



## A TREE-ARROWING GRAPH

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*Dedicated to the memory of Eric Milner*

**Abstract.** We answer a variant of a question of Rödl and Voigt by showing that, for a given infinite cardinal  $\lambda$ , there is a graph  $G$  of cardinality  $\kappa = (2^\lambda)^+$  such that for any colouring of the edges of  $G$  with  $\lambda$  colours, there is an induced copy of the  $\kappa$ -tree in  $G$  in the set theoretic sense with all edges having the same colour.

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### 1. Introduction

$\mathcal{G} = (V, E)$  is a graph with *vertex set*  $V$  and *edge set*  $E$ , where  $E \subseteq [V]^2$ . The graph  $\mathcal{H} = (W, F)$  is a *subgraph* of  $\mathcal{G}$  if  $W \subseteq V$  and  $F \subseteq E$ , it is an *induced subgraph* if  $F = E \cap [W]^2$ . If  $\lambda$  is a cardinal, the partition relation

$$\mathcal{G} \rightarrow (\mathcal{H})_\lambda^2, \tag{1}$$

means that if  $c : E \rightarrow \lambda$  is any colouring of the edges of  $\mathcal{G}$  with  $\lambda$  colours, then there is an induced copy of  $\mathcal{H}$  in  $\mathcal{G}$  in which all the edges have the same colour. There is a related notion  $\mathcal{G} \rightarrow (\mathcal{H})_\lambda^1$ , for vertex colourings of graphs. However, there is an essential difference since, for any given graph  $\mathcal{H}$  and any  $\lambda$ , there is

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some  $\mathcal{G}$  such that  $\mathcal{G} \rightarrow (\mathcal{H})_\lambda^1$  holds. This is not true for edge-colourings; Hajnal and Komjath [HK] proved the consistency of a negative answer, and Shelah [Sh] proved that a positive answer is also consistent. It is therefore of some interest to have instances of graphs  $\mathcal{H}$  such that (1) holds for some  $\mathcal{G}$ , and then, of course, one can ask for the smallest such  $\mathcal{G}$ .

Rödl and Voigt [RV] (see also [RV2]) proved a result of this kind by showing that for any infinite cardinal  $\lambda$  and a suitably large  $\kappa$ , there is a graph  $\mathcal{G}_\kappa$  of cardinality  $\kappa$  such that

$$\mathcal{G}_\kappa \rightarrow (\mathcal{T}_\kappa)_\lambda^2 \quad (2)$$

holds, where  $\mathcal{T}_\kappa$  is the tree in which every vertex has degree  $\kappa$  (see below). More precisely, ‘suitably large’ means that the ordinary partition relation

$$\text{cf}(\kappa) \rightarrow (\omega)_\lambda^3$$

holds so that, by [RV],  $\kappa \geq (2^{2^\lambda})^+$ ; in fact, they showed in this case that the ubiquitous *shift-graph* on  $\kappa$  works. Rödl and Voigt [RV] then asked, what is the smallest cardinal  $\kappa$  such that (2) holds? It is easily seen that (2) is false if  $\kappa \leq 2^\lambda$ , and they conjectured that it holds (for some suitable graph  $\mathcal{G}_\kappa$ ) if  $\kappa = (2^\lambda)^+$ . In this paper we prove that (2) holds with  $\mathcal{T}_\kappa$  replaced by  $\mathcal{T}(\kappa)$ , a related graph which we call *the transitive  $\kappa$ -tree* defined in the next section.

## 2. Preliminaries

For an infinite cardinal  $\kappa$  we denote by  ${}^{<\omega}\kappa$  the set of all increasing finite sequences of ordinals in  $\kappa$ . The *length* of an element  $s = \langle s_0, \dots, s_{n-1} \rangle \in {}^{<\omega}\kappa$  is denoted by  $\ell n(s) = n$ . Also, we define

$$\max(s) = \begin{cases} -1 & \text{if } s = \langle \rangle, \text{ the empty sequence,} \\ s_{\ell n(s)-1} & \text{if } \ell n(s) > 0. \end{cases}$$

If  $s = \langle s_0, \dots, s_{n-1} \rangle$  and  $t = \langle t_0, \dots, t_{m-1} \rangle$  are two elements of  ${}^{<\omega}\kappa$ , we write  $s \triangleleft t$  to denote the fact that  $s$  is a proper initial segment of  $t$ , that is  $n < m$  and  $s_i = t_i$  for  $i < n$ , and in this case we write  $s = t|n$ . We also write  $s = t_*$  if  $m = n + 1$  and  $s \triangleleft t$ . If  $s, t$  are distinct and  $\triangleleft$ -incomparable we write  $s \perp t$ . The  *$\kappa$ -tree of height  $\omega$*  is the graph  $\mathcal{T}_\kappa$  on  ${}^{<\omega}\kappa$  with edge set

$$E_\kappa = \{\{s, t\} : s, t \in {}^{<\omega}\kappa \wedge s = t_*\}.$$

We shall also consider a related graph, *the transitive  $\kappa$ -tree of height  $\omega$* , which is the graph  $\mathcal{T}(\kappa)$  on  ${}^{<\omega}\kappa$  with edge set

$$F_\kappa = \{\{s, t\} : s, t \in {}^{<\omega}\kappa \wedge s \triangleleft t\}.$$

We shall prove the following theorem.

**Theorem 2.1.** *Let  $\lambda$  be an infinite cardinal, and let  $\kappa = (2^\lambda)^+$ . Then there is a graph  $G_\kappa$  of cardinality  $\kappa$  such that*

$$G_\kappa \rightarrow (\mathcal{T})_\lambda^2,$$

where  $\mathcal{T}$  is  $\mathcal{T}(\kappa)$ .

**Remark.** *Instead of  $\kappa = (2^\lambda)^+$ , it is enough that  $\kappa$  be any regular cardinal such that  $|\alpha|^\lambda < \kappa$  holds for all  $\alpha < \kappa$ . The same proof works.*

The construction of a suitable  $\mathcal{G}_\kappa$  depends upon the following (slightly weaker version of a) theorem of Shelah [] (or more [, 3.5]):

( $\bullet$ ) *Let  $\lambda$  be an infinite cardinal,  $\kappa = (2^\lambda)^+$ ,  $S = \{\alpha < \kappa : \text{cf}(\alpha) = \lambda^+\}$ . Then there are a sequence  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  and a sequence  $\bar{h}^* = \langle h_\delta^* : \delta \in S \rangle$  such that  $C_\delta$  is a club in  $\delta$  having order type  $\lambda^+$ ,  $h_\delta^* : C_\delta \rightarrow 2$  and such that, for any club  $K$  in  $\kappa$ , there is a stationary subset  $B_K$  of  $S \cap K$  such that for each  $\delta \in B_K$  and each  $i < 2$ ,  $\min(C_\delta) \in K$  and the set*

$$D_K(\delta, i) = \{\alpha \in C_\delta \cap K : h_\delta^*(\alpha) = i \wedge \min(C_\delta \setminus (\alpha + 1)) \in K\}$$

*is cofinal in  $\delta$ .*

**Remarks.** 1. *The result is also true if 2, the range of each  $h_\delta^*$ , is replaced by  $\lambda$ ; also, if  $\kappa = \lambda^{++}$ , we can also require that  $D_K(\delta, i)$  be a stationary subset of  $\delta$  for each  $\delta \in B_K$  and  $i < \lambda$  (see []).*

2. *If  $2^\lambda > \lambda^+$ , then the following stronger assertion is true (see Shelah []):*  
 ( $\bullet\bullet$ ) *There is a sequence  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  such that  $C_\delta$  is a club in  $\delta$  having order type  $\lambda^+$  and, for any club  $K$  in  $\kappa$  and any stationary subset  $S' \subseteq S$ , there is a stationary subset  $B_K \subseteq S' \cap K$  such that  $C_\delta \subseteq K$  for each  $\delta \in B_K$ . Using this result instead of ( $\bullet$ ), the proof of Theorem 2.1 for the case when  $2^\lambda > \lambda^+$  may be slightly simplified.*

We will prove that Theorem 2.1 holds with the graph  $G_\kappa = (\kappa, \mathcal{E})$ , where

$$\mathcal{E} = \{\{\alpha, \beta\} : \beta \in S \wedge \min(C_\beta) < \alpha < \beta \wedge h_\beta^*(\sup(\alpha \cap C_\beta)) = 0\},$$

and the  $C_\beta$  and  $h_\beta^*$  are as described in ( $\bullet$ ).

### 3. The case $\mathcal{T} = \mathcal{T}(\kappa)$

We prove the result for the case of the transitive tree  $T(\kappa)$ .

*Proof:* Let  $c : \mathcal{E} \rightarrow \lambda$  be any  $\lambda$ -colouring of the edges of  $G_\kappa$ . For each  $\zeta \in \lambda$  consider the following two-person game  $\mathcal{G}_\zeta$ . The game has  $\omega$  moves. At the  $n$ -th stage the first player  $P_1$  chooses ordinals  $\alpha_n, \beta_n$ , and then the second player  $P_2$  chooses two ordinals  $\gamma_n, \delta_n$  so that

$$\alpha_n < \beta_n < \gamma_n < \delta_n < \kappa, \tag{3}$$

$$\delta_m < \alpha_n \quad (m < n). \tag{4}$$

The player  $P_2$  is declared the winner in a play of the game if he succeeds in choosing the  $\gamma_n$  so that

$$\{\gamma_m, \gamma_n\} \in \mathcal{E}, \quad c(\{\gamma_m, \gamma_n\}) = \zeta \quad (m < n < \omega), \quad (5)$$

and

$$\{\xi, \gamma_n\} \notin \mathcal{E} \text{ for } \xi \in (\alpha_m, \beta_m) \text{ and } m \leq n < \omega. \quad (6)$$

(As usual,  $(\alpha, \beta)$  denotes the open interval  $\{\xi : \alpha < \xi < \beta\}$  and  $[\alpha, \beta]$  is the corresponding closed interval.)

The proof of the theorem depends upon the following two facts:

**Fact A:** For some  $\zeta < \lambda$ ,  $P_2$  has a winning strategy for the game  $\mathcal{G}_\zeta$ .

**Fact B:** If  $P_2$  can win  $\mathcal{G}_\zeta$ , then the graph  $G_\kappa$  contains an induced copy of  $\mathcal{T}(\kappa)$  with all edges coloured  $\zeta$ .

*Proof of Fact B.* We assume that  $\zeta < \lambda$  and that the second player  $P_2$  has a winning strategy  $\sigma_\zeta$  for the game  $\mathcal{G}_\zeta$ . We shall define ordinals  $\alpha_s, \beta_s, \gamma_s, \delta_s$  for  $s$  a vertex of  $\mathcal{T}(\kappa)$  so that the following conditions are satisfied:

(a) For each  $s$  the sequence

$$\langle (\alpha_{s|i}, \beta_{s|i}, \gamma_{s|i}, \delta_{s|i}) : i < \ell n(s) \rangle$$

consists of the first  $2\ell n(s)$  moves in a proper play of the game  $\mathcal{G}_\zeta$  in which  $P_2$  uses the winning strategy  $\sigma_\zeta$ .

(b)  $\gamma_s \neq \gamma_t$  if  $s \neq t$ .

(c) If  $s \perp t$ , then  $\{\gamma_s, \gamma_t\} \notin \mathcal{E}$ .

Since (5) holds, these conditions imply that the map  $s \mapsto \gamma_s$  is an embedding of the tree  $\mathcal{T}(\kappa)$  into the graph  $G_\kappa$  and all the edges of the image have colour  $\zeta$ .

In fact, we shall choose the  $\alpha_s, \beta_s, \gamma_s, \delta_s$  so that (a) holds and so that the following condition is satisfied:

(d) For any vertices  $s, t$  of  $\mathcal{T}(\kappa)$ , if  $s \perp t$ , then

$$\begin{array}{ll} \text{EITHER (i)} & [\gamma_s, \delta_s] \subset \bigcup_{i \leq \ell n(t)} (\alpha_{t|i}, \beta_{t|i}), \\ \text{OR (ii)} & [\gamma_t, \delta_t] \subset \bigcup_{i \leq \ell n(s)} (\alpha_{s|i}, \beta_{s|i}). \end{array}$$

The conditions (a) and (d), and the fact that  $P_2$  is using the winning strategy  $\sigma_\zeta$ , ensure that (b) and (c) also hold.

We define  $\alpha_s, \beta_s, \gamma_s, \delta_s$  by induction on  $\max(s)$ . Let  $\alpha_\emptyset = 0$ ,  $\beta_\emptyset = 1$ , and then let  $(\gamma_\emptyset, \delta_\emptyset)$  be  $P_2$ 's response in the game  $\mathcal{G}_\zeta$  using his winning strategy  $\sigma_\zeta$ . Now let  $0 \leq \xi < \kappa$ , and suppose that we have suitably defined  $\alpha_s, \beta_s, \gamma_s, \delta_s$  for all vertices  $s$  of  $\mathcal{T}(\kappa)$  such that  $\max(s) < \xi$ . We need to define these when  $\max(s) = \xi$ .

Let  $\langle t_i : i < \theta(\xi) \rangle$  be an enumeration of all the nodes  $s$  of  $\mathcal{T}(\kappa)$  with  $\max(s) = \xi$ . Then  $1 \leq \theta(\xi) \leq 2^\lambda < \kappa$ . Now inductively choose the  $\alpha_{t_i}, \beta_{t_i}, \gamma_{t_i}, \delta_{t_i}$  for  $i < \theta(\xi)$  so that

$$\alpha_{t_i} = \delta_{(t_i)_*} + 1,$$

and if  $i = 0$ ,  $\beta_{t_i} = \alpha_{i_0} + 1$  and if  $i > 0$

$$\beta_{t_i} = \sup\{\delta_s + 2 : \max(s) < \xi \text{ or } s = t_j \text{ for some } j < i\}.$$

The corresponding pairs  $(\gamma_{t_i}, \delta_{t_i})$  are determined by the strategy  $\sigma_\zeta$ . With these choices it is easily seen that (a) continues to hold; we have to check that (d) also holds when  $s \perp t$  and  $\max(s) = \xi$  or  $\max(t) = \xi$ .

If  $\max(s) = \max(t) = \xi$ , then  $s = t_i$  and  $t = t_j$ , where say  $i < j$ . Then

$$\alpha_t = \delta_{t_*} + 1 < \beta_s < \gamma_s < \delta_s < \beta_t,$$

and so (d)(i) holds.

Suppose  $\max(s) < \xi = \max(t)$ . Then by the induction hypothesis, either (i) or (ii) of (d) holds when we replace  $t$  by  $t_*$ . Suppose first that (d)(i) holds. Then for some  $m \leq \ell n(t_*)$  we have that

$$\alpha_{t_*|m} < \gamma_s < \delta_s < \beta_{t_*|m}.$$

It follows that (d)(i) also holds for  $s$  and  $t$  since  $t|m = t_*|m$ . Now suppose that (d)(ii) holds so that, for some  $m \leq \ell n(s)$ ,

$$\alpha_{s|m} < \gamma_{t_*} < \delta_{t_*} < \beta_{s|m}.$$

Then, by the definitions of  $\alpha_t$  and  $\beta_t$ , it follows that

$$\alpha_t = \delta_{t_*} + 1 \leq \beta_s < \gamma_s < \delta_s < \beta_t,$$

so that again (d)(i) holds for  $s$  and  $t$ . Similarly, if  $\max(t) < \xi = \max(s)$ .  $\square$

*Proof of Fact A.* We have to show that  $P_2$  wins the game  $\mathcal{G}_\zeta$  for some  $\zeta < \lambda$ . Suppose for a contradiction that this is false. Since the games are open and hence determined, it follows that  $P_1$  has a winning strategy, say  $\tau_\zeta$ , for the game  $\mathcal{G}_\zeta$  for every  $\zeta < \lambda$ .

For convenience we write  $c(\{\alpha, \beta\}) = -1$  if  $\{\alpha, \beta\} \notin \mathcal{E}$ , so that  $c$  is defined on all pairs  $\{\alpha, \beta\} \in [\kappa]^2$ . For each bounded subset  $X \subseteq \kappa$  define an equivalence relation  $e_X$  on  $S \setminus (\sup(X) + 1)$  so that  $\beta e_X \gamma$  holds if and only if

- (i)  $\beta, \gamma \in S$  and  $\sup(X) < \beta, \gamma < \kappa$ ;
- (ii)  $c(\{\alpha, \beta\}) = c(\{\alpha, \gamma\})$  for all  $\alpha \in X$ ;
- (iii)  $X \cap C_\beta = X \cap C_\gamma$ , (iv) for  $\alpha \in X$ ,  $\alpha \leq \min(C_\beta) \Leftrightarrow \alpha \leq \min(C_\gamma)$ ,  
 $\text{tp}(\alpha \cap C_\beta) = \text{tp}(\alpha \cap C_\gamma)$  and  $h_\beta^*(\sup(\alpha \cap C_\beta)) = h_\gamma^*(\sup(\alpha \cap C_\gamma))$  (for  $\alpha > \min(C_\beta)$ ).

Note that the equivalence relation  $e_X$  has at most  $(\lambda^+)^{|X|} \leq 2^{\lambda|X|}$  classes. Also, if  $Y \subseteq X$ , then  $\beta e_X \gamma \Rightarrow \beta e_Y \gamma$ .

Since  $\kappa = (2^\lambda)^+$ , there is a continuous increasing sequence of ordinals  $\langle \rho_\eta : \eta < \kappa \rangle$  in  $\kappa$  such that the following two conditions hold:

- (o) If  $X \subseteq \rho_\eta$ ,  $|X| \leq \lambda$  and  $\rho_\eta < \beta < \kappa$ , then there is some  $\gamma \in (\rho_\eta, \rho_{\eta+1})$  such that  $\beta e_X \gamma$
- (oo)  $\rho_\eta$  is closed under  $\tau_\zeta$  for all  $\zeta < \lambda$ . In other words, if at the  $n$ -th stage of a play in the game  $\mathcal{G}_\zeta$ , player  $P_2$  chooses  $\gamma_n < \delta_n < \rho_\eta$ , then  $P_1$ 's response using  $\tau_\zeta$  is to choose  $\alpha_{n+1}, \beta_{n+1}$  so that  $\delta_n < \alpha_{n+1} < \beta_{n+1} < \rho_\eta$ .

Since  $K = \{\rho_\eta : \eta < \kappa\}$  is a club in  $\kappa$ , there is some  $\delta \in S$  such that  $\min(C_\delta) \in K$  and, for  $\varepsilon \in \{0, 1\}$ ,

$$A_\varepsilon = \{\alpha \in C_\delta \cap K : h_\delta^*(\alpha) = \varepsilon \wedge \min(C_\delta \setminus (\alpha + 1)) \in K\}$$

is an unbounded subset of  $\delta$ . Let  $C_\delta = \{i_\sigma : \sigma < \lambda^+\}$ , where  $i_0 < i_1 < \dots$ .

We claim that the following assertion holds for some  $\zeta < \lambda$ .

$(*)_\zeta$ : If  $X \subseteq \delta$ ,  $|X| \leq \lambda$ , then there are  $\sigma < \lambda^+$  and  $\gamma$  such that (a)  $\sup(X) < i_\sigma < \gamma < i_{\sigma+1}$ , (b)  $i_\sigma \in A_0$ , (c)  $\gamma e_X \delta$ , and (d)  $c(\gamma, \delta) = \zeta$ .

For suppose the claim is false. Then, for each  $\zeta < \lambda$  there is a counter-example  $X_\zeta$ . Let  $X = \bigcup \{X_\zeta : \zeta < \lambda\}$ . Then  $X \subseteq \delta$  and  $|X| \leq \lambda$  and so, for some  $\alpha \in A_0$ ,  $\sup(X) < \alpha < \delta$ . There are  $\eta < \kappa$  and  $\sigma < \lambda^+$  such that  $\alpha = \rho_\eta = i_\sigma$ , and therefore, by the choice of  $\rho_{\eta+1}$ , there is  $\gamma$  such that  $\rho_\eta < \gamma < \rho_{\eta+1}$  and  $\gamma e_X \delta$ . Since  $\alpha = i_\sigma \in A_0$ ,  $i_{\sigma+1} = \min(C_\delta \setminus (\alpha + 1)) \in K$ . So  $\rho_{\eta+1} \leq i_{\sigma+1}$ . Therefore,  $\sup(C_\delta \cap \gamma) = i_\sigma$ , and since  $\alpha = i_\sigma \in A_0$ , we have that  $h_\delta^*(\sup(C_\delta \cap \gamma)) = 0$ . Therefore,  $\{\gamma, \delta\}$  is an edge of  $G$  and there is some  $\zeta \in \lambda$  such that  $c(\gamma, \delta) = \zeta$ . But this contradicts the choice of  $X_\zeta \subseteq X$ , and hence  $(*)_\zeta$  holds for some  $\zeta < \lambda$ .

By induction on  $n < \omega$  we now choose ordinals  $\alpha_n, \beta_n, \gamma_n, \delta_n$  in  $\delta$  and  $\sigma(n) < \lambda^+$  so that the following conditions are satisfied:

- A:  $\langle (\alpha_m, \beta_m, \gamma_m, \delta_m) : m \leq n \rangle$  is an initial segment of a play in the game  $\mathcal{G}_\zeta$  in which  $P_1$  uses the winning strategy  $\tau_\zeta$ .
- B:  $\alpha_0, \beta_0 < \min(C_\delta)$ .
- C:  $\gamma_n = \min\{\gamma : \gamma > i_{\sigma(2n)} \wedge \gamma e_{X_n} \delta \wedge c(\gamma, \delta) = \zeta\}$ , where

$$X_n = \bigcup \{ \{ \alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell \} : \ell < n \} \cup \{ \alpha_n, \beta_n \} \cup \bigcup \{ \{ i_{\sigma(\ell)}, i_{\sigma(\ell)+1} \} : \ell < 2n \}.$$

- D:  $\delta_n = i_{\sigma(2n+1)}$ .
- E: For  $n > 0$ ,  $[\alpha_n, \beta_n] \subseteq (\delta_{n-1}, i_{\sigma(2n-1)+1})$ .
- F:  $i_{\sigma(n)}$  belongs to  $A_0$  or  $A_1$  according as  $n$  is even or odd and  $\sigma(n)+1 < \sigma(n+1)$ .

We have to prove that it is possible to choose the  $\alpha_n$  etc., so that these conditions are satisfied. Clearly (B) holds since, by (oo), the first moves by  $P_1$  using the strategy  $\tau_\zeta$  are  $\alpha_0 < \beta_0 < \rho_0$  and  $\rho_0 \leq \min(C_\delta) \in K$ . By  $(*)_\zeta$ , there are  $\sigma(0) < \lambda^+$

and  $\gamma$  such that  $i_{\sigma(0)} \in A_0$ ,  $i_{\sigma(0)} < \gamma < i_{\sigma(0)+1}$ ,  $\gamma e_{X_0} \delta$ , where  $X_0 = \{\alpha_0, \beta_0\}$  and  $c(\gamma, \delta) = \zeta$ ; let  $\gamma_0$  be the least such  $\gamma$ . Now let  $\sigma(1) > \sigma(0) + 1$  be minimal so that  $i_{\sigma(1)} \in A_1$ , and put  $\delta_0 = i_{\sigma(1)}$ . Now suppose that  $n > 0$  and that the  $\alpha_m, \beta_m, \gamma_m, \delta_m, \sigma(2m)$  and  $\sigma(2m+1)$  have been suitably defined for all  $m < n$ . Let  $\rho \in K$  be minimal such that  $\rho > \delta_{n-1}$ .  $P_1$  chooses  $\alpha_n, \beta_n$  using the strategy  $\tau_\zeta$  so that  $\delta_{n-1} < \alpha_n < \beta_n < \rho$ . Since  $\delta_{n-1} = i_{\sigma(2n-1)} \in A_1$ , it follows that  $i_{\sigma(2n-1)+1} \in K$  and hence  $\rho \leq i_{\sigma(2n-1)+1}$ . Now by  $(*)_\zeta$ , there are  $\sigma(2n)$  and  $\gamma$  so that  $i_{\sigma(2n)} \in A_0$ ,

$i_{\sigma(2n)} < \gamma < i_{\sigma(2n)+1}$ ,  $\gamma e_{X_n} \delta$  (where  $X_n$  is as described in (C)), and  $c(\gamma, \delta) = \zeta$ ; let  $\gamma_n$  be the least such  $\gamma$ . Note that, since  $i_{\sigma(2n)} \in A_0$ ,  $i_{\sigma(2n)+1} = \min(C_\delta \setminus (i_{\sigma(2n)} + 1)) \in K$ . Finally, choose a minimal ordinal  $\sigma(2n+1) > \sigma(2n) + 1$  so that  $\delta_n = i_{\sigma(2n+1)} \in A_1$ . This completes the definition of the  $\alpha_n$  etc., so that (A)-(F) hold.

By (C) it follows that  $c(\gamma_n, \delta) = \zeta$  for all  $n < \omega$ , and hence  $c(\gamma_m, \gamma_n) = \zeta$  holds for all  $m < n < \omega$  since  $\gamma_m \in X_n$  and  $\gamma_n e_{X_n} \delta$ . There is no edge of  $G_\kappa$  from  $\delta$  to  $(\alpha_0, \beta_0)$  since  $\beta_0 < \min(C_\delta)$ . Since  $\gamma_n e_{X_n} \delta$  and  $\beta_0 \in X_n$ , it follows that  $\beta_0 < \min(C_{\gamma_n})$  also, and so there is no edge from  $\gamma_n$  to  $(\alpha_0, \beta_0)$  either. By the construction, for  $0 < m < \omega$ ,  $i_{\sigma(2m-1)} < \alpha_m < \beta_m < i_{\sigma(2m-1)+1}$ , and hence  $C_\delta \cap (\alpha_m, \beta_m) = \emptyset$ . Therefore, for any  $\xi \in (\alpha_m, \beta_m)$ ,  $h_\delta^*(\sup(\xi \cap C_\delta)) = h_\delta^*(i_{\sigma(2m-1)}) = 1$  by (F), and so there is no edge of  $G$  from  $\delta$  to  $(\alpha_m, \beta_m)$ . If  $0 < m < n < \omega$ , then  $\gamma_n e_{X_n} \delta$  and therefore ,

$$\text{tp}(\alpha_m \cap C_{\gamma_n}) = \text{tp}(\alpha_m \cap C_\delta) = \text{tp}(\beta_m \cap C_\delta) = \text{tp}(\beta_m \cap C_{\gamma_n}).$$

Therefore, for any  $\xi \in (\alpha_m, \beta_m)$ , it follows that

$$h_{\gamma_n}^*(\sup(\xi \cap C_{\gamma_n})) = h_{\gamma_n}^*(\sup(\alpha_m \cap C_{\gamma_n})) = h_\delta^*(\sup(\alpha_m \cap C_\delta)) = 1$$

and so there are no edges of  $G$  from  $\gamma_n$  to  $(\alpha_m, \beta_m)$  either.

Thus we have produced a play in the game  $\mathcal{G}_\zeta$  in which  $P_1$  uses the strategy  $\tau_\zeta$  but the second player  $P_2$  wins! This contradicts the assumption that  $\sigma_\zeta$  is a winning strategy for the first player, and completes the proof.  $\square$

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