1} < \omega,\ a_l \subseteq {}^{n_l}2,\ |a_l|/2^{n_l} > 1 - 1/10^l\},$$

ordered by the end extension. If G_{Q_α} is Q_α-generic, let $\langle (\underset{\sim}{\bar{a}}^\alpha_\ell, \underset{\sim}{n}^\alpha_\ell) : \ell < \omega\rangle[G_{Q_\alpha}]$ be the ω–sequence such that every $p \in G_{Q_\alpha}$ is an initial segment of it. So we have defined the Q_α–name $\underset{\sim}{\bar{a}}^\alpha = \langle \underset{\sim}{a}^\alpha_\ell : \ell < \omega\rangle$ and similarly $\langle \underset{\sim}{n}^\alpha_\ell : \ell < \omega\rangle$. Let $\underset{\sim}{N}_\alpha = N[\underset{\sim}{\bar{a}}^\alpha]$.

Our aim is to prove that

Definition 2.6. For $\bar{Q} \in \mathcal{K}$ with $\alpha^* = \ell g(\bar{Q})$ let:

$(*)_{\bar{Q}}$ $\langle N_\alpha : \alpha < \lambda\rangle$ cover ${}^\omega 2$ in $\mathbf{V}^{P_{\alpha^*}}$, where $P_{\alpha^*} = \mathrm{Lim}(\bar{Q})$.

We eventually shall prove it not for every $\bar{Q}$, but for enough $\bar{Q}$'s (basically asking the A_α of cardinality $\geq \kappa$ to be closed enough).

Lemma 2.7. *For $\bar{Q} \in \mathcal{K}$ with $\gamma = \ell g(\bar{Q})$, a sufficient condition for $(*)_{\bar{Q}}$ is:*

$(**)_{\bar{Q}}$ *In $\mathbf{V}^{P_\gamma}$: there is no perfect tree $T \subseteq {}^{\omega >}2$ and $E \in [\lambda]^{\kappa^+}$ such that, for some $n < \omega$, $T \subseteq \mathrm{tree}_n[\bar{a}^\alpha]$ for all $\alpha \in E$.*

Proof. By induction on $\gamma \geq \lambda$. For $\gamma = \lambda$, trivial by properties of the Cohen forcing.

Suppose $\gamma > \lambda$ limit. Assume toward contradiction that

$$p \Vdash_{P_\gamma} \text{ " } \underset{\sim}{\eta} \notin \bigcup_{\alpha<\lambda} N[\underset{\sim}{\bar{a}}^\alpha] \text{ ".}$$

W.l.o.g.

$$p \Vdash_{P_\gamma} \text{“ } \underset{\sim}{\eta} \notin \mathbf{V}^{P_\beta} \text{ ”}$$

for every $\beta < \gamma$, hence by properties of FS iteration of c.c.c. forcing notions $\mathrm{cf}(\gamma) = \aleph_0$. So for each $\alpha < \lambda$ there are p_α, m_α such that

$$p \le p_\alpha \in P_\gamma, \qquad p_\alpha \Vdash \text{“ } \underset{\sim}{\eta} \in \lim(\mathrm{tree}_{m_\alpha}(\bar{a}^\alpha)) \text{ ”}.$$

Note that (by properties of c.c.c. forcing notions) $\langle \{\alpha < \lambda : p_\alpha \in P_\beta\} : \beta < \gamma\rangle$ is an increasing sequence of subsets of λ of length γ, so for some $\gamma_1 < \gamma$ there is $E \in [\lambda]^{\kappa^+}$ such that $p_\alpha \in P_{\gamma_1}$ for every $\alpha \in E$ and w.l.o.g. $m_\alpha = m$ for $\alpha \in E$. Note that for all but $< \kappa^+$ of the ordinals $\alpha \in E$ we have

$$p_\alpha \Vdash \text{“}|\{\beta \in E : p_\beta \in G_{P_{\gamma_1}}\}| = \kappa^+\text{”}.$$

Fix such α, let $G_{P_{\gamma_1}}$ be P_{γ_1}-generic over $\mathbf{V}$ subset of P_{γ_1} to which p_α belongs. Now in $\mathbf{V}[G_{P_{\gamma_1}}]$ let $E' = \{\beta \in E : p_\beta \in G_{P_{\gamma_1}}\}$ so $|E'| = \kappa^+$. Let $T^* = \bigcap_{\beta \in E'} \mathrm{tree}_m(\bar{a}^\beta)$. Note that, in $\mathbf{V}^{P_{\gamma_1}}$, T^* is a subtree of ${}^{\omega >}2$ and by $(**)$, T^* contains no perfect subtree. Hence $\lim(T^*)$ is countable, so absolute. But $p_\alpha \Vdash_{P_\gamma}$ “ $\underset{\sim}{\eta} \in \lim(T^*)$ ”, so $p_\alpha \Vdash$ “$\underset{\sim}{\eta} \in V^{P_{\gamma_1}}$ ”, a contradiction.

Assume now that $\gamma = \beta + 1 > \lambda$ and work in V^{P_β}. Choose $p, p_\alpha \in Q_\beta$ as before. Note that Q_β has a dense subset of cardinality $< \lambda$, so there is some $q \in Q_\beta$ and m such that $E = \{\alpha < \lambda : m_\alpha = n,\ p_\alpha \le q\}$ has cardinality $\ge \kappa^+$. Continue as above.

As we have covered the cases $\gamma = \lambda$, $\gamma > \lambda$ limit and $\gamma > \lambda$ successor, we have finished the proof. □

Discussion 2.8. Note that by Lemma 2.7 and Fact 2.3 it is enough to show that there is $\bar{Q} \in \mathcal{K}$ such that $\alpha^* = \ell g(\bar{Q})$ (where α^* is chosen as the length of the final iteration from 2.1 clause (b)), satisfying clauses (a)+(b)+(c) of Fact 2.3(2) + 2.3(6) and $(**)_{\bar{Q}}$. To prove the latter we need to impose more restrictions on the iteration.

Definition 2.9. $\mathcal{T}$, the set of blueprints, is the set of tuples

$$t = (w^t, \mathbf{n}^t, \mathbf{m}^t, \bar{\eta}^t, h^t_0, h^t_1, h^t_2, \bar{n}^t)$$

where:

(a) $w^t \in [\kappa]^{\aleph_0}$,
(b) $0 < \mathbf{n}^t < \omega$, $\mathbf{m}^t \le \mathbf{n}^t$
(c) $\bar{\eta}^t = \langle \eta^t_{\mathbf{n},k} : \mathbf{n} < \mathbf{n}^t, k < \omega\rangle$, $\eta^t_{\mathbf{n},k} \in {}^{w^t}2$ for $\mathbf{n} < \mathbf{n}^t, k < \omega$,
(d) h^t_0 is a partial function from $[0, \mathbf{n}^t)$ to ${}^\omega\kappa$, its domain includes the set $\{0, \dots, \mathbf{m}^t - 1\}$ (here we consider members of Q_α (for $\alpha < \lambda$) as integers[2]),
(e) h^t_1 is a partial function from $[0, \mathbf{n}^t)$ to $(0,1)_{\mathbb{Q}}$ (rationals), but for $\mathbf{n} \in [0, \mathbf{n}^t) \backslash \mathrm{dom}(h^t_1)$ we stipulate $h^t_1(\mathbf{n}) = 0$ and we assume $\sum\limits_{\mathbf{n} < \mathbf{n}^t} \sqrt{h^t_1(\mathbf{n})} < 1/10$.

[2] actually the case where each $h^t_0(\mathbf{n})$ is a constant function from ω to κ suffices, and so $\kappa < \lambda$ suffices instead $\kappa^{\aleph_0} < \lambda$

(f) h^t_2 is a partial function from $[0, \mathbf{n}^t)$ to ${}^{\omega>}2$,
(g) $\mathrm{dom}(h^t_0)$, $\mathrm{dom}(h^t_1)$ are disjoint with union $[0, \mathbf{n}^t)$,
(h) $\mathrm{dom}(h^t_2) = \mathrm{dom}(h^t_1)$,
(i) $\eta^t_{\mathbf{n}_1,k_1} = \eta^t_{\mathbf{n}_2,k_2} \Rightarrow \mathbf{n}_1 = \mathbf{n}_2$,
(j) for each $\mathbf{n} < \mathbf{n}^t$ we have: $\langle \eta^t_{\mathbf{n},k} : k < \omega\rangle$ is constant or with no repetitions; if it is constant and $\mathbf{n} \in \mathrm{dom}(h^t_0)$ then h^t_0 is constant.
(k) $\bar{n}^t = \langle n^t_k : k < \omega\rangle$ where $n^t_0 = 0$, $n^t_k < n^t_{k+1} < \omega$ and the sequence $\langle n^t_{k+1} - n^t_k : k < \omega\rangle$ goes to infinity. Let for $\ell < \omega$ and such $\bar{n}$, $k_{\bar{n}}(\ell) = k(\ell, \bar{n})$ be the unique k such that $n_k \le \ell < n_{k+1}$.

Discussion 2.10. The definitions of a blueprint $t \in \mathcal{T}$ (in Definition 2.9) and of iterations $\bar{Q} \in \mathcal{K}^3$ (defined in Definition 2.11 clause (c) below; the reader may first read it) contain the main idea of the proof, so though they have many clauses, the reader is advised to try to understand them.

In order to prove $(**)_{\bar{Q}}$ we will show in $\mathbf{V}^{P_{\alpha^*}}$ that if $E \in [\lambda]^{\kappa^+}$, and $n < \omega$, then $\bigcap\limits_{\alpha\in E} \mathrm{tree}_n(\bar{a}^\alpha)$ is a tree with finitely many branches. So let p be given, let $p \le p_\zeta \Vdash$ " $\beta_\zeta \in \underset{\sim}{E}$ " for $\zeta < \kappa^+$, $\beta_\zeta \notin \{\beta_\xi : \xi < \zeta\}$, we can assume p_ζ is in some pregiven dense set, and $\langle p_\zeta : \zeta < \kappa^+\rangle$ form a Δ-system (with some more "thinning" demands), $\mathrm{dom}(p_\zeta) = \{\alpha_{\mathbf{n},\zeta} : \mathbf{n} < \mathbf{n}^*\}$, $\alpha_{\mathbf{n},\zeta}$ is increasing with n, and $\alpha_{\mathbf{n},\zeta} < \lambda$ iff $\mathbf{n} < \mathbf{m}^*$. Let p'_ζ be p_ζ when $p_\zeta(\alpha_{\mathbf{n},\zeta})$ is increased a little, as described below.

It suffices to find $p^* \ge p$ such that $p^* \Vdash$ " $\underset{\sim}{A} =: \{\zeta < \omega : p'_\zeta \in \underset{\sim}{G}\}$ is large enough such that $\bigcap\limits_{\zeta\in \underset{\sim}{A}} \mathrm{tree}_m(\bar{a}^{\beta_\zeta})$ has only finitely many branches".

Because of "communication problems" the "large enough" is interpreted as of $\underset{\sim}{\Xi}^t_\alpha$-measure (again defined in 2.11 below).

The natural numbers $\mathbf{n} < \mathbf{n}^*$ such that $Q_{\alpha_{\mathbf{n},\zeta}}$ is a forcing notion of cardinality $< \kappa$, do not cause problems, as $h^t_0(\mathbf{n})$ tells us exactly what the condition $p_\zeta(\alpha_{\mathbf{n},\zeta})$ is. Still there are many cases of such $\langle p_\zeta : \zeta < \omega\rangle$ which fall into the same t; we possibly will get contradictory demands if $\alpha_{\mathbf{n}_1,\zeta_1} = \alpha_{\mathbf{n}_2,\zeta_2}$, $\mathbf{n}_1 \ne \mathbf{n}_2$. But the w^t, $\bar{\eta}^t$ are exactly built to make this case not to happen. That is, we have to assume $2^\kappa = \chi$ $(= |\alpha^*|)$ in order to be able for our iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \alpha^*\rangle$ to choose $\langle \eta_\alpha : \alpha < \alpha^*\rangle$, $\eta_\alpha \in {}^\kappa 2$ with no repetitions, so that if $v \subseteq \chi$, $|v| \le \aleph_0$ (e.g. $v = \{\alpha_{\mathbf{n},\zeta} : \mathbf{n} < \mathbf{n}^t, \zeta < \omega\}$) then for some $w = w^t \in [\kappa]^{\aleph_0}$ we have $\langle \eta_\alpha \restriction w : \alpha \in v\rangle$ is with no repetitions.

So the blueprint t describes such situation, giving as much information as we can, as long as the number of blueprints is not too large, $\kappa^{\aleph_0} = \kappa$ in our case.

If $Q_{\alpha_{\mathbf{n},\zeta}}$ is a partial random, we may get many candidates for $p_\zeta(\alpha_{\mathbf{n},\zeta}) \in$ Random and they are not all the same ones. We want that in many cases they will be in the generic set. Well, we can (using $h^t_1(\mathbf{n})$, $h^t_2(\mathbf{n})$) know that in some interval $({}^\omega 2)^{[h^t_2(\mathbf{n})]}$ the set $\lim p_\zeta(\alpha_{\mathbf{n},\zeta})$ is large, say of relative measure $\ge 1 - h^t_1(\mathbf{n})$, and we could have chosen the p_ζ's such that $\langle h^t_1(\mathbf{n}) : \mathbf{n} < \mathbf{n}^t\rangle$ is small enough, still the number of candidates is not bounded by

$1/h^t_1(\mathbf{n})$. Here taking limit by ultrafilters is not good enough, but using finitely additive measures is.

Well, we have explained w^t, $\bar\eta^t$, h^t_0, h^t_1, h^t_2, but what about the $\bar n^t = \langle n^t_k : k < \omega\rangle$? In the end (in §3) the specific demand on $\{\zeta : p'_\zeta \in G\}$ being large, is that for infinitely many $k < \omega$,

$$|\{\ell : n^t_k \le \ell < n^t_{k+1} \text{ and } p_\ell \in G\}|/(n^t_{k+1} - n^t_k)$$

is large, the n^t_k will be chosen such that it is increasing fast enough and $\langle p'_\ell(\beta_\ell) : \ell \in [n^t_k, n^t_{k+1})\rangle$ will be chosen such that for each $\varepsilon > 0$ for some $s < \omega$, for k large enough if the above fraction is $\ge \varepsilon$ then essentially ${}^k2 \cap \{\mathrm{tree}_m(\bar a^{\beta_\ell}) : n^t_k \le \ell < n^t_{k+1} \text{ and } p'_\ell \in G\}$ has $\le s$ members, this suffices.

What is our plan? We define $\mathcal{K}^3$, the class of suitable expanded iterations $\bar Q$ by choice of η_α (for $\alpha < \ell g(\bar Q)$) and names for finitely additive measures Ξ^t_α satisfying the demands natural in this context. You may wonder why we use Ξ-averages; this is like integral or expected value, and so "behave nicely" making the "probability computations" simpler. Then we show that we can find $\bar Q \in \mathcal{K}^3$ in which all obligations toward "cov(null) $\ge \lambda$" and $\mathrm{MA}_{<\kappa}$ hold.

The main point of §3 will be that we can carry the argument of "for some p^* we have $p^* \Vdash \{\ell < \omega : p'_\ell \in G\}$ is large" and why it gives $n < \omega$ & $E \in [\chi]^{\kappa^+} \Rightarrow \bigcap\limits_{\zeta\in A} \mathrm{tree}_n(\bar a^\zeta)$ has finitely many branches, thus proving theorem 0.1.

The reader may wonder how much the Ξ^t_α are actually needed. As explained above they are just a transparent way to express the property; this will be utilized in [**?**].

Definition 2.11. $\mathcal{K}^3$ is the class of sequences

$$\bar Q = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\Xi^t_\alpha)_{t\in\mathcal{T}} : \alpha \le \alpha^*, \beta < \alpha^*\rangle$$

(we write $\alpha^* = \ell g(\bar Q)$), such that:

(a) $\langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta : \alpha \le \alpha^*, \beta < \alpha^*\rangle$ is in $\mathcal{K}$,

(b) $\eta_\beta \in {}^{\kappa}2$ and $\forall \beta < \alpha < \alpha^*[\eta_\alpha \ne \eta_\beta]$,

(c) $\mathcal{T}$ is from Definition 2.9, and Ξ^t_α is a P_α-name of a finitely additive measure on ω (in $\mathbf{V}^{P_\alpha}$), increasing with α,

(d) We say that $\bar\alpha = \langle \alpha_l : l < \omega\rangle$ satisfies $(t, \mathbf{n})$ (for $\bar Q$) if:

- $\langle \alpha_l : l < \omega\rangle \in \mathbf{V}$ (of course),
- $t \in \mathcal{T}$, $\mathbf{n} < \mathbf{n}^t$,
- $\alpha_l \le \alpha_{l+1} < \alpha^*$,
- $\mathbf{n} < \mathbf{m}^t \Leftrightarrow (\forall l)(\alpha_l < \lambda) \Leftrightarrow (\exists k)(\alpha_k < \lambda)$,
- $\eta^t_{\mathbf{n},l} \subseteq \eta_{\alpha_l}$ (as functions),
- if $\mathbf{n} \in \mathrm{dom}(h^t_0)$, *then* $\mu_{\alpha_l} < \kappa$ and

$\Vdash_{P_{\alpha_l}}$ "$|Q_{\alpha_l}| < \kappa$ and $(h^t_0(\mathbf{n}))(l) \in \underset{\sim}{Q}_{\alpha_l}$ i.e. $(h^t_0(\mathbf{n}))(l) < \mu_{\alpha_l}$",

- if $\mathbf{n} \in \mathrm{dom}(h^t_1)$, *then* $\mu_{\alpha_l} \ge \kappa$ so $\Vdash_{P_{\alpha_l}}$ "Q_{α_l} has cardinality $\ge \kappa$", (so it is a partial random),

- if $\langle \eta^t_{\mathbf{n},k} : k \in \omega\rangle$ is constant, then $\forall l[\alpha_l = \alpha_0]$, if $\langle \eta^t_{\mathbf{n},k} : k < \omega\rangle$ is not constant, then $\forall l[\alpha_l < \alpha_{l+1}]$,

(e) if $\bar\alpha = \langle \alpha_l : l < \omega\rangle$ satisfies $(t,\mathbf{n})$ for $\bar Q$, $\bigwedge\limits_{l<\omega}(\alpha_l < \alpha_{l+1})$, $\mathbf{n} \in \mathrm{dom}(h^t_0)$ then

$\Vdash_{P_{\alpha^*}}$ "the following set has $\Xi^t_{\alpha^*}$- measure 1 : $\{k < \omega :$ if $l \in [n^t_k, n^t_{k+1})$ then $(h^t_0(\mathbf{n}))(l) \in \underset{\sim}{G}_{Q_{\alpha_l}}\}$",

(f) *if* $\bar\alpha = \langle \alpha_l : l < \omega\rangle$ satisfies $(t,\mathbf{n})$ for $\bar Q$, $\mathbf{n} \in \mathrm{dom}(h^t_1)$, $\forall l < \omega[\alpha_l < \alpha_{l+1}]$, and $\bar r = \langle \underset{\sim}{r}_l : l < \omega\rangle$ where for $l < \omega$, $\underset{\sim}{r}_l$ is a P_{α_l}-name of a member of Q_{α_l} such that (it is forced that)

$(*)$ $1 - h^t_1(\mathbf{n}) \le \mathrm{Leb}(\{\eta \in {}^\omega 2 : h^t_2(\mathbf{n}) \lhd \eta \in \lim(\underset{\sim}{r}_l)\})/2^{\ell g(h^t_2(\mathbf{n}))}$,

then for each $\varepsilon > 0$ we have

$\Vdash_{P_{\alpha^*}}$ " the following set has $\Xi^t_{\alpha^*}$-measure 1 : $\{k < \omega :$ in the set $\{\ell \in [n^t_k, n^t_{k+1}) : \underset{\sim}{r}_\ell \in G_{Q_{\alpha_\ell}}\}$ there are at least $(n^t_{k+1} - n^t_k) \times (1 - h^t_1(\mathbf{n})) \times (1-\varepsilon)$ elements$\}$ "

(g) *if* $\bar\alpha = \langle \alpha_l : l < \omega\rangle$ satisfies $(t,\mathbf{n})$ for $\bar Q$, $\mathbf{n} \in \mathrm{dom}(h^t_1)$, $\forall l[\alpha_l = \alpha]$, and $\underset{\sim}{r}, \underset{\sim}{r}_l$ are P'_{A_α}-names of members of Q_α satisfying $(**)^{\bar Q}_{\underset{\sim}{r},\langle \underset{\sim}{r}_l : l<\omega\rangle}$ (see below for the definition of $(**)$) *then*

$\Vdash_{P_{\alpha^*}}$ " if $\underset{\sim}{r} \in G_{Q_\alpha}$ then $(1 - h^t_1(\mathbf{n}))$ $\le Av_{\Xi^t_{\alpha^*}}(\langle |\{l \in [n^t_k, n^t_{k+1}) : \underset{\sim}{r}_l \in \underset{\sim}{G}_{Q_\alpha}\}|/(n^t_{k+1} - n^t_k) : k < \omega\rangle)$ ",

where

$(**)^{\bar Q}_{\underset{\sim}{r},\langle \underset{\sim}{r}_l : l<\omega\rangle}$ $\underset{\sim}{r}$, $\underset{\sim}{r}_l$ are P'_{A_α}–names of members of Q_α and, in $\mathbf{V}^{P_\alpha}$, for every $r' \in Q_\alpha$ satisfying $r \le r'$ we have

$$Av_{\Xi^t_\alpha}(\langle a_k(r') : k < \omega\rangle) \ge (1 - h^t_1(\mathbf{n}))$$

where

$(\boxtimes)$ $$a_k(r') = a_k(r', \bar r) = a_k(r', \bar r, \bar n^t) = \left(\sum_{\ell \in [n^t_k, n^t_{k+1})} \frac{\mathrm{Leb}(\lim(r') \cap \lim(r_\ell))}{\mathrm{Leb}(\lim(r'))}\right) \times \frac{1}{n^t_{k+1} - n^t_k}$$

(so $a_k(r', \bar r, \bar n) \in [0,1]_{\mathbb{R}}$ is well defined for $k < \omega$, $\bar r = \langle r_l : l < \omega\rangle$, $\{r, r_l\} \subseteq \mathrm{Random}$, $\bar n = \langle n_l : l < \omega\rangle$, $n_l < n_{l+1} < \omega$),

(h) $P'_{A_\alpha} \lessdot P_\alpha$, and $\beta \in A_\alpha$ & $|A_\beta| < \kappa \ \Rightarrow\ A_\beta \subseteq A_\alpha$,

(i) if $\Vdash_{P_\alpha}$ "$|\underset{\sim}{Q}_\alpha| \ge \kappa$", *then* $\Xi^t_\alpha \restriction \mathcal{P}(\omega)^{\mathbf{V}^{P_{A_\alpha}}}$ is a P_{A_α}-name[3], for every $t \in \mathcal{T}$.

Definition 2.12. (1) For $\bar Q \in \mathcal{K}^3$ and for $\alpha^* \le \ell g(\bar Q)$ let

$$\bar Q \restriction \alpha^* = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\mathcal{I}}_\beta, \eta_\beta, (\Xi^t_\alpha)_{t\in\mathcal{T}} : \alpha \le \alpha^*, \beta < \alpha^*\rangle,$$

(2) For $\bar Q^1, \bar Q^2 \in \mathcal{K}^3$ we say:

$$\bar Q^1 \le \bar Q^2 \quad \text{if} \quad \bar Q^1 = \bar Q^2 \restriction \ell g(\bar Q^1).$$

[3]Here the secret was whispered.

Fact 2.13. *(1) If $\bar{Q} \in \mathcal{K}^3$, $\alpha \le \ell g(\bar{Q})$, then $\bar{Q} \restriction \alpha \in \mathcal{K}^3$.*
(2) $(\mathcal{K}^3, \le)$ is a partial order.
(3) If a sequence $\langle \bar{Q}^\beta : \beta < \delta \rangle \subseteq \mathcal{K}^3$ is increasing, $\mathrm{cf}(\delta) > \aleph_0$, *then there is a unique $\bar{Q} \in \mathcal{K}^3$ which is the least upper bound, $\ell g(\bar{Q}) = \bigcup\limits_{\beta<\delta} \ell g(\bar{Q}^\beta)$ and $\bar{Q}^\beta \le \bar{Q}$ for all $\beta < \delta$.*

Proof. Easy (recall that it is well known that $({}^\omega 2)^{\mathbf{V}^{P_\delta}} = \bigcup\limits_{\beta<\delta} ({}^\omega 2)^{\mathbf{V}^{P_\beta}}$, so $\Xi^t_\delta = \bigcup\limits_{\beta<\delta} \Xi^t_\beta$ is a legal choice). □

Lemma 2.14. *Suppose that $\bar{Q}^n \in \mathcal{K}^3$, $\bar{Q}^n < \bar{Q}^{n+1}$, $\alpha_n = \ell g(\bar{Q}^n)$, $\delta = \sup\limits_{n<\omega}(\alpha_n)$. Then there is $\bar{Q} \in \mathcal{K}^3$ such that $\ell g(\bar{Q}) = \delta$ and $\bar{Q}^n \le \bar{Q}$ for $n \in \omega$.*

Proof. Note that the only problem is to define Ξ^t_δ for $t \in \mathcal{T}$, i.e., we have to extend $\bigcup\limits_{\alpha<\delta} \Xi^t_\alpha$ so that the following two conditions are satisfied, (they correspond to clauses (f) and (e) of Definition 2.11).

(a) we are given[4] $\mathbf{n} < \mathbf{n}^t$, $\mathbf{n} \in \mathrm{dom}(h^t_1)$, $\langle \alpha_l : l < \omega \rangle$, from $\mathbf{V}$ of course, satisfies $(t, \mathbf{n})$ for $\bar{Q}$ and is strictly increasing with limit δ and we are given $\langle \underset{\sim}{p}_l : l < \omega \rangle$ such that $\Vdash_{P_{\alpha_l}}$ "$\underset{\sim}{p}_l \in \underset{\sim}{Q}_{\alpha_l}$ and $1 - h^t_1(\mathbf{n}) \le \mathrm{Leb}(\{\eta \in {}^\omega 2 : h^t_2(\mathbf{n}) \lhd \eta \in \lim(\underset{\sim}{p}_l)\})/2^{\ell g(h^t_2(\mathbf{n}))}$". The demand is: for each $\varepsilon > 0$ we have $\Vdash_{P_\delta}$ "$\underset{\sim}{\Xi}^t_\delta(\underset{\sim}{C}) = 1$", where

$$\underset{\sim}{C} = \{k < \omega : \text{ in the set } \{\ell : \ell \in [n^t_k, n^t_{k+1}) \text{ and } p_\ell \in \underset{\sim}{G}_{Q_{\alpha_\ell}}\} \text{ there are at least } (n^t_{k+1} - n^t_k) \times (1 - h^t_1(\mathbf{n})) \times (1 - \varepsilon) \text{ elements}\}.$$

(b) If[4] $\mathbf{n} < \mathbf{n}^t$, $\mathbf{n} \in \mathrm{dom}(h^t_0)$, $\langle \alpha_\ell : \ell < \omega \rangle$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$ and is strictly increasing with limit δ, and $p_l \in Q_{\alpha_l}$, satisfy $p_l = h^t_0(\mathbf{n})(l)$ for $\ell < \omega$ (an ordinal $< \mu_{\alpha_l}$), *then* $\Vdash_{P_\delta}$ "$\underset{\sim}{\Xi}^t_\delta(\underset{\sim}{C}) = 1$" where

$$\underset{\sim}{C} = \{k < \omega : \text{ for every } l \in [n_k, n_{k+1}) \text{ we have } p_l \in \underset{\sim}{G}_{Q_{\alpha_l}}\}$$

As $\bigcup\limits_{\alpha<\delta} \Xi^t_\alpha$ is a (P_δ-name of a) member of $\mathcal{M}$, in $\mathbf{V}^{P_\delta}$ by 1.3(3) it suffices to prove

(∗) $\Vdash_{P_\delta}$ "if $\underset{\sim}{B} \in \bigcup\limits_{\alpha<\delta} \mathrm{dom}(\Xi^t_\alpha) = \bigcup\limits_{\alpha<\delta} \mathcal{P}(\omega)^{\mathbf{V}[P_\alpha]}$ and $\Xi^t_\alpha(\underset{\sim}{B}) > 0$ and $j^* < \omega$, and $\underset{\sim}{C}_j$ (for $j < j^*$) are from (a), (b) above *then* $\underset{\sim}{B} \cap \bigcap\limits_{j<j^*} \underset{\sim}{C}_j \ne \emptyset$".

Toward contradiction assume $q \in P_\delta$ force the negation so possibly increasing q we have: for some $\underset{\sim}{B}$ and for some $j^* < \omega$, for each $j < j^*$ we have the $\varepsilon_j > 0$, and $\mathbf{n}(j) < \mathbf{n}^t$, $\langle \alpha^j_l : l < \omega \rangle$, $\langle p^j_l : l < \omega \rangle$ involved in the definition of $\underset{\sim}{C}_j$ (in (a) or (b) above), q force: $\underset{\sim}{B} \in \bigcup\limits_{\alpha<\delta} \mathrm{dom}(\Xi^t_\alpha) = \bigcup\limits_{\alpha<\delta} \mathcal{P}(\omega)^{\mathbf{V}[P_\alpha]}$ and $(\bigcup\limits_{\alpha<\delta} \Xi^t_\alpha)(\underset{\sim}{B}) > 0$ and $\underset{\sim}{C}_j$ (for $j < j^*$) comes from (a) or (b) above, but

[4] in $\mathbf{V}$, so $\langle (\alpha_l, p_l) : l < \omega \rangle \in \mathbf{V}$, of course

$\underset{\sim}{B} \cap \bigcap\limits_{j<j^*} \underset{\sim}{C}_j = \emptyset$; as we can decrease ε, wlog $\varepsilon_j = \varepsilon$. Again w.l.o.g. for some $\alpha(*) < \delta$ we have $\underset{\sim}{B} \in \mathrm{dom}(\Xi^t_{\underset{\sim}{\alpha}(*)})$ is a $P_{\alpha(*)}$-name, and $\underset{\sim}{C}_j$ have the $\mathbf{n}(j) < \mathbf{n}^t$, $\langle \alpha^j_l : l < \omega\rangle$, $\langle p^j_l : l < \omega\rangle$ witnessing it is as required in (a) or (b) above. W.l.o.g. $q \in P_{\alpha(*)}$. Possibly increasing q (inside $P_{\alpha(*)}$ though) we can find $k < \omega$ such that $q \Vdash$"$k \in \underset{\sim}{B}$" and $\bigwedge\limits_{j<j^*} \bigwedge\limits_{l \in [n^t_k, n^t_{k+1})} \alpha^j_l > \alpha(*)$ and moreover such that $n^t_{k+1} - n^t_k$ is large enough compared to $1/\varepsilon$, j^* (just let $q \in G_{P_{\alpha(*)}} \subseteq P_{\alpha(*)}$, $G_{P_{\alpha(*)}}$ generic over $\mathbf{V}$ and think in $\mathbf{V}[G_{P_{\alpha(*)}}]$). Let $\{\alpha^j_l : j < j^*$ and $l \in [n^t_k, n^t_{k+1})\}$ be listed as $\{\beta_m : m < m^*\}$, in increasing order (so $\beta_0 > \alpha(*)$) (possibly $\alpha^{j(1)}_{l(1)} = \alpha^{j(2)}_{l(2)}$ & $(j(1), l(1)) \neq (j(2), l(2))$). Now we choose by induction on $m \leq m^*$ a condition $q_m \in P_{\beta_m}$ above q, increasing with m, where we stipulate $\beta_{m^*} = \delta$.

During this definition we "throw a dice" and prove that the probability of success (i.e. $q_{m^*} \Vdash$ "$k \in \underset{\sim}{C}_j$" for $j < j^*$) is positive, so there is q_{m^*} as desired hence we get the desired contradiction.

Case A: $m = 0$

Let $q_0 = q$

Case B: $m + 1$, and for some $\mathbf{n} < \mathbf{n}^t$, we have $\mathbf{n} \in \mathrm{dom}(h^t_0)$ and ζ and: if $j < j^*$ and $l < \omega$ then $\alpha^j_l = \beta_m \Rightarrow \mathbf{n}(j) = \mathbf{n}$ & $p^j_l = \zeta$ $(= h^t_0(\mathbf{n}(j))(l)) \in Q_{\beta_m}$).

In this case $\mathrm{dom}(q_{m+1}) = \mathrm{dom}(q_m) \cup \{\beta_m\}$, and

$$q_{m+1}(\beta) = \begin{cases} q_m(\beta) & \text{if } \beta < \beta_m (\text{so } \beta \in \mathrm{dom}(q_m)) \\ \zeta & \text{if } \beta = \beta_m \end{cases}$$

Case C: $m + 1$ and for some $\mathbf{n} < \mathbf{n}^t$, we have $\mathbf{n} \in \mathrm{dom}(h^t_1)$ and: $\alpha^j_l = \beta_m \Rightarrow \mathbf{n}(j) = \mathbf{n}$.

Work first in $\mathbf{V}[G_{P_{\beta_m}}]$, $q_m \in G_{P_{\beta_m}}$, $G_{P_{\beta_m}}$ generic over $\mathbf{V}$. The sets

$$\{\lim(\underset{\sim}{p}^j_l[G_{P_{\beta_m}}]) : \alpha^j_l = \beta_m \text{ (and } l \in [n^t_k, n^t_{k+1}), j < j^*)\}$$

are subsets of $({}^\omega 2)^{[h^t_2(\mathbf{n})]} = \{\eta \in {}^\omega 2 : h^t_2(\mathbf{n}) \lhd \eta\}$, we can define an equivalence relation E_m on $({}^\omega 2)^{[h^t_2(\mathbf{n})]}$:

$$\nu_1 E_m \nu_2 \quad \textbf{iff} \quad \nu_1 \in \lim(\underset{\sim}{p}^j_l[G_{P_{\beta_m}}]) \equiv \nu_2 \in \lim(\underset{\sim}{p}^j_l[G_{P_{\beta_m}}])$$

$$\text{whenever } \alpha^j_l = \beta_m.$$

Clearly E_m has finitely many equivalence classes, call them $\langle Z^m_i : i < i^*_m\rangle$, all are Borel (sets of reals) hence they are measurable; w.l.o.g. $\mathrm{Leb}(Z^m_i) = 0 \Leftrightarrow i \in [i^\otimes_m, i^*_m)$, so clearly $i^\otimes_m > 0$. For each $i < i^\otimes_m$ there is $r = r_{m,i} \in \underset{\sim}{Q}_{\beta_m}[G_{P_{\beta_m}}]$ such that

$$\lim(p^j_l[G_{P_{\beta_m}}]) \supseteq Z^m_i \Rightarrow r \geq p^j_l[G_{P_{\beta_m}}],$$

$$\lim(p^i_l[G_{P_{\beta_m}}]) \cap Z^m_i = \emptyset \ \Rightarrow\ (\lim r) \cap (\lim p^i_l[G_{P_{\beta_m}}]) = \emptyset.$$

We can also find a rational $a_{m,i} \in (0,1)_{\mathbb{R}}$ such that

$$a_{m,i} < \mathrm{Leb}(Z^m_i)/2^{\ell g(h^t_2(\mathbf{n}))} < a_{m,i} + \varepsilon/(2i^*_m).$$

We can find $q'_m \in G_{P_{\beta_m}}$, $q_m \le q'_m$ such that q'_m forces all this information (so for $\underset{\sim}{Z}{}^m_i$, $\underset{\sim}{r}_{m,i}$ we shall have P_{β_m}–names, but $a_{m,i}$, $i^\otimes_m$, i^*_m are actual objects). We then can find rationals $b_{m,i} \in (a_{m,i}, a_{m,i}+\varepsilon/2)$ such that $\sum\limits_{i<i^\otimes_m} b_{m,i} = 1$. Now we throw a dice choosing $i_m < i^\otimes_m$ with the probability of $i_m = i$ being $b_{m,i}$ and define q_{m+1} as:

$$\mathrm{dom}(q_{m+1}) = \mathrm{dom}(q'_m) \cup \{\beta_m\}$$

$$q_{m+1}(\beta) = \begin{cases} q'_m(\beta) & \text{if } \beta < \beta_m \text{ (so } \beta \in \mathrm{dom}(q'_m)) \\ \underset{\sim}{r}_{m,i_m} & \text{if } \beta = \beta_m \end{cases}$$

An important point is that this covers all cases (and in Case B the choice of (j,l) is immaterial) as for each β_m there is a unique $\mathbf{n} < \mathbf{n}^t$ and l such that $\eta_{\beta_m} \restriction w^t = \eta^t_{\mathbf{n},l}$ (see Definition 2.11 clause (b) and Definition 2.9 clause (i)). Basic probability computations (for $n^t_{k+1} - n^t_k$ independent experiments) show that for each j coming from clause (a), by the law of large numbers the probability of successes is $> 1 - 1/j^*$, successes meaning $q_{m^*} \Vdash$ "$k \in \underset{\sim}{C}_j$" (remember if j comes from clause (b) we always succeed). $\square$

Remark 2.15. In the definition of $t \in \mathcal{T}$ (i.e. 2.9) we can add $\eta^t_{\mathbf{n},\omega} \in {}^{w^t}2$ (i.e. replace $\langle \eta^t_{\mathbf{n},l} : l < \omega\rangle$ by $\langle \eta^t_{\mathbf{n},l} : l \le \omega\rangle$) and demand

(l) if $\zeta \in w^t$ then for every $n < \omega$ large enough, $\zeta \in \eta^t_{\mathbf{n},\omega} \equiv \zeta \in \eta^t_{\mathbf{n},\omega}$,

and in Definition 2.11 clause (d) use $\bar\alpha = \langle \alpha_l : l < \omega\rangle$ *but* this does not help here.

Lemma 2.16. *1) Assume*

(a) $\bar Q \in \mathcal{K}^3$, $\bar Q = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{r}_\beta, \eta_\beta, (\underset{\sim}{\Xi}{}^t_\alpha)_{t\in T} : \alpha \le \alpha^*, \beta < \alpha^*\rangle$,
(b) $A \subseteq \alpha^*$, $\kappa < |A| < \lambda$,
(c) $\eta \in ({}^\kappa 2)^V \setminus \{\eta_\beta : \beta < \alpha^*\}$,
(d) $(\forall \alpha \in A)[|A_\alpha| < \kappa \ \Rightarrow\ A_\alpha \subseteq A]$ *and* $P'_A \lessdot P_{\alpha^*}$, $\underset{\sim}{Q} = \underset{\sim}{Q}^{A,\bar Q}$ *is the* P_{α^*}*-name from 2.2(F)*(β) *and*

$$\text{if} \quad t \in \mathcal{T} \quad \text{then} \quad \underset{\sim}{\Xi}{}^t_{\alpha^*} \restriction \mathbf{V}^{P_A} \text{ is a } P_A\text{-name},$$

Then *there is*

$$\bar Q^+ = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\underset{\sim}{\Xi}{}^t_\alpha)_{t\in T} : \alpha \le \alpha^*+1, \beta < \alpha^*+1\rangle$$

from $\mathcal{K}^3$, *extending* $\bar Q$ *such that* $\underset{\sim}{Q}_{\alpha^*} = \underset{\sim}{Q}$, $A_{\alpha^*} = A$, $\eta_{\alpha^*} = \eta$.
2) If clauses (a)+(b)+(c) of part one hold then *we can find* A' *such that:* $A \subseteq A' \subseteq \alpha^*$, $|A'| \le (|A|+\kappa)^{\aleph_0}$ *(which is* $< \lambda$ *by Hypothesis 2.1) and such that* $\bar Q$, A', η *satisfy (a)+(b)+(c)+(d).*

Proof. 1) As before the problem is to define $\Xi^t_{\alpha^*+1}$. We have to satisfy clause (g) of Definition 2.11 for each fixed $t \in \mathcal{T}$. Let $\mathbf{n}^*$ be the unique $\mathbf{n} < \mathbf{n}^t$ such that $\eta \restriction w^t = \eta^t_{\mathbf{n},l}$. If $\mathbf{n}^* \in \mathrm{dom}(h^t_0)$ or $\langle \eta^t_{\mathbf{n}^*,l} : l < \omega\rangle$ not constant or there is no such $\mathbf{n}^*$ then we have nothing to do. So assume that $\alpha_l = \alpha^*$ for $l < \omega$, $\eta^t_{\mathbf{n}^*,l} = \eta \restriction w^t$ (for $l < \omega$). Let Γ be the set of all pairs $(\underset{\sim}{r}, \langle \underset{\sim}{r}_l : l < \omega\rangle)$ which satisfy the assumption $(**)^{\bar{Q}}_{\underset{\sim}{r},\langle \underset{\sim}{r}_l : l<\omega\rangle}$ of 2.11 clause (g). In $\mathbf{V}^{P_{\alpha^*+1}}$ we have to choose $\Xi^t_{\alpha^*+1}$ taking care of all these obligations. We work in $\mathbf{V}^{P_{\alpha^*}}$. By assumption (d) and Claim 1.6 it suffices to prove it for $\mathbf{V}^{P_A}$ so Q_{α^*} is $\mathrm{Random}^{\mathbf{V}^{P_A}}$ (see 2.3(7)). By 1.7 it is enough to prove condition (B) of 1.7. Suppose it fails. Then there are $\langle B_m : m < m(*)\rangle$ a partition of ω from $\mathbf{V}^{P_A}$, for simplicity $\Xi^t_{\alpha^*}(B_m) > 0$ for $m < m(*)$, and $(\underset{\sim}{r}^i, \langle \underset{\sim}{r}^i_l : l < \omega\rangle) \in \Gamma$ and $\mathbf{n}(i) = \mathbf{n}^* < \mathbf{n}^t$ for $i < i^* < \omega$ and $\varepsilon^* > 0$ and $r \in Q_{\alpha^*}$ which forces the failure (of (B) of 1.7) for these parameters; (the ε^* comes from 1.7). W.l.o.g. r forces that $\underset{\sim}{r}^i \in \underset{\sim}{G}_{Q_{\alpha^*}}$ for $i < i^*$ (otherwise we can ignore such $\underset{\sim}{r}^i$ as nothing is demanded on them in (g) of 2.11). So $r \geq r^i$ for $i < i^*$.

By the assumption, for each $i < i^*$ we have: for each $r' \geq r$ (hence $r' \geq r^i$ and $r' \in \mathrm{Random}$) and $i < i^*$ we have:

$$Av_{\Xi^t_{\alpha^*}}(\langle a^i_k(r') : k < \omega\rangle) \geq (1 - h^t_1(\mathbf{n}^*))$$

where (see 2.11(g)($\boxtimes$)) we let

$$a^i_k(r') = a_k(\underset{\sim}{r}, \langle \underset{\sim}{r}_l : l < \omega\rangle, \bar{n}^t) = \frac{1}{n^t_{k+1} - n^t_k} \sum_{l \in [n^t_k, n^t_{k+1})} \frac{\mathrm{Leb}(\lim(r') \cap \lim(r^i_l))}{\mathrm{Leb}(\lim(r'))}.$$

By 1.7 it suffices to prove the following

Lemma 2.17. *Assume Ξ is a finitely additive measure, $\langle B_0, \dots, B_{m^*-1}\rangle$ a partition of ω, $\Xi(B_m) = a_m$, $i^* < \omega$ and r, $r^i_l \in \mathrm{Random}$ for $i < i^*$, $l < \omega$ and $\bar{n}^* = \langle n^*_i : i < \omega\rangle$, $n^*_i < n^*_{i+1} < \omega$ are such that*

(*) *for every $r' \in \mathrm{Random}$, $r' \geq r$ and $i < i^*$ we have $Av_\Xi(\langle a^i_k(r') : k < \omega\rangle) \geq b_i$ where*

$$a^i_k(r') = a^i_k(r', \langle r^i_l : l < \omega\rangle, \bar{n}^*) = \frac{1}{n^*_{k+1} - n^*_k} \sum_{l=n^*_k}^{n^*_{k+1}-1} \frac{\mathrm{Leb}(\lim(r') \cap \lim(r^i_l))}{\mathrm{Leb}(\lim(r'))}.$$

Then *for each $\varepsilon > 0$, $k^* < \omega$ there is a finite $u \subseteq \omega \setminus k^*$ and $r' \geq r$ such that:*

(1) *$a_m - \varepsilon < |u \cap B_m|/|u| < a_m + \varepsilon$, for $m < m^*$*

(2) *for each $i < i^*$ we have*

$$\frac{1}{|u|} \sum_{k \in u} \frac{|\{l : n^*_k \leq l < n^*_{k+1} \text{ and } r' \geq r^i_l\}|}{n^*_{k+1} - n^*_k}$$

is $\geq b_i - \varepsilon$.

Proof. Let for $i < i^*$, $m < m^*$ and $r' \geq r$ (from Random):

$$c_{i,m}(r') = Av_{\Xi \restriction B_m}(\langle a^i_k(r') : k \in B_m \rangle) \in [0,1]_{\mathbb{R}}.$$

So clearly

$(*)_1$ for $r' \geq r$ (in Random)

$$\begin{aligned} b_i \leq Av_{\Xi}(\langle a^i_k(r') : k < \omega \rangle) &= \sum_{m<m^*} Av_{\Xi \restriction B_m}(\langle a^i_k(r') : k \in B_m \rangle)\ \Xi(B_m) \\ &= \sum_{m<m^*} c_{i,m}(r') a_m. \end{aligned}$$

There are $r^* \geq r$ and a sequence $\bar{c} = \langle c_{i,m} : i < i^*, m < m^* \rangle$ such that:

$(*)_2$ (a) $c_{i,m} \in [0,1]_{\mathbb{R}}$,

(b) $\sum\limits_{m<m^*} c_{m,i} a_m \geq b_i$,

(c) for every $r' \geq r^*$ there is $r'' \geq r'$ such that

$$(\forall i < i^*)(\forall m < m^*)[c_{i,m} - \varepsilon < c_{i,m}(r'') < c_{i,m} + \varepsilon].$$

[Why? Let $k^* < \omega$ be such that $1/k^* < \varepsilon/(10 \cdot l^* \cdot m^*)$ (so $k^* > 0$). Let $\Gamma = \{\bar{c} : \bar{c} = \langle c_{i,m} : i < i^*,\ m < m^* \rangle,\ c_{i,m} \in [0,1]_{\mathbb{R}}$ and $k^* c_{i,m}$ is an integer and $\sum\limits_{m<m^*} c_{i,m} a_m > b_i\}$. Clearly Γ is finite and let us list it as $\langle \bar{c}^s : s < s^* \rangle$. We try to choose by induction on $s \leq s^*$ a condition $r_s \in$ Random such that $r_0 = r$, $r_s \leq r_{s+1}$, and for no $r'' \geq r_{s+1}$ do we have

$$(\forall i < i^*)(\forall m < m^*)[c^s_{i,m} - \varepsilon < c_{i,m}(r'') < c^s_{i,m} + \varepsilon].$$

For $s = 0$ we have no problem. If we succeed to arrive to r_{s^*}, for $i < i^*$, $m < m^*$ we can define $c^*_{i,m} \in \{l/k^* : l \in \{0, \ldots, k^*\}\}$ such that $c_{i,m}(r_{s^*}) \leq c^*_{i,m} < c_{i,m}(r_{s^*}) + \varepsilon/(10 \cdot l^* \cdot m^*)$. By $(*)_1$ we have $b_i \leq \sum\limits_{m<m^*} c_{i,m}(r^*_{s^*}) a_m$. Clearly

$$\sum_{m<m^*} c^s_{i,m} a_m \geq \sum_{m<m^*} c_{i,m}(r^*_{s^*}) a_m$$

so $\bar{c}^* = \langle c^*_{i,m} : i < i^*,\ m < m^* \rangle \in \Gamma$, hence for some $s < s^*$, $\bar{c}^* = \bar{c}^s$. But then r^* contradicts the choice of r_{s+1}. Also by the above $\Gamma \neq \emptyset$. So we necessarily are stuck at some $s < s^*$, i.e. cannot find r_{s+1} as required. This means that $r_s, \bar{c}^s$ as needed in $(*)_2$, so $r^*, \bar{c}$ as required exist.]

Let $k^* < \omega$ be given. Now choose $s^* < \omega$ large enough and try to choose by induction on $s \leq s^*$, a condition $r_s \in$ Random and natural numbers (m_s, k_s) (flipping coins along the way) such that:

$(*)_3$ (a) $r_0 = r^*$,

(b) $r_{s+1} \geq r_s$,

(c) $c_{i,m} - \varepsilon < c_{i,m}(r_s) < c_{i,m} + \varepsilon$ for $i < i^*$, $m < m^*$,

(d) $k_s > k^*$, $k_{s+1} > k_s$,

(e) $k_s \in B_{m_s}$.

In stage s, given r_s, we define r_{s+1}, i_s, m_s, k_s as follows: choose $m_s < m^*$ randomly with the probability of $m_s = m$ being a_m. Next we can find a finite set $u_s \subseteq B_{m_s}$ with $\min(u_s) > \max\{k^* + 1, k_{s_1} + 1 : s_1 < s\}$ such that

$(*)$ if $i < i^*$ then $c_{i,m_s} - \varepsilon/2 < \frac{1}{|u_s|} \sum\limits_{k \in u_s} a^i_k(r_s) < c_{i,m_s} + \varepsilon/2$

We define an equivalence relation $\mathbf{e}_s$ on $\lim(r_s)$:

$$\eta_1 \mathbf{e}_s \eta_2 \text{ iff } (\forall i < i^*)(\forall k \in u_s)(\forall \ell \in [n^*_k, n^*_{k+1}))[\eta_1 \in \lim(r^i_\ell) \equiv \eta_2 \in \lim(r^i_\ell)].$$

The number of $\mathbf{e}_s$–equivalence classes is finite, and if $Y \in \lim(r_s)/\mathbf{e}_s$ satisfies $\mathrm{Leb}(Y) > 0$ choose $r_{s,Y} \in$ Random such that $\lim(r_{s,Y}) \subseteq Y$ and $r_{s,Y}$ satisfies clause (c) of $(*)_3$ (possible by clause (c) of $(*)_2$). Now choose r_{s+1} among $\{r_{s,Y} : Y \in \lim(r_s)/\mathbf{e}_s \text{ and } \mathrm{Leb}(Y) > 0\}$, with the probability of $r_{s,Y}$ being $\mathrm{Leb}(Y)/\mathrm{Leb}(\lim(r_s))$. Lastly choose $k_s \in u_s$, with all $k \in u_s$ having the same probability.

Now the expected value, assuming $m_s = m$, of

$$\frac{1}{n^*_{k_s+1} - n^*_{k_s}} \times |\{\ell : n^*_{k_s} \leq \ell < n^*_{k_s+1} \text{ and } r_{s+1} \geq r^i_\ell\}|$$

belongs to the interval $(c_{i,m} - \varepsilon/2, c_{i,m} + \varepsilon/2)$, because of the expected value of

$$\frac{1}{|u_s|} \sum_{k \in u_s} \frac{1}{n^*_{k+1} - n^*_k} \times |\{\ell : n^*_k \leq \ell < n^*_{k+1} \text{ and } r_{s+1} \geq r^i_\ell\}|$$

is in this interval (as

$$\sum \{\mathrm{Leb}(Y) : Y \in \lim(r_s)/\mathbf{e}_s,\ r_{s,Y} \geq r^i_l\} = \frac{\mathrm{Leb}(\lim(r_s) \cap \lim(r^i_l))}{\mathrm{Leb}(\lim(r_s))}$$

and see the choice of $a^i_k(-)$).

Let $r' = r_{s^*}$, $u = \{k_s : s \leq s^*\}$. Hence the expected value of

$$\frac{1}{|u|} \sum_{k \in u} \frac{1}{n^*_{k+1} - n^*_k} \times |\{\ell : n^*_k \leq \ell < n^*_{k+1} \text{ and } r' \geq r^i_\ell\}|$$

is $\geq \sum a_m(c_{i,m} - \varepsilon/2) \geq b_i - \varepsilon/2$.

As s^* is large enough with high probability (though just positive probability suffices), $(r_{s^*}, \{k_s : s < s^*\})$ are as required for (r', u); note: we do not know the variance but we have a bound for it not depending on s.

2) Straightforward. □

□

The following is needed later to show that there are enough cases of the Definition of t with clause (g) of Definition 2.11 being non trivial (i.e. $(**)$ there holds).

Lemma 2.18. *Assume*

(a) Ξ *is a finitely additive measure on* ω *and* $b \in (0,1]_{\mathbb{R}}$,
(b) $n^*_k < \omega$ *(for* $k < \omega$*),* $n^*_k < n^*_{k+1}$*, and* $\lim(n^*_{k+1} - n^*_k) = \infty$,
(c) r^*, $r_l \in$ Random *are such that:*
  $(*)$ $(\forall l < \omega)[\mathrm{Leb}(\lim(r^*) \cap \lim(r_l))/\mathrm{Leb}(\lim(r^*)) \geq b]$.

Then *for some* $r^\otimes \geq r^*$ *we have:*

$\otimes_{r^\otimes}$ *for every* $r' \geq r^\otimes$ *we have* $Av_\Xi(\langle a(r',k) : k < \omega\rangle) \geq b$ *where:*
$a_k(r') = a(r',k) = a_k(\lim r')$ *and for* $X \subseteq {}^\omega 2$ *we let*

$$a_k(X) = \frac{1}{n^*_{k+1} - n^*_k} \sum_{l \in [n^*_k, n^*_{k+1})} \frac{\mathrm{Leb}(X \cap (\lim r_l))}{\mathrm{Leb}(X)}.$$

Proof. Let

$$\mathcal{I} = \{r \in \mathrm{Random} : \ r \geq r^*, \text{ and } Av_\Xi(\langle a_k(r') : k < \omega\rangle) < b\}.$$

If $\mathcal{I}$ is not dense above r^* there is $r^\otimes \geq r^*$ (in Random) such that for every $r \geq r^\otimes$, $r \notin \mathcal{I}$ so $r^\otimes$ is as required, so assume toward contradiction that $\mathcal{I}$ is not dense above r^*. There is a maximal antichain $\mathcal{I}_1 = \{s_i : i < i^*\} \subseteq \mathcal{I}$ (maximal among those $\subseteq \mathcal{I}$), now $\mathcal{I}_1$ is a maximal antichain above r^* as $r \in \mathcal{I} \Rightarrow r \geq r^*$ and the previous sentence. Hence $\mathrm{Leb}(\lim r^*) = \sum\limits_{i<i^*} \mathrm{Leb}(\lim s_i)$; of course $|i^*| \leq \aleph_0$ as Random satisfies c.c.c. so w.l.o.g. $i^* \leq \omega$.

For any $j < i^*$ let $s^j = \bigcup\limits_{i<j} s_i$, note then $\lim(\bigcup\limits_{m<i} s_m) = \bigcup\limits_{m<i} \lim(s_m)$ and

$$a_k(s^j) = a_k(\bigcup_{m<i} s_m) = \sum_{i<j} \frac{\mathrm{Leb}(s_i)}{\mathrm{Leb}(\bigcup\limits_{m<j} s_m)} a_k(s_i)$$

hence

$$\begin{aligned} Av_\Xi(\langle a_k(s^j) : k < \omega\rangle) &= Av_\Xi(\langle a_k(\bigcup_{m<j} s_m) : k < \omega\rangle) \\ &= \sum_{i<j} \tfrac{\mathrm{Leb}(s_i)}{\mathrm{Leb}(\bigcup\limits_{m<j} s_m)} \times Av_\Xi(\langle a_k(s_i) : k < \omega\rangle) \\ &\leq \tfrac{\mathrm{Leb}(\lim(s_0))}{\mathrm{Leb}(\lim(\bigcup_{i<j} s_i))}(b - \varepsilon) + \sum_{0<i<j} \tfrac{\mathrm{Leb}(\lim(s_i))}{\mathrm{Leb}(\lim(\bigcup_{m<j} s_m))} b \\ &= b - \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon \end{aligned}$$

where $\varepsilon = b - Av_\Xi(\langle a_k(s_0) : k < \omega\rangle)$ so $\varepsilon > 0$.

Let j be large enough such that $\frac{\mathrm{Leb}(\lim(r^*) \setminus \lim(s^j))}{\mathrm{Leb}(\lim(r^*))} < \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon$. So

$$\begin{aligned} &Av_\Xi(\langle a_k(r^*) : k < \omega\rangle) = \\ &\quad \tfrac{\mathrm{Leb}(\lim(r^*) \setminus \lim(s^j))}{\mathrm{Leb}(\lim(r^*))} Av_\Xi(\langle a_k(\lim(r^*) \setminus \lim(s^j)) : k < \omega\rangle) \\ &\qquad + \tfrac{\mathrm{Leb}(\lim(s^j))}{\mathrm{Leb}(\lim(r^*))} Av(\langle a_k(s^j) : k < \omega\rangle) \\ &\quad \leq \tfrac{\mathrm{Leb}(\lim(r^*) \setminus \lim(s^j))}{\mathrm{Leb}(r^*)} \times 1 + \tfrac{\mathrm{Leb}(\lim(s^j))}{\mathrm{Leb}(\lim(r^*))} \times (b - \mathrm{Leb}(\lim(s_0))\varepsilon) \\ &\quad < \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon + (b - \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon) = b \end{aligned}$$

contradicting assumption (c). □

Claim 2.19. *Assume*

(a) $\bar{Q} \in \mathcal{K}^3$, $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{r}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t \in T} : \alpha \leq \alpha^*, \beta < \alpha^*\rangle$,
(b) $A \subseteq \alpha^*$ *and* $|A| < \kappa$ *and* $\mu < \kappa$ *are such that* $\beta \in A$ & $|A_\beta| < \kappa \ \Rightarrow$ $A_\beta \subseteq A$,
(c) $\eta \in {}^\kappa 2 \setminus \{\eta_\beta : \beta < \alpha^*\}$,

(d) $\underset{\sim}{Q}$ *is a* P_{α^*}*-name of a forcing notion with set of elements* μ*, and is really definable in* $\mathbf{V}[\langle \underset{\sim}{\mathcal{I}}_\alpha : \alpha \in A\rangle]$ *from* $\langle \underset{\sim}{\mathcal{I}}_\alpha : \alpha \in A\rangle$ *and parameters from* $\mathbf{V}$.

Then *there is*

$$\bar{Q}^+ = \langle P\alpha, \underset{\sim}{Q}_\alpha, A_\beta, \mu_\beta, \underset{\sim}{\mathcal{I}}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t\in\mathcal{T}} : \alpha \le \alpha^*+1, \beta < \alpha^*+1\rangle$$

from $\mathcal{K}^3$ *extending* $\bar{Q}$ *such that* $\underset{\sim}{Q}_{\alpha^*} = \underset{\sim}{Q}$, $A_{\alpha^*} = A$, $\eta_{\alpha^*} = \eta$ *and* $\mu_{\alpha^*} = \mu$.

Proof. Straight. □

Remark 2.20. If Q is the Cohen forcing we can make one step toward $\{A \subseteq \omega : \Xi^t_{\alpha^*+1}(A) = 1\}$ being a selective filter but not needed at present.

3. Continuation of the proof of Theorem 0.1

We need the following lemma.

Lemma 3.1. *Suppose that* $\bar{\varepsilon} = \langle \varepsilon_l : l < \omega\rangle$ *is a sequence of positive reals and* $\bar{Q} \in \mathcal{K}^3$ *has length* α. *The following set* $\mathcal{I}_{\bar{\varepsilon}} \subseteq P_\alpha$ *is dense:*

$\mathcal{I}_{\bar{\varepsilon}} = \{p \in P'_\alpha :$ *there are* m *and* α_l, ν_l *(for* $l < m$*) such that*
(a) $\mathrm{dom}(p) = \{\alpha_0, \dots, \alpha_{m-1}\}$, $\alpha_0 > \alpha_1 > \ldots > \alpha_{m-1}$,
(b) *if* $|Q_{\alpha_l}| < \kappa$, *then* $p(\alpha_l)$ *is an ordinal* ,
(c) *if* Q_{α_l} *is a partial random, then* $\Vdash_{P_{\alpha_l}}$ "$p(\alpha_l) \subseteq ({}^\omega 2)^{[\nu_l]}$ *and* $\mathrm{Leb}(\lim(p(\alpha_l))) \ge (1-\varepsilon_l)/2^{\ell g(\nu_l)}$"$\}$

Proof. By induction on α for all possible $\bar{\varepsilon}$. □

Discussion 3.2. 1) By the previous sections it follows that it is enough to prove that if $\bar{Q} \in \mathcal{K}^3$, $P_\alpha = \mathrm{Lim}(\bar{Q})$, *then* in $\mathbf{V}^{P_\alpha}$ the following sufficient condition holds:

$(**)_{\bar{Q}}$ In $\mathbf{V}^{P_\alpha}$: there is no perfect tree $T \subseteq {}^{\omega>}2$, $m \in \omega$ and $E \in [\lambda]^{\kappa^+}$ such that $T \subseteq \mathrm{tree}_m[\bar{a}^\alpha]$ for all $\alpha \in E$.

2) Note that if we just want to prove $\Vdash_{P_\alpha}$ "$\mathfrak{b} \le \kappa$" life is easier: Ξ^t_α is a zero-one measure (so essentially an ultrafilter) and we interpret for $\alpha < \lambda$, the forcing notion Q_α as $({}^{\omega>}\omega, \lhd)$ with generic real $\underset{\sim}{\eta}_\alpha$ and replace below $(**)_{\bar{Q}}$ by

$(**)^+_{\bar{Q}}$ in V^{P_α} there is no $\eta^* \in {}^\omega\omega$ such that $\{\alpha < \lambda : (\forall \ell < \omega)(\underset{\sim}{\eta}_\alpha(\ell) \le \eta^*(\ell)\}$ has cardinality $\ge \kappa^+$.

In the proof below, $\underset{\sim}{T}$ is replaced by $\underset{\sim}{\eta}$, and $p'_\zeta(\alpha_\zeta)$ is $s^\frown\langle\zeta\rangle$.

3) We can make the requirements on the Δ-system stronger: make it indiscernible also over some $A \subseteq \alpha$ of cardinality $< \kappa$, where $\underset{\sim}{T}$ is a P_A-name, $p^* \in P_A$, and w.l.o.g. the heart is $\subseteq A$.

4) Here the existence of h^t_2 help; we can use 3.1 with $\sum\limits_{\ell<\omega} \varepsilon_\ell$ very small.

Lemma 3.3. *If* $P_\alpha = \mathrm{Lim}(\bar{Q})$, $\alpha = \ell g(\bar{Q})$ *and* $\bar{Q} \in \mathcal{K}^3$, then $(**)_{\bar{Q}}$ *from 2.7.*

Proof. Suppose that $p^* \Vdash_{P_\alpha}$ "$\underset{\sim}{T}, m, \underset{\sim}{E}$ form a counterexample to $(**)_{\bar{Q}}$", wlog $p^* \in P'_\alpha$. Let $\bar{\varepsilon} = \langle \varepsilon_l : l < \omega \rangle$ be such that $\varepsilon_l \in (0,1)_{\mathbb{R}}$ and $\sum\limits_{l<\omega} \sqrt{\varepsilon_l} < 1/10$. For each $\zeta < \kappa^+$ let $p_\zeta \geq p^*$ be such that $p_\zeta \in \mathcal{I}_{\bar{\varepsilon}}$ $(\subseteq P'_\alpha)$ witnessed by $\langle \nu^\zeta_\beta : \beta \in \mathrm{dom}(p_\zeta)$ and $|Q_\beta| \geq \kappa \rangle$ (on $\mathcal{I}_{\bar{\varepsilon}}$ see 3.1) and

$$p_\zeta \Vdash_{P_\alpha} \text{"}\alpha_\zeta \text{ is the } \zeta\text{-th element of } \underset{\sim}{E}\text{"}.$$

So clearly $\alpha_\zeta < \lambda$. W.l.o.g., by thinning out, we can assume that:

- $\mathrm{dom}(p_\zeta) = \{\gamma^\zeta_i : i < i^*\}$ with γ^ζ_i increasing with i, let $v^\zeta_0 = \{i < i^* : |Q_{\gamma^\zeta_i}| < \kappa\}$, then $v^\zeta_0 = v_0$ is fixed for all $\zeta < \kappa^+$, and let $v_1 = i^* \setminus v_0$,
- $\mathrm{dom}(p_\zeta)$ $(\zeta < \kappa^+)$ form a Δ–system, with the heart Δ, so $\Delta \supseteq \mathrm{dom}(p^*)$,
- $\alpha_\zeta \in \mathrm{dom}(p_\zeta)$, $\alpha_\zeta = \gamma^\zeta_z$ for a fixed $z < i^*$,
- $(\mathrm{dom}(p_\zeta), \Delta, <)$ are isomorphic for $\zeta < \kappa^+$,
- if $i \in v_0$, then $p_\zeta(\gamma^\zeta_i) = \gamma_i$, for $\zeta < \kappa^+$,
- if $i \in v_1$, then $\nu^\zeta_{\gamma^\zeta_i} = \nu_i$, (recall $\nu^\zeta_{\gamma^\zeta_i} \in {}^{\omega>}2$ is given by the definition of $\mathcal{I}_{\bar{\varepsilon}}$),
- $p_\zeta(\alpha_\zeta) = s^*$ for $\zeta < \kappa^+$ with $s^* = \langle (n_l, a_l) : l < m^* \rangle$, w.l.o.g. $m^* > m$ (where m is from "the counterexample to $(**)_{\bar{Q}}$") and $m^* > 10$,
- for each $i < i^*$ the sequence $\langle \gamma^\zeta_i : \zeta < \kappa^+ \rangle$ is constant or strictly increasing,
- the sequence $\langle \alpha_\zeta : \zeta < \kappa^+ \rangle$ is with no repetitions (as if p_{ζ_1}, p_{ζ_2} are compatible and $\zeta_1 < \zeta_2 < \lambda$ then $\alpha_{\zeta_1} \neq \alpha_{\zeta_2}$).

Now we are interested only in the first ω conditions, i.e., we consider $\zeta < \omega$. For every such ζ let $p'_\zeta \geq p_\zeta$ be such that $\mathrm{dom}(p'_\zeta) = \mathrm{dom}(p_\zeta)$, $p'_\zeta(\gamma) = p_\zeta(\gamma)$ except for $\gamma = \alpha_\zeta$ in which case we extend $p_\zeta(\alpha_\zeta) = s^*$ in the following way. We put $\ell g(p'_\zeta(\alpha_\zeta)) = \ell g(s^*) + 1 = m^* + 1$, $p'_\zeta(\alpha_\zeta) = s^* \frown \langle (j^0_\zeta, a_\zeta) \rangle$. Before we define j^0_ζ, a_ζ choose an increasing sequence of integers $\bar{s} = \langle s_l : l < \omega \rangle$, $s_0 = 0$, such that $s_{k+1} - s_k = |[^{j_k}2]^{2^{j_k}(1-8^{-m^*})}|$ (i.e. it is the number of subsets of $^{j_k}2$ with $2^{j_k}(1-8^{-m^*})$ elements), where $j^* = 3n_{m^*-1} + 1$ (i.e. we define j^* from the first coordinate in the last pair in s^*) and we let $j_k = j^* + k!!$, and let $j^0_\zeta = j_k$ when $\zeta \in [s_k, s_{k+1})$. Now for $\zeta \in [s_k, s_{k+1})$ define a_ζ such that

$$\{a_\zeta : \zeta \in [s_k, s_{k+1})\} = [^{j_k}2]^{2^{j_k}(1-8^{-m^*})}$$

(so necessarily without repetitions). For $\varepsilon^* > 0$ we define a P_α-name by

$$\underset{\sim}{A}_{\varepsilon^*} = \{k < \omega : |\{\zeta \in [s_k, s_{k+1}) : p'_\zeta \in \underset{\sim}{G}_{P_\alpha}\}|/(s_{k+1} - s_k) > \varepsilon^*\}.$$

For the proof of 3.3 we need:

Subclaim 3.4. *There is a condition $p^\otimes \geq p^*$ which forces that for some $\varepsilon^* > 0$ the set $\underset{\sim}{A}_{\varepsilon^*}$ is infinite .*

Proof. Choose $\varepsilon^* > 0$ small enough. First we define a suitable blueprint $t \in \mathcal{T}$,

$$t = (w^t, \mathbf{n}^t, \mathbf{m}^t, \bar{\eta}^t, h^t_0, h^t_1, h^t_2, \bar{n}^t).$$

Let

$$w^t = \{\min\{\beta < \kappa : \eta_{\gamma^{\zeta(1)}_{i(1)}}(\beta) \neq \eta_{\gamma^{\zeta(2)}_{i(2)}}(\beta)\} : \zeta(1), \zeta(2) < \omega \text{ and } i(1), i(2) < i^* \text{ and } \gamma^{\zeta(1)}_{i(1)} \neq \gamma^{\zeta(2)}_{i(2)}\ \}.$$

Let $\mathbf{n}^t = i^*$, $\mathrm{dom}(h^t_0) = v_0$, $\mathrm{dom}(h^t_1) = \mathrm{dom}(h^t_2) = v_1$ and $n^t_l = s_l$. If $\mathbf{n} \in v_0$, then $h^t_0(\mathbf{n})(l) = \gamma_{\mathbf{n}}$ and $\eta^t_{\mathbf{n},\zeta} = \eta_{\gamma^\zeta_{\mathbf{n}}} \restriction w^t$. If $\mathbf{n} \in v_1$, then $h^t_1(\mathbf{n}) = \varepsilon_n$, $h^t_2(\mathbf{n}) = \nu_{\mathbf{n}}$.

We now define a condition $p^\otimes$, it will be in P_α, $\mathrm{dom}(p^\otimes) = \Delta$, $p^* \leq p^\otimes$; remember $\mathrm{dom}(p^*) \subseteq \Delta$ as for each ζ we have $p^* \leq p_\zeta$. If $\gamma \in \Delta$ then for some $\mathbf{n} < \mathbf{n}^t$, we have $\bigwedge\limits_{\zeta<\omega} \gamma^\zeta_{\mathbf{n}} = \gamma$. If $\mathbf{n} \in v_0$ we let $p^\otimes(\gamma) = h^t_0(\mathbf{n})$, so trivially in $\mathbf{V}^{P_\gamma}$

$$p^\otimes(\gamma) \Vdash_{Q_\gamma} \text{“}\underset{\sim}{\Xi}^t_{\gamma+1}(\{\zeta < \omega : h^t_0(\mathbf{n}) \in G_{Q_\gamma}\}) = 1 \text{ if } \mathbf{n} \in \mathrm{dom}(h^t_0)(= v_0)\text{ ”}$$

If $\mathbf{n} \in v_1$, then define a P_γ-name for a member of Q_γ as follows. Consider $\underset{\sim}{r}^{\mathbf{n}}_\zeta = \underset{\sim}{p}'_\zeta(\gamma)$ for $\zeta < \omega$. Let $\underset{\sim}{r}$ be the member $({}^\omega 2)^{[h^t_2(n)]}$ of Q_γ. Working in $\mathbf{V}^{P'_{A_\alpha}}$, by Lemma 2.18 there is $\underset{\sim}{r}^*_\gamma \geq \underset{\sim}{r}$ from $\underset{\sim}{Q}_\gamma$ such that for every $r' \geq r^*_\gamma$ in Q_γ we have

$(**)_{r',\varepsilon}$ $Av_{\Xi^t_\alpha}(\langle a^{\mathbf{n}}_k(r') : k < \omega\rangle) \geq (1 - h^t_1(\mathbf{n})) = (1 - \varepsilon_{\mathbf{n}})$ where

$$a^{\mathbf{n}}_k(r') =: \frac{1}{n^t_{k+1} - n^t_k} \sum_{l \in [n^t_k, n^t_{k+1})} \frac{\mathrm{Leb}(\lim(r') \cap \lim(r^{\mathbf{n}}_l))}{\mathrm{Leb}(\lim(r'))}.$$

Hence the assumption of condition (g) in Definition 2.11 holds, hence in $\mathbf{V}^{P_\gamma}$ we have:

$$r^*_\gamma \Vdash_{Q_\gamma} \text{“ } Av_{\underset{\sim}{\Xi}^t_{\gamma+1}}(\langle |\{\ell \in [n^t_{k+1} - n^t_k) : p_\ell(\gamma) \in \underset{\sim}{G}_{Q_\gamma}\}|/(n^t_{k+1} - n^t_k) : k \in \omega\rangle) \geq 1 - \varepsilon_n \text{ ”}.$$

So there is a P_γ-name $\underset{\sim}{r}^*_\gamma$ of such a condition. In this case let $p^\otimes(\gamma) = \underset{\sim}{r}^*_\gamma$, so we have finished defining $p^\otimes$, clearly it has the right domain.

Now suppose that $\mathbf{n} < \mathbf{n}^t$, $\mathbf{n} \in v_1$ is such that $\gamma^\zeta_{\mathbf{n}} \notin \Delta$. Define $\bar{\beta} = \langle \beta_\zeta : \zeta < \omega\rangle$, $\beta_\zeta = \gamma^\zeta_{\mathbf{n}}$. Then $\bar{\beta}$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$. By our assumption the assumption of clause (f) in Definition 2.11 is satisfied, hence in $\mathbf{V}^{P_\alpha}$, for any $\varepsilon > 0$:

$$\Vdash_{P_\alpha} \text{“ } \underset{\sim}{\Xi}^t_\alpha\Big(\{k\colon \frac{|\{l \in [n^t_k, n^t_{k+1})\colon p_l(\gamma^l_n) \in \underset{\sim}{G}_{Q_{\gamma^l_n}}\}|}{(n^t_{k+1} - n^t_k)} \geq (1-\varepsilon_n)\cdot(1-\varepsilon)\}\Big) = 1 \text{ ”}.$$

Now for each $\mathbf{n} \in v_1$, as

$$(1 - \varepsilon_{\mathbf{n}}) \cdot (1 - \varepsilon) \leq Av_{\Xi^t_\alpha}(\langle \frac{|\{\ell : n^t_k \leq \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1} - n^t_k} : k < \omega\rangle)$$

and $\varepsilon > 0$ was arbitrary, clearly

$$(*)_{\mathbf{n}} \text{ in } \mathbf{V}^{P_\alpha},\ \sqrt{2\varepsilon_n} \geq \Xi^t_\alpha\Big(\{k<\omega : 1-\sqrt{2\varepsilon_n} \geq \tfrac{|\{\ell: n^t_k \leq \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1}-n^t_k}\}\Big).$$

Let

$$\underset{\sim}{A}'_{\varepsilon^*} = \{k<\omega :\ \text{if } \zeta \in [s_k, s_{k+1}) \text{ and } i \in v_o \text{ then } p_\zeta \restriction \{\gamma^\zeta_i\} \in \underset{\sim}{G}_{P_\alpha}\},$$

clearly $\Xi^t_\alpha(\underset{\sim}{A}'_{\varepsilon^*}) = 1$.

Let $\varepsilon^* < 1 - \sum\limits_{\mathbf{n}} \sqrt{2\varepsilon_{\mathbf{n}}}$ and $\varepsilon^* > 0$, so

$$\underset{\sim}{A}_{\varepsilon^*} \cup (\omega \setminus A'_{\varepsilon^*}) = \{k<\omega : \varepsilon^* < \tfrac{|\bigcap\limits_{\mathbf{n}\in v_1} \{l \in [n^t_k, n^t_{k+1}) \text{ and } p'_\zeta(\gamma^\zeta_{\mathbf{n}}) \in G_{Q_{\gamma_{\mathbf{n}}}}\}|}{(n^t_{k+1}-n^t_k)}\}$$

$$\supseteq \{k<\omega :\ \text{if } \mathbf{n} \in v_1 \text{ then } \tfrac{|\{\ell: n^t_k \leq \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1}-n^t_k} \geq 1 - \sqrt{2\varepsilon_{\mathbf{n}}}\}$$

$$= \omega \setminus \bigcup_{\mathbf{n}\in v_1} \{k<\omega : \tfrac{|\{\ell: n^t_k \leq \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1}-n^t_k} < 1 - \sqrt{2\varepsilon_{\mathbf{n}}}\},$$

hence $\Xi^t_\alpha(\underset{\sim}{A}_{\varepsilon^*} \cup (\omega \setminus \underset{\sim}{A}'_{\varepsilon^*})) \geq 1 - \sum\limits_{\mathbf{n}\in v} \sqrt{2\varepsilon_{\mathbf{n}}} \geq \varepsilon^* > 0$, but

$$\Xi^t_\alpha(\omega \setminus \underset{\sim}{A}'_{\varepsilon^*}) = 1 - \Xi^t_\alpha(\underset{\sim}{A}'_{\varepsilon^*}) = 1 - 1 = 0$$

hence necessarily A_{ε^*} is infinite.

This suffice for 3.4. □

Let $p^\otimes$ be as in the claim 3.4, let G_{P_α} be a generic subset of P_α to which $p^\otimes$ belongs and we shall work in $\mathbf{V}[G_{P_\alpha}]$. So $A = \underset{\sim}{A}_{\varepsilon^*}[G]$ is infinite. For $k \in A$, let $b_k = \{\zeta \in [s_k, s_{k+1}) : p'_\zeta \in G_{P_\alpha}\}$. We know that $|b_k| > (s_{k+1} - s_k) \times \varepsilon^*$. Note that if $k \in A$, then $T \cap {}^{j_k}2 \subseteq \bigcap_{\zeta\in b_k} a_\zeta$ as $\ell g(s^*) = m^* > m$. To reach a contradiction it is enough to show that for infinitely many $k \in A$ there is a bound on the size of $c_k = T \cap {}^{j_k}2$ which does not depend on k.

Now $|b_k|/(s_{k+1} - s_k)$ is at most the probability that if we choose a subset of ${}^{(j_k)}2$ with $2^{j_k}(1 - 8^{-m^*})$ elements, it will include $T \cap {}^{(j_k)}2$; now if $k \in A$ this probability has a lower bound ε^* not depending on k, and this implies a bound on $|T \cap {}^{(j_k)}2|$ not depending on k. More formally, for a fixed $k < \omega$ we have:

$$\begin{array}{rcl} |b_k| & = & |\{a_\zeta : \zeta \in [s_k, s_{k+1}),\ \zeta \in b_k\}| \\ & \leq & |\{a_\zeta : \zeta \in [s_k, s_{k+1}),\ (T \cap {}^{j_k}2) \subseteq a_\zeta\}| \\ & \leq & |\{a \subseteq ({}^{j_k}2) : (T \cap {}^{j_k}2) \subseteq a \text{ and } |a| = 2^{j_k}(1 - 8^{-m^*})\}| \\ & = & |\{a \subseteq ({}^{j_k}2) \setminus (T \cap {}^{j_k}2) : |a| = 2^{j_k} \times 8^{-m^*}\}| \end{array}$$

$$= \begin{pmatrix} 2^{j_k} - |T \cap {}^{(j_k)}2| \\ 2^{j_k} \cdot 8^{-m^*} \end{pmatrix}$$

similarly $s_{k+1}-s_k = \begin{pmatrix} 2^{j_k} \\ 2^{j_k}\cdot 8^{-m^*} \end{pmatrix}$. Hence

$$\begin{aligned} |b_k|/(s_{k+1}-s_k) &\leq \begin{pmatrix} 2^{j_k}-|T\cap {}^{j_k}2| \\ 2^{j_k}\cdot 8^{-m^*} \end{pmatrix} \Big/ \begin{pmatrix} 2^{j_k} \\ 2^{j_k}\cdot 8^{-m^*} \end{pmatrix} \\ &= \prod_{i<|T\cap {}^{(j_k)}2|} (2^{j_k}-2^{j_k}8^{-m^*}-i)/\prod_{i<|T\cap {}^{(j_k)}2|}(2^{j_k}-i) \\ &= \prod_{i<|T\cap {}^{(j_k)}2|} \left(1-\tfrac{2^{j_k}8^{-m^*}}{2^{j_k}-i}\right). \end{aligned}$$

Let $i_k(*) = \min\{|T\cap {}^{(j_k)}2|, 2^{j_k-1}\}$ so

$$\begin{aligned} \varepsilon^* \leq \tfrac{|b_k|}{s_{k+1}-s_k} &\leq \prod_{i<|T\cap {}^{j_k}2|} (1-\tfrac{2^{j_k}8^{-m^*}}{2^{j_k}-i}) \\ &\leq \prod_{i<i_k(*)} (1-\tfrac{2^{j_k}8^{-m^*}}{2^{j_k}}) = (1-8^{-m^*})^{i_k(*)}. \end{aligned}$$

So we can find a bound to $i_k(*)$ not depending on k:

$$i_k(*) \leq (\log(1/\varepsilon^*)/\log(1/(1-8^{-m^*}),$$

remember $m^*>10$ so $1-8^{-m^*}\in(0,1)_{\mathbb{R}}$. So for k large enough,

$$|T\cap {}^{(j_k)}2| = i_k(*) \leq \log(1/\varepsilon^*)/\log(1/(1-8^{-m^*})).$$

This finishes the proof. □

Theorem 3.5. *Under Hypothesis 2.1 there is $\bar{Q}\in\mathcal{K}^3$, $\ell g(\bar{Q})=\chi=\delta^*$ (if clause (α) of 2.1(b) holds) or $\ell g(\bar{Q})=\chi\times\chi\times\lambda^+$ (if clause (β) of 2.1(b) holds) such that in $\mathbf{V}^{P_{\lim\bar{Q}}}$ we have $MA_{<\kappa}+\mathrm{cov}(\mathrm{null})=\lambda$.*

Proof. First assume clause (α) of 2.1. By 2.3(2) and 2.3(6) it suffices to find an iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\Xi^t_\alpha)_{t\in\mathcal{T}} : \alpha\leq\chi, \beta<\chi\rangle \in \mathcal{K}^3$$

(see definition 2.11) satisfying clauses (a)+(b)+(c) of 2.3(2)+(6) (as the only property missing, $\mathrm{cov}(\mathrm{null})\leq\lambda$, holds by 2.7 + 3.3.

Let $\mathcal{K}^-_3=\{\bar{Q}\in\mathcal{K}_3 : \ell g(\bar{Q})<\chi\}$.

Now choose $\bar{Q}^\xi\in\mathcal{K}^-_3$ for $\xi<\chi$ increasing with ξ (see definition 2.12) by induction on χ. Now if $\mathrm{cf}(\xi)>\aleph_0$ use 2.13(3), if $\mathrm{cf}(\xi)=\aleph_0$ use 2.14. Bookkeeping give us sometimes a case $\underset{\sim}{Q}$ of 2.3(6)(c) as assignment, we can find suitable $A\subseteq\ell g(\bar{Q}^\xi)$ by 2.3(4) and then apply 2.19 to get $\bar{Q}^{\xi+1}$. For other ξ, bookkeeping gives us a case of 2.3(2)(b) as assignment $A\subseteq\ell g(\bar{Q}^\xi)$, such that $|A|<\lambda$. Now we apply 2.16(2) (with $\bar{Q}$, A there standing for $\bar{Q}'$, A here) and get A' as there. Now apply 2.16(1) with $\bar{Q}'$, A' here standing for $\bar{Q}$, A' here standing for $\bar{Q}$, A there (and η any member of ${}^\kappa 2\setminus\{\eta_\beta:\beta<\ell g(\bar{Q}')\}$ possible as $\ell g(\bar{Q}')<\chi$ as $\bar{Q}'\in\mathcal{K}^3$) and get $\bar{Q}^{\xi+1}$ (corresponding to $\bar{Q}^+$ there).

Second assume clause (β) of 2.1(b). We just should be more careful in our bookkeeping, particularly in the beginning let $\langle\eta_\alpha:\alpha<\chi\times\chi\times\lambda^+\rangle$ be an enumeration of ${}^\kappa 2$ with no repetition. □

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