

## THE CONDITION IN THE TRICHOTOMY THEOREM IS OPTIMAL

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ABSTRACT. We show that the assumption  $\lambda > \kappa^+$  in the Trichotomy Theorem cannot be relaxed to  $\lambda > \kappa$ .

### 1. INTRODUCTION

The Trichotomy Theorem specifies three alternatives for the structure of an increasing sequence of ordinal functions modulo an ideal on an infinite cardinal  $\kappa$  — provided the sequence has regular length  $\lambda$  and  $\lambda$  is at least  $\kappa^{++}$ .

The natural context of the Trichotomy Theorem is, of course, pcf theory, where a sequence of ordinal functions on  $\kappa$  usually has length which is larger than  $\kappa^{+\kappa}$ . However, the trichotomy theorem has already been applied in several proofs to sequences of length  $\kappa^{+n}$ , ( $n \geq 2$ ) (see [4], [1] and [3]).

Therefore, a natural question to ask is, whether the Trichotomy Theorem is valid also for sequences of length  $\kappa^+$ , namely, whether the lower bound on the length of the sequence can be lowered by one cardinal.

Below we show that the assumption  $\lambda \geq \kappa^{++}$  in the Trichotomy Theorem is tight. For every infinite  $\kappa$ , we construct an ideal  $I$  over  $\kappa$  and  $<_I$ -increasing sequence  $\bar{f} \subseteq \text{On}^\kappa$  so that all three alternatives in the Trichotomy theorem are violated by  $\bar{f}$ .

### 2. THE COUNTER-EXAMPLE

Let  $\kappa$  be an infinite cardinal. Denote by  $\text{On}^\kappa$  the class of all functions from  $\kappa$  to the ordinal numbers.

Let  $I$  be an ideal over  $\kappa$ . We write  $f <_I g$ , for  $f, g \in \text{On}^\kappa$ , if  $\{i < \kappa : f(i) \geq g(i)\} \in I$  and we write  $f \leq_I g$  if  $\{i < \kappa : f(i) > g(i)\} \in I$ . A sequence  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \text{On}^\kappa$  is  $<_I$ -increasing if  $\alpha < \beta < \lambda$  implies that  $f_\alpha <_I f_\beta$  and is  $<_I$ -decreasing if  $\alpha < \beta < \lambda$  implies that  $f_\beta <_I f_\alpha$ .

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A function  $f \in \text{On}^\kappa$  is a least upper bound mod  $I$  of a sequence  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \text{On}^\kappa$  if  $f_\alpha <_I f$  for all  $\alpha < \lambda$  and whenever  $f_\alpha <_I g$  for all  $\alpha$  then  $f \leq_I g$ . A function  $f \in \text{On}^\kappa$  is an *exact upper bound* of  $\bar{f}$  if  $f_\alpha \leq f$  for all  $\alpha < \lambda$ , and whenever  $g <_I f$ , there exists  $\alpha < \lambda$  such that  $g <_I f_\alpha$ . For subsets  $t, s$  of  $\kappa$ , write  $t \subseteq_I s$  if  $s - t \in I$ .

The dual filter  $I^*$  of an ideal  $I$  over  $\kappa$  is the set of all complements of members of  $I$ . The relations  $\leq_I, <_I$  and  $\subseteq_I$  will also be written as  $\leq_{I^*}, <_{I^*}$  and  $\subseteq_{I^*}$ .

Let us quote the theorem under discussion:

**Theorem 1.** (*The Trichotomy Theorem*)

Suppose  $\lambda \geq \kappa^{++}$  is regular,  $I$  is an ideal over  $\kappa$  and  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  is a  $<_I$ -increasing sequence of ordinal functions on  $\kappa$ . Then  $\bar{f}$  satisfies one of the following conditions:

- (1)  $\bar{f}$  has an exact upper bound  $f$  with  $\text{cf}(f(i)) > \kappa$  for all  $i < \kappa$ ;
- (2) there are sets  $S(i)$  for  $i < \kappa$  satisfying  $|S(i)| \leq \kappa$  and an ultrafilter  $U$  over  $\kappa$  extending the dual of  $I$  so that for all  $\alpha < \lambda$  there exists  $h_\alpha \in \prod_{i < \kappa} S(i)$  and  $\beta < \lambda$  such that  $f_\alpha <_U h_\alpha <_U f_\beta$ .
- (3) there is a function  $g : \kappa \rightarrow \text{On}$  such that the sequence  $\bar{t} = \langle t_\alpha : \alpha < \lambda \rangle$  does not stabilize modulo  $I$ , where  $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$ .

Proofs of the Trichotomy Theorem are found in [5], II,1.2, in [3] or in the future version of [2].

**Theorem 2.** For every infinite  $\kappa$  there exists an ultrafilter  $U$  over  $\kappa$  and a  $<_U$ -increasing sequence  $\bar{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle \subseteq \text{On}^\kappa$  such conditions 1, 2 and 3 in the Trichotomy Theorem fail for  $\bar{f}$ .

*Proof.* Let  $\lambda = \kappa^+$ .

Let us establish some notation.

We recall that every ordinal has an expansion in base  $\lambda$ , namely can be written as a unique finite sum  $\sum_{k \leq l} \lambda^{\beta_k} \alpha_k$  so that  $\beta_{k+1} < \beta_k$  and  $\alpha_k < \lambda$ . We limit ourselves from now on to ordinal  $\zeta < \lambda^\omega$ . For such ordinals, the expansion in base  $\kappa$  contains only finite powers of  $\lambda$  (that is, every  $\beta_k$  is a natural number).

We agree to write an ordinal  $\zeta = \lambda^l \alpha_l + \lambda^{l-1} \alpha_{l-1} + \dots + \alpha_0$  simply as a finite sequence  $\alpha_l \alpha_{l-1} \dots \alpha_0$ . We identify an expansion with  $\lambda$  digits with one with  $n > l$  digits by adding zeroes on the left. If  $\zeta = \alpha_l \alpha_{l-1} \dots \alpha_0$ , we call  $\alpha_k$ , for  $k \leq l$ , the  $k$ -th digit in the expansion of  $\zeta$ .

For  $\alpha < \lambda$  and an integer  $l$ , define:

$$A_\alpha^l = \{\alpha_l \alpha_{l-1} \dots \alpha_0 : \alpha_k < \alpha \text{ for all } k \leq l\} \quad (1)$$

$A_\alpha^l$  is the set of all ordinals below  $\lambda^\omega$  whose expansion in base  $\lambda$  contains  $l + 1$  or fewer digits from  $\alpha$ .

**Fact 3.** For all  $\alpha < \lambda$  and  $l < \omega$ ,

- (1)  $\bigcup_{\alpha < \lambda} A_\alpha^l = \lambda^{l+1}$
- (2) The ordinal  $\sum_{k=0}^l \lambda^k = \overbrace{\alpha \alpha \dots \alpha}^{l+1}$  is the maximal element in  $A_{\alpha+1}^l$
- (3) if  $\zeta = \alpha_l \alpha_{l-1} \dots \alpha_0 \in A_\zeta^l$  is not maximal in  $A_{\alpha+1}^l$ , then the immediate successor of  $\zeta$  in  $A_{\alpha+1}^n$  is obtained from  $\zeta$  as follows: let  $k$  be the first  $k \leq l$  for which  $\alpha_k < \alpha$ . Replace  $\alpha_k$  by  $\alpha_k + 1$  and replace  $\alpha_m$  by 0 for all  $m < k$

Fix a partition  $\{X_n : n < \omega\}$  of  $\kappa$  with  $|X_n| = \kappa$  for all  $n$ . Let  $n(i)$ , for  $i < \kappa$ , be the unique  $n$  for which  $i \in X_n$ .

By induction on  $\alpha < \kappa^+$ , define  $f_\alpha : \kappa \rightarrow \text{On}$  so that:

- (1)  $f_\alpha(i) \in A_{\alpha+1}^{n(i)} - A_\alpha^{n(i)}$
- (2) For all  $n, l < \omega$  and finite, strictly increasing, sequences  $\langle \alpha_k : k \leq l \rangle \subseteq \lambda$  it holds that for every sequence  $\langle \zeta_k : k \leq l \rangle$  which satisfies  $\zeta_k \in A_{\alpha_k+1}^n - A_{\alpha_k}^n$ , there are  $\kappa$  many  $i \in X_n$  for which  $\bigwedge_{k \leq l} f_{\alpha_k}(i) = \zeta_k$ .

The first item above says that  $f_\alpha(i)$  is an ordinal below  $\lambda^\omega$  whose expansion in base  $\lambda$  has  $\leq n(i)$  digits, at least one of which is  $\alpha$ . The second item says that every possible finite sequence of values  $\langle \zeta_k : k < l \rangle$  is realized  $\kappa$  many times as  $\langle f_{\alpha_k}(i) : k < l \rangle$  for an arbitrary increasing sequence  $\langle \alpha_k : k < l \rangle$ .

The induction required to define the sequence is straightforward.

Define now, for every  $\alpha < \kappa^+$ , a function  $g_\alpha : \kappa \rightarrow \text{On}$  as follows:

$$g_\alpha(i) = \min[(A_{\alpha+1}^{n(i)} \cup \{\lambda^{n(i)+1}\}) - f_\alpha(i)] \quad (2)$$

Since  $f_\alpha(i) < \lambda^{n(i)+1}$  for  $i \in X_n$ , the definition is good. If  $f_\alpha(i)$  is not maximal in  $A_{\alpha+1}^{n(i)}$ , then  $g_\alpha(i)$  is the *immediate successor* of  $f_\alpha(i)$  in  $A_{\alpha+1}^{n(i)}$ . Let us make a note of that:

**Fact 4.** There are no members of  $A_{\alpha+1}^{n(i)}$  between  $f_\alpha(i)$  and  $g_\alpha(i)$

We have defined two sequences:

$$\begin{aligned}\bar{f} &= \langle f_\alpha : \alpha < \lambda \rangle \\ \bar{g} &= \langle g_\alpha : \alpha < \lambda \rangle\end{aligned}$$

Next we wish to find an ideal modulo which  $\bar{f}$  is  $<_I$ -increasing and  $\bar{g}$  is a  $<_I$ -decreasing sequence of upper bounds of  $\bar{f}$ .

**Claim 5.** *For every finite increasing sequence  $\alpha_0 < \alpha_1 < \dots < \alpha_l < \lambda$  there exists  $i < \kappa$  such that for all  $k < l$*

$$f_{\alpha_k}(i) < f_{\alpha_{k+1}}(i) < g_{\alpha_{k+1}}(i) < g_{\alpha_k}(i) \tag{3}$$

*Proof.* Suppose  $\alpha_0 < \alpha_1 < \dots < \alpha_l < \lambda$  is given and choose  $n > l$ . Let  $\zeta_0 = \overbrace{\alpha_0 \alpha_0 \dots \alpha_0}^{l+1} \in A_{\alpha_0}^n$ . Let  $\zeta_{k+1}$  be obtained from  $\zeta_k$  by replacing the first  $l + 1 - k$  digits of  $\zeta_k$  by  $\alpha_{k+1}$ :

$$\begin{aligned}\alpha_0 \alpha_1 \dots \alpha_k \alpha_{k+1} \dots \alpha_l &= \zeta_l \\ &\vdots \\ \alpha_0 \alpha_1 \dots \alpha_{k-1} \alpha_k \dots \alpha_k &= \zeta_k \\ &\vdots \\ \alpha_0 \alpha_1 \dots \alpha_1 \dots \alpha_1 &= \zeta_1 \\ \alpha_0 \alpha_0 \dots \alpha_0 \dots \alpha_0 &= \zeta_0\end{aligned}$$

Thus  $\zeta_0 < \zeta_1 < \dots < \zeta_l$  and  $\zeta_k \in A_{\alpha_k}^l \subseteq A_{\alpha_k}^n$  is not maximal in  $A_{\alpha_k}^n$  (because  $n > l$ ). Let  $\xi_k$  be the immediate successor of  $\zeta_k$  in  $A_{\beta_k}^n$ .

By Fact 3 above, we have

$$\begin{aligned}\overbrace{1 \ 0 \ \dots \ \dots \ 0 \ \dots \ \dots \ 0}^{l+1} &= \xi_0 \\ (\alpha_0 + 1) 0 \ \dots \ \dots \ \dots \ 0 &= \xi_1 \\ &\vdots \\ \alpha_0 \alpha_1 \dots (\alpha_{k-1} + 1) 0 \dots 0 &= \xi_k \\ &\vdots \\ \alpha_0 \alpha_1 \ \dots \ \dots \ (\alpha_{l-1} + 1) 0 &= \xi_l\end{aligned}$$

Therefore  $\zeta_0 < \zeta_1 < \dots < \zeta_l < \xi_l < \xi_{l-1} < \dots < \xi_0$ . To complete the proof it remains to find some  $i \in X_n$  for which  $f_{\alpha_k}(i) = \zeta_k$  for

$k \leq l$ , and, consequently, by the definition (2) above,  $g_{\alpha_k}(i) = \xi_k$ . The existence of such  $i \in X_n$  is guaranteed by the second condition in the definition of  $\bar{f}$ .  $\square$

For  $\alpha < \beta < \lambda$ , let

$$C_{\alpha,\beta} = \{i < \kappa : f_\alpha(i) < f_\beta(i) < g_\beta(i) < g_\alpha(i)\} \quad (4)$$

**Claim 6.**  $\{C_{\alpha,\beta} : \alpha < \beta < \kappa^+\}$  has the finite intersection property

*Proof.* Suppose that  $\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_l, \beta_l$  are given and  $\alpha_k < \beta_k < \lambda$  for  $k \leq l$ . Let  $\langle \gamma_m : m < m(*) \rangle$  be the increasing enumeration of  $\bigcup_{k \leq l} \{\alpha_k, \beta_k\}$ . To show that  $\bigcap_{k \leq l} C_{\alpha_k, \beta_k}$  is not empty, it suffices to find some  $i < \kappa$  for which the sequence  $g_{\gamma_m}(i)$  is decreasing in  $m$  and  $f_{\gamma_m}(i)$  is increasing in  $m$ . The existence of such an  $i < \kappa$  follows from Claim 5.  $\square$

Let  $U$  be any ultrafilter extending  $\{C_{\alpha,\beta} : \alpha < \beta < \lambda\}$ . Since for every  $\alpha < \beta$  it holds that  $f_\alpha <_U f_\beta <_U g_\beta <_U g_\alpha$ , we conclude that  $\bar{f}$  is  $<_U$ -increasing, that  $\bar{g}$  is  $<_U$ -decreasing and that every  $g_\alpha$  is an upper bound of  $\bar{f} \bmod U$ .

**Claim 7.** There is no exact upper bound of  $\bar{f} \bmod U$ .

*Proof.* It suffices to check that there is no  $h \in \text{On}^\kappa$  that satisfies  $f_\alpha <_U h <_U g_\alpha$  for all  $\alpha < \kappa^+$ . Suppose, then, that  $h \in \text{On}^\kappa$  satisfies this. Since  $h <_U g_0$ , we may assume that  $g(i) < \lambda^{n(i)+1}$  for all  $i < \kappa$  (by changing  $h$  on a set outside of  $U$ ).

Let  $i < \kappa$  be arbitrary. Since  $\bigcup_{\alpha < \lambda} A_\alpha^{n(i)} = \lambda^{n(i)+1}$ , there is some  $\alpha(i)$  so that  $h(i) \in A_{\alpha(i)}^{n(i)}$ . By regularity of  $\lambda$  it follows that there is some  $\alpha(*) < \lambda$  such that  $h(i) \in A_{\alpha(*)}^{n(i)}$  for all  $i < \kappa$ . By our assumption about  $h$ ,  $f_{\alpha(*)} <_U h <_U g_{\alpha(*)}$ . Thus, there is some  $i < \kappa$  for which  $f_{\beta(*)}(i) < h(i) < g_{\beta(*)}(i)$ . However, all three values belong to  $A_{\alpha(*)+1}^{n(i)}$ , while by Fact 4 there are no members of  $A_{\alpha(*)+1}^{n(i)}$  between  $f_{\beta(*)}(i)$  and  $g_{\alpha(*)}(i)$  — a contradiction.  $\square$

**Claim 8.** there are no sets  $S(i) \subseteq \text{On}$  for  $i < \kappa$  which satisfy condition 2 in the trichotomy for  $\bar{f}$  and  $U$ .

*Proof.* Suppose that  $S(i)$ , for  $i < \kappa$ , and  $h_\alpha \in \prod_{i < \kappa} S(i)$  satisfy 2. in the Trichotomy Theorem. Find  $\alpha < \lambda$  such that  $S(i) \subseteq A_\alpha^{n(i)}$  for all  $i$ . Thus  $f_\alpha <_U h_\alpha <_U g_\alpha$  — contradiction to 4.  $\square$

**Claim 9.** there is no  $g : \kappa \rightarrow \text{On}$  such that  $g, \bar{f}$  and the dual of  $U$  satisfy condition 3. in the Trichotomy Theorem.

*Proof.* Let  $g : \kappa \rightarrow On$  be arbitrary, and let  $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$ . As  $\bar{f}$  is  $<_U$ -increasing, for every  $\alpha < \beta < \lambda$  necessarily  $t_\alpha \subseteq_U t_\beta$ . Since  $U$  is an ultrafilter, every  $\subseteq_U$ -increasing sequence of sets stabilizes.  $\square$

$\square$

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