

**EXISTENTIALLY CLOSED LOCALLY FINITE GROUPS**  
**SH312**

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**ABSTRACT.** We investigate this class of groups originally called ulf (universal locally finite groups) of uncountable cardinality. We prove that for every locally finite group  $G$  there is a canonical existentially closed extension of the same cardinality, unique up to isomorphism and increasing with  $G$ . Also we get, e.g. existence of complete members (i.e. with no non-inner automorphisms) in many cardinals (provably in ZFC). The main point here is having a parallel to stability theory in the sense of investigating definable types though the class is very unstable.

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*Date:* June 9, 2017.

1991 *Mathematics Subject Classification.* Primary: 03C55, 03C50, 03C60; Secondary: 03C98, 03C45.

*Key words and phrases.* model theory, applications of model theory, groups, locally finite groups, canonical closed extension, endo-rigid group.

Partially supported by the ISF, Israel Science Foundation.

I would like to thank Alice Leonhardt for the beautiful typing.

First Typed - 02/Oct/18.

## § 0. INTRODUCTION

## § 0(A). Background.

On lf (locally finite) groups and exlf (existentially closed locally finite) groups, see the book by Kegel-Wehrfritz [KW73]; exlf groups were originally called ulf (= universal locally finite) groups, we change as the word “universal” has been used in this context with a different meaning, see Definition 0.21 and Claim 0.14.

Recall

**Definition 0.1.** 1)  $G$  is a lf (locally finite) group if  $G$  is a group and every finitely generated subgroup is finite.

2)  $G$  is an exlf (existentially closed lf) group (in [KW73] it is called ulf, universal locally finite group) when  $G$  is a locally finite group and for any finite groups  $K \subseteq L$  and embedding of  $K$  into  $G$ , the embedding can be extended to an embedding of  $L$  into  $G$ .

3) Let  $\mathbf{K}_{\text{lf}}$  be the class of lf (locally finite) groups (partially ordered by  $\subseteq$ , being a subgroup) and let  $\mathbf{K}_{\text{exlf}}$  be the class of existentially closed  $G \in \mathbf{K}_{\text{lf}}$ .

In particular there is one and only one exlf group of cardinality  $\aleph_0$ . Hall proved that every lf group can be extended to an exlf group, as follows. It suffices for a given lf group  $G$  to find  $H \supseteq G$  such that if  $K \subseteq L$  are finite and  $f$  embeds  $K$  into  $G$ , then some  $g \supseteq f$  embed  $L$  into  $H$ . To get such  $H$ , for finite  $K \subseteq G$  let  $E_{G,K} = \{(a,b) : a,b \in G \text{ and } aK = bK\}$  and let  $G^\oplus$  be the group of permutations  $f$  of  $G$  such that for some finite  $K \subseteq G$  we have  $a \in G \Rightarrow aE_{G,K}f(a)$ ; now  $b \in G$  should be identified with  $f_b \in G^\oplus$  where  $f_b$  is defined by  $f_b(x) = xb$  hence  $f_b \in G^\oplus$  because if  $b \in K \subseteq G$  then  $a \in G \Rightarrow f_b(a) = ab \in abK = aK$  and  $f_{b_2} \circ f_{b_1}(x) = (xb_1)b_2 = x(b_1b_2) = f_{b_1b_2}(x)$ . Now  $H = G^\oplus$  is essentially as required.

The proof gives a canonical extension. This means for example that every automorphism of  $G$  can be extended to an automorphism of  $G^\oplus$  and, moreover, we can do it uniformly so preserving isomorphisms. Still we may like to have more; (for a given lf infinite group  $G$ ) the extension  $G^\oplus$  defined above is of cardinality  $2^{|G|}$  rather than the minimal value -  $|G| + \aleph_0$  (not to mention having to repeat this  $\omega$  times in order to get an exlf extension). Also if  $G_1 \subseteq G_2$  then the connection between  $G_1^\oplus$  and  $G_2^\oplus$  is not clear, i.e. failure of “naturality”. A major point of the present work is a construction of a canonical existentially closed extension of  $G$  which has those two additional desirable properties, see e.g. 3.15.

Note that in model theoretic terminology the exlf groups are the  $(\mathbf{D}, \aleph_0)$ -homogeneous groups, with  $\mathbf{D}$  the set of isomorphism types of finite groups or more exactly complete qf (= quantifier free) types of finite tuples generating a finite group, see e.g. [Shea, §2]. We use quantifier free types as we use embeddings (rather than, e.g. elementary embeddings). Let  $\mathbf{D}(G)$  be the set of qf-complete types of finite sequences from the group  $G$ .

Let  $\mathbf{K}_{\text{exlf}}$  be the class of exlf groups. By Grossberg-Shelah [GS83], if  $\lambda = \aleph_0$  then no  $G \in \mathbf{K}_{\aleph}^{\text{exlf}} := \{H \in \mathbf{K}_{\text{exlf}} : |H| = \lambda\}$  is universal in it, i.e., such that every other member is embeddable into it. But if  $\kappa$  is a compact cardinal and  $\lambda > \kappa$  is strong limit of cofinality  $\aleph_0$  then there is a universal exlf in cardinality  $\lambda$ , (this is a special case of a general theorem).

Wehrfritz asked about the categoricity of the class of exlf groups in any  $\lambda > \aleph_0$ . This was answered by Macintyre-Shelah [MS76] which proved that in every  $\lambda > \aleph_0$

there are  $2^\lambda$  non-isomorphic members of  $\mathbf{K}_\lambda^{\text{exlf}}$ . This was disappointing in some sense: in  $\aleph_0$  the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

A natural and frequent question on a class of structures is the existence of rigid members, i.e. ones with no non-trivial automorphism. Now any exlf group  $G \in \mathbf{K}_{\text{exlf}}$  has non-trivial automorphisms - the inner automorphisms (recalling it has a trivial center). So the natural question is about complete members where a group is called complete iff it has no non-inner automorphism.

Concerning the existence of a complete, locally finite group of cardinality  $\lambda$ : Hickin [Hic78] proved one exists in  $\aleph_1$  (and more, e.g. he finds a family of  $2^{\aleph_1}$  such groups pairwise far apart, i.e. no uncountable group is embeddable in two of them). Thomas [Tho86] assumed G.C.H. and built one in every successor cardinal (and more, e.g. it has no Abelian or just solvable subgroup of the same cardinality). Related are Giorgetta-Shelah [GS84], Shelah-Zigler [SZ79], which investigate  $\mathbf{K}_{G_*}$  getting similar results where

- (\*) assume  $G_*$  an existentially closed countable group we let
  - (a)  $\mathbf{K}_{G_*}$  is the class of groups  $G$  such that every finitely generated subgroup of  $G$  is embeddable into  $G_*$
  - (b)  $\mathbf{K}_{G_*}^{\text{excl}}$  is the class of groups  $G$  which are  $\mathbb{L}_{\infty, \aleph_0}$ -equivalent to  $G_*$  (excl stands for existentially closed); equivalently  $G \in \mathbf{K}_{G_*}$ , every finitely generated subgroup of  $G_*$  is embeddable into  $G$  and if  $\bar{a}, \bar{b} \in {}^n G$  realize the same qf type in  $G$  then some inner automorphism of  $G$  maps  $\bar{a}$  to  $\bar{b}$
- (\*\*) we can replace “group  $G_*$ ” by any other structure.

Giorgetta-Shelah [GS84] build in cardinality continuum  $G \in \mathbf{K}_{\text{exlf}}$  with no uncountable Abelian subgroup and similarly for  $\mathbf{K}_{G_*}^{\text{excl}}$ ,  $G_*$  as in (\*) and also for the similarly defined  $\mathbf{K}_{F_*}^{\text{excl}}$ ,  $F_*$  an existentially closed countable fixed division ring. Shelah-Zigler [SZ79] build, for  $G_*$  as in (\*) and  $\lambda > \aleph_0$ ;  $N_\lambda^\ell \in \mathbf{K}_{G_*}^{\text{exlf}}$  of cardinality  $\lambda$  for  $\ell = 1, 2$  such that  $N_\lambda^1$  has no Abelian group of cardinality  $\lambda$  and every subgroup of cardinality  $\lambda$  has a free subgroup of the same cardinality; moreover, there are  $2^\lambda$  pairwise non-isomorphic  $N$  like  $N_\lambda^\ell$ .

In 1985 the author wrote notes (in Hebrew) for proving that there are anti-prime constructions and complete exlf groups when, e.g.,  $\lambda = \mu^+, \mu^{\aleph_0} = \mu$ ; using black boxes and “anti-prime” construction, i.e. using definable types as below; here we exclusively use qf (quantifier free) types; this was announced in [She87, pg.418], but the work was not properly finished. To do so is our aim here.

Meanwhile Dugas-Göbel [DG93, Th.2] prove that for  $\lambda = \aleph^{\aleph_0}$  and  $G_0 \in \mathbf{K}_{\leq \lambda}^{\text{lf}}$  there is a complete  $G \in \mathbf{K}_{\lambda^+}^{\text{exlf}}$  extending  $G_0$ ; moreover  $2^{\lambda^+}$  pairwise non-isomorphic ones. Then Braun-Göbel [BG03] got better results for complete locally finite  $p$ -groups. Those constructions build an increasing continuous chain  $\langle G_\alpha : \alpha < \lambda^+ \rangle$ , each  $G_\alpha$  of cardinality  $\lambda$ , such that  $G_{\alpha+1}$  is the wreath product of  $G_\alpha$  and suitable Abelian locally finite groups,  $G = \{G_\alpha : \alpha < \lambda^+\}$  is the desired group. This gives a tight control over the group and implies, e.g. that only few (i.e.  $\leq \lambda$ ) members commute with  $G_0$ . Here we are interested in groups  $G'$  which are “more existentially closed”, e.g. “for every  $G' \subseteq G$  of cardinality  $< |G|$ , there are  $|G|$  elements commuting with it”; such properties are called “being full”, note that

fullness implies that a restriction on the cardinal is necessary and not so without it, see 5.5.

We show that though the class  $\mathbf{K}_{\text{exlf}}$  is very “unstable” there is a large enough set of definable types so we can imitate stability theory and have reasonable control in building exlf groups, using quantifier free types. This may be considered a “correction” to the non-structure results discussed above.

In §1 we present somewhat abstractly our results relying on the existence of a dense and closed so called  $\mathfrak{S}$ , a set of schemes of definitions of the relevant types. So before we turn to explaining our results we deal with the so called schemes, needed for explaining them.

### § 0(B). Schemes.

We deal with a class  $\mathbf{K}$  of structures, usually it is the class of locally finite groups, but some of the results holds for suitable universal classes, see §6.

Central here are so-called schemes. For models theorists they are for a given  $G \in \mathbf{K}_{\text{lf}}$  and finite sequence  $\bar{a} \subseteq G$  (realizing a suitable quantifier free type) a definition of a complete (quantifier free) type over  $G$  so realized in some extension of  $G$  from  $\mathbf{K}_{\text{lf}}$ , which does not split over  $\bar{a}$ ; alternatively you may say that they are definitions of a complete-free type quantifier over  $G$  which does not split over  $\bar{a}$  and its restriction.

For algebraists they are our replacement of free products  $G_1 *_{G_0} G_2$ , but  $\mathbf{K}_{\text{lf}}$  is not closed under free product, in fact, fail amalgamation. So we are interested in replacements in the cases  $G_0$  is finite, also we waive symmetry.

**Convention 0.2.** 1)  $\mathbf{K}$  a universal class of structures (i.e. all of the same vocabulary, closed under isomorphisms and  $M \in \mathbf{K}$  iff every finite generated substructure belongs to  $\mathbf{K}$ ; usually  $\mathbf{K} = \mathbf{K}_{\text{lf}}$ ).

2)  $G, H, \dots \in \mathbf{K}$ .

**Definition 0.3.** For  $H \in \mathbf{K}, n < \omega$ , a set  $A \subseteq H$  and  $\bar{a} \in {}^n H$  let  $\text{tp}(\bar{a}, A, H) = \text{tp}_{\text{bs}}(\bar{a}, A, H)$  be the basic type of  $\bar{a}$  in  $H$  over  $A$ , that is:

$$\{\varphi(\bar{x}, \bar{b}) : \varphi \text{ is a basic (atomic or negation of atomic) formula in the variables } \bar{x} = \langle x_\ell : \ell < n \rangle \text{ and the parameters } \bar{b}, \text{ a finite sequence from } A, \text{ which is satisfied by } \bar{a} \text{ in } H\}.$$

So if  $\mathbf{K}$  is a class of groups without loss of generality  $\varphi$  is  $\sigma(\bar{x}, \bar{b}) = e$  or  $\sigma(\bar{x}, \bar{b}) \neq e$  for some group-term  $\sigma$ , a so called “word”, (for  $\mathbf{K}_{\text{of}}$  we also have  $\sigma_1(\bar{x}, \bar{b}) < \sigma_2(\bar{x}, \bar{b})$ ) but we may write  $p(\bar{y}) = \text{tp}_{\text{bs}}(\bar{b}, A, H)$  or  $p(\bar{z}) = \text{tp}_{\text{bs}}(\bar{c}, A, H)$  or just  $p$  when the sequence of variables is clear from the context.

2) We say  $p(\bar{x})$  is an  $n$ -*bs-type* over  $G$  when it is a set of basic formulas in the variables  $\bar{x} = \langle x_\ell : \ell < n \rangle$  and parameters from  $G$ , such that  $p(\bar{x})$  is consistent, which means: if  $K \subseteq G$  is f.g. and  $q(\bar{x})$  is a finite subset of  $p(\bar{x})$  and  $q(\bar{x})$  is over  $K$  (i.e. all the parameters appearing in  $q(\bar{x})$  are from  $K$ ) then  $q(\bar{x})$  is realized in some  $L \in \mathbf{K}$  extending  $K$ . We say  $\bar{a}$  realizes  $p$  in  $H$  if  $G \subseteq H$  and  $\varphi(\bar{x}, \bar{b}) \in p \Rightarrow H \models \varphi[\bar{a}, \bar{b}]$ .

3)  $\mathbf{S}_{\text{bs}}^n(G) = \{\text{tp}_{\text{bs}}(\bar{a}, G, H) : G \subseteq H, H \text{ is from } \mathbf{K} \text{ and } \bar{a} \in {}^n H\}$  and  $\mathbf{S}_{\text{bs}}(G) = \bigcup_n \mathbf{S}_{\text{bs}}^n(G)$ ; if  $\mathbf{K}$  is not clear from the context we should write  $\mathbf{S}_{\text{bs}}^n(G, \mathbf{K}), \mathbf{S}_{\text{bs}}(G, \mathbf{K})$ .

**Observation 0.4.** For every  $p \in \mathbf{S}_{\text{bs}}^n(M)$  and  $M \in \mathbf{K}$  there are  $N, \bar{a}$  such that  $M \subseteq N \in \mathbf{K}, \bar{a} \in {}^n N$  realizes  $p, G_N = \text{cl}(G_M + \bar{a}, N)$  and if  $M \subseteq N' \in \mathbf{K}$  and  $\bar{a}'$  realizes  $p$  in  $N'$  then there is  $N'' \subseteq N'$  and an isomorphism  $f$  from  $N$  onto  $N''$  extending  $\text{id}_M$  such that  $f(\bar{a}) = \bar{a}'$ .

*Remark 0.5.* 0) In 0.4 we shall later use the convention of 0.15(1),(3).

1) We are particularly interested in types which are definable in some sense over small sets.

2) We can define “ $p \in \mathbf{S}_{\text{bs}}^n(M)$ ” syntactically, because for a set  $p$  of basic formulas  $\varphi(\bar{x}, \bar{a}), \bar{a}$  from  $M$  which is complete (i.e. if  $\varphi(\bar{x}, \bar{a})$  is an atomic formula over  $M$  then  $\varphi(\bar{x}, \bar{a}) \in p$  or  $\neg\varphi(\bar{x}, \bar{a}) \in p$ ), we have  $p \in \mathbf{S}_{\text{bs}}^n(M)$  iff for every f.g.  $N \subseteq M$  we have  $p|N := \{\varphi(\bar{x}, \bar{a}) \in p : \bar{a} \subseteq N\} \in \mathbf{S}_{\text{bs}}^n(N)$ .

3) Why do we use below types which do not split over a finite subgroup and the related set of schemes? As we like to get a canonical extension of  $M \in \mathbf{K}$  it is natural to use a set of types closed under automorphisms of  $M$ , and as their number is preferably  $\leq \|M\|$ , it is natural to demand that any such type is, in some sense, definable over some finite subset of  $M$ .

As in [She71]:

**Definition 0.6.** We say that  $p = \text{tp}_{\text{bs}}(\bar{a}, G, H) \in \mathbf{S}_{\text{bs}}^n(G)$  does not split over  $K \subseteq G$  when for every  $m < \omega$  and  $\bar{b}_1, \bar{b}_2 \in {}^m G$  satisfying  $\text{tp}_{\text{bs}}(\bar{b}_1, K, G) = \text{tp}_{\text{bs}}(\bar{b}_2, K, G)$  we have  $\text{tp}_{\text{bs}}(\bar{b}_1 \hat{\ } \bar{a}, K, H) = \text{tp}_{\text{bs}}(\bar{b}_2 \hat{\ } \bar{a}, K, H)$ .

**Definition 0.7.** 1) Let  $\mathbf{D}(\mathbf{K}) = \bigcup_n \mathbf{D}_n(\mathbf{K})$ , where  $\mathbf{D}_n(\mathbf{K}) = \{\text{tp}_{\text{bs}}(\bar{a}, \emptyset, M) : \bar{a} \in {}^n M \text{ and } M \in \mathbf{K}\}$ .

2) Assume<sup>1</sup>  $p(\bar{x})$  is a  $k$ -type, that is,  $\bar{x} = \langle x_\ell : \ell < k \rangle$  and for some  $p'(\bar{x})$  we have  $p(\bar{x}) \subseteq p'(\bar{x}) \in \mathbf{D}_k(\mathbf{K})$  and  $m < \omega$ . We let  $\mathbf{D}_{p(\bar{x}), m}(\mathbf{K}) = \mathbf{D}_m(p(\bar{x}), \mathbf{K})$  be the set of  $q(\bar{x}, \bar{y}) \in \mathbf{D}_{k+m}(\mathbf{K})$  such that  $q(\bar{x}, \bar{y}) \supseteq p(\bar{x})$ , which means that there is  $M \in \mathbf{K}$  and  $\bar{a} \in {}^k M$  realizing  $p(\bar{x})$  and  $(\bar{a}, \bar{b})$  realizing  $q(\bar{x}, \bar{y})$  in  $M$ , i.e.  $\ell g(\bar{a}) = k, \ell g(\bar{b}) = m$  and  $\bar{a} \hat{\ } \bar{b}$  realizes  $q(\bar{x}, \bar{y})$ .

3) In part (2) let  $\mathbf{D}_{p(\bar{x})}(\mathbf{K}) = \cup \{\mathbf{D}_m(p(\bar{x}), \mathbf{K}) : m < \omega\}$ .

*Remark 0.8.* Below  $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$  is a scheme to fully define a type  $q(\bar{z}) \in \mathbf{S}_{\text{bs}}^n(M)$  for a given parameter  $\bar{a} \in {}^k M$  such that  $q(\bar{z})$  does not split over  $\bar{a}$ . Sometimes  $\mathfrak{s}$  is not unique but if, e.g.,  $M \in \mathbf{K}_{\text{exlf}}$  it is.

**Definition 0.9.** 1) Let  $\Omega[\mathbf{K}]$  be the set of schemes, i.e.  $\cup \{\Omega_{n,k}[\mathbf{K}] : k, n < \omega\}$  where  $\Omega_{n,k}[\mathbf{K}]$  is the set of  $(k, n)$ -schemes  $\mathfrak{s}$  which means, see below.

1A) We say  $\mathfrak{s}$  is a  $(k, n)$ -scheme when for some  $p(\bar{x}) = p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$  with  $\ell g(\bar{x}_{\mathfrak{s}}) = k$ , (and  $k_{\mathfrak{s}} = k(\mathfrak{s}) = k, n_{\mathfrak{s}} = n(\mathfrak{s}) = n$ ) we have:

- (a)  $\mathfrak{s}$  is a function with domain  $\mathbf{D}_{p(\bar{x})}(\mathbf{K})$  such that for each  $m$  it maps  $\mathbf{D}_{p(\bar{x}), m}(\mathbf{K})$  into  $\mathbf{D}_{k+m+n}(\mathbf{K})$
- (b) if  $s(\bar{x}, \bar{y}) \in \mathbf{D}_{p(\bar{x}), m}(\mathbf{K})$  and  $r(\bar{x}, \bar{y}, \bar{z}) = \mathfrak{s}(s(\bar{x}, \bar{y}))$  then  $r(\bar{x}, \bar{y}, \bar{z}) \upharpoonright (k+m) = s(\bar{x}, \bar{y})$ ; that is, if  $(\bar{a}, \bar{b}, \bar{c})$ , i.e.  $\bar{a} \hat{\ } \bar{b} \hat{\ } \bar{c}$ , realizes  $r(\bar{x}, \bar{y}, \bar{z})$  in  $M \in \mathbf{K}$  so  $k = \ell g(\bar{a}), m = \ell g(\bar{b}), n = \ell g(\bar{c})$ , then  $\bar{a} \hat{\ } \bar{b}$  realizes  $s(\bar{x}, \bar{y})$  in  $M$ ; see 1.2(1)

<sup>1</sup>This is used to define the set  $\mathfrak{S}$  of schemes; for this section the case  $p(\bar{x}) = p'(\bar{x})$  is enough as we can consider all the completions but the general version is more natural in counting a set  $\mathfrak{S}$  of schemes and in considering actual examples.

(c) in clause (b), moreover if  $\bar{b}' \in {}^{\omega}M$ ,  $\text{Rang}(\bar{b}') \subseteq \text{Rang}(\bar{a} \hat{\ } \bar{b})$  then  $\bar{a} \hat{\ } \bar{b}' \hat{\ } \bar{c}$  realizes the type  $\mathfrak{s}(\text{tp}_{\text{bs}}(\bar{a} \hat{\ } \bar{b}', \emptyset, M))$ ; this is to avoid  $\mathfrak{s}$ 's which define contradictory types<sup>2</sup>.

2) Assume  $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$  and  $M \in \mathbf{K}$  and  $\bar{a} \in {}^kM$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ ; we let  $q_{\mathfrak{s}}(\bar{a}, M)$  be the unique  $r(\bar{z}) = r(z_{\mathfrak{s}}) \in \mathbf{S}_{\text{bs}}^n(M)$  such that for any  $\bar{b} \in {}^{\omega}M$  letting  $r_{\bar{b}}(\bar{x}, \bar{y}, \bar{z}) := \mathfrak{s}(\text{tp}_{\text{bs}}(\bar{a} \hat{\ } \bar{b}, \emptyset, M))$  we have  $r_{\bar{b}}(\bar{a}, \bar{b}, \bar{z}) \subseteq r(\bar{z})$ .

3) We call  $\mathfrak{s}$  full when  $p_{\mathfrak{s}}(\bar{x}) \in \mathbf{D}_{k(\mathfrak{s})}(\mathbf{K})$ .

4) For technical reasons we allow  $\bar{x}_{\mathfrak{s}} = \langle x_{\mathfrak{s},\ell} : \ell \in u \rangle$ ,  $u \subseteq \mathbb{N}$ ,  $|u| = k_{\mathfrak{s}}$  and in this case  ${}^{k(\mathfrak{s})}M$  will mean  ${}^uM = \{a_{\ell} : \ell \in u\}$  and we do not pedantically distinguish between  $u$  and  $k_{\mathfrak{s}}$ . Similarly for  $n_{\mathfrak{s}}$  and  $\bar{z}$ , the reason is 1.1, 1.6(4).

**Convention 0.10.**  $\mathfrak{S}$  will denote a subset of  $\Omega[\mathbf{K}]$ .

§ 0(C). **The Results.** In particular (in the so-called first avenue, see below):

**Theorem 0.11.** *Let  $\lambda$  be any cardinal  $\geq |\mathfrak{S}|$ .*

1) *For every  $G \in \mathbf{K}_{\leq \lambda}^{\text{lf}}$  there is  $H_G \in K_{\lambda}^{\text{exlf}}$  which is  $\lambda$ -full over  $G$  (hence over any  $G' \subseteq G$ ; see Definition 1.15) and  $\mathfrak{S}$ -constructible over it (see 1.19).*

2) *If  $H \in \mathbf{K}_{< \lambda}^{\text{lf}}$  is  $\lambda$ -full over  $G \in \mathbf{K}_{\leq \lambda}^{\text{lf}}$  then  $H_G$  from above can be embedded into  $H$  over  $G$ , see 1.23(4).*

This is proved by 1.23 + §2. So in some sense  $H_G$  is prime over  $G$ , that is, it is prime but not among the members of  $\mathbf{K}_{\lambda}^{\text{exlf}}$ , i.e. for a different class. Still we would like to have canonicity so uniqueness. There are some additional avenues helpful toward this.

The second avenue tries to get results which are nicer by assuming  $\mathfrak{S}$  is so called symmetric which is the parallel of being stable in this context. Under this assumption we prove the existence of a canonical closure of a locally finite group to an exlf one. This is done in 1.12, 1.13.

The third avenue is without assuming “ $\mathfrak{S}$  is symmetric” but using a more complicated construction, for which we have similar, somewhat weaker results using special linear orders. The failure of symmetry seems to draw you to order the relevant pairs  $(\mathfrak{s}, \bar{a})$  for  $G$ . That is, trying to repeat the construction in 1.12(2), without symmetry we have to well order or at least linearly order  $\text{def}(G) = \text{def}_{\mathfrak{S}}(G)$  which is essentially the set of relevant complete quantifier types over  $G$  over a finite set of parameters, see Definition 1.1; this suffices by 1.8(9). At first glance we have to linearly order  $\text{def}(G)$ , but we take a list of  $\text{def}(G)$ , with each appearing  $\lambda$  times and linearly order it such that it does not induce a linear order of  $\text{def}(G)$ . See below. So we prove (in 1.30, 1.31, 1.33)

**Theorem 0.12.** 1) *We can for every lf group  $G$ , define  $G^{cl}$  such that:*

- (a) *if  $G \in K_{\leq \lambda}^{\text{lf}}$  then  $G \subseteq G^{cl} \in K_{\lambda}^{\text{exlf}}$*
- (b)  *$G^{cl}$  is unique up to isomorphism over  $G$ .*

<sup>2</sup>But some  $\mathfrak{s}$ 's satisfying clauses (a),(b) of 0.9(1A) but failing clause (c) this may give a consistent type in an interesting class of cases.

2) Also<sup>3</sup> essentially it commutes with extensions, i.e.  $G_1 \subseteq G_2 \Rightarrow G_1^{cl} \subseteq G_2^{cl}$ , pedantically

(c) if  $G_1 \subseteq G_2$  and  $G_1^{cl}$  is as above then there is an embedding  $h$  of  $G_1^{cl}$  into  $G_2^{cl}$  such that  $h(G_1^{cl}) \cap G_2 = G_1$

(c)' restricting ourselves to  $\{G \in \mathbf{K}_{\text{lf}} : \text{every } x \in G \text{ is a singleton}\}$  we have:

(b)''  $G^{cl}$  is really unique

(c)''  $G_1 \subseteq G_2 \Rightarrow G_1^{cl} \subseteq G_2^{cl}$ .

To stress the generality in addition to the class  $\mathbf{K}_{\text{lf}}$  of lf-groups we use  $\mathbf{K}_{\text{olf}}$ , the class of ordered locally finite groups (see 0.15); for them the proof of the existence of a suitable  $\mathfrak{S}$  is easier. Naturally for  $\mathbf{K}_{\text{olf}}$  we certainly do not have a symmetric  $\mathfrak{S}$ .

In §2 we show that  $\mathfrak{S}$  as needed in §1 exists, but not necessarily symmetric and define and investigate some specific schemes used later; also we define and investigate NF, a relative of free amalgamation. In §3 we find a fourth avenue which is more specific to the class of lf groups. We show that we can induce symmetry, i.e. define symmetric constructions even for non-symmetric  $\mathfrak{S}$  hence get somewhat better results, see 3.15. In particular we construct reasonable closures.

In §4(A), we show that we can find amalgamation preserving commuting and so can get a new relative NF<sup>3</sup> of NF. In §4(B) we deal with some related schemes (of types). In §4(C) we deal with types with infinitely many variables.

In §5 we prove the existence of a complete group  $G_* \in \mathbf{K}_{\lambda}^{\text{exlf}}$  when  $\lambda = \mu^+$ ,  $\mu = \mu^{\aleph_0}$ . Moreover, we prove the existence of a complete extension  $G_* \in \mathbf{K}_{\lambda}^{\text{exlf}}$  of an arbitrary  $G \in \mathbf{K}_{\leq \mu}^{\text{lf}}$ .

Some of the definitions and claims work also in quite a general framework, but it is not clear at present how interesting this is. Still we consider some expansions of  $\mathbf{K}_{\text{lf}}$ , and comment on them in §6.

We here also consider the partial order  $\leq_{\mathfrak{S}}$  on  $\mathbf{K}$ , where  $G_1 \leq_{\mathfrak{S}} G_2$  means that every finite  $\bar{a} \subseteq G_2$  realizes over  $G_2$  a type from  $\text{def}_{\mathfrak{S}}(G_1)$ . Note that on  $(\mathbf{K}, \leq_{\mathfrak{S}})$  we may generalize stability theory, in particular when  $\mathfrak{S}$  is symmetric (see §1) or when we use the symmetrized version (see §3). In particular, we can investigate orthogonality, parallelism, super-stability, and indiscernible sets which  $\Delta$ -converge ([She87] or [Shel]). A class somewhat similar to  $\mathbf{K}_{\text{lf}}$ , for an existentially closed countable group  $L$  is  $\mathbf{K}_L$ , the class of groups  $G$  such that every f.g. subgroup is embeddable into  $L$ . We further investigate  $\mathbf{K}_{\text{lf}}$  in [Shee] and in more general direction in a work in preparation with G. Paolini.

We thank Omer Zilberboim and Gianluca Paolini for some help in the proofs of this paper and a referee for many useful comments to clarify this paper.

## § 0(D). Preliminaries.

**Definition 0.13.** 1) Let  $\mathbf{K}_{\lambda}^{\text{lf}}$  be the class of  $G \in \mathbf{K}_{\text{lf}}$  of cardinality  $\lambda$ , let  $\mathbf{K}_{\lambda}^{\text{exlf}}$  be the class of  $G \in \mathbf{K}_{\text{exlf}}$  of cardinality  $\lambda$ ; see Definition 0.1.

2) Let  $\text{fsb}(M)$  be the set of f.g. (finitely generated) sub-structures of  $M$ .

Note that  $\mathbf{K}_{\text{exlf}}$  is the same  $\mathbf{K}_{\text{ulf}}$  as defined by Hall as proved in Macintyre-Shelah [MS76], Wood [Woo72]; that is:

<sup>3</sup>See on this in 3.14.

**Claim 0.14.** *The following conditions on a locally finite group  $G$  are equivalent:*

(A)  $G$  is ulf which means:

- (a) every finite group is embeddable into  $G$
- (b) if  $H_1, H_2$  are isomorphic finite subgroups of  $G$ , then for some  $x \in G$ , conjugation by  $x$  maps  $H_2$  onto (here equivalently into)  $H_1$ , i.e.  $x^{-1}H_2x = H_1$

(B)  $G \in \mathbf{K}_{\text{exlf}}$ .

*Proof.* (B)  $\Rightarrow$  (A)

Clause (A)(a): let  $H$  be a finite group, let  $H_1 = \{e_H\} \subseteq H$  so a sub-group of  $H$  and let  $H_2 = H$  and let  $h_1 : H_1 \rightarrow G$  be defined by  $h_1(e_H) = e_G$ . So by clause (B) there is an extension  $h_2$  of  $h_1$  embedding  $H_2 = H$  into  $G$ , so  $h_2(H)$  is as required.

Clause (A)(b): let  $H_1, H_2 \subseteq G$  be finite sub-groups and let  $H_3 \subseteq G$  be the finite subgroup which  $H_1 \cup H_2$  generates. There is a finite group  $H_4$  extending  $H_3$  such that: any partial automorphism of  $H_3$  is included in some conjugation in  $H_4$ . Let  $h_3 : H_3 \rightarrow H_3 \subseteq G$  be the identity, hence by Clause (B) recalling  $G \in \mathbf{K}_{\text{exlf}}$ , there is an embedding  $h_4$  of  $H_4$  into  $G$  extending  $h_3$ .

So in  $h_4(H_4) \subseteq G$  there is a conjugation as required.

(A)  $\Rightarrow$  (B):

Let  $H_1 \subseteq H_2$  be finite groups and  $h_1$  be an embedding of  $H_1$  into  $G$ . Let  $H_4 \supseteq H_2$  be a finite group such that any automorphism of  $H_1$  is included in an inner automorphism of  $H_4$ . By Clause (A)(a) there is an embedding  $h_4$  of  $H_4$  into  $G$ . By Clause (A)(b) there is  $x \in G$  such that  $H'_4 := x^{-1}h_4(H_4)x \subseteq G$  is equal to  $h_1(H_1)$ .

Recalling 0.23(7)  $h'_4 = (\square_x \upharpoonright h_4(H_4)) \circ h_4$  embeds  $H_4$  into  $G$  and maps  $H_1$  onto  $h_1(H_1)$ ; but the embedding  $h'_4$  does not necessarily extend  $h_1$ . However, by clause (A)(b), for some  $y \in h'_4(H_4)$ ,  $\square_y h'_4$  embeds  $H_4$  (hence  $H_2$ ) and extends  $h_1$  as required.  $\square_{0.14}$

We may use the class  $\mathbf{K}_{\text{olf}}$  of linearly ordered lf groups, it is closely related and some issues are more transparent for it;  $\mathbf{K}_{\text{olf}}$  is defined as follows.

**Definition 0.15.** 1) Let  $\mathbf{K}_{\text{olf}}$  be the class of structures  $M$  which are an expansion of a lf group  $G = G_M$  by a linear order  $<_M$ , also this class is partially ordered by  $M_1 \subseteq M_2, M_1$  a sub-structure of  $M_2$ .

2) We say that  $M \in \mathbf{K}_{\text{olf}}$  is existentially closed as in 0.13(2) and define  $\mathbf{K}_{\lambda}^{\text{olf}}$  as in 0.1(2).

3) If  $M \in \mathbf{K}_{\text{lf}}$  then we let  $G_M = M$ .

*Remark 0.16.* For  $\mathbf{K}_{\text{lf}}$  conceivably there is a symmetric dense  $\mathfrak{S}$ , hence a very natural canonical exlf-closure. Without it we can either use a somewhat less natural one (using linear orders, see end of §1) or “make it symmetric by brute force” (see §3). But for the class  $\mathbf{K}_{\text{olf}}$  we can use only the linear orders, so every  $M$  has a canonical existentially closed extension, but it is more difficult to make it unique up to isomorphism. We shall in 6.2 introduce another class,  $\mathbf{K}_{\text{clf}}$ , locally finite groups with choice.



**Convention 0.17.** 1) Except in §6,  $\mathbf{K}$  is the class  $\mathbf{K}_{\text{lf}}$  of locally finite groups or  $\mathbf{K}_{\text{olf}}$  of ordered locally finite groups (we may use  $\leq_{\mathbf{K}}$  but here  $\mathbf{K}$  is partially ordered by  $\subseteq$ , being a substructure) and see 0.16.

2) Let xlf-group mean a member of  $\mathbf{K}$ . Let  $\mathbf{K}_{\text{ec}}$  be the class of existentially closed members of  $\mathbf{K}$ .

3) In §2, §3, §4, §5 we use only  $\mathbf{K}_{\text{lf}}$ ; in §1 you can restrict yourself to  $\mathbf{K} = \mathbf{K}_{\text{lf}}$  but in §6 we have further cases on which we comment.

The following definition is for the more general framework.

**Definition 0.18.** 1) For  $M, N \in \mathbf{K}$  let  $M \leq_{\text{fsb}} N$  mean that if  $K \subseteq L$  are f.g.,  $K \subseteq M, L \subseteq N$ , then there is an embedding of  $L$  into  $M$  over  $K$ .

2) For  $M, N \in \mathbf{K}$  let  $M \leq_{\Sigma_1} N$  means that  $M \subseteq N$  and if  $\bar{a} \in {}^{\ell g(\bar{y})}M, \bar{b} \in {}^{\ell g(\bar{x})}N$  and  $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_{\mathbf{K}})$  is quantifier free and  $N \models \varphi[\bar{b}, \bar{a}]$  then for some  $\bar{b}' \in {}^{\ell g(\bar{x})}M$  we have  $M \models \varphi[\bar{b}', \bar{a}]$ .

3) Let  $M_\ell \in \mathbf{K}, \bar{a}_\ell \in {}^{n(\ell)}(M_\ell)$  for  $\ell = 1, 2$ . We say that a relation on  $M_1 \times M_2$  is quantifier-free definable in  $(M_1, \bar{a}_1, M_2, \bar{a}_2)$  when it is a Boolean combination of finitely many simple ones, where  $R$  is a simple  $n$ -place relation on  $M_1 \times M_2$  when  $R$  is the set of  $n$ -tuples  $((b_0, c_0), \dots, (b_{n-1}, c_{n-1}))$  such that  $b_i \in M_1, c_i \in M_2$  for  $i < n$  and

$$M_1 \models \varphi_1[b_0, \dots, b_{n-1}, \bar{a}_1]$$

$$M_2 \models \varphi_2[c_0, \dots, c_{n-1}, \bar{a}_2]$$

for some quantifier-free formulas  $\varphi_1, \varphi_2$  in  $\mathbb{L}(\tau_{\mathbf{K}})$  and finite sequences  $\bar{a}_1, \bar{a}_2$  from  $M_1, M_2$  respectively.

*Remark 0.19.* 1) Note that 0.18(3) is not actually used, but just indicate the form of definability used.

2) Note that  $\leq_{\Sigma_1}$  for  $\mathbf{K}_{\text{lf}}$  and  $\mathbf{K}_{\text{olf}}$  is the same as  $\leq_{\text{fsb}}$ . For other classes, see §6, if the vocabulary is finite and we deal with locally finite structures they are still the same. Otherwise, by our choice of “does not split” we have to use  $\leq_{\text{fsb}}$ . But if we prefer to use  $\leq_{\Sigma_1}$  we have to strengthen the definition of “does not split” to make the proof of 1.10(1) work.

**Convention 0.20.** Let  $M_1, M_2 \in \mathbf{K}, M_1 \subseteq M_2$  and  $\bar{a} \in {}^n(M_2)$ , so  $\bar{a} = (a_0, a_1, a_2, \dots, a_{n-1})$ .

1) Denote by  $cl(M_1 + \bar{a}, M_2)$  the sub-structure generated by  $M_1 \cup \bar{a} = M_1 \cup \{a_0, a_1, \dots, a_{n-1}\}$  in  $M_2$ .

2) For a group  $G$  and  $A \subseteq G$  let

- $\mathbf{C}_G(A) = \{g \in G : G \models \text{“}ag = ga\text{” for every } a \in A\}$
- $\mathbf{Z}(G) = \mathbf{C}_G(G)$
- $\mathbf{N}_G(A) = \{c \in G : c^{-1}Ac = A\}$ .

4) For a group  $G$ ,  $\text{aut}(G)$  is the group of automorphisms of  $G$  and  $\text{inner}(G)$  is the normal subgroup of  $\text{aut}(G)$  consisting of the inner automorphisms of  $G$ .

A side issue here is:

**Definition 0.21.** 1) For a class  $\mathbf{K}$  of structures (of a fixed vocabulary) we say  $M \in \mathbf{K}$  is  $\lambda$ -universal in  $\mathbf{K}$  when every  $N \in \mathbf{K}$  of cardinality  $\lambda$  can be embedded into it.

2) We say  $M \in \mathbf{K}$  is  $(\leq \lambda)$ -universal in  $\mathbf{K}$  when every  $N \in \mathbf{K}$  of cardinality  $\leq \lambda$  can be embedded into  $M$ .

3) We say  $M \in \mathbf{K}$  is universal when it is  $\lambda$ -universal for  $\lambda = \|M\|$ .

4) Assume  $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}}), K_{\mathfrak{k}}$  as a class of  $\tau$ -structures (for some vocabulary  $\tau = \tau_{\mathfrak{k}}$ ), closed under isomorphism, and  $\leq_{\mathfrak{k}}$  a partial order on  $K_{\mathfrak{k}}$  preserved under isomorphisms. Above “ $M \in K_{\mathfrak{k}}$  is  $\lambda$ -universal in  $\mathfrak{k}$ ” means that if  $N \in K_{\mathfrak{k}}$  has cardinality  $\lambda$  then there is a  $\leq_{\mathfrak{k}}$ -embedding  $f$  of  $N$  into  $M$ , i.e.  $f$  is an isomorphism from  $N$  onto some  $N' \leq_{\mathfrak{k}} M$ . Similarly in the other variants.

The problem of the existence of universal members of  $\mathbf{K}_{\lambda}^{\text{lf}}$  is connected to

*Question 0.22.* Fixing  $\kappa$  and an ideal  $J$  on  $\kappa$ , what is  $\lambda_{\mu, \kappa}(J, \mathbf{K})$ , which is the minimal cardinal (or  $\infty$ )  $\lambda$  which is  $> \mu$  and there is no sequence  $\langle (G_{\alpha}, \bar{a}_{\alpha}) : \alpha < \lambda \rangle$  such that  $G_{\alpha} \in \mathbf{K}_{\leq \mu}, \bar{a}_{\alpha} \in {}^{\kappa}(G_{\alpha})$  and there are no  $H \in \mathbf{K}$  and  $\alpha < \beta < \lambda$  and embeddings  $f_1, f_2$  of  $G_{\alpha}, G_{\beta}$  respectively into  $H$  such that  $\{i < \kappa : f(a_{\alpha, i}) \neq a_{\beta, i}\} \in J$ .

*Notation 0.23.* 1) Let  $G, H, K$  denote members of  $\mathbf{K}$ .

2) Let  $p, q, r$  and  $s$  denote types.

3)  $\mathfrak{s}$  denotes a scheme of defining types, here qf.

4)  $t$  denotes a member of some  $\text{def}(G)$ , i.e. a pair  $(\mathfrak{s}, \bar{a})$  which defines a type in  $\mathbf{S}_{\text{bs}}^{n(\mathfrak{s})}(G)$ .

5) For  $A \subseteq M$  let  $\text{cl}(A, M) = \langle A \rangle_M$  be the closure of the set  $A$  under the functions of  $M$ , i.e. the sub-structure of  $M$  which  $A$  generates when  $M$  is, as usual, a group.

6) We may write, e.g.  $A+B, A+\bar{a}, \sum_{i < \alpha} \bar{a}_i$  instead of  $A \cup B, A \cup \text{Rang}(\bar{a}), \bigcup_{i < \alpha} \text{Rang}(\bar{a}_i)$ .

7) For a group  $G$  and  $x \in G$  let  $\square_x$  be conjugation by  $x$ , that is, the mapping  $y \mapsto x^{-1}yx$  for  $y \in G$ .

## § 1. DEFINABLE TYPES

What is accomplished in §1 and under what assumptions? We have to assume that there are dense  $\mathfrak{S} \in \Omega[\mathbf{K}]$  to get existentially closed  $H$  (see §2). Still there are  $\mathfrak{S}$ 's and any  $\mathfrak{S}$  can be extended to a closed one, preserving density. For any  $\mathfrak{S}$ , the partial order  $\leq_{\mathfrak{S}}$  on  $\mathbf{K}$  is quite reasonable: not fully so called a.e.c. still close enough. In  $(\mathbf{K}, \leq_{\mathfrak{S}})$  for regular  $\lambda$  we can find over any  $G \in K_{\leq \lambda}$  a prime  $H$  among the  $H \in \mathbf{K}_{\lambda}$  extending  $G$  which are so-called  $(\lambda, \mathfrak{S})$ -full over it, see 1.23. Also we can find such  $H$  quite definable in three ways. First avenue is to allow a well order. Second avenue is to assume  $\mathfrak{S}$  is symmetric, then  $H$  is canonical and commutes with extensions (1.13, 1.16, 1.23, 1.17). Third avenue relies on linear order. We still get uniqueness, but rely on linear ordering of  $\text{def}(G)$  and the commutation with extension is problematic. However, we may use pair  $(I, E)$ ,  $I$  a linear order,  $E$  an equivalence relation on  $I$  and “dedicate” each equivalence class to some  $t \in \text{def}(G)$ , so can avoid linearly ordering  $\text{def}(G)$ , see 1.30, 1.33; see more in §3.

## § 1(A). The Framework.

**Definition 1.1.** 1) For  $G \in \mathbf{K}$  let  $\text{def}(G)$  be the set of pairs  $t = (\mathfrak{s}, \bar{a}) = (\mathfrak{s}_t, \bar{a}_t)$  such that  $\mathfrak{s} \in \Omega[\mathbf{K}]$  and  $\bar{a} \in {}^{\omega}G$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$  and let  $q_t(G) = q_{\mathfrak{s}_t}(\bar{a}_t, G)$  and  $p_t(\bar{x}_t) = p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ ,  $k(t) = k(\mathfrak{s})$ ,  $n(t) = n(\mathfrak{s})$ .

2) We say  $\mathfrak{s}_1, \mathfrak{s}_2$  are disjoint when  $\bar{x}_{\mathfrak{s}_1}, \bar{x}_{\mathfrak{s}_2}$  are disjoint as well as  $\bar{z}_{\mathfrak{s}_1}, \bar{z}_{\mathfrak{s}_2}$  recalling 0.9(4). Similarly for  $t_1, t_2 \in \text{def}(G)$ .

3) We say  $\mathfrak{s}_1, \mathfrak{s}_2$  are congruent, written  $\mathfrak{s}_1 \equiv \mathfrak{s}_2$  when we get  $\mathfrak{s}_2$  from  $\mathfrak{s}_1$  by replacing  $\bar{x}_{\mathfrak{s}_1}, \bar{z}_{\mathfrak{s}_1}$  by other sequences of variables,  $\bar{x}_{\mathfrak{s}_2}, \bar{z}_{\mathfrak{s}_2}$  (again with no repetitions, of the same length respectively, of course). Similarly for  $t_1, t_2 \in \text{def}(G)$  (the aim is to be able to get disjoint congruent copies; we do not always remember to replace a scheme by some congruent copy).

4) We say  $\mathfrak{S}$  is invariant when: if  $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$  are congruent then  $\mathfrak{s}_1 \in \mathfrak{S} \Leftrightarrow \mathfrak{s}_2 \in \mathfrak{S}$ .

5) The invariant closure of  $\mathfrak{S}$  is defined naturally. Let  $|\mathfrak{S}|$  mean its cardinality up to congruency, that is,  $|\mathfrak{S}/\equiv|$ ; if not said otherwise we use invariant  $\mathfrak{S}$ .

6) We define the (equivalence) relation  $\approx_G$  on  $\text{def}(G)$  by  $t_1 \approx_G t_2$  iff  $t_1, t_2 \in \text{def}(G)$  and  $q_{t_1}(G) = q_{t_2}(G)$ .

**Claim 1.2.** 1) If  $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$  and  $G \in \mathbf{K}$ ,  $\bar{a} \in {}^k M$  then indeed  $q_{\mathfrak{s}}(\bar{a}, G) \in \mathbf{S}_{\text{bs}}^n(G)$  so exist and is unique and does not split over  $\bar{a}$ , see Definition 0.9(2); if  $\bar{a}$  is empty, i.e.  $k_{\mathfrak{s}} = 0$  we may write  $q_{\mathfrak{s}}(G)$ .

1A) If  $G_1 \subseteq G_2 \subseteq \mathbf{K}$  and  $t \in \text{def}(G_1)$  then  $t \in \text{def}(G_2)$  and  $q_t(G_1) \subseteq q_t(G_2)$ .

2) Assume  $G \subseteq H \in \mathbf{K}$  and  $G$  is existentially closed or just  $G \leq_{\Sigma_1} H \in \mathbf{K}$ . If  $t_1, t_2 \in \text{def}(G)$  then  $q_{t_1}(G) = q_{t_2}(G)$  iff  $q_{t_1}(H) = q_{t_2}(H)$ .

3) Let  $K \subseteq G \in \mathbf{K}$ ,  $G$  be existentially closed or just every  $r \in \mathbf{S}_{\text{bs}}^{<\omega}(K)$  is realized in  $G$ ,  $K$  is finite, and  $p \in \mathbf{S}_{\text{bs}}^n(G)$ .

The type  $p$  does not split over  $K$  iff there are  $\mathfrak{s} \in \Omega[\mathbf{K}]$  and a finite sequence  $\bar{a}$  from  $K$  (even listing  $K$ ) realizing  $p_{\mathfrak{s}}(\bar{x})$  such that  $p = q_{\mathfrak{s}}(\bar{a}, M)$ .

4) If  $G \subseteq H$ ,  $\mathfrak{s} \in \Omega[\mathbf{K}]$ ,  $\bar{a} \in {}^{k(\mathfrak{s})}G$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$  and  $\bar{c} \in {}^{n(\mathfrak{s})}H$  realizes  $q_{\mathfrak{s}}(\bar{a}, G)$  in  $H$  and  $\sigma(\bar{z}_{\mathfrak{s}}, \bar{x}_{\mathfrak{s}})$  is a group-term then  $\sigma^H(\bar{c}, \bar{a}) \in G \Rightarrow \sigma^H(\bar{c}, \bar{a}) \in \text{cl}(\bar{a}, G)$ .

4A) In (4), if  $\bar{a}' = \bar{a} \hat{\ } \bar{a}''$  then  $\sigma^H(\bar{c}, \bar{a}'') \in G \Rightarrow \sigma^H(\bar{c}, \bar{a}'') \in \text{cl}(\bar{a}'', G)$  because  $p$  also does not split over  $\bar{a}^*$  if  $\bar{a}^* \subseteq G$ ,  $\bar{a} \subseteq \text{cl}(\bar{a}^*, H)$ .

*Proof.* 1) Let  $K_* \subseteq G$  be the subgroup of  $G$  generated by  $\bar{a}$ .

First, there are  $H$  and  $\bar{c}$  such that:

$$(*)_{H, \bar{c}}^1 \quad G \subseteq H \in \mathbf{K} \text{ and } \bar{c} \in {}^n H \text{ such that } \text{tp}(\bar{c}, G, H) = q_{\mathfrak{s}}(\bar{a}, G).$$

Why? For every  $K \in \mathbf{K}^* := \{K \subseteq G : K \text{ finite extending } K_*\}$  we can choose a pair  $(H_K, \bar{c}_K)$  such that:  $K \subseteq H_K \in \mathbf{K}$ ,  $H_K$  is finite,  $\bar{c}_K \in {}^n(H_K)$ ,  $H_K$  is generated by  $K \cup \bar{c}_K$  and for some  $\bar{b}$  listing  $K$ ,  $\text{tp}_{\text{bs}}(\bar{a} \hat{\ } \bar{b} \hat{\ } \bar{c}_K, \emptyset, H_K) = \mathfrak{s}(\text{tp}_{\text{bs}}(\bar{a} \hat{\ } \bar{b}, \emptyset, G))$ .

[Why? By Definition 0.9(1A)(b).] Now for every  $K_1 \subseteq K_2$  from  $\mathbf{K}_*$  we can choose an embedding  $f_{K_2, K_1}$  from  $H_{K_1}$  into  $H_{K_2}$  extending  $\text{id}_{K_1}$  and mapping  $\bar{c}_{K_1}$  to  $\bar{c}_{K_2}$ . [Why? By Definition 0.9(1A)(c).]

As  $H_{K_1}$  is generated by  $K_1 \cup \bar{c}$ , this mapping is unique. Now if  $K_1 \subseteq K_2 \subseteq K_3$  are from  $\mathbf{K}_*$  then  $f_{K_3, K_2} \circ f_{K_2, K_1}$  is an embedding of  $H_{K_1}$  into  $H_{K_3}$  extending  $\text{id}_{K_1}$  and mapping  $\bar{c}_{K_1}$  to  $\bar{c}_{K_3}$ ; hence by the previous sentence  $f_{K_3, K_2} \circ f_{K_2, K_1} = f_{K_3, K_1}$ . Hence  $\langle H_{K_1}, f_{K_2, K_1} : K_1 \subseteq K_2 \text{ are from } \mathbf{K}_* \rangle$  has a direct limit, i.e. we can find a group  $H$  and  $\bar{f} = \langle f_K : K \in \mathbf{K}_* \rangle$  such that  $f_K$  embed  $H_K$  into  $H$  and for every  $K_1 \subseteq K_2$  from  $\mathbf{K}_*$  we have  $f_{K_1} = f_{K_2} \circ f_{K_2, K_1}$ . Without loss of generality  $H = \cup \{f_K(H_K) : K \in \mathbf{K}_*\}$  hence  $H$  is a locally finite group and  $\{f_K : K \in \mathbf{K}_*\}$  embeds  $G$  into  $H$ , so without loss of generality  $G \subseteq H$  and  $f_K \upharpoonright K = \text{id}_K$  for  $K \in \mathbf{K}_*$ . Letting  $\bar{c} = f_K(\bar{c}_K)$  for any  $K \in \mathbf{K}_*$ , clearly  $(H, \bar{c})$  is as required in  $(*)_{H, \bar{c}}^1$ .

$$(*)_2 \quad \text{tp}_{\text{bs}}(\bar{c}, G, H) \text{ belongs to } \mathbf{S}_{\text{bs}}^n(G).$$

[Why? By the definitions of  $\mathbf{S}_{\text{bs}}^n(G)$  because  $G \subseteq H \in \mathbf{K}$  and  $\bar{c} \in {}^n H$ .]

$$(*)_3 \quad q_{\mathfrak{s}}(\bar{a}, G) \text{ is unique and does not split over } \bar{a}.$$

[Why? See Definition 0.9(1A)(c).]

1A) See Definition 0.9(2).

2) For  $\ell = 1, 2$  we have  $q_{t_\ell}(G) \subseteq q_{t_\ell}(H)$ , moreover,  $q_{t_\ell}(G) = \{\varphi(\bar{z}_{n(t)}, \bar{b}) \in q_{t_\ell}(H) : \bar{b} \subseteq G\}$ . For the other direction, note that  $\bar{a}_{t_1}, \bar{a}_{t_2} \subseteq G$  and assume  $q_{t_1}(H) \neq q_{t_2}(H)$ , hence there are  $m$  and  $\bar{b} \in {}^m H$  and a basic formula  $\varphi(\bar{y}_m, \bar{z}_n)$  such that  $\varphi(\bar{b}, \bar{z}_n) \in q_{t_1}(H)$ ,  $\neg \varphi(\bar{b}, \bar{z}_n) \in q_{t_2}(H)$ . Now there is  $\bar{b}' \in {}^m G$  such that  $\text{tp}_{\text{bs}}(\bar{b}', \bar{a}_{t_1} \hat{\ } \bar{a}_{t_2}, G) = \text{tp}_{\text{bs}}(\bar{b}, \bar{a}_{t_1} \hat{\ } \bar{a}_{t_2}, H)$  because  $G \leq_{\Sigma_1} H$  and our choice of  $\mathbf{K}$ . As  $q_{t_\ell}(H)$  does not split over  $\bar{a}_{t_\ell}$ , clearly  $\varphi(\bar{b}', \bar{z}_n) \in q_{t_\ell}(H) \Leftrightarrow \varphi(\bar{b}, \bar{z}_n) \in q_{t_\ell}(H)$  for  $\ell = 1, 2$ .

Together with an earlier sentence,  $\varphi(\bar{b}', \bar{z}_n) \in q_{t_1}(H)$ ,  $\neg \varphi(\bar{b}', \bar{z}_n) \in q_{t_2}(H)$  hence by the first sentence in the proof of 1.2(2) we have  $\varphi(\bar{b}', \bar{z}_n) \in q_{t_1}(G)$  and  $\neg \varphi(\bar{b}', \bar{z}_n) \in q_{t_2}(G)$  hence  $q_{t_2}(G) \neq q_{t_2}(G)$  so we are also done with the “other” direction.

3) The implication “if” holds by 1.2(1). For the other direction assume  $p$  does not split over  $K$ . As  $K$  is finite, let  $k = |K|$  and let  $\bar{a} \in {}^n K \subseteq {}^n G$  list  $K$ .

We now define  $\mathfrak{s}$  by:

- (a)  $p_{\mathfrak{s}} = \text{tp}_{\text{bs}}(\bar{a}, \emptyset, K)$  so  $k_{\mathfrak{s}} = k$
- (b)  $q = \mathfrak{s}(s(\bar{x}, \bar{y}))$  iff for some  $\bar{b} \in {}^m G$  we have:
  - $s(\bar{x}, \bar{y}) = \text{tp}(\bar{a} \hat{\ } \bar{b}, \emptyset, G)$
  - $q = \text{tp}_{\text{bs}}(\bar{a} \hat{\ } \bar{b} \hat{\ } \bar{c}, \emptyset, G)$  for some  $\bar{c} \in {}^n G$  realizing  $p \upharpoonright (\bar{a} \hat{\ } \bar{b})$ .

Now  $\mathfrak{s}$  is well defined because on the one hand  $p$  does not split over  $\bar{a}$ , and on the other hand  $G$  is existentially closed or just every  $r \in \mathbf{S}_{\text{bs}}^{<\omega}(K)$  is realized in  $G$ .

4) By 1.2(2) without loss of generality  $G$  is existentially closed, assume  $\sigma^H(\bar{c}, \bar{a}) \in G$  and let  $b = \sigma^H(\bar{c}, \bar{a})$ . If  $b \notin \text{cl}(\bar{a}, G)$  there is  $b' \in G \setminus \{b\}$  realizing  $\text{tp}_{\text{bs}}(b, K, G)$  because  $\mathbf{K}$  has disjoint amalgamation for finite members. As  $q_{\mathfrak{s}}(\bar{a}, G)$  does not split over  $\bar{a}$  and  $b', b \in G$  realize the same type over  $\bar{a}$  it follows that  $H \models “(\sigma(\bar{c}, \bar{a}) = b) \equiv (\sigma(\bar{c}, \bar{a}) = b’)”$ , an obvious contradiction.

4A) Should be clear. □<sub>1.2</sub>

**Example 1.3.** There is  $\mathfrak{s} \in \Omega[\mathbf{K}]$  such that:

- (a)  $k_{\mathfrak{s}} = 0$  and  $n_{\mathfrak{s}} = 1$ ;
- (b) if  $G \subseteq H \in \mathbf{K}$  and  $a \in H$ , then:  $a$  realizes  $q_{\mathfrak{s}}(\langle \cdot \rangle, G)$  iff  $a \in H \setminus G$  has order 2 and commute with every member of  $G$ .

**Definition 1.4.** 1) For  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  we define the two place relation  $\leq_{\mathfrak{S}}$  on  $\mathbf{K}$  as follows:  $M \leq_{\mathfrak{S}} N$  iff  $M \subseteq N$  (and they belong to  $\mathbf{K}$ ) and for every  $n < \omega$  and  $\bar{c} \in {}^n N$  we can find  $k < \omega$  and  $\bar{a} \in {}^k M$  and  $\mathfrak{s} \in \mathfrak{S}$  such that  $p_{\mathfrak{s}}(\bar{x}) \subseteq \text{tp}_{\text{bs}}(\bar{a}, \emptyset, M) \in \mathbf{D}_k(\mathbf{K})$  and  $\text{tp}_{\text{bs}}(\bar{c}, M, N) = q_{\mathfrak{s}}(\bar{a}, M)$  recalling  $q_{\mathfrak{s}}(\bar{a}, M) \in \mathbf{S}_{\text{bs}}^n(M)$ .

2) For  $M \in \mathbf{K}$  and  $\mathfrak{S} \subseteq \mathfrak{S}[\mathbf{K}]$  let

- (a)  $\mathbf{S}_{\mathfrak{S}}^n(M) = \{q_{\mathfrak{s}}(\bar{a}, M) : \mathfrak{s} \in \mathfrak{S} \text{ satisfies } n_{\mathfrak{s}} = n \text{ and } \bar{a} \in {}^{k(\mathfrak{s})} M \text{ realizes } p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})\}$
- (b)  $\text{def}_{\mathfrak{S}}(M) = \{t \in \text{def}(M) : \mathfrak{s}_t \in \mathfrak{S}\}$
- (c)  $\mathbf{S}_{\mathfrak{S}}(M) = \cup \{\mathbf{S}_{\mathfrak{S}}^n(M) : n < \omega\}$ .

3) We say  $M \in \mathbf{K}$  is  $\mathfrak{S}$ -existentially closed when for every  $\mathfrak{s} \in \mathfrak{S}$ , finite<sup>4</sup>  $G \subseteq M$  and  $\bar{a} \in {}^{\omega} G$  realizing  $p_{\mathfrak{s}}(\bar{x})$  the type  $q_{\mathfrak{s}}(\bar{a}, G)$  is realized in  $M$ ; (this is equivalent to being existentially closed if  $\mathfrak{S}$  is dense, see Definition 1.6(2) below).

**Definition 1.5.** We say  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is symmetric when : if  $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{S}$ ,  $M \subseteq N$  are from  $\mathbf{K}$  and  $\bar{c}_{\ell} \in {}^{n(\mathfrak{s}_{\ell})} N$  realizes  $q_{\mathfrak{s}_{\ell}}(\bar{a}_{\ell}, M)$  in  $N$  (so  $\bar{a}_{\ell} \in {}^{k(\mathfrak{s}_{\ell})} M$  realizes  $p_{\mathfrak{s}_{\ell}}(\bar{x}_{\mathfrak{s}_{\ell}})$ ) and  $M_{\ell} = \text{cl}(M + \bar{c}_{\ell}, N) \subseteq N$  for  $\ell = 1, 2$  then  $\bar{c}_1$  realizes  $q_{\mathfrak{s}_1}(\bar{a}_1, M_2)$  in  $N$  iff  $\bar{c}_2$  realizes  $q_{\mathfrak{s}_2}(\bar{a}_2, M_1)$  in  $N$ .

**Definition 1.6.** 1) We say  $\mathfrak{S}$  is closed when it is dominating-closed and composition-closed, see below and invariant of course.

1A)  $\mathfrak{S}$  is composition-closed when if  $H_0 \subseteq H_1 \subseteq H_2 \in \mathbf{K}$ ,  $\bar{a}_{\ell} \in {}^{n(\ell)}(H_{\ell})$  for  $\ell = 0, 1, 2$  and  $\text{tp}_{\text{bs}}(\bar{a}_{\ell+1}, H_{\ell}, H_{\ell+1}) = q_{\mathfrak{s}_{\ell}}(\bar{a}_{\ell}, H_{\ell}) \in \mathbf{S}_{\mathfrak{S}}^{n(\ell+1)}(H_{\ell})$  and  $H_{\ell+1} = \text{cl}(H_{\ell} + \bar{a}_{\ell}, H_{\ell+1})$ ,  $\mathfrak{s}_{\ell} \in \mathfrak{S}$  for  $\ell = 0, 1$  then  $\text{tp}_{\text{bs}}(\bar{a}_1 \hat{\ } \bar{a}_2, H_0, H_2) = q_{\mathfrak{s}}(\bar{a}_0, H_0)$  for some  $\mathfrak{s} \in \mathfrak{S} \cap \Omega_{n(1)+n(2), n(0)}[\mathbf{K}]$ .

1B)  $\mathfrak{S}$  is dominating-closed when: if  $H_0 \subseteq H_1 \in \mathbf{K}$ ,  $\bar{a}_1 \in {}^{k(1)}(H_0)$ ,  $\bar{c}_1 \in {}^{n(1)}(H_1)$ ,  $\text{tp}_{\text{bs}}(\bar{c}_1, H_0, H_1) = q_{\mathfrak{s}}(\bar{a}_1, H_0) \in \mathbf{S}_{\mathfrak{S}}^{n(1)}(H_0)$  and  $\bar{c}_2 \in {}^{n(2)}(H_1)$  and  $\bar{a}_2 \in {}^{k(2)}(H_0)$ ,  $\text{Rang}(\bar{a}_2) \supseteq \text{Rang}(\bar{a}_1)$  and  $\bar{c}_2 \subseteq \text{cl}(\bar{a}_2 + \bar{c}_1, H_1)$  then  $\text{tp}(\bar{c}_2, H_0, H_1) = q_{\mathfrak{s}}(\bar{a}_2, H_0)$  for some  $\mathfrak{s} \in \mathfrak{S}$ .

2) We say  $\mathfrak{S}$  is weakly dense when: every  $\mathfrak{S}$ -existentially closed  $G \in \mathbf{K}$  is existentially closed.

3) We say  $\mathfrak{S}$  is dense when: for every  $G_0 \subseteq H \in \mathbf{K}$ ,  $G_0 \subseteq G_1 \in \mathbf{K}$ ,  $G_0, G_1$  are finite and  $\bar{c} \in {}^n(G_1)$  there is  $p(\bar{z}) \in \mathbf{S}_{\mathfrak{S}}^n(H)$  which extends  $\text{tp}_{\text{bs}}(\bar{c}, G_0, G_1)$ . Moreover  $p(\bar{z}) = q_{\mathfrak{s}}(\bar{a}, H)$  for some  $\mathfrak{s} \in \mathfrak{S}$  and  $\bar{a}$  from  $G_0$ .

<sup>4</sup>For general  $\mathbf{K}$ : we use finitely generated  $G \subseteq M$ ; generally this change is needed.

4) For disjoint  $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{S}$  define  $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$  with  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = p_{\mathfrak{s}_1}(\bar{x}_{\mathfrak{s}_1}) \cup p_{\mathfrak{s}_2}(\bar{x}_{\mathfrak{s}_2})$ , recalling  $\bar{x}_{\mathfrak{s}_1}, \bar{x}_{\mathfrak{s}_2}$  are disjoint, as follow: if  $G_0 \subseteq G_1 \subseteq G_2$  are from  $\mathbf{K}$  and  $\bar{a}_{\ell} \in {}^{k(\mathfrak{s}_{\ell})}G_0$ ,  $\bar{a}_{\ell}$  realizes  $p_{\mathfrak{s}_{\ell}}(\bar{x}_{\mathfrak{s}_{\ell}})$  in  $G_0 \in \mathbf{K}$  and  $\bar{c}_{\ell} \in {}^{n(\mathfrak{s}_{\ell})}(G_{\ell+1})$  realizes  $q_{\mathfrak{s}_{\ell}}(\bar{a}_{\ell}, G_{\ell})$  for  $\ell = 1, 2$  then  $\bar{c}_1 \hat{\ } \bar{c}_2$  realizes  $q_{\mathfrak{s}}(\bar{a}_1 \hat{\ } \bar{a}_2, G_0)$  in  $G_2$ .

4A) For (disjoint)  $t_1, t_2 \in \text{def}(G)$  we define  $t_1 \oplus t_2 = t_1 \oplus_G t_2$  similarly.

5) We define  $\bigoplus_{k < m} \mathfrak{s}_k, \bigoplus_{k < m} t_k$  similarly using associativity, see 1.8(5).

6) Let  $\mathfrak{s}_1 \leq \mathfrak{s}_2$  means: if  $G \in \mathbf{K}, \bar{a}_2 \in {}^{u(2)}G$  realizes  $p_{\mathfrak{s}_2}(\bar{x}_{\mathfrak{s}_2}), G \subseteq H, \bar{c}_2 \in {}^{n(\mathfrak{s}_2)}H$  realizes  $q_{\mathfrak{s}_2}(\bar{a}_2, G)$  then  $\text{dom}(\bar{x}_{\mathfrak{s}_1}) \subseteq u(2)$  and  $\bar{c}_2 \upharpoonright \text{dom}(\bar{z}_{\mathfrak{s}_2})$  realizes  $q_{\mathfrak{s}_1}(\bar{a}_2 \upharpoonright k(\mathfrak{s}_1), G)$  and  $p_{\mathfrak{s}_2}(\bar{x}_{\mathfrak{s}_2}) \upharpoonright \bar{x}_{\mathfrak{s}_1} = p_{\mathfrak{s}_1}(\bar{x}_{\mathfrak{s}_1})$ .

7) Let  $\mathfrak{s}_1 \leq_{\bar{h}} \mathfrak{s}_2$  means that  $\bar{h} = (h', h''), h'$  is a one-to-one function from  $\text{dom}(\bar{x}_{\mathfrak{s}_1})$  into  $\text{dom}(\bar{x}_{\mathfrak{s}_2})$  and  $h''$  is a one-to-one function from  $\text{dom}(\bar{z}_{\mathfrak{s}_1})$  into  $\text{dom}(\bar{z}_{\mathfrak{s}_2})$  such that: if  $\text{tp}_{\text{bs}}(\bar{c}_2, G, H) = q_{\mathfrak{s}_2}(\bar{a}_2, G)$  and  $\bar{a}_1 = \langle a_{2, h''(\ell)} : \ell \in \text{dom}(\bar{a}_1) \rangle$  and  $\bar{c}_1 = \langle c_{2, h(\ell)} : \ell \in \text{dom}(\bar{c}_2) \rangle$  then  $\text{tp}_{\text{bs}}(\bar{c}_1, G, H) = q_{\mathfrak{s}_1}(\bar{a}_1, G, H)$ . Similarly  $t_1 \leq_{\bar{h}} t_2$  for  $t_1, t_2 \in \text{def}(G)$ . If  $h' \cup h''$  is well defined we may write  $h' \cup h''$  instead of  $\bar{h}$ .

*Remark 1.7.* 0) Concerning 1.6(7) the point of disjoint  $\mathfrak{s}_1, \mathfrak{s}_2$  and congruency is to avoid using it. So we may ignore it as well as 1.9(2),(3), 3.4(3), 3.5(4), 3.6(5).

1) Note that the operation  $\mathfrak{s}_1 \oplus \mathfrak{s}_2$  is not necessarily commutative, e.g. for  $\mathbf{K}_{\text{of}}$  it cannot be.

2) In e.g. Definition 1.6(1A), in general  $\mathfrak{s}$  is not uniquely determined by the relevant information  $\text{tp}_{\text{bs}}(\bar{a}_1 \hat{\ } \bar{a}_2 \hat{\ } \bar{c}_1 \hat{\ } \bar{c}_2, H_0, H_2)$  and the lengths of  $\bar{a}_1, \bar{a}_2, \bar{c}_1, \bar{c}_2$  but if  $H_1$  is existentially closed, it is. We could have written the definition in a computational form.

3) So  $\mathfrak{s}_1 \leq \mathfrak{s}_1$  means  $\mathfrak{s}_1 \leq_{\bar{h}} \mathfrak{s}_2$  with  $h_{\ell}$  the identity for  $\ell = 1, 2$ .

**Definition/Claim 1.8.** 1) For any  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  we can define its closure as the minimal closed (and invariant, of course)  $\mathfrak{S}_1 \subseteq \Omega[\mathbf{K}]$  which includes it, see 1.6(1); we denote it by  $cl(\mathfrak{S}) = cl(\mathfrak{S}; \mathbf{K})$ .

2) Similarly for dominating-closure  $\text{docl}(\mathfrak{S})$  and composition-closure  $\text{cocl}(\mathfrak{S})$ .

3) Those closures preserve density and countability (and being invariant), and have the obvious closure properties.

4) Also dominating-closure preserve being composition closed.

5) The operation  $\oplus$  on  $\Omega[\mathbf{K}]$  is well defined and associative. If  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is closed under  $\oplus$ , for transparency, then  $\mathfrak{S}$  is symmetric (see 1.5) iff the operation  $\oplus$  on  $\mathfrak{S}$  is commutative (when defined). Similarly for  $\text{def}_{\mathfrak{S}}(G)$ .

6)  $\Omega[\mathbf{K}]$  has cardinality  $\leq 2^{\aleph_0}$ ; generally  $\leq 2^{|\tau(\mathbf{K})| + \aleph_0}$ .

7)  $\leq_{\mathfrak{S}}$  is a transitive relation on  $\mathbf{K}$ , if  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is closed.

8) If  $H_0 \subseteq H_1 \subseteq H_2, \mathfrak{s} \in \Omega[\mathbf{K}]$  and  $\text{tp}_{\text{bs}}(\bar{c}, H_1, H_2) = q_{\mathfrak{s}}(\bar{a}, H_1)$  and  $\bar{a} \in {}^{k(\mathfrak{s})}H_0$  then  $\text{Rang}(\bar{c}) \cap H_1 = \text{Rang}(\bar{c}) \cap H_0$ .

9) Assume  $\mathfrak{S}$  is dense and closed. If  $G \subseteq H \in \mathbf{K}$  and  $G$  is finite then  $G \leq_{\mathfrak{S}} H$ .

10) If  $\mathfrak{s} = \mathfrak{s}_0 \oplus \dots \oplus \mathfrak{s}_{n-1}$  and  $i(0) < \dots < i(k-1) < n$  and  $\mathfrak{s}' = \mathfrak{s}_{i(0)} \oplus \dots \oplus \mathfrak{s}_{i(k-1)}$  then  $\mathfrak{s}' \leq \mathfrak{s}$ .

*Proof.* Natural, noting that (8) is specific for our present  $\mathbf{K}$ , see 1.2(4). □<sub>1.8</sub>

**Claim 1.9.** 0) *The operation  $\oplus$  is well defined, that is:*

- (a) *if  $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$  are disjoint then  $\mathfrak{s}_1 \oplus \mathfrak{s}_2 \in \Omega[\mathbf{K}]$  is well defined;*
- (b) *if  $t_1, t_2 \in \text{def}(G)$  are disjoint then  $t_1 \oplus t_2 \in \text{def}(G)$ .*

1) The operation  $\oplus$  on disjoint pairs from  $\text{def}(G)$  respects congruency, see Definition 1.1(3). If  $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$  then  $(\mathfrak{s}_1/\equiv) \oplus (\mathfrak{s}_2/\equiv)$  is well defined, i.e. if  $\mathfrak{s}'_\ell, \mathfrak{s}''_\ell$  are congruent to  $\mathfrak{s}_\ell$  for  $\ell = 1, 2$  and  $\mathfrak{s}' = \mathfrak{s}'_1 \oplus \mathfrak{s}'_2, \mathfrak{s}'' = \mathfrak{s}''_1 \oplus \mathfrak{s}''_2$  are well defined (equivalently for  $\ell = 1, 2$  the two schemes  $\mathfrak{s}'_\ell, \mathfrak{s}''_\ell$  are disjoint) then  $\mathfrak{s}', \mathfrak{s}''$  are congruent. (So we may forget to be pedantic about this.)

2) If  $(\mathfrak{s}, \bar{a}) = (\mathfrak{s}_1, \bar{a}_1) \oplus_G (\mathfrak{s}_2, \bar{a}_2)$  then  $(\mathfrak{s}_\ell, \bar{a}_\ell) \leq (\mathfrak{s}, \bar{a})$ .

3) If in  $\text{def}(G)$  we have  $(\mathfrak{s}_\ell, \bar{a}_\ell) \leq_{h_\ell} (\mathfrak{s}'_\ell, \bar{a}'_\ell)$  for  $\ell = 1, 2$  and  $\text{Dom}(h_1) \cap \text{Dom}(h_2) = \emptyset$ ,  $\text{Rang}(h_1) \cap \text{Rang}(h_2) = \emptyset$  then  $(\mathfrak{s}_1, \bar{a}_1) \oplus (\mathfrak{s}_2, \bar{a}_2) \leq_{h_1 \cup h_2} (\mathfrak{s}'_1, \bar{a}'_1) \oplus (\mathfrak{s}'_2, \bar{a}'_2)$ . Similarly for  $\bar{h}_1, \bar{h}_2$ .

*Proof.* Straightforward. □<sub>1.9</sub>

**Claim 1.10.** Assume  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is dominating-closed and  $G_0 \subseteq G_1 \in \mathbf{K}$  and  $G_0 \leq_{\mathfrak{S}} G_2$  and, for transparency,  $G_1 \cap G_2 = G_0$  and<sup>5</sup>  $G_0 \leq_{\Sigma_1} G_2$  (holds if  $G_0$  is existentially closed in  $\mathbf{K}$ ).

1) There is  $G_3 \in \mathbf{K}$  such that  $G_1 \leq_{\mathfrak{S}} G_3$  and  $G_2 \subseteq G_3$  and  $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$  and  $G_1 \leq_{\Sigma_1} G_3$ .

2) Above  $G_3$  above is unique up to isomorphism over  $G_1 \cup G_2$ .

3) If  $\mathfrak{S}$  is symmetric and  $G_0 \leq_{\mathfrak{S}} G_1$  in part (1) then also  $G_2 \leq_{\mathfrak{S}} G_3$ .

*Proof.* Straightforward, e.g.

1) Let  $\bar{c} = \langle c_\alpha : \alpha < \alpha(*) \rangle$  list the elements of  $G_2$ , and for every finite  $u \subseteq \alpha(*)$  let  $\bar{x}_u = \langle x_\alpha : \alpha \in u \rangle$  and  $p_u^0(\bar{x}_u) = \text{tp}_{\text{bs}}(\bar{c}|u, G_0, G_2)$  hence by assumption, there is  $\mathfrak{s}_u \in \mathfrak{S}$  (up to congruency) and  $\bar{a}_u \in {}^{k(\mathfrak{s}_u)}(G_0)$  such that  $p_u^0(\bar{x}) = q_{\mathfrak{s}_u}(\bar{a}_u, G_0)$  so  $\text{dom}(\bar{x}_{\mathfrak{s}_u}) = u$ . We define  $p_u^1(\bar{x}_u) \in \mathbf{S}(G_1)$  as  $q_{\mathfrak{s}_u}(\bar{a}_u, G_1)$ . We define  $G_3$  as a group extending  $G_1$  generated by  $G_1 \cup \{c_\alpha : \alpha < \alpha(*)\}$  such that  $\bar{c}|u$  realizes  $p_u^1(\bar{x}_u)$  for every finite  $u \subseteq \alpha(*)$ . But for this to work we have to prove that for finite  $u \subseteq v \subseteq \alpha(*)$  we have  $p_u^1(\bar{x}_u) \subseteq p_v^1(\bar{x}_v)$ . This is straightforward recalling 1.2(1A).

Lastly,  $G_1 \leq_{\Sigma_1} G_3$  is easy, too. □<sub>1.10</sub>

*Remark 1.11.* 1) We may consider an alternative definition of  $\leq_{\mathfrak{S}}$ :

- <sub>1</sub>  $G \leq_{\mathfrak{S}} H$  iff for every finite  $A \subseteq H$  there are  $\bar{c} \in {}^{\omega}H, \bar{a} \in {}^{\omega}G$  and  $\mathfrak{s} \in \mathfrak{S}$  such that:  $\bar{a}$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}), \bar{c}$  realizes  $q_{\mathfrak{s}}(\bar{a}, G)$  in  $H$  and  $A \subseteq \text{Rang}(\bar{c})$ .

An even weaker version is:

- <sub>2</sub> as in •<sub>1</sub> but “ $A \subseteq \text{Rang}(\bar{c})$ ” is replaced by  $A \subseteq \text{cl}(G \cup \bar{c}, H)$ .

2) But, e.g. for •<sub>1</sub>, to prove  $\leq_{\mathfrak{S}}$  is transitive we need a stronger version of composition-closed: if  $G_0 \subseteq G_1 \subseteq G_2$  and for  $\ell = 0, 1, \bar{c}_\ell \in {}^{n(\ell)}(G_{\ell+1})$  realizes  $q_{\mathfrak{s}_\ell}(\bar{a}_\ell, G_\ell)$  and  $\text{Rang}(\bar{b}_0) \subseteq \text{Rang}(\bar{a}_1)$  then for some  $\mathfrak{s} \in \mathfrak{S}, p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = p_{\mathfrak{s}_0}(\bar{x}_{\mathfrak{s}_0})$  and  $\bar{a}_1 \hat{\ } \bar{a}_2$  realizes  $q_{\mathfrak{s}}(\bar{a}_0, G_0)$ .

3) In any case for closed  $\mathfrak{S}$  the three definitions are equivalent, i.e. those in •<sub>1</sub>, in •<sub>2</sub> and in 1.4(1).

4) Does the operation  $\oplus_G$  respect  $\approx_G$ , see Definition 1.1, i.e. if  $t_1 \approx_G t'_1$  and  $t_2 \approx_G t'_2$  then  $t_1 \oplus_G t_2 \approx_G t'_1 \oplus_G t'_2$ ?; all this assuming the operations are well defined, i.e. the disjointness demands from 1.6(4) are satisfied. We do not see a reason for this to hold.

<sup>5</sup>If  $G_2 = \langle G_0 \cup A \rangle, A$  finite then for part (1) this is not necessary.

§ 1(B). **Constructions.**

Before we present the more systematic construction from [She90, Ch.IV], we give a self-contained direct definition and proof for the existence of a canonical existentially closed extension of  $G \in \mathbf{K}$  when  $\mathfrak{S}$  is symmetric, i.e. the “second avenue” in §0B. We shall deal with the non-symmetric case later.

**Definition 1.12.** Assume  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is symmetric.

- 1) We say  $H$  is a  $\mathfrak{S}$ -closure of  $G$  when there is a sequence  $\langle G_n : n < \omega \rangle$  such that  $G_0 = G$ ,  $H = \cup\{G_n : n < \omega\}$  and  $G_{n+1}$  is a one-step  $\mathfrak{S}$ -closure of  $G_n$ , see below.
- 2) We say that  $H$  is a one-step  $\mathfrak{S}$ -closure of  $G$  when:

- (a)  $G \subseteq H$  are from  $\mathbf{K}$ ;
- (b)  $S := \text{def}(G) = \{(\mathfrak{s}, \bar{a}) : \mathfrak{s} \in \mathfrak{S} \text{ and } \bar{a} \in {}^{\omega}G \text{ realizes } p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})\}$  and let  $t = (\mathfrak{s}_t, \bar{a}_t) = (\mathfrak{s}(t), \bar{a}(t))$  for  $t \in S$ ;
- (c)  $\bar{c}_t \in {}^{n(\mathfrak{s}(t))}H$  realizes  $q_{\mathfrak{s}_t}(\bar{a}_t, G)$  for  $t \in S$ ;
- (d)  $H$  is generated by  $\{\bar{c}_t : t \in S\} \cup G$ ;
- (e)  $\bar{c}_t$  realizes  $q_{\mathfrak{s}_t}(\bar{c}_t, \text{cl}(\cup\{\bar{c}_s : s \in S \setminus \{t\}\} \cup G, H))$  inside  $H$  for every  $t \in S$ .

**Claim 1.13.** Let  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  be symmetric.

- 1) For every  $G \in \mathbf{K}$  there is a one-step  $\mathfrak{S}$ -closure  $H$  of  $G$ .
- 2) For every  $G \in \mathbf{K}$  there is an  $\mathfrak{S}$ -closure  $H$  of  $G$ .
- 3) In both parts (1) and (2) we have  $|G| \leq |H| \leq |G| + |\mathfrak{S}| + \aleph_0$ .
- 4) In both parts (1) and (2),  $H$  is unique up to isomorphism over  $G$ .
- 5) If the pair  $(G_\ell, H_\ell)$  is as in part (1), or as in part (2) for  $\ell = 1, 2$  and  $G_1 \subseteq G_2$  then  $H_1$  can be embedded into  $H_2$  over  $G_1$ .
- 6) In both parts (1) and (2) there is a set theoretic class function  $\mathbf{F}$  computing  $H$  from  $G$ , pedantically for every  $G \in \mathbf{K}$  and ordinal  $\alpha$  not in the transitive closure  $\text{tr} - \text{cl}(G)$  of  $G$ ,  $\mathbf{F}_\alpha(G)$  is well defined such that:

- (A) (a)  $\mathbf{F}_\alpha(G) \in \mathbf{K}_{\text{if}}$  is of cardinality  $\leq |G| + \aleph_0 + |\alpha|$
- (b) if  $\alpha = 0$  then  $\mathbf{F}_\alpha(G) = G$
- (c) the sequence  $\langle \mathbf{F}_\beta(G) : \beta \leq \alpha \rangle$  is increasing continuous
- (d)  $\mathbf{F}_{\alpha+1}(G)$  is a one step closure of  $\mathbf{F}_\alpha(G)$
- (B) if  $G_1 \subseteq G_2 \wedge G_2 \cap \mathbf{F}_\alpha(G_1) = G_1 \wedge \bigwedge_{\ell=1}^2 \emptyset = (\alpha + 1) \cap \text{tr} - \text{cl}(G_\ell)$  then  $\mathbf{F}_\alpha(G_1) \subseteq \mathbf{F}_\alpha(G_2)$ ; this is “naturality”; an alternative is 0.12(2).

7) In fact we do not have to use the axiom of choice.

*Proof.* Should be clear (alternatively, below we do more). □<sub>1.13</sub>

*Remark 1.14.* Similarly in §3.

**Definition 1.15.** 1) We say  $N$  is  $(\lambda, \mathfrak{S})$ -full over  $M$  when:  $M \subseteq N$  and if  $M \subseteq M_1 \subseteq N$  and  $M_1 = \text{cl}(M + A, N)$  for some  $A \subseteq M_1$  of cardinality  $< \lambda$  and  $\mathfrak{s} \in \mathfrak{S}$  and  $\bar{a} \in {}^{k(\mathfrak{s})}M_1$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$  in  $M_1$  then  $q_{\mathfrak{s}}(\bar{a}, M_1)$  is realized in  $N$ .



- 2) We may write “ $N$  is  $\mathfrak{S}$ -full over  $M$ ” when  $\lambda = \|N\|$  is regular or, in general, when<sup>6</sup> there is a list  $\langle a_\alpha : \alpha < \|N\| \rangle$  of  $N$  such that for every  $\alpha < \|N\|$  and  $\mathfrak{s} \in \mathfrak{S}$  we have: if  $M_\alpha = \text{cl}(M + \{a_\beta : \beta < \alpha\}, N)$  and  $\bar{a} \in {}^{k(\mathfrak{s})}M_\alpha$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$  then the type  $q_{\mathfrak{s}}(\bar{a}, M_\alpha)$  is realized in  $N$  by  $\|N\|$  elements.
- 3) We may omit  $\mathfrak{S}$  when  $\mathfrak{S} = \Omega[\mathbf{K}]$ .

**Claim 1.16.** *Let  $\mathfrak{S}$  be symmetric.*

- 1) If  $\mathfrak{S} \subseteq \mathfrak{S}(\mathbf{K})$  is closed (see 1.6(1)) then  $(\mathbf{K}, \leq_{\mathfrak{S}})$  is a weak a.e.c. with amalgamation<sup>7</sup> (even canonical), see [Shea, 1.2] or [She09, Ch.I], i.e. in the Definition of a.e.c. we have Ax 0,(I),(II),(III),(V) but LST $(\mathbf{K}, \leq_{\mathfrak{S}})$  may be  $\infty$  and we omit Ax(IV), see 1.18 below.
- 2) If  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is dense and closed (see 1.6) then for every  $M \in \mathbf{K}_\lambda$  there is an existentially closed  $N \in \mathbf{K}_\lambda$  which  $\leq_{\mathfrak{S}}$ -extends it, in fact any  $\mathfrak{S}$ -closure of  $M$  can serve.
- 3) If  $N$  is  $(\lambda, \mathfrak{S})$ -full over  $M_1$  and  $M_0 \subseteq M_1$ , then  $N$  is  $(\lambda, \mathfrak{S})$ -full over  $M_0$ ; also in Definition 1.15 without loss of generality  $\bar{a}$  is from  $M \cup A$ , i.e.  $\bar{a} \in {}^{k(\mathfrak{s})}(M \cup A)$ .
- 4) If  $M \in \mathbf{K}_{\leq \lambda}$  then there is a model  $N$ ,  $(\lambda, \mathfrak{S})$ -full over  $M$  of cardinality  $\leq \lambda + \|M\| = \lambda$ ; moreover if  $\mathfrak{S}$  is dense, then  $M$  is existentially closed.
- 5) In (4), we can add: if  $N' \in \mathbf{K}$  is  $(\lambda, \mathfrak{S})$ -full over  $M$  then we can find an embedding of  $N$  into  $N'$  over  $M$ .

*Proof.* 1) Easy.

2) Easy by 1.13 and see more below.

3) Easy.

4) We choose  $G_n \in \mathbf{K}$  by induction on  $n$  such that:

- (a)  $G_0 = M$ ;
- (b)  $G_{n+1} \supseteq G_n$  is as in Definition 1.12 but each  $t$  appears  $\lambda$  times, i.e.
- $G_{n+1} = \text{cl}(\cup\{\bar{c}_{t,\alpha}^n : t \in \text{def}_{\mathfrak{S}}(G_n) \text{ and } \alpha < \lambda\} \cup G_n, G_{n+1})$  where
  - $\text{tp}_{\text{bs}}(\bar{c}_{t,\alpha}^n, G_{n,t,\alpha}, G_{n+1}) = q_t(\bar{a}_t, G_{n,t,\alpha})$  where
  - $G_{n,t,\alpha} = \text{cl}(\cup\{\bar{c}_{t_1,\alpha_1}^n : t_1 \in \text{def}_{\mathfrak{S}}(G_n), \alpha_1 < \lambda \text{ but } (t_1, \alpha_1) \neq (t, \alpha)\} \cup G_n, G_{n+1})$ .

Let  $\hat{G} = \bigcup_n G_n$  and we shall show that  $\hat{G}$  is  $(\lambda, \mathfrak{S})$ -full over  $G$ . We can ignore the case  $\lambda = \aleph_0$  being obvious. Assume  $A \subseteq \hat{G}$ ,  $|A| < \lambda$  and  $t_* \in \text{def}_{\mathfrak{S}}(\hat{G})$  and  $\bar{a}_{t_*} \subseteq \text{cl}(G_0 + A, \hat{G})$ , hence we can find  $\bar{S}$  such that:

- (\*) (a)  $\bar{S} = \langle S_n : n < \omega \rangle$ ;
- (b)  $S_n \subseteq \text{def}_{\mathfrak{S}}(G_n) \times \lambda$  and  $\bigcup_m S_m$  has cardinality  $< |A|^+ + \aleph_0$ ;
- (c) if  $(t, \alpha) \in S_n$  then  $\bar{a}_t \subseteq \text{cl}(\cup\{\bar{c}_{t_1,\alpha_1}^m : m < n \text{ and } (t_1, \alpha_1) \in S_m\} \cup G_0, G_n)$ ;
- (d)  $A \subseteq \bigcup_n A_n \cup G_0$  where  $A_n = \cup\{\bar{c}_{t,\alpha}^m : (t, \alpha) \in S_m \text{ and } m < n\}$ ;

<sup>6</sup>For the case  $\mathfrak{S}$  is not symmetric and  $\lambda$  is singular, if we like to have “prime”, (as in 1.16(5)) we should add: for every pair  $t = (\mathfrak{s}, \bar{a})$  as in 1.15(2), for every large enough  $\mu < \lambda$ , for every  $\alpha < \mu^+$  for some  $\bar{c} \subseteq M_{\mu^+}$  requires  $q_{\mathfrak{s}}(\bar{a}, M_\alpha)$  is realized in  $M_{\mu^+}$ ; also we can in 1.23(1) have such  $\mathcal{A}$ , i.e. strengthen (d) there as here so weakens the assumption in 1.23(4).

<sup>7</sup>Not enough for quoting results.

(e) for some  $n_*$ ,  $(t_*, 0) \in S_{n_*}$ .

We have to prove that  $q_{t_*}(cl(A \cup G, \hat{G}))$  is realized in  $M$ . Choose  $\alpha_*$  such that  $(t_*, \alpha_*) \notin S_{n_*}$  and prove by induction on  $n \geq n_*$  that  $\bar{c}_{t_*, \alpha_*}^{n_*}$  realizes  $q_{t_*}(cl(A_n \cup G_0, \hat{G}))$ .

For  $n = n_*$  this is obvious, so assume this holds for  $n$  and we shall prove for  $n + 1$ .

For this it suffices to prove, for every finite  $u \subseteq S_n$  that  $\bar{c}_{t_*, \alpha_*}^{n(*)}$  realizes  $q_{t_*}(cl(A_n \cup G_0 \cup \{\bar{c}_{t, \alpha}^n : (t, \alpha) \in u\}, \hat{G}))$ ; we prove this by induction on  $|u|$ . Now if  $|u| = 0$  this holds by the induction hypothesis on  $n$  and if  $|u| > 0$ , let  $\beta \in u$  and use the induction for  $u' = u \setminus \{\beta\}$  and  $\mathfrak{S}$  being symmetric.

5) We can find a list  $\langle (n_\zeta, t_\zeta, \alpha_\zeta) : \zeta < \lambda \rangle$  of  $\{(n, t, \alpha) : n < \omega \text{ and } (t, \alpha) \in S_n\}$  such that  $\bar{a}_{t_\zeta} \subseteq cl(\cup\{\bar{c}_{t_\xi, \alpha_\xi}^{n_\xi} : \xi < \zeta\} \cup M, N)$ .

Now choose  $f(\bar{c}_{t_\zeta, \alpha_\zeta}^{n_\zeta}) \subseteq N'$  by induction on  $\zeta$ . □<sub>1.16</sub>

**Discussion 1.17.** 1) So by 1.13(2), 1.16(2) if there is a symmetric closed dense  $\mathfrak{S}$  then for every lf group  $G$  there is a “nice” extension of  $G$  to an existentially closed one  $\hat{G}$ , that is we have:

- (a) uniqueness (by 1.13(4))
- (b) cardinality  $\leq |\theta| + |\mathfrak{S}|$  (by 1.13(3))
- (c) extending  $G$  (see 1.12(1))
- (d) being existentially closed (see 1.16(2)).

2) Fixing  $\lambda$  and demanding  $G \in \mathbf{K}_{\leq \lambda}$  we can add

- (e)  $\hat{G}$  is  $(\lambda, \mathfrak{S})$ -full over  $M$
- (f) if  $H \supseteq G$  is  $(\lambda, \mathfrak{S})$ -full then there is an embedding of  $\hat{G}$  into  $H$  over  $G$ .

**Discussion 1.18.** Concerning 1.16(1), if we assume  $\langle G_\alpha : \alpha \leq \delta + 1 \rangle$  is  $\subseteq$ -increasing continuous and  $\alpha < \delta \Rightarrow G_\alpha \leq_{\mathfrak{S}} G_{\delta+1}$ , does it follow that  $G_\delta \leq_{\mathfrak{S}} G_{\delta+1}$ ? This is Ax(IV) of the definition of a.e.c. Well, if  $\delta$  has uncountable cofinality and each  $G_\alpha$  is existentially closed then yes. The point is that the relevant types do not split over finite sets. If we deal with “not split over countable sets” we need  $cf(\delta) \geq \aleph_2$ , etc.

So  $(\mathbf{K}, \leq_{\mathfrak{S}})$  is not an a.e.c. in general failing Ax(IV); in fact, e.g. we may prove for the maximal  $\mathfrak{S}$  that this axiom fails, see the proof of 5.1.

Now we turn to constructions not necessarily assuming “ $\mathfrak{S}$  is symmetric” presenting the “first avenue” in §0(B).

**Definition 1.19.** 1) We say that  $\mathcal{A} = \langle G_i, \bar{a}_j, w_j, K_j : i \leq \alpha, j < \alpha \rangle$  is an  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction (for  $\mathbf{K}$ ) when :

- (a)  $G_i$  for  $i \leq \alpha$  is an  $\leq_{\mathfrak{S}}$ -increasing continuous sequence of members of  $\mathbf{K}$ ;
- (b)  $G_{i+1}$  is generated by  $G_i \cup \bar{a}_i, \bar{a}_i$  a finite sequence;
- (c)  $w_i$  is a finite subset of  $i$ ;
- (d)  $K_i \subseteq G_i$  is finite;
- (d)<sup>+</sup> moreover  $K_i \subseteq \langle G_0 + \sum_{j \in w_i} \bar{a}_j \rangle_{G_i}$ ; we may add “ $K_j$  generated by  $\cup\{\bar{a}_j : j \in w_i\} \cup (K_i \cap G_i)$ ”;

(e)  $\text{tp}_{\text{bs}}(\bar{a}_i, G_i, G_{i+1}) \in \mathbf{S}_{\mathfrak{S}}^{\ell g(\bar{a}_i)}(G_i)$  as witnessed by  $K_i$ , i.e. it is  $q_{\mathfrak{s}}(\bar{a}, G_i)$  for some  $\bar{a} \in {}^{\omega}K_i$  realizing  $p_{\mathfrak{s}}$  for some  $\mathfrak{s} \in \mathfrak{S}$ .

2) We may say above that  $G_\alpha$  is  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -constructible over  $G_0$ ; and may also say that  $\mathcal{A}$  is an  $\mathfrak{S}$ -construction over  $G_0$ . We let  $\alpha = \ell g(\mathcal{A}), G_i = G_i^{\mathcal{A}}, \bar{a}_i = \bar{a}_i^{\mathcal{A}}, w_j = w_j^{\mathcal{A}}, K_j = K_j^{\mathcal{A}}$ .

3) We say above that  $\mathcal{A}$  is a definite  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction when for every  $j < \alpha$  we have also  $t_j = t_j^{\mathcal{A}} \in \text{def}(G_j^{\mathcal{A}})$  such that  $\bar{a}_{t_j} \in {}^{\omega}K_j$  and  $\bar{a}_j^{\mathcal{A}}$  realizes  $q_{t_j}(G_j)$  (note that in 1.19(1)(e) we have “for some  $\mathfrak{s}_j$ ”, so  $\mathcal{A}$  does not determine the  $\mathfrak{s}$ ’s (or here the  $t_j$ ; so every  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction can be expanded to a definite one, but not necessarily uniquely).

4) We say  $\mathcal{A}$  is a  $\lambda$ -full definite  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction when  $\alpha$  is divisible by  $\lambda$  and for every  $i < \alpha$  and  $t \in \text{def}(G_i)$ , the set  $\{j : j \in (i, \alpha) \text{ and } t_j^{\mathcal{A}} = t\}$  is an unbounded subset of  $\alpha(*)$  of order type divisible by  $\lambda$ .

**Discussion 1.20.** We may replace 1.19(1)(e) by “ $\text{tp}_{\text{bs}}(\bar{a}_i, G_i, G_{i+1})$  does not split over  $K_i$ ”, this is like the case  $\mathbf{F}_{\aleph_0}^p$  in [She90, Ch.IV, Def.2.6, pg.168] and [She90, Ch.IV, Lemma 2.20, pg.168] and is equal to  $\mathbf{F}_{\aleph_0}^{\text{nsf}}$  in [She15, §1, 1.1-1.12], both for first order theories, but we seemingly lose the following:

**Observation 1.21.** 1) If  $\mathcal{A}$  is a  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction and  $G_0^{\mathcal{A}} \subseteq G$  and  $G \cap G_{\ell g(\mathcal{A})}^{\mathcal{A}} = G_0^{\mathcal{A}}$  then there is an  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction  $\mathcal{B}$  with  $G_0^{\mathcal{B}} = G, \ell g(\mathcal{B}) = \ell g(\mathcal{A})$  and  $G_{\ell g(\mathcal{B})}^{\mathcal{B}} = \langle G_{\ell g(\mathcal{A})}^{\mathcal{A}} \cup G \rangle_{G_{\ell g(\mathcal{A})}^{\mathcal{A}}}$ .

2) Like (1) but with definite  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -constructions and then add in the end  $t_j^{\mathcal{B}} = t_j^{\mathcal{A}}$  for  $j < \ell g(\mathcal{A})$ .

3) For the definite version, see 1.19(3), we get even uniqueness in (2).

**Discussion 1.22.** In 1.24 below, we may consider (see [She98, Ch.IV, §1]):

Ax(V.1): If  $(q, G, L) \in \mathbf{F}, G \subseteq H \in \mathbf{K}; \bar{a}, \bar{b} \in {}^{\omega}H; q = \text{tp}_{\text{bs}}(\bar{a} \bar{b}, G, H)$  and  $p = \text{tp}_{\text{bs}}(\bar{a}, \langle G + \bar{b} \rangle_H, H)$  then  $(p, \langle G + \bar{b} \rangle_H, L) \in \mathbf{F}$ .

Ax(V.2): A notational variant of (V1) so we ignore it.

The following claim (together with §2, the existence of countable dense  $\mathfrak{S}$ ) proves Theorem 0.11.

**Claim 1.23.** 1) If  $G \in \mathbf{K}$  is of cardinality  $\leq \lambda$  and  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is closed and dense and of cardinality  $\leq \lambda$  (if  $\lambda \geq 2^{\aleph_0}$  this follows) then there is an  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction  $\mathcal{A}$  such that:

- (a)  $\alpha^{\mathcal{A}} = \lambda$ ;
- (b)  $G_0^{\mathcal{A}} = G$ ;
- (c)  $G_\lambda^{\mathcal{A}} \in \mathbf{K}$  is existentially closed of cardinality  $\lambda$ ;
- (d)  $\mathcal{A}$  is  $\lambda$ -full, that is for every  $\mathfrak{s} \in \mathfrak{S}$  and  $\bar{a} \in {}^{k(\mathfrak{s})}(G_\lambda^{\mathcal{A}})$  realizing  $p_{\mathfrak{s}}(\bar{x})$ , for  $\lambda$  ordinals  $\alpha < \lambda$  we have:  $\text{tp}_{\text{bs}}(\bar{a}_\alpha, G_\alpha^{\mathcal{A}}, G_{\alpha+1}^{\mathcal{A}}) = q_{\mathfrak{s}}(\bar{a}, G_\alpha^{\mathcal{A}})$ .

2) Assume  $\lambda \geq \|G\| + |\mathfrak{S}|$  is regular. Then we can find  $H \in \mathbf{K}_\lambda$  which is  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -constructible over  $G$ , is  $(\lambda, \mathfrak{S})$ -full over  $M$  and is embeddable over  $M$  into any  $N'$  which is  $(\lambda, \mathfrak{S})$ -full over  $G$ , in fact  $G_\lambda$  from part (1) is as required.

3) If  $\mathfrak{S}$  is symmetric and is closed and  $H_1, H_2$  are  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -constructible over  $G$  and  $(\lambda, \mathfrak{S})$ -full over  $G$  and of cardinality  $\lambda$  then  $H_1, H_2$  are isomorphic over  $G$ .

4) If  $\lambda \geq \|G\|$  and  $\mathcal{A}$  is an  $\mathbf{F}_{\aleph_0}^{\text{sch}} - \mathfrak{S}$ -construction of  $H$  over  $G$  and  $lg(\mathcal{A}) = \lambda$  then for every  $H' \in \mathbf{K}$  which is  $(\lambda, \mathfrak{S})$ -full over  $G$ , we have  $H$  is embeddable into  $H'$  over  $G$ .

*Proof.* By [She90, Ch.VI,§3] as all the relevant axioms there apply (see below or [She90, Ch.IV,§1,pg.153]) or just check directly. Of course, we can use a monster  $\mathfrak{C}$  for groups, but use only sets  $A$  such that  $cl(A, \mathfrak{C}) = \langle A \rangle_{\mathfrak{C}}$  is locally finite, and we use quantifier-free types. □1.23

Now we make the connection to [She90, Ch.IV].

**Definition/Claim 1.24.** 1) Let  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  be closed and below let  $\lambda = \lambda(\mathbf{F}_{\mathfrak{S}})$  be  $\aleph_0$ . Then  $\mathbf{F} = \mathbf{F}_{\mathfrak{S}}$  is defined as the set of triples  $(p, G, A)$  such that  $A$  is finite, for some  $B \subseteq G \in \mathbf{K}$  we have  $A \subseteq B$ ,  $cl(B) = cl(B, G) = G \in \mathbf{K}$ ,  $p \in \mathbf{S}_{\mathfrak{S}}^{<\omega}(cl(B))$  is  $q_{\mathfrak{s}}(\bar{b}, cl(B))$  for some  $\mathfrak{s} \in \mathfrak{S}$ ,  $\bar{b} \subseteq cl(A)$  over  $A$ ; we may restrict ourselves to the case  $B = cl(B, G) = G$ . Note that: as here we do not have a monster model  $\mathfrak{C}$  we can either demand  $B \in \mathbf{K}$  or demand  $B \subseteq G \in \mathbf{K}$  but then it is more natural to write  $(p, G, A)$  instead of  $(p, A)$ .

2)  $\mathbf{F}$  satisfies the axioms (from [She90, Ch.IV,§1] written below in the present notation) except possibly V, VI, VIII, X.1, X.2, XI.1, XI.2.

3) If  $\mathfrak{S}$  is symmetric then  $\mathbf{F}$  satisfies also Ax(VI).

4) If  $\mathfrak{S}$  is dense then  $\mathbf{F}$  satisfies also Ax(X.1).

*Remark 1.25.* If  $\mathfrak{S}$  is compact (see 1.6(5)), then  $\mathbf{F}$  satisfies Ax(VIII), i.e.

Ax(VIII) when  $\mathfrak{S}$  is compact: If  $\langle G_i : i \leq \delta + 1 \rangle$  is  $\subseteq$ -increasing continuous in  $\mathbf{K}$ ,  $\bar{L} \subseteq G_0$  finite,  $p \in \mathbf{S}_{\mathfrak{S}}(G_\lambda)$  and  $i < \delta \Rightarrow (p \upharpoonright G_i, G_i, L) \in \mathbf{F}$  then  $(p, G_\delta, L) \in \mathbf{F}$ .

[Why? By the Definition; also holds when  $cf(\delta) > \aleph_0$ .]

*Proof.* Isomorphism - Ax(I): preservation under isomorphism.

Obvious.

Concerning trivial  $\mathbf{F}$ -types:

Ax(II1): If  $K \subseteq L \subseteq G \in \mathbf{K}$ ,  $|L| < \lambda$ ,  $K$  is finite,  $\bar{a} \in {}^{\omega}K$  and  $p = \text{tp}_{\text{bs}}(\bar{a}, L, G)$  then  $(p, G, K) \in \mathbf{F}$ .

[Why? Trivially; recall  $\lambda = \aleph_0$ .]

Axiom(II2)-(II3)-(II4): irrelevant here.

Concerning monotonicity:

Ax(III1): If  $L \subseteq G_1 \subseteq G_2$  and  $(p, G_2, L) \in \mathbf{F}$  then  $(p \upharpoonright G_1, G_1, L) \in \mathbf{F}$ .

[Why? Because if  $\bar{a} \in {}^{\omega}L$ ,  $L \subseteq G_1 \subseteq G_2 \in \mathbf{K}$  and  $q_{\mathfrak{s}}(\bar{a}, G_2)$  is well defined and equal to  $p$ , then  $q_{\mathfrak{s}}(\bar{a}, G_1) = q_{\mathfrak{s}}(\bar{a}, G_2) \upharpoonright G_1$ , see Claim 1.2(1A).]

Ax(III2): If  $L \subseteq L_1 \subseteq G$ ,  $|L_1| < \lambda$ , i.e.  $L_1$  is finite and  $(p, G, L) \in \mathbf{F}$  then  $(p, G, L_1) \in \mathbf{F}$ .

[Why? By the definition.]

Ax(IV): If  $\bar{a}, \bar{b} \in {}^{\omega}H$ ,  $L \subseteq G \subseteq H$ ,  $(\text{tp}_{\text{bs}}(\bar{b}, G, H), G, L) \in \mathbf{F}$  and  $\text{Rang}(\bar{a}) \subseteq \text{Rang}(\bar{b})$  then  $(\text{tp}_{\text{bs}}(\bar{a}, G, H), G, L) \in \mathbf{F}$ .

[Why? Straightforward as  $\mathfrak{S}$  is domination closed, see Definition 1.6(1B).]

Concerning transitivity and symmetry:

Ax(VI): ( $\mathfrak{S}$  is symmetric). If  $G \subseteq H \in \mathbf{K}$ ,  $\bar{a}, \bar{b} \in {}^{\omega}H$  and  $L_1, L_2 \subseteq G$  are finite and  $(\text{tp}_{\text{bs}}(\bar{b}, \langle G + \bar{a} \rangle_H, H), \langle G + \bar{a} \rangle_H, L_1) \in \mathbf{F}$  and  $(\text{tp}_{\text{bs}}(\bar{a}, G, H), G, L_2) \in \mathbf{F}$  then  $(\text{tp}_{\text{bs}}(\bar{a}, \langle G + \bar{b} \rangle_H, H), \langle G + \bar{b} \rangle_H, L_1) \in \mathbf{F}$ .

[Why? By  $\mathfrak{S}$  being symmetric when we claim this axiom, i.e. in 1.24(3).]

Ax(VII): If  $G \subseteq H \in \mathbf{K}$ ,  $\bar{a}, \bar{b} \in {}^{\omega}H$ ,  $(\text{tp}_{\text{bs}}(\bar{a}, \langle G + \bar{b} \rangle_H, H), \langle G + \bar{b} \rangle_H, L) \in \mathbf{F}$  and  $(\text{tp}_{\text{bs}}(\bar{b}, G, H), G, L) \in \mathbf{F}$  hence  $L \subseteq G$  is finite, then  $(\text{tp}_{\text{bs}}(\bar{a} \wedge \bar{b}, G, H), G, L) \in \mathbf{F}$ .

[Why? By  $\mathfrak{S}$  being composition-closed, see Definition 1.6(1A).]

Concerning continuity:

Ax(IX): irrelevant as  $\lambda = \aleph_0$ .

Concerning existence:

Ax(X.1): If  $L_1 \subseteq G \in \mathbf{K}$ ,  $L_1 \subseteq L_2$  finite,  $\bar{a} \in {}^{\omega}(L_2)$  then for some  $p$  extending  $\text{tp}_{\text{bs}}(\bar{a}, L_1, L_2)$  and finite  $L \subseteq G$  we have  $(p, G, L) \in \mathbf{F}$ , moreover without loss of generality  $L = L_1$ .

[Why? By  $\mathfrak{S}$  being dense.]

Ax(X.2): irrelevant and follows by the moreover in Ax(X.1).

Ax(XI.1): If  $p \in \mathbf{S}_{\text{bs}}(G_1)$ ,  $(p, G_1, L) \in \mathbf{F}$  hence  $p \in \mathbf{S}_{\mathfrak{S}}^n(G_1)$  for some  $n$  and  $G_1 \subseteq G_2$  then there is  $q \in \mathbf{S}_{\text{bs}}^n(G_2)$  extending  $p$  such that  $(q, G_1, L_2) \in \mathbf{F}$  for  $L_2$ , so  $\bar{q} \in \mathbf{S}_{\mathfrak{S}}^n(G_2)$ ; moreover, in fact,  $L_2 = L$  is O.K.

[Why? Use the same  $\mathfrak{s} \in \mathfrak{S}$ .]

Ax(XI.2): irrelevant and really follows by the moreover in (XI.1).  $\square_{1.24}$

**Definition 1.26.** A sequence  $\mathbf{I} = \langle \bar{a}_s : s \in I \rangle$  in  $G \in \mathbf{K}$  is  $\kappa$ -convergent when for some  $m, s \in I \Rightarrow \bar{a}_s \in {}^m G$  and for every finite  $K \subseteq G$  and some  $q \in \mathbf{S}^m(K)$  for all but  $< \kappa$  members  $s$  of  $\mathbf{I}$ ,  $q = \text{tp}_{\text{bs}}(\bar{a}_s, K, G)$ .

*Remark 1.27.* 1) So  $\mathbf{F}_{\mathfrak{S}}$ -constructions preserve “ $\mathbf{I}$  is  $\kappa$ -convergent”. Moreover, if  $\mathbf{I}$  is  $\kappa$ -convergent in  $G \in \mathbf{K}$  and  $G \leq_{\mathfrak{S}} H$ , where  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  then  $\mathbf{I}$  is  $\kappa$ -convergent in  $H$ .

2) We can assume  $I$  is a linear order with no last member and of cofinality  $\geq \kappa$  and replace “all but  $< \kappa$  of the  $s \in I$ ” by “every large enough  $s \in I$ ”. See more in [Shed, §(1C)].

### § 1(C). Using Order.

We now turn to the third avenue of §(0B) to deal with the general and not necessarily symmetric case. Can we get uniqueness for non-symmetric  $\mathfrak{S}$ ? Can we get every automorphism extendable, etc.? The answer is that at some price, yes. A major point in the construction was the use of linear well ordered index set ( $\lambda$  in 1.23(1) or  $\alpha^{\mathcal{A}}$  in general). But actually we can use linear non-well ordered index sets, so those index sets can have automorphisms which help us toward uniqueness. The solution here is not peculiar to locally finite groups.

**Definition 1.28.** We say  $(I, E)$  is  $\lambda$ -suitable when (we may omit  $\lambda$  when  $\lambda = |I|$ , we may write  $(I, P_i)_{i < \lambda}$  with  $\langle P_i : i < \lambda \rangle$  listing the  $E$ -equivalence classes (with no repetitions)):

- (a)  $I$  is a linear order;
- (b)  $E$  is an equivalence relation on  $I$  with  $\lambda$  equivalence classes;
- (c) every permutation of  $I/E$  is induced by some automorphism of the linear order which preserves equivalence and non-equivalence by  $E$ ;
- (d) each  $E$ -equivalence class has cardinality  $|I|$ .

**Claim 1.29.** *Let  $T = \text{Th}(\mathbb{R}, <, E)$  where  $\text{Th}$  stands for “the first order theory of”,  $E := \{(a, b) : a, b \in \mathbb{R} \text{ and } a - b \in \mathbb{Q}\}$ ; so  $(A, <, E) \models T$  iff  $(A, <)$  is a dense linear order with neither first nor last element,  $E$  an equivalence relation with each equivalence class a dense subset of  $A$  and with infinitely many equivalence classes.*

1) *If  $\lambda = \lambda^{<\lambda}$  and  $(I, E)$  is a saturated model of  $T$  of cardinality  $\lambda$ , then  $(I, E)$  is suitable<sup>8</sup>*

2) *For every  $\lambda$  the  $(I, P_i)_{i < \lambda}$  from [Shec, §2] (see history there) is  $\lambda$ -suitable and  $|I| = \lambda$ .*

3) *There is a definable sequence  $\langle (I_\lambda, P_i^\lambda)_{i < \lambda} : \lambda \text{ an infinite cardinal} \rangle$  such that  $(I_\lambda, P_i^\lambda)_{i < \lambda}$  is  $\lambda$ -suitable and is increasing with  $\lambda$  and this definition is absolute.*

*Proof.* 1) Obvious.

2),3) See there. □<sub>1.29</sub>

**Claim 1.30.** *Assume*

- (A)  $G \in \mathbf{K}$  is of cardinality  $\lambda$ ;
- (B)  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is closed and dense;
- (C) (a)  $i(*) \leq \lambda$  and  $\mathcal{S} = \{t_i = (\mathfrak{s}_i, \bar{a}_i) : i < i(*)\}$  lists  $\text{def}_{\mathfrak{S}}(G)$ , i.e. the pairs  $(\mathfrak{s}, \bar{a})$ , as in clause (d) of 1.23(1) or 1.23(3);
- (b) each such pair appears exactly once;
- (c) let  $t_i = (\mathfrak{s}_*, \langle \rangle)$  for  $i \in [i(*), \lambda)$  so  $\mathfrak{s}_* \in \mathfrak{S}, k_{\mathfrak{s}_*} = 0, n_{\mathfrak{s}_*} = 1, i(*) = \|\text{def}_{\mathfrak{S}}(G)\|, \mathfrak{s}_*$  is from 1.3; so  $\bar{a}_i = \langle \rangle$ ;
- (D)  $(I, P_i)_{i < \lambda}$  is  $\lambda$ -suitable, see Definition 1.28.

Then we can find  $H, \mathbf{c} = \langle \bar{c}_r : r \in I \rangle$  (the ordered one step  $(\lambda, \mathfrak{S})$ -closure), such that:

- (a)  $H \in \mathbf{K}$  is a  $\leq_{\mathfrak{S}}$ -extension of  $G$ ;
- (b)  $\bar{c}_r \in {}^{n(t_i)}H$  if  $r \in P_i, i < \lambda$ ;
- (c)  $H$  is generated by  $G \cup \{\bar{c}_r : r \in I\}$ ;
- (d) if  $i < \lambda$  and  $r \in P_i$  then  $\bar{c}_r$  realizes in  $H$  over  $\text{cl}(G \cup \{\bar{c}_s : s <_I r\}, H)$  the type defined by  $(\mathfrak{s}_i, \bar{a}_i)$ ;
- (e) every automorphism of  $G$  can be extended to an automorphism of  $H$ .

*Proof.* Straightforward; e.g. to define  $H$  we should choose  $q_{r_0, \dots, r_{n-1}}$  for every  $r_0 <_I \dots <_I r_{n-1}$  by induction on  $n$  such that in the end  $q_{r_0, \dots, r_{n-1}} = \text{tp}_{\text{bs}}(\bar{c}_{r_0} \hat{\ } \dots \hat{\ } \bar{c}_{r_{n-1}}, G, H)$ , by clause (d), and prove that:

- (\*) if  $m \leq n$  and  $h : \{0, \dots, m-1\} \rightarrow \{0, \dots, n-1\}$  is increasing then  $q_{r_{h(0)}, \dots, r_{h(m-1)}} \leq_h q_{r_0, \dots, r_{n-1}}$ .

<sup>8</sup>By similar arguments, if  $\lambda \geq 2^\mu$  then there is a  $\mu$ -suitable  $(I, P_i)_{i < \mu}$  but  $|I/E| = \mu < \lambda$ . We can use any model of cardinality  $\lambda$  which is strongly  $\mu^+$ -sequence homogeneous; this means that every partial automorphism of cardinality  $\leq \mu$  can be extended to an automorphism.

Note that clause (e) follows by clauses (a)-(d) above recalling clause (c) of Definition 1.28.

Why? Let  $\pi$  be an automorphism of  $G$ , for each  $i < \lambda$  we have  $(\mathfrak{s}_i, \bar{a}_i) \in \mathcal{S}$  and also  $(\mathfrak{s}_i, \pi(\bar{a}_i)) \in \mathcal{S}$ , so by the choice of  $\langle (\mathfrak{s}_i, \bar{a}_i) : i < \lambda \rangle$  there is a unique  $j < \lambda$  such that  $i \geq i(*) \Rightarrow j = i$  and  $(\pi(\bar{a}_i), \mathfrak{s}_i) = (\bar{a}_j, \mathfrak{s}_j)$ , so let  $j = \hat{\pi}(i)$ . So  $\hat{\pi}$  is a permutation of  $\lambda$ . By “ $(I, P_i)_{i < \lambda}$  is  $\lambda$ -suitable” there is an automorphism  $\check{\pi}$  of the linear order  $I$  such that  $i < \lambda \Rightarrow \check{\pi}(P_i) = P_{\hat{\pi}(j)}$ . Clearly there is a unique automorphism  $\dot{\pi}$  of  $H$  such that  $\pi = \dot{\pi} \upharpoonright G$  and  $\dot{\pi}(\bar{c}_i) = \bar{c}_{\hat{\pi}(i)}$ .  $\square_{1.30}$

**Definition 1.31.** 1) We say  $H$  is an ordered one-step  $(\lambda, \mathfrak{S})$ -closure of  $G$ , pedantically the ordered one step  $(I, E) - \mathfrak{S}$ -closure of  $G$ , when  $G, H, \mathbf{c}$  are as in 1.30. 2) We say  $H$  is an ordered  $(\lambda, \mathfrak{S})$ -closure of  $G$ , pedantically the ordered  $(I, E) - \mathfrak{S}$ -closure of  $G$  when:

- (a)  $H = \bigcup_n H_n$
- (b)  $H_0 = G$
- (c)  $H_{n+1}$  is the one step  $(I, E) - \mathfrak{S}$ -closure of  $H_n$ .

*Remark 1.32.* 1) In what way is 1.30 weaker? We have to choose the listing of  $\text{def}(G)$  in clause (C). Also for  $G_1 \subseteq G_2$  it is not clear why  $H_1 \subseteq H_2$ , where  $(G_\ell, H_\ell)$  is as above. But see 1.29(3).

2) On naturality see Paolini-Shelah [PS18].

**Conclusion 1.33.** *The parallel of parts (2)-(6) of 1.13 holds.*

*Proof.* Straightforward, for part (6) of 1.13 use 1.29(3).  $\square_{1.33}$

## § 2. THERE ARE ENOUGH REASONABLE SCHEMES

## § 2(A). There is a Dense Set of Schemes.

We like to find  $\mathfrak{S}$ 's as in §1 for  $\mathbf{K}_{\text{lf}}$ , in particular to prove that there are dense  $\mathfrak{S}$ , so we have to look in details at amalgamations of lf-groups under special assumptions.

Recall the well known: for finite groups  $G_0 \subseteq G_\ell \in \mathbf{K}$  for  $\ell = 1, 2$  we can amalgamate  $G_1, G_2$  over  $G_0$  by embedding into suitable finite permutations group; see the proof of the theorem of Hall, explained in the second paragraph of §(0A).

Concerning the  $\mathbf{K}_{\text{of}}$  versions of 2.2, see later in 6.7.

**Convention 2.1.**  $\mathbf{K}$  is  $\mathbf{K}_{\text{lf}}$ .

**Definition 2.2.** 1) Let  $\mathbf{X}_{\mathbf{K}} = \mathbf{X}(\mathbf{K})$ , the set of amalgamation tries, be the set of  $\mathbf{x}$  such that:  $\mathbf{x}$  is a quintuple  $(G_0, G_1, G_2, \mathbf{I}_1, \mathbf{I}_2) = (G_{\mathbf{x},0}, G_{\mathbf{x},1}, \dots)$  satisfying:

- (a)  $G_0 \subseteq G_\ell \in \mathbf{K}$  for  $\ell = 1, 2$ ;
- (b)  $\mathbf{I}_\ell$  is a set of representatives of the left  $G_0$ -cosets in  $G_\ell$ , i.e.  $\langle gG_0 : g \in \mathbf{I}_\ell \rangle$  is a partition of  $G_\ell$  (so without repetitions) for  $\ell = 1, 2$ ;
- (c)  $e_{G_{\mathbf{x},0}} \in \mathbf{I}_{\mathbf{x},1} \cap \mathbf{I}_{\mathbf{x},2}$ .

2) For  $\mathbf{x}$  as above let

- (a)  $\mathcal{U} = \mathcal{U}_{\mathbf{x}} = \{(g_0, g_1, g_2) : g_\ell \in G_\ell \text{ for } \ell = 0, 1, 2 \text{ and } g_1 \in \mathbf{I}_1, g_2 \in \mathbf{I}_2\}$ ;
- (b) for  $\ell = 1, 2$  and  $g \in \mathbf{I}_\ell$  let  $\mathcal{U}_g^\ell = \mathcal{U}_{\mathbf{x},g}^\ell := \{(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}} : g_\ell = g\}$ ;
- (c) if  $G_1 \cap G_2 = G_0$  then we let  $\mathbf{j}_{\mathbf{x}} = \mathbf{j}_{\mathbf{x},1} \cup \mathbf{j}_{\mathbf{x},2}$ , see below;
- (d) for  $\ell = 0, 1, 2$  let  $\mathbf{j}_\ell = \mathbf{j}_{\mathbf{x},\ell}$  be the following embedding of  $G_\ell$  into  $\text{per}(\mathcal{U}_{\mathbf{x}})$ , the group of permutations of  $\mathcal{U}_{\mathbf{x}}$ , so let  $g \in G_\ell$  and we should define  $\mathbf{j}_\ell(g)$ , so let  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$  and we define  $(g'_0, g'_1, g'_2) = (\mathbf{j}_\ell(g))(g_0, g_1, g_2)$  from  $\mathcal{U}_{\mathbf{x}}$  as follows:

$\underline{\ell=0}$ :  $g'_0 = g_0g$  in  $G_0$  and  $g'_1 = g_1, g'_2 = g_2$ ;

$\underline{\ell=1}$ :  $g'_1g'_0 = g_1g_0g$  in  $G_1$  and  $g'_2 = g_2$ ;

$\underline{\ell=2}$ :  $g'_2g'_0 = g_2g_0g$  in  $G_2$  and  $g'_1 = g_1$ .

3) Let  $G_{\mathbf{x}} = G_{\mathbf{x},3}$  be the subgroup of  $\text{Sym}(\mathcal{U}_{\mathbf{x}})$  which  $\text{Rang}(\mathbf{j}_{\mathbf{x},1}) \cup \text{Rang}(\mathbf{j}_{\mathbf{x},2})$  generates where  $G_{\mathbf{x}} \models "f_1f_2 = f_3"$  means that for every  $u \in \mathcal{U}_{\mathbf{x}}$ ,  $f_3(u) = f_2(f_1(u))$ , i.e. we look at the permutation as acting from the right.

4) Let  $\leq_{\mathbf{X}(\mathbf{K})}$  be the following partial order on  $\mathbf{X}_{\mathbf{K}} : \mathbf{x} \leq_{\mathbf{X}(\mathbf{K})} \mathbf{y}$  iff:

- (a)  $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathbf{K}}$ ;
- (b)  $G_{\mathbf{x},0} = G_{\mathbf{y},0}$ ;
- (c)  $G_{\mathbf{x},\ell} \subseteq G_{\mathbf{y},\ell}$  for  $\ell = 1, 2$ ;
- (d)  $\mathbf{I}_{\mathbf{x},\ell} = \mathbf{I}_{\mathbf{y},\ell} \cap G_{\mathbf{x},\ell}$  for  $\ell = 1, 2$ .

5) We say  $(f_1, f_2)$  embeds  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$  into  $\mathbf{y} \in \mathbf{X}_{\mathbf{K}}$  when:

- (a)  $f_\ell$  embeds  $G_{\mathbf{x},\ell}$  into  $G_{\mathbf{y},\ell}$  for  $\ell = 1, 2$ ;
- (b)  $f_1 \upharpoonright G_{\mathbf{x},0} = f_2 \upharpoonright G_{\mathbf{y},0}$  maps  $G_{\mathbf{x},0}$  onto  $G_{\mathbf{y},0}$ .



6) We say  $(f_1, f_2)$  is an isomorphism from  $\mathbf{x} \in \mathbf{X}_K$  onto  $\mathbf{y} \in \mathbf{X}_K$  when above  $f_\ell$  is onto  $G_{\mathbf{y},\ell}$  for  $\ell = 1, 2$ .

**Observation 2.3.** Let  $\mathbf{x}$  be as in Definition 2.2, i.e. it is an amalgamation try.

0) If  $G_0 \subseteq G_\ell \in \mathbf{K}$  for  $\ell = 1, 2$  then for some  $\mathbf{x} \in \mathbf{X}_K$  we have  $G_{\mathbf{x},\ell} = G_\ell$  for  $\ell = 0, 1, 2$ .

1) In Definition 2.2(2), for  $\ell = 0, 1, 2$  if  $g \in G_{\mathbf{x},\ell}$  then  $\mathbf{j}_{\mathbf{x},\ell}(g)$  is a permutation of  $\mathcal{U}_{\mathbf{x}}$ , in fact, its restriction to  $\mathcal{U}_{g_1}^{3-\ell}$  is a permutation for each  $g_1 \in G_{3-\ell}$ .

2) Moreover in part (1) the mapping  $\mathbf{j}_{\mathbf{x},\ell}$  embeds the group  $G_{\mathbf{x},\ell}$  into the group of permutation of  $\mathcal{U}_{\mathbf{x}}$  hence into  $G_{\mathbf{x}}$ .

3) The mapping  $\mathbf{j}_{\mathbf{x},0}$  is equal to  $\mathbf{j}_{\mathbf{x},1}|_{G_{\mathbf{x},0}}$  and also to  $\mathbf{j}_{\mathbf{x},2}|_{G_{\mathbf{x},0}}$ .

4) If  $G_{\mathbf{x},\ell}$  is finite for  $\ell = 0, 1, 2$  then  $|G_{\mathbf{x}}| \leq (|G_{\mathbf{x},1}| \times |G_{\mathbf{x},2}| / |G_{\mathbf{x},0}|)!$

5) If  $\mathbf{x}$  is an amalgamation try and  $G_{\mathbf{x},0} \subseteq G'_\ell \subseteq G_{\mathbf{x},\ell}$  so  $G'_\ell$  is a subgroup of  $G_{\mathbf{x},\ell}$ , for  $\ell = 1, 2$  then for one and only one amalgamation try  $\mathbf{y}$  we have  $G_{\mathbf{y},0} = G_{\mathbf{x},0}, G_{\mathbf{y},\ell} = G'_\ell$  for  $\ell = 1, 2$  and  $\mathbf{I}_{\mathbf{y},\ell} = \mathbf{I}_{\mathbf{x},\ell} \cap G'_\ell$  so  $\mathbf{y} \leq_{\mathbf{X}(K)} \mathbf{x}$ .

6) Moreover in part (5), if  $\mathbf{z}$  is an amalgamation try with  $(G_{\mathbf{z},0}, G_{\mathbf{z},1}, G_{\mathbf{z},2}) = (G_{\mathbf{x},0}, G'_1, G'_2)$  then for some  $\mathbf{x}'$ , the pair  $(\mathbf{x}', \mathbf{z})$  is like  $(\mathbf{x}, \mathbf{y})$  in (5) and  $(G_{\mathbf{x}',0}, G_{\mathbf{x}',1}, G_{\mathbf{x}',2}) = (G_{\mathbf{x},0}, G_{\mathbf{x},1}, G_{\mathbf{x},2})$ .

7) In part (5) there is a unique homomorphism  $f$  from  $(\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2))_{\text{Sym}(\mathcal{U}_{\mathbf{x}})}$  onto  $G_{\mathbf{y}}$  such that  $\ell \in \{1, 2\} \wedge g \in G'_\ell \Rightarrow \mathbf{j}_{\mathbf{y},\ell}(g) = f(\mathbf{j}_{\mathbf{x},\ell}(g))$ .

8) In part (5), if  $G'_1, G'_2$  are finite then  $(\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2))_{G_{\mathbf{x}}}$  has at most  $(n_*)^{m_*}$  members where  $n_* = |G'_1| \times |G'_2| \times |G_{\mathbf{x},0}|^3$  and  $m_* = (n_*)^{|G'_1|+|G'_2|}$ .

*Proof.* Straightforward. E.g.:

2) E.g. let  $\ell = 1$  and  $f, h \in G_1$ . For  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$  let  $(\mathbf{j}_1(f))(g_0, g_1, g_2) = (g'_0, g'_1, g'_2)$  and  $(\mathbf{j}_1(h))(g'_0, g'_1, g'_2) = (g''_0, g''_1, g''_2)$  hence

$$(*)_1 \quad (\mathbf{j}_1(h))(\mathbf{j}_1(f))(g_0; g_1, g_2) = (g''_0, g''_1, g''_2).$$

Then  $g_2 = g'_2$  and  $g'_2 = g''_2$  and in  $G_1$  we have  $g_1 g_0 f = g'_1 g'_0$  and  $g'_1 g'_0 h = g''_1 g''_0$ , hence  $g_2 = g''_2$  and  $g''_1 g''_0 = g'_1 g'_0 h = (g_1 g_0 f) h = (g_1 g_0)(fh)$ , so by the definition of  $\mathbf{j}_1(fh)$  we have

$$(*)_2 \quad \mathbf{j}_1(fh)(g_0, g_1, g_2) = (g''_0, g''_1, g''_2).$$

By  $(*)_1 + (*)_2$  we have

$$(*)_3 \quad \mathbf{j}_1(fh)(g_0, g_1, g_2) = (\mathbf{j}_1(h))(\mathbf{j}_1(f))(g_0, g_1, g_2).$$

As this holds for every  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$  we have  $G_{\mathbf{x}} \models \text{“}\mathbf{j}_1(fh) = \mathbf{j}_1(f)\mathbf{j}_1(h)\text{”}$ .

4) Clearly  $|G_{\mathbf{x},\ell}| = |\mathbf{I}_{\mathbf{x},\ell}| \times |G_{\mathbf{x},0}|$  for  $\ell = 1, 2$  hence  $|\mathcal{U}_{\mathbf{x}}| = |\mathbf{I}_{\mathbf{x},1}| \times |\mathbf{I}_{\mathbf{x},2}| \times |G_{\mathbf{x},0}| = (|G_{\mathbf{x},1}|/|G_{\mathbf{x},0}|) \times (|G_{\mathbf{x},2}|/|G_{\mathbf{x},0}|) \times |G_{\mathbf{x},0}| = |G_{\mathbf{x},1}| \times |G_{\mathbf{x},2}|/|G_{\mathbf{x},0}|$ .

Hence  $|G_{\mathbf{x}}| \leq |\text{Sym}(\mathcal{U}_{\mathbf{x}})| = (|\mathcal{U}_{\mathbf{x}}|)! = (|G_{\mathbf{x},1}| \times |G_{\mathbf{x},2}|/|G_{\mathbf{x},0}|)!$  as stated).

7) First, why there is such a homomorphism? If  $b \in \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$  then  $b$  is a permutation of  $\mathcal{U}_{\mathbf{x}}$  which maps the set  $\mathcal{U}_{\mathbf{y}} = G_0 \times \mathbf{I}_{\mathbf{y},1} \times \mathbf{I}_{\mathbf{y},2}$  onto itself. It follows that every  $b \in G' := (\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2))_{\text{Sym}(\mathcal{U}_{\mathbf{x}})}$  maps the set  $\mathbf{U}_{\mathbf{y}} = G_0 \times \mathbf{I}_{\mathbf{y},1} \times \mathbf{I}_{\mathbf{y},2}$  onto itself. Hence the mapping with domain  $G'$  defined by  $f(b) = b|_{\mathcal{U}_{\mathbf{y}}}$  is a homomorphism from  $G'$  into  $\text{Sym}(\mathcal{U}_{\mathbf{y}})$ . However, for each  $b \in \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$  we have  $b|_{\mathcal{U}_{\mathbf{y}}}$  belongs to  $G_{\mathbf{y},3}$  so  $b \in G' \Rightarrow f(b) \in G_{\mathbf{y},3}$ , hence  $f$  is as required.

Second, why  $f$  is unique? Because  $\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$  generates  $G'$  and on it  $f$  is determined.

8) Let  $G_0 = G_{\mathbf{x},0}$ . We define  $E = \{((g'_0, g'_1, g'_2), (g''_0, g''_1, g''_2)) \in \mathcal{U}_{\mathbf{x}} \times \mathcal{U}_{\mathbf{x}} : G_0 g'_1 G'_1 = G_0 g''_1 G''_1 \text{ and } G_0 g'_2 G'_2 = G_0 g''_2 G''_2\}$ , this is an equivalence relation on  $\mathcal{U}_{\mathbf{x}}$ , each equivalence class has  $\leq (|G'_1| \times |G'_2| \times |G_{\mathbf{x},0}|^3) = n_*$  members.

[Why? As if  $(g'_0, g'_1, g'_2) \in (g_0, g_1, g_2)/E$  then  $g'_0 \in G_0, g'_1 \in G_0 g_1 G'_1, g'_2 \in G_0 g_2 G'_2$  and  $|G_0 g_\ell G'_\ell| \leq |G_0| \times |G'_\ell|$ .]

Also each of the permutations of  $\mathcal{U}_{\mathbf{x}}$  from  $\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$  maps each  $E$ -equivalence class onto itself. Hence for  $n \in [1, n_*]$  there are  $\leq m_n^* := n!^{|G'_1|+|G'_2|-1}$  isomorphism types of structures of the form:  $N = (|N|, F_f^N)_{f \in G'_1 \cup G'_2}$ , where  $|N|$ , the universe, has exactly  $n$  elements and is an  $E$ -equivalence class, and for each  $f \in G'_1 \cup G'_2$  we have:  $F_f^N$  is a permutation of this equivalence class and  $F_{e(G_0)}^N$  is the identity. Clearly as  $\sum_{n \leq n_*} (n!)^{|G'_1|+|G'_2|-1} \leq (n_*)^{|G'_1|+|G'_2|} = m_*$ , the subgroup  $\langle \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2) \rangle_{G_{\mathbf{x}}}$  of  $G_{\mathbf{x}}$  has at most  $(n_*)^{m_*}$  members. Of course<sup>9</sup>, the argument gives better bounds, e.g. the number of relevant  $N$ 's is much smaller and using a finer  $E$ .  $\square_{2.3}$

**Claim 2.4.** In Definition 2.2,  $\mathbf{j}_{\mathbf{x},1}(G_1) \cap \mathbf{j}_{\mathbf{x},2}(G_2) = \mathbf{j}_{\mathbf{x},\ell}(G_0)$ .

*Proof.* Assume that  $a_\ell \in G_\ell$  and  $b_\ell = \mathbf{j}_{\mathbf{x},\ell}(a_\ell)$  for  $\ell = 1, 2$ . It suffices to show that: if  $b_1 = b_2$  then  $a_1, a_2 \in G_{\mathbf{x},0}$  and  $a_1 = a_2$ . We check to what  $b_\ell$  maps the triple  $(e, e, e) \in \mathcal{U}_{\mathbf{x}}$ : by the definition of  $\mathbf{j}_{\mathbf{x},1}, \mathbf{j}_{\mathbf{x},2}$  we have:

- $b_1((e, e, e)) = (g'_0, g_1, e) \in \mathcal{U}_{\mathbf{x}}$  where  $G_1 \models g_1 g'_0 = b_1$ ;
- $b_2((e, e, e)) = (g''_0, e, g_2) \in \mathcal{U}_{\mathbf{x}}$  where  $G_2 \models g_2 g''_0 = b_2$ .

So if  $b_1 = b_2$  then  $(g'_0, g_1, e) = b_1((e, e, e)) = b_2((e, e, e)) = (g''_0, e, g_2)$ , hence  $g'_0 = g''_0 \wedge g_1 = e \wedge e = g_2$ ; this implies that  $g'_0 = b_1, g''_0 = b_2$  hence  $g'_0 = g''_0$ , also  $g''_0 \in G_0$  together  $a_1 = a_2$  so we are done.  $\square_{2.4}$

**Definition 2.5.** 1) Let<sup>10</sup>  $\text{NF}_{\text{fin}}(G_0, G_1, G_2, G_3)$  means that  $G_\ell \subseteq G_3 (\in \mathbf{K})$  for  $\ell < 3$  and  $\text{NF}_{\text{fin}}(G_0, G_1, G_2, \langle G_1 \cup G_2 \rangle_{G_3})$ , see below.

2) Let  $\text{NF}_{\text{fin}}(G_0, G_1, G_2, G_3)$  mean that:

- (a)  $G_0 \subseteq G_\ell \subseteq G_3 \in \mathbf{K}$  are finite groups for  $\ell = 1, 2$ ;
- (b)  $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$ ;
- (c) if  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$  and  $G_0 = G_{\mathbf{x},0}, G_1 \subseteq G_{\mathbf{x},1}, G_2 \subseteq G_{\mathbf{x},2}$  then there is a homomorphism  $\mathbf{f}$  from  $G_3$  into  $G_{\mathbf{x}}$  such that  $\mathbf{f}|_{G_\ell} = \mathbf{j}_{\mathbf{x},\ell}|_{G_\ell}$  for  $\ell = 1, 2$ ;
- (d) if  $a \in G_3 \setminus \{e_{G_3}\}$  then for some  $\mathbf{x}, \mathbf{f}$  as above we have  $\mathbf{f}(a) \neq e_{G_3}$ .

*Remark 2.6.* Note the choice “ $G_\ell \subseteq G_{\mathbf{x},\ell}$ ” rather than  $G_\ell = G_{\mathbf{x},\ell}$  in clause (c) of 2.5.

Now the amalgamation in Definition 2.5 is very nice but do we have existence, in  $\mathbf{K}_{\text{if}}$  of course? The following Claim 2.7(3) answers positively.

**Claim 2.7.** 1) In clause (c) of Definition 2.5(2), the homomorphism  $\mathbf{f}$  is unique. 1A) If  $\text{NF}_{\text{fin}}(G_0^i, G_1^i, G_2^i, G_3^i)$  for  $i = 1, 2$  and  $\mathbf{f}_\ell$  is an isomorphism from  $G_\ell^1$  onto  $G_\ell^2$  such that  $\mathbf{f}_0 \subseteq \mathbf{f}_\ell$  for  $\ell = 0, 1, 2$  then there is one and only one isomorphism  $\mathbf{f}_3$  from  $G_3^1$  onto  $G_3^2$  extending  $\mathbf{f}_1 \cup \mathbf{f}_2$ .

2) In Definition 2.5, necessarily  $G_1 \cap G_2 = G_0$ .

<sup>9</sup>See more in [Shee].

<sup>10</sup>NF stands for non-forking.

3) If  $G_0 \subseteq G_\ell \in \mathbf{K}$  are finite for  $\ell = 1, 2$  then we can find  $\bar{f}, \bar{H}$  such that

- (a)  $\bar{f} = \langle f_0, f_1, f_2 \rangle$ ;
- (b)  $\bar{H} = \langle H_\ell : \ell \leq 3 \rangle$ ;
- (c)  $\text{NF}_{\text{fin}}(H_0, H_1, H_2, H_3)$ ;
- (d)  $f_\ell$  is an isomorphism from  $G_\ell$  onto  $H_\ell$  for  $\ell = 0, 1, 2$ ;
- (e)  $f_0 \subseteq f_1$  and  $f_0 \subseteq f_2$ .

*Proof.* 1), 1A) Obvious.

2) By Claim 2.4 recalling clause (c) of 2.5(2).

3) Follows by 2.3(8) but we elaborate. Let  $\bar{G} = \langle G_\ell : \ell = 0, 1, 2 \rangle$  and

- (\*)<sub>1</sub> let  $\mathbf{X}_{\bar{G}} := \{\mathbf{x} \in \mathbf{X}_{\mathbf{K}} : G_{\mathbf{x},0} = G_0 \text{ and } G_{\mathbf{x},\ell} \text{ is a lf group extending } G_\ell \text{ for } \ell = 1, 2\}$ ;
- (\*)<sub>2</sub> for  $\mathbf{x} \in \mathbf{X}_{\bar{G}}$  let:  $n_{\bar{G}}(\mathbf{x}) =$  the number of elements of  $\langle \mathbf{j}_{\mathbf{x},1}(G_1) \cup \mathbf{j}_{\mathbf{x},2}(G_2) \rangle_{G_{\mathbf{x},3}}$ .

We define  $\mathbf{X}_{\bar{G}}^{\text{mx}}$  as the set of  $\mathbf{x}$ 's such that:

- (\*)<sub>3</sub> (a)  $\mathbf{x} \in \mathbf{X}_{\bar{G}}$ ;
- (b) if  $\mathbf{y} \in \mathbf{X}_{\bar{G}}$  and  $\mathbf{x} \leq \mathbf{y}$  then  $n_{\bar{G}}(\mathbf{x}) = n_{\bar{G}}(\mathbf{y})$ ;
- (\*)<sub>4</sub> if  $\mathbf{x}, \mathbf{z} \in \mathbf{X}_{\bar{G}}^{\text{mx}}$  and  $\mathbf{x} \leq_{\mathbf{X}(\mathbf{K})} \mathbf{z}$ , then  $n_{\bar{G}}(\mathbf{x}) \leq n_{\bar{G}}(\mathbf{z})$ .

[Why? Because by 2.3(7) there is a homomorphism from  $G_{\mathbf{z}} = \langle \mathbf{j}_{\mathbf{z},1}(G_1) \cup \mathbf{j}_{\mathbf{z},2}(G_2) \rangle$  onto  $G_{\mathbf{x}} = \langle \mathbf{j}_{\mathbf{x},1}(G_1) \cup \mathbf{j}_{\mathbf{x},2}(G_2) \rangle$ .]

- (\*)<sub>5</sub> for every  $\mathbf{x} \in \mathbf{X}_{\bar{G}}$  there is  $\mathbf{y} \in \mathbf{X}_{\bar{G}}^{\text{mx}}$  such that  $\mathbf{x} \leq \mathbf{y}$ ; hence  $\mathbf{X}_{\bar{G}}^{\text{mx}} \neq \emptyset$ .

[Why? By (\*)<sub>4</sub> and 2.3(8).]

- (\*)<sub>6</sub>  $(\mathbf{X}_{\bar{G}}, \leq_{\mathbf{X}(\mathbf{K})})$  has amalgamation, that is
  - if  $\mathbf{x}_0 \leq_{\mathbf{X}(\mathbf{K})} \mathbf{x}_\ell$  for  $\ell = 1, 2$  then we can find  $\mathbf{x}_3$  and  $(f_1^\ell, f_2^\ell)$  for  $\ell = 1, 2$  such that:
    - (a)  $\mathbf{x}_3 \in \mathbf{X}_{\mathbf{K}}$
    - (b)  $\mathbf{x}_0 \leq_{\mathbf{X}(\mathbf{K})} \mathbf{x}_3$
    - (c)  $(f_1^\ell, f_2^\ell)$  embeds  $\mathbf{x}_\ell$  into  $\mathbf{x}_3$  over  $\mathbf{x}_0$   
(over  $\mathbf{x}_0$  means:  $f_1^\ell \upharpoonright G_{\mathbf{x}_0,1} = \text{id}_{G_{\mathbf{x}_0,1}}, f_2^\ell \upharpoonright G_{\mathbf{x}_0,2} = \text{id}_{G_{\mathbf{x}_0,2}}$ ).

[Why? For  $\ell = 1, 2$ , we use the disjoint amalgamation for finite groups, i.e. find  $(G_\ell, f_\ell^1, f_\ell^2)$  such that:

- <sub>1</sub>  $G_\ell$  is a finite group extending  $G_{\mathbf{x}_0,0}$
- <sub>2</sub>  $f_\ell^1$  embeds  $G_{\mathbf{x}_1,\ell}$  into  $G_\ell$  over  $G_{\mathbf{x}_0,0}$
- <sub>3</sub>  $f_\ell^2$  embeds  $G_{\mathbf{x}_2,\ell}$  into  $G_\ell$  over  $G_{\mathbf{x}_0,0}$
- <sub>4</sub>  $f_\ell^1(G_{\mathbf{x}_1,\ell}) \cap f_\ell^2(G_{\mathbf{x}_2,\ell}) = G_{\mathbf{x}_0,0}$ .

Note that  $f_\ell^1(\mathbf{I}_{\mathbf{x}_1,\ell}) \cap f_\ell^2(\mathbf{I}_{\mathbf{x}_2,\ell}) = \{e_{G_{\mathbf{x}_0,0}}\}$ , moreover, working inside  $G_\ell, \langle gG_{\mathbf{x}_0,0} : g \in f_\ell^1(\mathbf{I}_{\mathbf{x}_1,\ell}) \cup f_\ell^2(\mathbf{I}_{\mathbf{x}_2,\ell}) \rangle$  is a sequence of pairwise disjoint sets. Hence there is  $\mathbf{I}_\ell \subseteq G_\ell$  extending  $f_\ell^1(\mathbf{I}_{\mathbf{x}_1,\ell}) \cup f_\ell^2(\mathbf{I}_{\mathbf{x}_2,\ell})$  such that  $\langle gG_{\mathbf{x}_0,0} : g \in \mathbf{I}_\ell \rangle$  is a partition of  $G_\ell$ .

Define  $\mathbf{x}_3$  by:

- <sub>1</sub>'  $G_{\mathbf{x}_3,0} = G_{\mathbf{x}_0,0}$

- <sub>2</sub>'  $G_{\mathbf{x}_3, \ell} = G_\ell$  for  $\ell = 1, 2$
- <sub>3</sub>'  $\mathbf{I}_{\mathbf{x}_3, \ell} = \mathbf{I}_\ell$  for  $\ell = 1, 2$ .

Now check that  $\mathbf{x}_3, (f'_1, f'_2)$  for  $\iota = 1, 2$  are as required.]

(\*)<sub>7</sub> if  $\mathbf{y} \in \mathbf{x}_G^{\text{mx}}$  then  $\text{NF}_{\text{fin}}(G_{\mathbf{y}, 0}, G_{\mathbf{y}, 1}, G_{\mathbf{y}, 2}, G_{\mathbf{y}, 3})$ .

[Should be clear now.]

Alternatively<sup>11</sup>, use 2.15 below and 2.3(8). □<sub>2.7</sub>

We give now further basic properties, mainly connecting it to non-splitting (in 2.8(4)).

**Claim 2.8.** *Assume  $\text{NF}_{\text{fin}}(G_0, G_1, G_2, G_3)$  hence  $\text{NF}_{\text{fin}}(G_0, G_1, G_2, G_3) \Leftrightarrow G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$ .*

- 1) *Symmetry: Also  $\text{NF}_{\text{fin}}(G_0, G_2, G_1, G_3)$  holds.*
- 2) *Monotonicity: If  $G_0 \subseteq G'_\ell \subseteq G_\ell$  for  $\ell = 1, 2$  and  $G'_1 \cup G'_2 \subseteq G'_3 \subseteq G_3$  then  $\text{NF}_{\text{fin}}(G_0, G'_1, G'_2, G'_3)$ .*
- 3) *Uniqueness: if  $\text{NF}_{\text{fin}}(G'_0, G'_1, G'_2, G'_3)$  hence  $G'_3 = \langle G'_1 \cup G'_2 \rangle_{G'_3}$ ,  $f_\ell$  is an isomorphism from  $G'_\ell$  into  $G_\ell$  for  $\ell = 0, 1, 2$  such that  $f_1 \upharpoonright G'_0 = f_0 = f_2 \upharpoonright G'_0$  and  $f_0$  is onto  $G_0$ , then there is an embedding  $f_3$  of  $G'_3$  into  $G_3$  extending  $f_1 \cup f_2$  (unique, of course; it is onto if and only if  $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$  and  $f_\ell$  is onto  $G_\ell$  for  $\ell = 1, 2$ ).*
- 4) *Definability: If  $\bar{a} \in {}^\omega \langle G_2 \rangle$  then  $\text{tp}_{\text{bs}}(\bar{a}, G_1, G_3)$  does not split over  $G_0$ .*

*Proof.* Straightforward but we elaborate.

- 1) Use the symmetry in the definition (recall that in §2 we have  $\mathbf{K} = \mathbf{K}_{\text{lf}}$  not  $\mathbf{K}_{\text{olf}}$ !)
- 2) By 2.3(7) and use the uniqueness in 2.7(1). Alternatively use 2.15 below and 2.3(8).
- 3) Easily, too.
- 4) Obvious by parts (2) and (3). □<sub>2.8</sub>

Now above the restriction of  $G_1, G_2$  to be finite is undesirable.

**Definition 2.9.** Let  $\text{NF}_f(G_0, G_1, G_2, G_3)$  or “ $G_1, G_2$  are  $\text{NF}_f$ -stably amalgamated over  $G_0$  inside  $G_3$ ” mean that:

- (a)  $G_\ell \in \mathbf{K}$  for  $\ell \leq 3$
- (b)  $G_0$  is finite
- (c)  $G_0 \subseteq G_\ell \subseteq G_3$  for  $\ell = 1, 2$  and  $G_1 \cap G_2 = G_0$
- (d) if  $G'_1, G'_2$  are finite groups and  $G_0 \subseteq G'_\ell \subseteq G_\ell$  for  $\ell = 1, 2$  and  $G'_3 = \langle G'_1 \cup G'_2 \rangle_{G_3}$  then  $\text{NF}_{\text{fin}}(G_0, G'_1, G'_2, G'_3)$ .

**Claim 2.10.** *Stable Amalgamation over Finite Claim 1) Existence: If  $G_0 \in \mathbf{K}$  is finite and  $G_0 \subseteq G_\ell \in \mathbf{K}$  for  $\ell = 1, 2$  and for transparency  $G_1 \cap G_2 = G_0$  then for some  $G_3$  we have  $\text{NF}_f(G_0, G_1, G_2, G_3)$  and  $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$ .*

- 2) *Uniqueness: In part (1),  $G_3$  is unique up to isomorphism over  $G_1 \cup G_2$ .*
- 3) *Monotonicity: If  $G_0 \subseteq G'_\ell \subseteq G_\ell$  for  $\ell = 1, 2$  and  $\text{NF}_\ell(G_0, G_1, G_1, G_2)$  then  $\text{NF}_f(G_0, G'_1, G'_2, G_3)$ .*
- 4) *Symmetry:  $\text{NF}_f(G_0, G_1, G_2, G_3)$  holds iff  $\text{NF}_f(G_0, G_2, G_1, G_3)$  holds.*
- 5) *Definability: If  $\text{NF}_f(G_0, G_1, G_2, G_3)$ , then  $G_1 \leq_{\Omega[\mathbf{K}]} G_3$ .*

<sup>11</sup>Or see [Shee].

*Proof.* 1) Straightforward by 2.7(1) and 2.10(2),(3), i.e. existence follows from the existence for finite  $G_1, G_2$  using uniqueness and monotonicity.

2) - 5) easy too; holds by 2.8(3), 2.8(1), 2.8(1), 2.8(3), i.e. uniqueness, monotonicity, symmetry and definability respectively.  $\square_{2.10}$

Now we go back to the major problem left in §1.

**Claim 2.11.** *There is one and only one full  $\mathfrak{s} \in \Omega[\mathbf{K}]$  such that  $q = q_{\mathfrak{s}}(\bar{a}, G_1)$  when:*

- (a)  $G_1 \in \mathbf{K}$  is existentially closed
- (b)  $q(\bar{x}) \in \mathbf{S}_{\mathfrak{S}_{\text{atdf}}}^n(G)$  is  $\text{tp}_{\text{bs}}(\bar{c}, G_1, G_3)$ , see below
- (c)  $\text{NF}_f(G_0, G_1, G_2, G_3)$
- (d)  $\bar{c} \in {}^n(\mathfrak{s})(G_2)$  and  $\bar{a} \in {}^k(\mathfrak{s})(G_0)$  generate  $G_0$ .

*Proof.* By 2.10 and 1.2(3).  $\square_{2.11}$

**Definition 2.12.** Let  $\mathfrak{S}_{\text{df}} \subseteq \Omega[\mathbf{K}]$  be the closure of  $\mathfrak{S}_{\text{atdf}}$ , see Definition 1.6 where  $\mathfrak{S}_{\text{atdf}} \subseteq \mathfrak{S}(\mathbf{K})$  is the set of  $\mathfrak{s} \in \Omega[\mathbf{K}]$  as in 2.11.

**Claim 2.13.** 1)  $\mathfrak{S}_{\text{df}}$  is well defined, see Definition 2.12, 2.8(3).

2)  $\mathfrak{S}_{\text{df}}$  is dense (see Definition 1.6(2)), closed and countable.

*Proof.* 1) Obvious.

2)  $\mathfrak{S}_{\text{df}}$  is dense: holds by 2.10 and 2.11 recalling Definition 2.9, 2.12.

$\mathfrak{S}_{\text{df}}$  is closed: by its definition.

$\mathfrak{S}_{\text{df}}$  is countable: as  $\mathfrak{S}_{\text{atdf}}$  is by 2.8(3), 2.10(2) recalling 1.8(3).  $\square_{2.13}$

**Discussion 2.14.** Is  $\mathfrak{S}_{\text{df}}$  symmetric? Not clear, however, in the end of §1 we have circumvented this and we shall in §3 circumvent this in another way.

**Claim 2.15.** 1) Assume  $G_0 \subseteq G_\ell \in \mathbf{K}$  and  $G_\ell$  is existentially closed for  $\ell = 1, 2$  and  $G_0$  finite.

*Then* we can find  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$  such that  $G_\ell = G_{\mathbf{x}, \ell}$  for  $\ell = 0, 1, 2$  and  $(\mathbf{j}_{\mathbf{x}, 0}(G_0) \subseteq \mathbf{j}_{\mathbf{x}, \ell}(G_\ell) \leq_{\mathfrak{S}_{\text{df}}} G_{\mathbf{x}})$  and  $\text{NF}_f(\mathbf{j}_{\mathbf{x}, 0}(G_0), \mathbf{j}_{\mathbf{x}, 1}(G_1), \mathbf{j}_{\mathbf{x}, 2}(G_2), G_{\mathbf{x}})$ .

2) Assume  $G_0 \subseteq G'_\ell \subseteq G_\ell$  and  $G'_\ell$  finite (or just  $(G_\ell : G'_\ell) = |G_\ell|$ ) and  $\mathbf{y} \in \mathbf{X}_{\mathbf{K}}$ ,  $G_{\mathbf{y}, 0} = G_0$  and  $G_{\mathbf{y}, \ell} = G'_\ell$  for  $\ell = 1, 2$ . *Then* in part (1) we can demand that  $\mathbf{x}$  extends  $\mathbf{y}$ .

*Remark 2.16.* If  $G_0 \subseteq G_\ell \in \mathbf{K}$  for  $\ell = 1, 2$  then we can find infinite  $G'_1, G'_2 \in \mathbf{K}$  extending  $G_1, G_2$  respectively as  $\mathbf{K}$  is closed under (finite) product (for  $\mathbf{K}_{\text{of}}$  use lexicographic order).

*Proof.* 1) By the definitions it is easy. That is, for  $\ell = 1, 2$  we can choose  $\mathbf{I}_\ell$  as in 2.2(1)(b) satisfying:

- (\*) if  $G'_\ell \subseteq G_\ell$  is finite and extends  $G_0$  and  $\mathbf{I}' \subseteq G'_\ell$  is such that  $e_{G_0} \in \mathbf{I}'$  and  $\langle gG_0 : g \in \mathbf{I}' \rangle$  is a partition of  $G'_\ell$  then we can find  $g^* \in G_\ell$  such that  $\{g^*g : g \in \mathbf{I}'\} \subseteq \mathbf{I}_\ell$ .

Now think.

2) Similarly.  $\square_{2.15}$

## § 2(B). Constructing Reasonable Schemes.

We now give some examples of  $\mathfrak{s} \in \Omega[\mathbf{K}]$ .

**Definition 2.17.** 1) Let  $\mathfrak{s}_{\text{cg}}$  be the  $\mathfrak{s}$  from 2.18(2) below.

2) Let  $\mathfrak{s}_{\text{gl}}$  be the  $\mathfrak{s}$  from 2.18(3) below.

**Claim 2.18.** 1) For every  $G \in \mathbf{K}_{\text{lf}}$  there are  $G^+$  and  $a$  such that  $G \subseteq G^+ \in \mathbf{K}_{\text{lf}}$ ,  $G^+ = \langle G \cup \{a\} \rangle_{G^+}$ ; in  $G^+$  the element  $a$  does not commute with any  $b \in G \setminus \{e_G\}$ ,  $a$  has order 2 and the sets  $G, a^{-1}Ga$  commute in  $G^+$  and their intersection is  $\{e_G\}$ .

2) There is unique  $\mathfrak{s} \in \Omega[\mathbf{K}]$  such that  $k_{\mathfrak{s}} = 0, n_{\mathfrak{s}} = 1, p_{\mathfrak{s}}$  is empty and in part (1) above  $\text{tp}_{\text{bs}}(a, G, G^+)$  is  $q_{\mathfrak{s}}(\langle \rangle, G)$ .

3) There is  $\mathfrak{s} \in \Omega[\mathbf{K}]$  with  $k_{\mathfrak{s}} = 1, n_{\mathfrak{s}} = 4, p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = \{x_0 = x_0^{-1} \wedge x_0 \neq e\}$  such that: if  $G \in \mathbf{K}_{\text{lf}}$  and  $a \in G$  realizes  $p_{\mathfrak{s}}(x_0)$  then there are  $G^+, \bar{c}$  such that  $G \subseteq G^+ = \langle G \cup \bar{c} \rangle_{G^+}$ ,  $\text{tp}_{\text{bs}}(\bar{c}, G, G^+) = q_{\mathfrak{s}}(\langle a \rangle, G)$  and  $c_{\ell}$  realizes  $q_{\mathfrak{s}_{\text{cg}}}(\langle \rangle, G)$  in  $G^+$  for  $\ell < n_{\mathfrak{s}}$  and  $a \in \langle \bar{c} \rangle_{G^+}$ .

*Proof.* We first make a less specific construction for any  $G \in \mathbf{K}$ .

For  $n \geq 2$  let  $\mathcal{U}_n = G \times n = \{(g, \iota) : g \in G, \iota < n\}$ . For finite  $K \subseteq G$  let  $E_K := \{((g_1, \iota_1), (g_2, \iota_2)) : g_1, g_2 \in G \text{ and } \iota_1, \iota_2 < n \text{ and } g_1K = g_2K\}$ , this is an equivalence relation on  $\mathcal{U}_n$ , each equivalence class has  $\leq n \times |K|$  elements. For  $\bar{a} \in {}^>G$  let  $E_{\bar{a}} = E_K$  when  $K = \langle \text{Rang}(\bar{a}) \rangle_G$  which is finite.

For  $\bar{a} \in {}^nG$  and  $\pi$  a permutation of  $\{0, \dots, n-1\}$  let  $h_{\bar{a}, \pi}$  be the following function from  $\mathcal{U}_n$  into  $\mathcal{U}_n$ :

$$(*)_1 \quad h_{\bar{a}, \pi}((g, \iota)) = (ga_{\iota}, \pi(\iota)).$$

Clearly

$$(*)_2 \quad h_{\bar{a}, \pi} \text{ is a permutation of } \mathcal{U}_n \text{ which maps every } E_{\bar{a}}\text{-equivalence class onto itself.}$$

Let  $H$  be the group of permutations of  $\mathcal{U}_n$  generated by  $\{h_{\bar{a}, \pi} : \bar{a} \in {}^nG \text{ and } \pi \text{ is a permutation of } \{0, \dots, n-1\}\}$ , now by  $(*)_2$  it is easy to see that  $H \in \mathbf{K}_{\text{lf}}$  where, as in earlier cases,

$$\bullet \quad H \models \text{“}h = h_1h_2\text{” iff } x \in \mathcal{U}_n \Rightarrow h(x) = h_2(h_1(x)).$$

Now for  $\iota < n$  let  $\mathbf{j}_{\iota}$  be the following function from  $G$  into  $H$ :

$$(*)_3 \quad \mathbf{j}_{\iota}(a) = h_{\bar{b}, \pi} \text{ when } \pi = \text{the identity and } b_k \text{ is } a \text{ if } k = \iota \text{ and is } e_G \text{ otherwise.}$$

Now

$$(*)_4 \quad \text{for } \iota < n, \mathbf{j}_{\iota} \text{ is an embedding of } G \text{ into } H.$$

[Why? Check.]

We let  $G^*, \mathbf{j}_*$  be such that  $G^* \supseteq G$  and  $\mathbf{j}_*$  is an isomorphism from  $G^*$  onto  $H$  extending  $\mathbf{j}_0$ .

For later use note:

$$(*)_5 \quad \text{for transparency we can use existentially closed } G.$$

Also

- (\*)<sub>6</sub> (a)  $G \leq_{\Omega[\mathbf{K}]} G^+$  equivalently  $\mathbf{j}_0(G) = \mathbf{j}_*(G) \leq_{\Omega[\mathbf{K}]} H$ ;  
 (b) if  $A \subseteq G$  and for  $\ell < m$  we have  $\bar{a}_\ell \in {}^n A$  and  $\pi_\ell$  is a permutation of  $\{0, \dots, n-1\}$  then  $p = \text{tp}_{\text{bs}}(\langle h_{\bar{a}_\ell, \pi_\ell} : \ell < m \rangle, \mathbf{j}_*(G), H)$  does not split over  $A$ ;  
 (c) if above  $A$  is finite and  $\bar{a}$  lists  $A$  then for some  $\mathfrak{s} \in \Omega[\mathbf{K}]$ ,  $p = q_{\mathfrak{s}}(\bar{a}, \mathbf{j}_*(G))$ .

[E.g. why clause (c) holds? By 2.10(2) recalling (\*).<sub>5</sub>.]

Now we prove each part.

- 1) Let  $n = 2$  and  $\pi$  be the permutation of  $\{0, 1\}$  such that  $\pi(0) = 1, \pi(1) = 0$ , and let  $a = \mathbf{j}_*^{-1}(h_{\langle e_G, e_G \rangle, \pi})$ .
- 2) Should be clear by (\*).<sub>6</sub>(c).
- 3) First note that

- $\oplus_1 \mathbf{j}_*^{-1}(h_{\bar{a}, \pi})$  realizes  $q_{\mathfrak{s}_{\text{cg}}}(\langle \rangle, G)$  in  $G^*$  when for some  $k \in \{1, \dots, n-1\}$  we have
- <sub>1</sub>  $\pi$  is a permutation of  $\{0, \dots, n-1\}$  and has order two
  - <sub>2</sub>  $\pi(0) = k$
  - <sub>3</sub>  $\pi(k) = 0$
  - <sub>4</sub>  $\bar{a} \in {}^n G$  satisfies  $a_{\pi(\iota)} = a_\iota^{-1}$  for  $\iota < n$
  - <sub>5</sub> if  $\pi(\iota) \neq \iota, \iota < n$  then  $a_\iota = e_G$ , (or just  $a_0$  belongs to the center of  $G$ ).

[Why? By (\*).<sub>2</sub> and the choice of  $H$  clearly  $h_{\bar{a}, \pi} \in H$  and inspecting (\*).<sub>1</sub>, easily  $h_{\bar{a}, \pi}$  has order two. By the choice of  $\mathbf{j}_*, \pi$  as  $\pi(0) = k, \pi(k) = 0$  and  $a_k = e_G = a_0$ , for  $g \in G$  we get  $H \models "h_{\bar{a}, \pi}^{-1} \mathbf{j}_0(g) h_{\bar{a}, \pi} = \mathbf{j}_k(g)"$ . However, for every  $g_1, g_2 \in G$  the elements  $\mathbf{j}_0(g_1), \mathbf{j}_k(g_2)$  of  $H$  commute as  $h_{\bar{a}_1, \pi_1}, h_{\bar{a}_0, \pi_2}$  commute in  $H$ , e.g. when  $\pi_1 = \text{id}_n = \pi_2$  and  $\bigwedge_{\ell < n} (a_{1, \ell} = e \vee a_{2, \ell} = e)$ . Lastly,  $g_1, g_2 \in G \wedge \mathbf{j}_0(g_1) = \mathbf{j}_0(g_2) \Rightarrow g_1 = e_G = g_2$ . Together we are done.]

Let  $n = 3$  and for  $\ell < 4$  let  $g_\ell \in H$  be  $h_{\bar{a}_\ell, \pi_\ell}$  where  $\pi_\ell, \bar{a}_\ell$  are defined by (recall  $a \in G$  is given and has order 2):

- $\oplus_2$  for  $\ell < 4$  let  $\pi_\ell$  be such that:
- $\underline{\ell = 0, 3}$ : the orbits are  $\{0, 1\}, \{2\}$ ;
  - $\underline{\ell = 1, 2}$ : the orbits are  $\{0, 2\}, \{1\}$ .
- $\oplus_3$  let  $\bar{a}_\ell = \langle a_{\ell, i} : i < 3 \rangle$  be  $\langle e, e, e \rangle, \langle e, a, e \rangle, \langle e, e, e \rangle, \langle e, e, e \rangle$  for  $\ell = 0, 1, 2, 3$ .

Now

- $\oplus_4 c_\ell := \mathbf{j}_*^{-1}(h_{\bar{a}_\ell, \pi_\ell})$  realizes  $q_{\mathfrak{s}_{\text{cg}}}(\langle \rangle, G)$  for  $\ell < 4$ .

[Why? We apply  $\oplus_1$  with  $k$  being 1 for  $\ell = 0, 3$  and 2 for  $\ell = 1, 2$ . So we have to check •<sub>1</sub> – •<sub>4</sub> for each  $\ell$ ; now •<sub>1</sub> + •<sub>2</sub> + •<sub>3</sub> holds by inspecting  $\oplus_2$  and the choice of  $k$  and of  $\pi_\ell$ .

Lastly, for •<sub>4</sub> + •<sub>5</sub> note that  $a, e$  has order 2 and  $a_{\ell, 0} = e_G = a_{\ell, k}$  by inspecting  $\oplus_3$ .]

- $\oplus_5 \text{tp}_{\text{bs}}(\langle c_0, c_1, c_2, c_3 \rangle, G, G^*)$  does not split over  $\langle a \rangle$ , moreover is  $q_t(\langle a \rangle, G)$  for some  $t \in \Omega[\mathbf{K}]$ .

[Why? Just think recalling (\*).<sub>6</sub>.]

Lastly,

$$\oplus_6 G^+ \models "c_0 c_1 c_2 c_3 = a".$$

[Why? This is equivalent to  $H \models h_{\bar{a}_0, \pi_0} h_{\bar{a}_1, \pi_1} h_{\bar{a}_2, \pi_2} h_{\bar{a}_3, \pi_3} = \mathbf{j}_0(a)$ . By the definition of the product we check how each  $(g, \ell) \in \mathcal{Z}_n$  is mapped (see above, so  $h_{\bar{a}_0, \pi_0}$  is applied first) applying  $h_{\bar{a}_\ell, \pi_\ell}$  in turn:

$$(g, 0) \mapsto (ge, 1) \mapsto (gea, 1) \mapsto (geae, 1) \mapsto (geaee, 0) = (ga, 0) = \mathbf{j}_0(a)(g, 0)$$

and

$$(g, 1) \mapsto (ge, 0) \mapsto (gee, 2) \mapsto (geee, 0) \mapsto (geeee, 1) = (g, 1) = \mathbf{j}_0(a)(g, 1)$$

$$(g, 2) \mapsto (ge, 2) \mapsto (gee, 0) \mapsto (geee, 2) \mapsto (geeee, 2) = (g, 2) = \mathbf{j}_0(a)(g, 2).$$

So we are done.] □<sub>2.18</sub>

The following will be used in the proof of existence of complete existentially closed  $G$ .

**Claim 2.19.** *1) If (A) then (B) where:*

- (A) (a)  $G_n \subseteq G_{n+1} \in \mathbf{K}$  for  $n < \omega$  and  $I$  a set;
  - (b)  $a_n^t \in G_{n+1}$  and let  $b_n^t = a_0^t \dots a_n^t$  in  $G_{n+1}$  for  $n < \omega, t \in I$ ;
  - (c)  $\bar{a}_n = \langle a_n^t : t \in I \rangle, \bar{b}_n = \langle b_n^t : t \in I \rangle$ ;
  - (d) (α)  $\text{tp}_{\text{bs}}(\bar{a}_n, G_n, G_{n+1})$  is increasing<sup>12</sup> with  $n$ ;
  - (β)  $\text{cl}(\bar{a}_n, G_{n+1}) \cap G_n = \{e_{G_n}\}$ ; if  $I = \{t\}$  and  $a_n^t$  has order  $k(t)$  this means that for every  $i \in \{1, \dots, k(t)\}$  we have:  
 $G_{n+1} \models "(a_n^t)^i = e_{G_1}"$  iff  $(a_n^t)^i \in G_n$  iff  $i = k(t)$ ;
  - (e)  $a_n^t$  commutes with every  $c \in G_n$ ;
  - (f)  $G_\omega = \cup \{G_n : n < \omega\}$  hence  $\in \mathbf{K}$ ;
- (B) for some  $\bar{b}_\omega, G_{\omega+1}$  we have:
- (a)  $G_{\omega+1} \supseteq G_\omega$  belongs to  $\mathbf{K}$ ;
  - (b)  $\bar{b}_\omega = \langle b_\omega^t : t \in I \rangle$  and  $b_\omega^t \in G_{\omega+1}$ ;
  - (c)  $G_{\omega+1} = \text{cl}(G_\omega \cup \{b_\omega^t : t \in I\}, G_{\omega+1})$ ;
  - (d) if  $n < \omega$  then  $p_n = \text{tp}_{\text{bs}}(\bar{b}_\omega, G_n, G_{\omega+1}) = \text{tp}_{\text{bs}}(\bar{b}_n, G_n, G_{n+1})$ .

2) If we have (A) except omitting (A)(d)(β), still we have:

- (B)' (a) – (c) as above;
- (d)  $\bar{b}_\omega \upharpoonright u$  realizes  $\text{tp}_{\text{bs}}(\bar{b}_n!, G_n! \upharpoonright u, G_{n+1})$  in  $G_{\omega+1}$  when  $u \subseteq I$  is finite and  $n$  is large enough.

*Proof.* 1) Letting  $p_n(\bar{x}) = \text{tp}_{\text{bs}}(\bar{b}_n, G_n, G_{n+1})$ , it is enough to prove:

$$(*)_1 \ p_n \subseteq p_{n+1}.$$

For this it is enough to prove, letting  $\bar{y} = \langle y_t : t \in I \rangle$ ,

<sup>12</sup>So by (A)(e) this is equivalent to " $\text{tp}(\bar{a}_n, \emptyset, G_{n+1})$  is constant".



- (\*)<sub>2</sub> if  $\sigma(\bar{y}, \bar{z})$  is a group-term and  $\bar{c} \in {}^{\ell g(\bar{z})}(G_n)$  then  $G_{n+2} \models \text{“}\sigma(\bar{b}_{n+1}, \bar{c}) = e\text{”}$   
iff  $G_{n+1} \models \text{“}\sigma(\bar{b}_n, \bar{c}) = e\text{”}$ .

Towards proving (\*)<sub>2</sub> note:

- <sub>2.1</sub>  $\bar{c}$  and  $\bar{b}_n$ , and hence  $\sigma(\bar{b}_n, \bar{c})$  are from  $G_{n+1}$ ,
- <sub>2.2</sub>  $a_{n+1}^t$  commutes with every  $c_i (i < \ell g(\bar{c}))$  and with  $b_n^s$  for  $s \in I$ .

By clause (A)(b) of the assumption of the claim,

- <sub>2.3</sub>  $b_{n+1}^t = b_n^t a_{n+1}^t$  and  $b_n^t = b_{n-1}^t a_n^t$  stipulating  $b_{-1}^t = e$ .

Similarly,

- <sub>2.4</sub>  $\bar{c}$  and  $\bar{b}_{n-1}$  are from  $G_n$ ;
- <sub>2.5</sub>  $a_n^t$  commute with every  $c_i (i < \ell g(\bar{c}))$  and with  $b_{n-1}^s$  for  $s \in I$ .

Hence for some group term  $\sigma_*(\bar{x})$ :

- <sub>2.6</sub>  $G_{n+2} \models \text{“}\sigma(\bar{b}_{n+1}, \bar{c}) = \sigma(\bar{b}_n, \bar{c})\sigma_*(\bar{a}_{n+1})\text{”}$ ;
- <sub>2.7</sub>  $G_{n+1} \models \text{“}\sigma(\bar{b}_n, \bar{c}) = \sigma(\bar{b}_{n-1}, \bar{c})\sigma_*(\bar{a}_n)\text{”}$ .

Hence by clauses (A)(d)( $\alpha$ ), ( $\beta$ ):

- <sub>2.8</sub>  $\sigma_*(\bar{a}_n) \in G_n$  iff  $\sigma_*(\bar{a}_n) = e_{G_n}$  iff  $\sigma_*(\bar{a}_{n+1}) = e_{G_n}$  iff  $\sigma_*(\bar{a}_{n+1}) \in G_{n+1}$ ;
- <sub>2.9</sub> if  $\sigma_*(\bar{a}_n) \notin G_n$ , hence  $\sigma_*(\bar{a}_{n+1}) \notin G_{n+1}$ , then both statements in (\*)<sub>2</sub> fail because:
  - ( $\alpha$ )  $\sigma(\bar{b}_n, \bar{c})$  is from  $G_{n+1}$  and  $\sigma_*(\bar{a}_{n+1}) \notin G_{n+1}$  so  $\sigma(\bar{b}_{n+1}, \bar{c}) \notin G_{n+1}$  and thus  $\sigma(\bar{b}_{n+1}, \bar{c}) \neq e_{G_n}$ ;
  - ( $\beta$ ) similarly  $\sigma(\bar{b}_n, \bar{c}) \notin G_n$  and thus  $\sigma(\bar{b}_n, \bar{c}) \neq e_{G_n}$ ;
- <sub>2.10</sub> if  $\sigma_*(\bar{a}_n) \in G_n$  hence  $\sigma_*(\bar{a}_n) = e = \sigma_*(\bar{a}_{n+1})$ , then  $\sigma(\bar{b}_{n+1}, \bar{c}) = \sigma(\bar{b}_n, \bar{c})$  and again we are done.

Together (\*)<sub>2</sub> holds.

2) Similarly (and the same as part (1) when  $G_n$  is existentially closed for every  $n$ ) but we elaborate. Without loss of generality  $I$  is finite; letting  $p_n(\bar{y}) = \text{tp}_{\text{bs}}(\bar{a}_n, G_n)$ , we need:

- (\*)<sub>1</sub> if  $\bar{c}$  is a finite sequence from  $G_\omega$  then the sequence  $\langle \text{tp}_{\text{bs}}(\bar{b}_n \hat{\ } \bar{c}, \emptyset, G_{n+1}) : n < \omega \rangle$  is eventually constant.

Let  $K_n = \text{cl}(\bar{a}_n, G_{n+1})$ , so by clause (A)(d)( $\alpha$ ) of the assumption  $|K_n|$  is constant, finite and  $K_n \cap G_n$  is  $\subseteq$ -increasing with  $n$ . Hence for some  $K_*, n(*)$  we have  $n \geq n(*) \Rightarrow K_n \cap G_n = K_*$  and let  $k(*) = |K_*|$ . Without loss of generality  $n(*) \geq k(*)$ ; so it is enough to prove

- (\*)<sub>2</sub> if  $\bar{y} = \langle y_t : t \in I \rangle$  and  $\sigma(\bar{y}, \bar{z})$  is a group term,  $\bar{c} \in {}^{\ell g(\bar{z})}(G_n)$  and  $n \geq n(*)$ , then  $G_{n+1} \models \text{“}\sigma(\bar{b}_n, \bar{c}) = e\text{”}$  iff  $G_{n+k(*)+1} \models \text{“}\sigma(\bar{b}_{n+k(*)}, \bar{c}) = e\text{”}$ .

As in part (1) we can prove that for some group term  $\sigma_*(\bar{y})$  we have

- ▮ if  $n \geq n(*)$  then  $G_{n+2} \models \sigma(\bar{b}_{n+1}, \bar{c}) = \sigma(\bar{b}_n, \bar{c})\sigma_*(\bar{a}_{n+1})$ .

Case 1: “ $\sigma_*(\bar{a}_n) \notin G_n$  for some, equivalently every,  $n \geq n(*)$ .”

In this case  $G_{n+1} \models “\sigma(\bar{b}_n, \bar{c}) \neq e”$  for every  $n \geq n(*)$ .

Case 2: “ $\sigma_*(\bar{a}_n) \in G_n$  for some, equivalently every,  $n \geq n(*)$ .”

In this case there is  $b$  such that  $\sigma_*(\bar{a}_n, \bar{c}) = b$  for every  $n \geq n(*)$ . So for every  $n \geq n(*)$  by induction on  $m$  we can prove  $\sigma(\bar{b}_{n+m}, \bar{c}) = \sigma(\bar{b}_n, \bar{c}) \cdot b^m$ . But necessarily  $b \in K_*$  hence  $b$  has order dividing  $|K_*| = k(*)$ . Hence  $n \geq n(*) \Rightarrow \sigma(\bar{b}_{n+k(*)}, \bar{c}) = \sigma(\bar{b}_n, \bar{c})$  and thus  $n_2 > n_1 \geq n(*) \wedge k(*) \mid (n_2 - n_1) \Rightarrow \sigma(\bar{b}_{n_2-2}, \bar{c}) = \sigma(\bar{b}_{n_1}, \bar{c})$ , and so we can finish easily.  $\square_{2.19}$

**Definition/Claim 2.20.** 1) For  $k = 2, 3, \dots$  let  $\mathfrak{s}_{ab(k)}$  be the unique  $\mathfrak{s} \in \Omega[\mathbf{K}_{\text{lf}}]$  such that:

- (a)  $n(\mathfrak{s}) = 1, k(\mathfrak{s}) = 0$ ;
- (b) if  $G \subseteq H$  and  $c \in H$  realizes  $q_{\mathfrak{s}}(\langle \rangle, G) = \text{tp}_{\text{bs}}(c, G, H)$  then  $c$  commutes with every  $a \in G$ ;
- (c) also for every  $m < \omega, a^m = e_H$  iff  $a^m \in G$  iff  $k \mid m$ .

2) Assume  $K \in \mathbf{K}_{\text{lf}}$  is finite and  $\bar{c} \in {}^{|K|}K$  list it. Then let  $\mathfrak{s} = \mathfrak{s}_{ab}(\bar{c}, K)$  be the unique  $\mathfrak{s} \in \Omega[\mathbf{K}_{\text{lf}}]$  such that:

- (a)  $n(\mathfrak{s}) = \ell g(\bar{c}), k(\mathfrak{s}) = 0$  so  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = \emptyset$ ;
- (b) if  $G \subseteq H \in \mathbf{K}_{\text{lf}}$  and  $\bar{c}' \in {}^{n(\mathfrak{s})}H$  then the following are equivalent:
  - ( $\alpha$ )  $\text{tp}(\bar{c}', G, H) = q_{\mathfrak{s}}(\langle \rangle, G)$ ,
  - ( $\beta$ )  $\bar{c}'$  commutes with  $G$ , realizes  $\text{tp}(\bar{c}, \emptyset, K)$  and  $\langle \bar{c}' \rangle_H \cap G = \{e\}$ .

**Claim 2.21.** Assume  $\text{NF}_f(G_0, G_1, G_2, G_3)$  and  $a \in G_1 \setminus G_0, b \in G_2 \setminus G_0$ . Then  $a, b$  commute in  $G_3$  iff  $a \in \mathbf{C}_{G_1}(G_0), b \in \mathbf{C}_{G_2}(G_0)$  and  $G_0$  is commutative.

*Remark 2.22.* 1)  $\text{NF}_f$  is from Definition 2.9.

2) Recall  $g^{[a]} = a^{-1}ga$ .

*Proof.* Without loss of generality  $G_1, G_2$  are existentially closed (by monotonicity of  $\text{NF}_f$ , see 2.10(3) and existence of existentially closed extensions).

First assume

$$\oplus a \in \mathbf{N}_{G_1}(G_0) \text{ and } b \in \mathbf{N}_{G_2}(G_0).$$

By 2.15, without loss of generality we can find  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$  such that  $G_3 = G_{\mathbf{x}}, G_{\mathbf{x}, \ell} = G_{\ell}$  for  $\ell < 3$  and let  $f_a = \mathbf{j}_{\mathbf{x}, 1}(a), f_b = \mathbf{j}_{\mathbf{x}, 2}(b)$ ; we shall use the fact that: we have some freedom in the choice of  $\mathbf{x}$ , see 2.15.

Let  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$  and we should see whether  $f_b \circ f_a((g_0, g_1, g_2)) = f_a \circ f_b((g_0, g_1, g_2))$ ; there are unique  $a', h_a, b', h_b$  such that:

- (\*)<sub>0</sub> (a)  $g_1 a = a' h_a$  with  $h_a \in G_0, a' \in \mathbf{I}_{\mathbf{x}, 1}$ ;
- (b)  $g_2 b = b' h_b$  with  $h_b \in G_0, b' \in \mathbf{I}_{\mathbf{x}, 2}$ .

Now

$$(*)_1 f_a((g_0, g_1, g_2)) = (h_a g_0^{[a]}, a', g_2).$$

[Why? As  $g_1 g_0 a = g_1 a g_0^{[a]} = a'(h_a g_0^{[a]})$ , noting that  $g_0^{[a]} \in G_0$  because we are assuming that  $a$  normalize  $G_0$  inside  $G_1$ .]

$$(*)_2 \quad f_b((h_a g_0^{[a]}, a', g_2)) = (h_b h_a^{[b]} g_0^{[a][b]}, a', b').$$

[Why? As  $g_2(h_a g_0^{[a]})b = g_2 b(h_a^{[b]} g_0^{[a][b]}) = b'(h_b h_a^{[b]} g_0^{[a][b]})$ .]

So,

$$(*)_3 \quad (f_b \circ f_a)((g_0, g_1, g_2)) = (h_b h_a^{[b]} g_0^{[a][b]}, a', b').$$

Now,

$$(*)_4 \quad f_b((g_0, g_1, g_2)) = (h_b g_0^{[b]}, g_1, b').$$

[Why? As  $g_2 g_0 b = g_2 b g_0^{[b]} = b' h_b g_0^{[b]}$ .]

$$(*)_5 \quad f_a((h_b g_0^{[b]}, g_1, b')) = (h_a h_b^{[a]} g_0^{[b][a]}, a', b').$$

[Why? As  $g_1(h_b g_0^{[b]})a = g_1 a(h_b^{[a]} g_0^{[b][a]}) = a'(h_a h_b^{[a]} g_0^{[b][a]})$ .]

Hence,

$$(*)_6 \quad (f_a \circ f_b)((g_0, g_1, g_2)) = (h_a h_b^{[a]} g_0^{[b][a]}, a', b').$$

Together we can deduce:

$$(*)_7 \quad (f_b \circ f_a)(g_0, g_1, g_2) = (f_a \circ f_b)(g_0, g_1, g_2) \text{ iff } h_b h_a^{[b]} g_0^{[a][b]} = h_a h_b^{[a]} g_0^{[b][a]} \text{ in } G_0.$$

Now, not assuming  $\oplus$  we shall prove the claim by cases (using  $(*)_7$  when  $\oplus$  holds).

$\oplus_1$   $a, b$  commute in  $G_3$  when:

- <sub>1</sub>  $a$  commutes with  $G_0$  in  $G_1$ ,
- <sub>2</sub>  $b$  commutes with  $G_0$  in  $G_2$ ,
- <sub>3</sub>  $G_0$  is commutative.

[Why? Note that the assumption  $\oplus$  holds (by  $\bullet_1 + \bullet_2$ ), and so let  $\mathbf{x} \in \mathbf{X}_K$  be as above. For any  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$ , we can apply  $(*)_7$  thus  $h_a, h_b \in G_0$  are well defined, by  $(*)_0$ . Now as  $h_b, h_a, g_0 \in G_0$  and  $a \in \mathbf{C}_{G_1}(G_0), b \in \mathbf{C}_{G_1}(G_0)$  and  $G_0$  is commutative, by the present assumptions, clearly  $h_b h_a^{[b]} g_0^{[a][b]} = h_b h_a g_0 = h_a h_b g_0 = h_a h_b^{[a]} g_0^{[b][a]}$ . As  $G_3 = G_{\mathbf{x}}, G_{\mathbf{x}}$  is a group of permutations of  $\mathcal{U}_{\mathbf{x}}$  and  $(*)_7$  holds for any  $(g_0, g_1, g_3) \in \mathcal{U}_{\mathbf{x}}$ , clearly  $f_a, f_b \in G_{\mathbf{x}}$  commute, so we are done.]

$\oplus_2$   $a, b$  do not commute in  $G_3$  when:

- $a$  commutes with  $G_0$ ,
- $b$  commutes with  $G_0$ ,
- $G_0$  is not commutative.

[Why? Choose  $h_1, h_2 \in G_0$  which do not commute and let  $(g_0, g_1, g_2) = (e_{G_0}, e_{G_1}, e_{G_2}) = (e, e, e)$ ; note that  $ah_\ell^{-1} \notin G_0, bh_\ell^{-1} \notin G_0$  for  $\ell = 1, 2$ .

Above we could have chosen  $\mathbf{x} \in \mathbf{X}_K$  such that  $(G_{\mathbf{x}, \ell} = G_\ell$  for  $\ell < 3$  and)  $ah_1^{-1} \in \mathbf{I}_{\mathbf{x}, 1}, bh_2^{-1} \in \mathbf{I}_{\mathbf{x}, 2}$ . Again  $\oplus$  holds hence  $(*)_7$  holds for any relevant  $\mathbf{x}, g_0, g_1, g_2$ . Recall  $(g_0, g_1, g_2) := (e, e, e)$ , so  $g_1 a = ea = a = (ah_1^{-1})h_1$ . So in  $(*)_0(a)$ , we get  $a' = ah_1^{-1}$  and  $h_a = h_1$ . Similarly in  $(*)_0(b)$  we get  $b' = bh_2^{-1}, h_b = h_2$ . So  $f_a, f_b \in G_{\mathbf{x}}$  do not commute by  $(*)_7$ , because we get  $h_b h_a^{[b]} g_0^{[a][b]} = h_b h_a g_0 =$

$h_2h_1g_0 \neq h_1h_2g_0 = h_ah_bg_0 = h_ah_b^{[a]}g_0^{[b][a]}$ , the inequality as  $G_0 \models h_1h_2 \neq h_2h_1$  and we are done by  $(*)_7$ .]

$\oplus_3$   $a, b$  does not commute in  $G_3$  when:

- $a$  normalizes  $G_0$  in  $G_1$ ,
- $b$  normalizes  $G_0$  in  $G_2$ ,
- $b$  does not commute with  $G_0$  in  $G_2$ .

[Why? Again  $\oplus$  holds hence we can apply  $(*)_7$  for any relevant  $\mathbf{x}, g_0, g_1, g_2$ . Let  $h_1 \in G_0$  be such that it does not commute with  $b$  in  $G_2$  and let  $h_2 = e_{G_0}$ . Choose above  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$  such that  $ah_1^{-1} \in \mathbf{I}_{\mathbf{x},1}$  and  $b = bh_2^{-1} \in \mathbf{I}_{\mathbf{x},2}$  and let  $(g_0, g_1, g_2) = (e, e, e)$ . Again in  $(*)_0$  we get  $a' = ah_1^{-1}, h_a = h_1$  and  $b' = bh_2^{-1}, h_b = h_2 = e$ . Now  $h_bh_a^{[b]}g_0^{[a][b]} = eh_a^{[b]}e = h_a^{[b]} \neq h_a = h_aee = h_ah_b^{[a]}g_0^{[b][a]}$ , the inequality by the choice of  $h_a = h_1$ .]

$\oplus_4$   $a, b$  do not commute in  $G_3$  when:

- $a$  normalizes  $G_0$  in  $G_1$ ,
- $b$  normalizes  $G_0$  in  $G_2$ ,
- $a$  does not commute with  $G_0$  in  $G_2$ .

[Why? Like  $\oplus_3$ .]

Next

$\oplus_5$   $a, b$  does not commute in  $G_3$  when:

- $a \in G_1 \setminus G_0$  does not normalize  $G_0$ .

Why? Choose  $h \in G_0$  such that  $a^{-1}ha \notin G_0$  hence  $ha \notin aG_0$  and, of course,  $ha \notin G_0$  as  $a \notin G_0, h \in G_0$  and similarly  $bh^{-1} \in G_2 \setminus G_0$ . Let  $a' = ha$  so  $a' \neq a$  because  $h \neq e$ .

Choose above  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}_{\text{if}}}$  such that  $bh^{-1} \in \mathbf{I}_{\mathbf{x},2}$  and  $a, a' \in \mathbf{I}_{\mathbf{x},1}$ . Why can we choose such  $\mathbf{I}_{\mathbf{x},1}$ ? Because  $a' = ha \in G_1 \setminus G_0, a \in G_1 \setminus G_0$  and  $aG_0 \neq a'G_0$ , as otherwise for some  $h_1 \in G_0$  we have  $a' = ah_1$ , and so  $a^{-1}ha = a^{-1}a' = h_1 \in G_0$ , contradicting the choice of  $h$ .

Let  $f_a, f_b$  be as above for this choice of  $\mathbf{x}$ .

Now consider  $(e, e, e) \in \mathcal{U}_{\mathbf{x}}$  so

$$(*)'_1 \quad f_a((e, e, e)) = (e, a, e).$$

[Why? As  $a \in \mathbf{I}_{\mathbf{x},1}$ .]

$$(*)'_2 \quad f_b((e, a, e)) = (h, a, bh^{-1}).$$

[Why? Because  $bh^{-1} \in \mathbf{I}_{\mathbf{x},2}, h \in G_0$ .]

$$(*)'_3 \quad (f_b \circ f_a)(e, e, e) = (h, a, bh^{-1}).$$

[Why? By  $(*)'_1 + (*)'_2$ .]

$$(*)'_4 \quad f_b((e, e, e)) = (h, e, bh^{-1}).$$

[Why? Because  $bh^{-1} \in \mathbf{I}_{\mathbf{x},2}$  and  $h \in G_0$ .]

$$(*)'_5 \quad f_a((h, e, bh^{-1})) = (e, a', bh^{-1}).$$

[Why? As  $eha = ha = a' = a'e$  and  $a' \in \mathbf{I}_{\mathbf{x},1}$ .]

$$(*)'_6 (f_a \circ f_b)((e, e, e) = (e, a', bh^{-1})).$$

[Why? By  $(*)'_4 + (*)'_5$ .]

By  $(*)'_3 + (*)'_6$ , as  $a' \neq a$  the triple  $(e, e, e)$  exemplifies  $\mathbf{j}_{\mathbf{x},1}(a), \mathbf{j}_{\mathbf{x},2}(b)$  do not commute in  $G_{\mathbf{x}}$ .

Lastly,

$\oplus_6$   $a, b$  do not commute in  $G_3$  when :

- $b \in G_2 \setminus G_0$  does not normalize  $G_0$ .

[Why? As in  $\oplus_5$ .]

As we have covered all the cases we are done. □<sub>2.21</sub>

**Claim 2.23.** *Assume  $\mathfrak{S} \subseteq \Omega[\mathbf{K}_{\text{lf}}]$  and  $G_1 \leq_{\mathfrak{S}} G_2, G_1$  is existentially closed and  $d \in G_2$ . If conjugation by  $d$  (in  $G_2$ ) maps  $G_1$  onto itself then for some  $c \in G_1$  we have  $a \in G_1 \Rightarrow c^{-1}ac = d^{-1}ad$ , i.e.  $dc^{-1}a = adc^{-1}$ , i.e.  $dc^{-1}$ ,  $a$  commute in  $G_1$  so  $dc^{-1}$  commute with  $G_1$ .*

*Proof.* Easy. Clearly there is  $(\mathfrak{s}, \bar{a}) \in \text{def}(G_1)$  such that  $\text{tp}_{\text{bs}}(d, G_1, G_2) = q_{\mathfrak{s}}(\bar{a}, G_1)$ , hence if  $b, b_1, c_1 \in G_1$  and  $\text{tp}_{\text{bs}}(\langle b_1, c_1 \rangle, \bar{a}, G_1) = \text{tp}_{\text{bs}}(\langle b, d^{-1}bd \rangle, \bar{a}, G_1)$  then  $d^{-1}b_1d = c_1$ . Having disjoint amalgamation we have  $x \in G_1 \Rightarrow d^{-1}xd \in \text{cl}(\bar{a} \hat{\ } \langle x \rangle, G_1)$ . We can continue or note that if there is no  $c \in G_1$  as desired, then every existentially closed  $G$  has a non-inner automorphism, contradiction. □<sub>2.23</sub>

§ 3. SYMMETRIZING

Our intention is to start with  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  which may contain  $\mathfrak{s}_1, \mathfrak{s}_2$  failing symmetry but have the nice conclusion as for symmetric  $\mathfrak{S}$ . Towards this we define the operation  $\otimes$ , related to  $\oplus$  defined in Definition 1.6(4),(4A), and  $\mathfrak{S}-\otimes$ -constructions (close but not the same as the constructions in Definition 1.12, 1.19, 1.24) and  $\mathfrak{S}-\oplus$ -constructions.

Note that  $\mathfrak{S}_{\text{atdf}}$  has “quasi symmetry”, i.e. when the parameter (= base of amalgamation) is the same, but when we allow increasing the base this is not clear. Now  $\otimes$  is like  $\oplus$  when we insist on it being symmetric. We use the construction here in §4,§5 where we sometimes give more details. Recall  $\text{def}(G)$  for  $G \in \mathbf{K}$  is from Definition 1.1(1).

Recall

**Definition 3.1.** For  $t \in \text{def}(G)$  let  $q_t(G) = q_{\mathfrak{s}_t}(\bar{a}_t, G)$  and  $n_t = n_{\mathfrak{s}_t}, k_t = k_{\mathfrak{s}_t}$  and see Definition 1.1(6).

**Definition 3.2.** 1) On  $\text{def}(G)$  we define a (partial) operation  $\otimes$  by  $t_1 \otimes t_2 = (\mathfrak{s}_{t_1} \otimes \mathfrak{s}_{t_2}, \bar{a}_{t_1} \hat{\ } \bar{a}_{t_2})$ , see below.

2)  $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2$  means that  $\mathfrak{s}_1, \mathfrak{s}_2$  are disjoint<sup>13</sup>,  $\bar{x}_{\mathfrak{s}} = \bar{x}_{\mathfrak{s}_1} \hat{\ } \bar{x}_{\mathfrak{s}_2}, \bar{z}_{\mathfrak{s}} = \bar{z}_{\mathfrak{s}_1} \hat{\ } \bar{z}_{\mathfrak{s}_2}$ , so  $k(\mathfrak{s}) = k(\mathfrak{s}_1) + k(\mathfrak{s}_2), n(\mathfrak{s}) = n(\mathfrak{s}_1) + n(\mathfrak{s}_2)$  and:

▮ if  $H \subseteq H^+ \in \mathbf{K}, \bar{a}_\ell \in {}^{k(\mathfrak{s}_\ell)}H$  realizes  $p_{\mathfrak{s}_\ell}(\bar{x}_{\mathfrak{s}_\ell})$  in  $H$  and  $\bar{c}_\ell \in {}^{n(\mathfrak{s}_\ell)}(H^+)$  for  $\ell = 1, 2$ , then  $\bar{c}_1 \hat{\ } \bar{c}_2$  realizes  $q_{\mathfrak{s}}(\bar{a}_1 \hat{\ } \bar{a}_2, H)$  iff:

(a)  $\bar{c}_\ell$  realizes  $q_{\mathfrak{s}_\ell}(\bar{a}_\ell, H)$  in  $H^+$ , for  $\ell = 1, 2$ ;

(b) if  $\sigma(\bar{z}_1, \bar{z}_2, \bar{y})$  is a group-term,  $\ell g(\bar{z}_1) = n(\mathfrak{s}_1), \ell g(\bar{z}_2) = n(\mathfrak{s}_2)$  and  $\bar{b} \in {}^{\ell g(\bar{y})}(H)$ , then  $(\alpha) \Leftrightarrow (\beta)$  where:

( $\alpha$ )  $H^+ \models “\sigma(\bar{c}_1, \bar{c}_2, \bar{b}) = e_H”$ ,

( $\beta$ )  $(\sigma(\bar{z}_1, \bar{c}_2, \bar{b}) = e) \in q_{\mathfrak{s}_1}(\bar{a}_1, H^+)$  and  $(\sigma(\bar{c}_1, \bar{z}_2, \bar{b}) = e) \in q_{\mathfrak{s}_2}(\bar{a}_2, H^+)$ .

**Claim 3.3.** 1) If  $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$  then  $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2$  belongs to  $\Omega[\mathbf{K}]$ .

2) If  $G \in \mathbf{K}$  and  $t_1, t_2 \in \text{def}(G)$  then  $t = t_1 \otimes t_2 \in \text{def}(G)$ .

*Proof.* Straightforward. □<sub>3.3</sub>

**Definition 3.4.** 1) Let  $\approx_G^*$  be the following two-place relation on  $\text{def}(G) : (\mathfrak{s}_1, \bar{a}_1) \approx_G^* (\mathfrak{s}_2, \bar{a}_2)$  if both are in  $\text{def}(G)$  and  $G \subseteq G^+ \in \mathbf{K} \Rightarrow q_{\mathfrak{s}_1}(\bar{a}_1, G^+) = q_{\mathfrak{s}_2}(\bar{a}_2, G^+)$ , (compare with  $\approx_G$  from 1.1(6)).

2) For  $t_1, t_2 \in \text{def}(G)$  let  $t_1 \leq t_2$  means  $\text{dom}(\bar{x}_{t_1}) \subseteq \text{dom}(\bar{x}_{t_2}), \text{dom}(\bar{z}_{t_1}) \subseteq \text{dom}(\bar{z}_{t_2})$  and  $\bar{a}_{t_1} = \bar{a}_{t_2} \upharpoonright \text{dom}(\bar{x}_{t_1})$ , and if  $G \subseteq G_1 \subseteq G_2$  and  $\bar{c}_2$  realizes  $q_{t_2}(G_1)$  in  $G_2$  then  $\bar{c}_2 \upharpoonright \text{dom}(\bar{z}_{t_1})$  realizes  $q_{t_1}(G)$  in  $G_2$ .

3)  $t_1 \leq_{\bar{h}} t_2$  is defined similarly as in 1.6(7).

**Claim 3.5.** 0)  $\approx_G^*$  is an equivalence relation on  $\text{def}(G)$ .

1) If  $(\mathfrak{s}, \bar{a}) \in \text{def}(G_1)$  and  $G_1 \subseteq G_2 \in \mathbf{K}$  then  $q_{\mathfrak{s}}(\bar{a}, G_1) \subseteq q_{\mathfrak{s}}(\bar{a}, G_2)$  and  $(\mathfrak{s}, \bar{a}) \in \text{def}(G_2)$ .

2) If  $G \in \mathbf{K}$  and  $(\mathfrak{s}_\ell, \bar{a}_\ell) \in \text{def}(G)$  for  $\ell = 1, 2$ , then the satisfaction of  $(\mathfrak{s}_1, \bar{a}_1) \approx_G^* (\mathfrak{s}_2, \bar{a}_2)$  depends just on  $\mathfrak{s}_1, \mathfrak{s}_2$  and  $\text{tp}_{\text{bs}}(\bar{a}_1 \hat{\ } \bar{a}_2, \emptyset, G)$ .

3) *Transitivity:* in Definition 3.4(2),  $\leq$  is indeed a partial order.

4) Moreover if  $(\mathfrak{s}_1, \bar{a}_1) \leq_{\bar{h}_1} (\mathfrak{s}_2, \bar{a}_2) \leq_{\bar{h}_2} (\mathfrak{s}_3, \bar{a}_3)$  then  $(\mathfrak{s}_1, \bar{a}_1) \leq_{\bar{h}_2 \circ \bar{h}_1} (\mathfrak{s}_3, \bar{a}_3)$ .

<sup>13</sup>As we use only invariant  $\mathfrak{S}$ , this is not a real restriction.

*Proof.* Easy. □<sub>3.5</sub>

**Claim 3.6.** 0) The operation  $\otimes$  on disjoint pairs respects congruency (see Definition 1.1(3), Claim 1.9(1)).

1) The operation  $\otimes$  respects  $\approx_G^*$ , i.e. if  $t_1 \approx_G^* t'_1$  and  $t_2 \approx_G^* t'_2$  then  $t_1 \otimes t_2 \approx_G^* t'_1 \otimes t'_2$  assuming the operations are well defined, of course.

2) If  $(\mathfrak{s}, \bar{a}) = (\mathfrak{s}_1, \bar{a}_1) \otimes (\mathfrak{s}_2, \bar{a}_2)$ , then  $(\bar{\mathfrak{s}}_\ell, \bar{a}_\ell) \leq (\mathfrak{s}, \bar{a})$ .

3) If in  $\text{def}(G)$  we have  $t_\ell \leq t'_\ell$  for  $\ell = 1, 2$  and  $t'_1 \otimes t'_2$  is well defined (i.e.  $t'_1, t'_2$  are disjoint) then  $t_1 \otimes t_2 \leq t'_1 \otimes t'_2$ .

4) The operation  $\otimes$  is associative and is symmetric, e.g. symmetry means: if  $G \subseteq G^+$  and  $(\mathfrak{s}_\ell, \bar{a}_\ell) \in \text{def}(G)$  and  $\bar{c}_\ell \hat{\wedge} \bar{c}_{3-\ell}^{\ell}$  realizes  $q_{t_\ell}(G)$  in  $G^+$ , where  $t_\ell = (t_\ell, \bar{b}_\ell) = (\mathfrak{s}_\ell, \bar{a}_\ell) \otimes (\mathfrak{s}_{3-\ell}, \bar{a}_{3-\ell})$ , (so assuming disjointness for transparency), for  $\ell = 1, 2$ , then  $\text{tp}_{\text{bs}}(\bar{c}_1 \hat{\wedge} \bar{c}_2^1, G, G^+) = \text{tp}_{\text{bs}}(\bar{c}_1^2 \hat{\wedge} \bar{c}_2^2, G, G^+)$ .

5) If in  $\text{def}(G)$  we have  $(\mathfrak{s}_\ell, \bar{a}_\ell) \leq_{h_\ell} (\mathfrak{s}'_\ell, \bar{a}'_\ell)$  for  $\ell = 1, 2$  and  $\text{Dom}(h_1) \cap \text{Dom}(h_2) = \emptyset$ ,  $\text{Rang}(h_1) \cap \text{Rang}(h_2) = \emptyset$  then  $(\mathfrak{s}_1, \bar{a}_1) \otimes (\mathfrak{s}_2, \bar{a}_2) \leq_{h_1 \cup h_2} (\mathfrak{s}'_1, \bar{a}'_1) \otimes (\mathfrak{s}'_2, \bar{a}'_2)$ .

*Proof.* Straightforward. □<sub>3.6</sub>

**Remark 3.7.** 1) Also the operation  $\oplus$  satisfies the parallels of 3.6(1),(2),(3) and the first demand in (4).

2) We may phrase 3.6(5) as in 3.6(3) and vice versa.

**Definition 3.8.** Assume  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is closed.

1) We say  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is  $\otimes$ -closed when (recalling it is invariant) if  $\mathfrak{s}_\ell \in \mathfrak{S}$  for  $\ell = 1, 2$  are disjoint then  $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2 \in \mathfrak{S}$ .

2) The  $\otimes$ -closure of  $\mathfrak{S}$  is the  $\subseteq$ -minimal  $\otimes$ -closed  $\mathfrak{S}' \subseteq \Omega[\mathbf{K}]$  such that  $\mathfrak{S} \subseteq \mathfrak{S}'$ .

3) Let  $G_3 = G_1 \bigotimes_{G_0}^{\mathfrak{S}} G_2$  or  $G_3 = \otimes_{\mathfrak{S}}(G_0, G_1, G_2)$  mean:

- (\*) (a)  $G_0 \leq_{\mathfrak{S}} G_2 \subseteq G_3 \in \mathbf{K}$  and  $G_0 \leq_{\mathfrak{S}} G_1 \subseteq G_3$  and  $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$
- (b) if  $\text{tp}_{\text{bs}}(\bar{c}_\ell, G_0, G_\ell) = q_{\mathfrak{s}_\ell}(\bar{a}_\ell, G_0)$  so  $\bar{c}_\ell \in \omega^>(G_\ell)$ ,  $\bar{a}_\ell \in \omega^>(G_0)$  for  $\ell = 1, 2$ , then  $\text{tp}_{\text{bs}}(\bar{c}_1 \hat{\wedge} \bar{c}_2, G_0, G_3) = q_{\mathfrak{s}}(\bar{a}_1 \hat{\wedge} \bar{a}_2, G_0)$  when  $(\mathfrak{s}, \bar{a}_1 \hat{\wedge} \bar{a}_2) = (\mathfrak{s}, \bar{a}_1) \otimes (\mathfrak{s}_2, \bar{a}_2)$ ; note that without loss of generality  $\mathfrak{s}_1, \mathfrak{s}_2$  are disjoint, (i.e. as in the proof of 1.10).

4)  $\text{NF}_{\mathfrak{S}}^2(G_0, G_1, G_2, G_3)$  means that  $G_0 \leq_{\mathfrak{S}} G_\ell \leq_{\mathfrak{S}} G_3$  for  $\ell = 1, 2$  and the demands in (3) hold except that possibly  $G_3 \neq \langle G_1 \cup G_2 \rangle_{G_3}$ .

**Claim 3.9.** Assume  $\mathfrak{S}$  is closed and moreover  $\otimes$ -closed.

1)  $G_3 = \otimes_{\mathfrak{S}}(G_0, G_1, G_2)$  iff  $\text{NF}_{\mathfrak{S}}^2(G_0, G_1, G_2, G_3)$  and  $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$ .

2) (disjointness):  $\text{NF}_{\mathfrak{S}}^2(G_0, G_1, G_2, G_3)$  implies  $G_1 \cap G_2 = G_0$ .

3) (uniqueness): If  $G_3^\iota = \otimes_{\mathfrak{S}}(G_0^\iota, G_1^\iota, G_2^\iota)$  for  $\iota = 1, 2$  and  $f_\ell$  is an isomorphism from  $G_\ell^1$  onto  $G_\ell^2$  for  $\ell = 1, 2$  and  $G_0^1 = G_0^2$ ,  $f_1 \upharpoonright G_0^1 = f_2 \upharpoonright G_0^2$  and  $G_0$  is existentially closed<sup>14</sup> then there is one and only one isomorphism from  $G_3^1$  onto  $G_3^2$  extending  $f_1 \cup f_2$  (which is well defined by (2)).

<sup>14</sup>Why? The problem is that  $G \leq_{\mathfrak{S}} H \in \mathbf{K}$  does not imply the existence of  $\bar{t} = \langle t_{\bar{c}} : \bar{c} \in \omega^>H \rangle$  such that  $t_{\bar{c}} \in \text{def}(G)$ ,  $\text{tp}_{\text{bs}}(\bar{c}, G, H) = q_{\bar{t}}(G)$  and if  $\bar{c}^1, \bar{c}^2 \in \omega^>H$ ,  $h : \text{lg}(\bar{c}^1) \rightarrow \text{lg}(\bar{c}^2)$  and  $\bar{c}^2 = \langle c_{h(i)}^2 : i < \text{lg}(\bar{c}^1) \rangle$  then  $t_{\bar{c}^1} \leq_h t_{\bar{c}^2}$ . Moreover, even if there is such  $\bar{t}$  we can “amalgamate for it” but this is not enough as  $\bar{t}$  is not necessarily unique, which may give different results. Why 3.9(3) is O.K.? As in Definition 3.8(3) we ask “for every  $\mathfrak{s}_1, \mathfrak{s}_2$ ”. In other words if  $G_0 \subseteq G_1, G_0 \subseteq G_2$  and  $t_1, t_2 \in \text{def}(G_0)$ ,  $\text{tp}_{\text{bs}}(\bar{c}_\ell, G_0, G_2) = q_{t_\ell}(G_0)$  for  $\ell = 1, 2$  but  $q_{t_1}(G_1) \neq q_{t_2}(G_1)$  we can amalgamate as in 3.8(3).

- 4) (symmetry):  $\text{NF}_{\mathfrak{S}}^2(G_0, G_1, G_2, G_3)$  iff  $\text{NF}_{\mathfrak{S}}^2(G_0, G_2, G_1, G_3)$ .  
 5) (monotonicity): If  $\text{NF}_{\mathfrak{S}}^2(G_0, G_1, G_2, G_3)$  and  $G_0 \subseteq G'_\ell \subseteq G_\ell$  for  $\ell = 1, 2$  then  $\text{NF}_{\mathfrak{S}}^2(G_0, G'_1, G'_2, G_3)$ .  
 6) (existence): If  $G_0 \leq_{\mathfrak{S}} G_\ell$  for  $\ell = 1, 2$  and  $G_0$  is existentially closed and  $G_1 \cap G_2 = G_0$  then for some  $G_3 \in \mathbf{K}$  we have  $\text{NF}_{\mathfrak{S}}^2(G_0, G_1, G_2, G_3)$ .

*Remark 3.10.* For parts (3) and (6) of 3.9 recall: for such  $G$ , if  $t_1, t_2 \in \text{def}(G)$ ,  $q_{t_1}(G) = q_{t_2}(G)$  and  $G \subseteq G^+ \in \mathbf{K}$  then  $q_{t_1}(G^+) = q_{t_2}(G^+)$ .

*Proof.* Straightforward, e.g. for disjointness (= part (2)) use Claim 1.2(4).  $\square_{3.9}$

Alternative to §1 from 1.12 on is: we repeat it with changes being that we use  $\otimes$  instead of  $\oplus$  and we incorporated the  $\lambda$ -fullness, also in 3.12(3) we choose another version. We have not sorted out whether we can generalize 1.16(5) based on 1.15 and 1.23(2).

**Definition 3.11.** 1) We say that  $\mathcal{A}$  is a one step  $(\lambda, \mathfrak{S}) - \otimes$ -construction when  $\mathcal{A} = (G, H, \langle \bar{c}_\alpha, t_\alpha : \alpha < \alpha(\mathcal{A}) = \alpha_{\mathcal{A}} \rangle)$  satisfies:

- (a)  $G \subseteq H \in \mathbf{K}$
- (b)  $t_\alpha \in \text{def}_{\mathfrak{S}}(G)$  for  $\alpha < \alpha(\mathcal{A})$ ;
- (c) if  $\alpha_0, \dots, \alpha_{n-1} < \alpha(\mathcal{A})$  with no repetitions then  $\bar{c}_{\alpha_0} \hat{\ } \dots \hat{\ } \bar{c}_{\alpha_{n-1}}$  realizes  $q_t(G_0)$  in  $H$  where  $t = t_{\alpha_0} \otimes \dots \otimes t_{\alpha_{n-1}} \in \text{def}(G)$ ;
- (d)  $H = \langle \cup \{ \bar{c}_\alpha : \alpha < \alpha(\mathcal{A}) \} \cup G \rangle_H$ ;
- (e)  $\langle t_\alpha : \alpha < \alpha(\mathcal{A}) \rangle$  lists  $\text{def}_{\mathfrak{S}}(G)$  each appearing exactly  $\lambda$  times.

2) In (1) we may use any index set instead of  $\alpha(\mathcal{A})$ , e.g.  $\text{def}_{\mathfrak{S}}(G)$  itself when  $\lambda = 1$ ,  $\text{def}_{\mathfrak{S}}(G) \times \lambda$  in general.

3) We say  $\mathcal{A}$  is an  $\alpha(\mathcal{A})$ -step- $(\lambda, \mathfrak{S}) - \otimes$ -construction or  $(\alpha(\mathcal{A}), \lambda, \mathfrak{S}) - \otimes$ -construction when:

- (a)  $\mathcal{A} = (G_\alpha, \langle \bar{c}_{\beta,s}, t_{\beta,s} : s \in S_\beta \rangle : \alpha \leq \alpha(\mathcal{A}), \beta < \alpha(\mathcal{A})$ )
- (b)  $(G_\alpha : \alpha \leq \alpha(\mathcal{A}))$  is increasing continuous (in  $\mathbf{K}$ )
- (c)  $(G_\alpha, G_{\alpha+1}, \langle \bar{c}_{\alpha,s}, t_{\alpha,s} : s \in S_\alpha \rangle)$  is a one step  $(\lambda, \mathfrak{S}) - \otimes$ -construction.

4) In part (3), let  $G_\alpha^{\mathcal{A}} = G_\alpha[\mathcal{A}]$  be  $G_\alpha$ , etc., and in part (1) let  $G^{\mathcal{A}} = G[\mathcal{A}]$  be  $G$ , etc.

5) In part (3) if  $\alpha(\mathcal{A}) = \omega$  then we may omit it; also for every  $\alpha < \alpha(\mathcal{A})$  the sequence  $(G_\alpha^{\mathcal{A}}, G_{\alpha+1}^{\mathcal{A}}, \langle \bar{c}_{\alpha,s}, t_{\alpha,s} : s \in S_\alpha^{\mathcal{A}} \rangle)$  is called the  $\alpha$ -th step of  $\mathcal{A}$ .

**Definition 3.12.** 1) We say  $H$  is a  $\lambda$ -full one step  $\mathfrak{S} - \otimes$ -closure of  $G$  when there is a one step  $(\lambda, \mathfrak{S}) - \otimes$ -construction  $\mathcal{A}$  such that  $G[\mathcal{A}] = G$ ,  $H[\mathcal{A}] = H$ . We may say  $H$  is  $\lambda$ -full one step  $\mathfrak{S} - \otimes$ -constructible over  $G$ ; similarly in part (2).

2) We say  $H$  is  $\lambda$ -full  $\alpha$ -step  $\mathfrak{S}$ -closure over  $G$  or  $H$  is  $(\alpha, \lambda, \mathfrak{S})$ -closure of  $G$  when there is a  $(\alpha, \lambda, \mathfrak{S}) - \otimes$ -construction  $\mathcal{A}$  with  $G = G_0^{\mathcal{A}}$ ,  $H = G_{\ell g(\mathcal{A})}^{\mathcal{A}}$ .

3) We say  $G_*$  is  $(\delta, \lambda, \mathfrak{S}) - \otimes$ -full over  $G$  when for some  $\bar{G} = \langle G_i : i \leq \delta \rangle$  increasing continuous sequence in  $\mathbf{K}$ ,  $G_0 = G$ ,  $G_\delta = G_*$  and  $G_{i+1}$  is  $(1, \lambda, \mathfrak{S}) - \otimes$ -full over  $G_i$  which means some  $G' \subseteq G_{i+1}$  is a one step  $(\lambda, \mathfrak{S}) - \otimes$ -construction over  $G_i$ . If  $\delta = \omega$  one may omit it writing  $(\lambda, \mathfrak{S}) - \otimes$  instead of  $(\delta, \lambda, \mathfrak{S})$ .

4) We may in part (3) replace  $\otimes$  by  $\oplus$ .



**Claim 3.13.** Assume  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  is  $\otimes$ -closed,  $\alpha$  an ordinal,  $\lambda$  a cardinal.

- 1) If  $G \in \mathbf{K}$  then there is a one step  $(\lambda, \mathfrak{S}) - \otimes$ -construction  $\mathcal{A}$  over  $G$  (i.e.  $G_0^{\mathcal{A}} = G$ ) of cardinality  $\leq \lambda + |G| + |\mathfrak{S}|$  and  $\geq \lambda$ .
- 2) If in part (1),  $\mathcal{A}_1, \mathcal{A}_2$  are one step- $(\lambda, \mathfrak{S}) - \otimes$ -constructions over  $G$  then  $H[\mathcal{A}_1], H[\mathcal{A}_2]$  are isomorphic over  $G$ .
- 3) For any  $G \in \mathbf{K}$  there is an  $(\alpha, \lambda, \mathfrak{S}) - \otimes$ -construction  $\mathcal{A}$  over  $G$  and  $G_\alpha[\mathcal{A}]$  is unique up to isomorphism over  $G$ .
- 4) If  $\mathfrak{S}$  is dense,  $H$  is an  $(\alpha, \lambda, \mathfrak{S}) - \otimes$ -closure of  $G$  and  $\alpha$  is a limit ordinal then  $H$  is existentially closed and is  $(\alpha, \lambda, \mathfrak{S})$ -full over  $G$ .

*Proof.* Straightforward, as in 1.23(3). □3.13

**Discussion 3.14.** Essentially we know that if “ $G_1 \subseteq G_2$ ” implies the  $(\alpha, \lambda, \mathfrak{S})$ -closure of  $G_1$  is a subgroup of the  $(\alpha, \lambda, \mathfrak{S})$ -closure of  $G_2$ .

But we have a delicate problem: what if the  $(\alpha, \lambda, \mathfrak{S})$ -closure of  $G_1$  is not disjoint to  $G_2 \setminus G_1$ ?

We have similar problems with “the algebraic closure of a field” or “the field of quotients of a field”, but there if  $G_1 \subseteq G_2$  then the closure  $G_1^+$  of  $G_1$  inside  $G_2$  is definable (from  $G_2, G_1$  and  $G_2^+$ ). Here this is not true, but clearly this is not a serious problem. Ways to circumvent this appear in 0.12(2), 1.13(2) and below.

**Claim 3.15.** 1) We can choose  $\hat{G} \in \mathbf{K}_{\text{exlf}}$  such that  $\hat{G}$  extends  $G \in \mathbf{K}_{\text{lf}}, G_1 \cong G_2 \Rightarrow \hat{G}_1 \cong \hat{G}_2$  and every embedding  $f : G_1 \rightarrow G_2 \in \mathbf{K}_{\text{lf}}$  can be extended to  $\hat{f} : \hat{G}_1 \rightarrow \hat{G}_2$  canonically.

1A) Moreover  $G_1 \subseteq G_2 \Rightarrow \hat{G}_1 \subseteq \hat{G}_2$  but pedantically see (2).

2) There is a set theoretic class function  $\mathbf{F}$ , that computes from  $G \in \mathbf{K}, \alpha \in \text{Ord}, \lambda \in \text{Card}, \gamma \in \text{Ord}$  and  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  a group  $H = \mathbf{F}(G, \alpha, \mathfrak{S}, \gamma)$  such that:

- (a)  $\mathbf{F}(G, \alpha, \mathfrak{S}, \gamma) \in \mathbf{K}$  extends  $G$ , moreover;
- (b)  $\mathbf{F}(G, \alpha, \gamma, \mathfrak{S})$  is an  $(\alpha, \lambda, \mathfrak{S})$ -closure of  $G$ ;
- (c) [uniqueness]: if  $G_1, G_2 \in \mathbf{K}$  and  $g$  is an isomorphism from  $G_1$  onto  $G_2$  and  $H_\ell = \mathbf{F}(G_\ell, \alpha, \mathfrak{S}, \gamma)$  for  $\ell = 1, 2$  then there is an isomorphism  $g$  from  $H_1$  onto  $H_2$  extending  $g$ ;
- (d) we have  $H_1 \subseteq H_2$  and  $G_1 = H_1 \cap G_2$  when  $G_1 \subseteq G_2 \in \mathbf{K}, \gamma > \alpha$  and<sup>15</sup>  $\gamma > \sup(\text{Ord} \cap \text{tr-cl}(G_\ell))$  for  $\ell = 1, 2$  and  $H_\ell = \mathbf{F}(G_\ell, \alpha, \mathfrak{S}, \gamma)$ ;
- (e) if we restrict ourselves to  $G \in \mathbf{K}' = \{G \in \mathbf{K} : \text{if } x \in G \text{ then } x \text{ is a singleton}\}$  then  $G_1 \subseteq G_2 \Rightarrow \mathbf{F}(G, \alpha, \mathfrak{S}) \subseteq \mathbf{F}(G, \alpha, S, 0)$ .

\* \* \*

In §4,§5 we intend to use also some relative of those constructions, including:

**Definition 3.16.** Assume  $\bar{H} = \langle H_i : i < \delta \rangle$  is  $\subseteq$ -increasing in  $\mathbf{K}$  and  $H_\delta = \cup\{H_i : i < \delta\}$ , (we shall use  $\delta = \omega$ ). We say  $\mathcal{A}$  is a one step atomic  $\mathfrak{S} - \otimes$ -construction above  $\bar{H}$ , when (and we may say  $\bar{H}$  is weakly atomically  $\mathfrak{S} - \otimes$ -constructible over  $\bar{H}$ , omitting  $\bar{H}$  means for some  $\bar{H}$  of length  $\omega$  and we may replace  $\alpha_{\mathcal{A}} = \alpha(\mathcal{A})$  by any index set)  $\mathcal{A}$  has the following objects satisfying the following additional conditions:

<sup>15</sup>Recalling tr-cl is the (set-theoretic) transitive closure.

- (A)  $(\bar{H}, H_\delta, H, \langle \bar{c}_\alpha, t_{\alpha,i}, \alpha < \alpha_{\mathcal{A}}, i < \delta \rangle)$ ;
- (B)  $H_\delta \subseteq H \in \mathbf{K}$ ;
- (C)  $t_{\alpha,i} \in \text{def}_{\mathfrak{S}}(H_i)$ ;
- (D)  $H = \langle \cup\{\bar{c}_\alpha : \alpha < \alpha_{\mathcal{A}}\} \cup H_\delta \rangle_H$ ;
- (E)  $\bar{c}_\alpha$  realizes  $q_{t_{\alpha,i}}(H_i)$  in  $H$  for  $\alpha < \alpha_{\mathcal{A}}, i < \delta$ ;
- (F)  $\bar{c}_{\alpha,i} \subseteq H_{i+1}$  realizes  $q_{t_{\alpha,i}}(H_i)$  for  $i < \delta, \alpha < \alpha_{\mathcal{A}}$  and moreover;
- (F)<sup>+</sup> assuming  $\alpha(0) < \dots < \alpha(n-1) < \alpha_{\mathcal{A}}$  and  $\ell g(\bar{x}_\alpha) = \ell g(\bar{c}_\alpha)$  and  $\varphi = \varphi(\bar{x}_{\alpha(0)}, \dots, x_{\alpha(n-1)}, \bar{y})$  we have<sup>16</sup>  
 $\varphi(\bar{x}_{\alpha(0)}, \dots, \bar{x}_{\alpha(n-1)}, \bar{b}) \in \text{tp}_{\text{at}}(\bar{c}_{\alpha(0)} \hat{\ } \dots \hat{\ } \bar{c}_{\alpha(n-1)}, G_\delta, H)$   
iff  $\bar{b} \subseteq {}^{\ell g(\bar{y})}G_\delta$  and for every permutation  $\pi$  of  $n$ ,  
 $(\forall^\infty i(0 < \delta)(\forall^\infty i(1 < \delta), \dots, (\forall^\infty i(n-1) < \delta)$   
 $\varphi[\bar{c}_{\alpha(0),i(\pi(0))}, \bar{c}_{\alpha(1),i(\pi(1))}, \dots, \bar{c}_{\alpha(n-1),i(\pi(n-1))}, \bar{b}]$   
(used in the proof of (\*)<sub>5.2</sub> stage C in the proof of 5.1); note that  $\varphi$  is not necessarily atomic.

*Remark 3.17.* 1) We may consider replacing clause (F)<sup>+</sup> by:

- (F)<sup>\*</sup>  $\bar{c}_{\alpha(0)} \hat{\ } \dots \hat{\ } \bar{c}_{\alpha(n-1)}$  realizes  $q_{t_{\alpha(0)} \otimes \dots \otimes t_{\alpha(n-1)}}$  for  $\alpha(0) < \dots < \alpha(n-1) < \alpha_{\mathcal{A}}$ .

2) In this alternative version we do not need the existence of  $\bar{c}_{\alpha,i} \subseteq H_{i+1}$ , so it is easier to prove existence but the version above is the one we actually use. In particular the version in (1) would create problems in (\*)<sub>5.7</sub> in the proof of 5.1; we may try to take care of this by changing the definition of  $L_\beta^*$  there.

3) A sufficient condition for having the assumptions of 3.16 appear in 2.19.

**Observation 3.18.** Let  $\mathfrak{S}$  be closed and  $\otimes$ -closed. Assume  $\langle G_i : i \leq \alpha \rangle$  is  $\subseteq$ -increasing continuous in  $\mathbf{K}$ .

- 1) In 3.11(1) we can prove  $G_{\mathcal{A}} \leq_{\mathfrak{S}} H_{\mathcal{A}}$  and in 3.11(2), we can prove  $\langle G_{\alpha}^{\mathcal{A}} : \alpha \leq \alpha_{\mathcal{A}} \rangle$  is  $\leq_{\mathfrak{S}}$ -increasing continuous.
- 2) In 3.16, if  $\bar{H}$  is  $\leq_{\mathfrak{S}}$ -increasing then we have  $i < \delta \Rightarrow H_i \subseteq_{\mathfrak{S}} H$ .
- 3) Assume  $S$  is a set of limit ordinals  $< \delta$ ,  $\langle G_i : i \leq \delta \rangle$  is a  $\subseteq$ -increasing continuous sequence of members of  $\mathbf{K}$  and  $G_{i+1}$  is a one step  $\mathfrak{S} - \otimes$ -constructible over  $G_i$  for  $i \in \delta \setminus S$  and  $G_{i+1}$  is weakly one step  $\mathfrak{S} - \otimes$ -constructible over  $\bar{G} \upharpoonright C_i$  for some unbounded  $C_i \subseteq i \setminus S$  for each  $i \in S$ , (hence  $i$  is a limit ordinal). Then  $i < j \leq \delta \wedge i \notin S \Rightarrow G_i \leq_{\mathfrak{S}} G_j$ .

*Remark 3.19.* The idea of  $\mathfrak{s}_1 \otimes \mathfrak{s}_2$  can be applied to one  $\mathfrak{s}$  (and is used in the end of the proof of  $\boxplus_1$  in stage B the proof of Theorem 5.1).

Toward this in §4(B) we shall deal with finding such amalgamations and  $\mathfrak{s}$ 's.

**Definition/Claim 3.20.** Assume  $\mathfrak{s} \in \Omega[\mathbf{K}_{\text{lf}}]$  and  $H_1 \subseteq H_2 \in \mathbf{K}$  are finite,  $\bar{a} \in {}^{k(\mathfrak{s})}(H_1)$ ,  $\bar{c} \in {}^{n(\mathfrak{s})}(H_2)$  and  $\bar{a}, \bar{c}$  generate  $H_1, H_2$  respectively, and  $\bar{a}$  realizes  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$  in  $H_1$  and  $\bar{c}$  realizes  $q_{\mathfrak{s}}(\bar{a}, H_1)$  in  $H_2$ . Assume further  $K$  is a group of automorphisms of  $H_2$  mapping  $H_1$  onto itself. Then there is a one and only one  $\mathfrak{t}$  such that:

- (a)  $\mathfrak{t} \in \Omega[\mathbf{K}_{\text{lf}}]$
- (b)  $k(\mathfrak{t}) = k(\mathfrak{s})$  and  $p_{\mathfrak{t}}(\bar{x}_{\mathfrak{t}}) = \text{tp}_{\text{qf}}(\bar{a}, \emptyset, H_1)$

<sup>16</sup>Yes!  $\text{tp}_{\text{at}}$  and not  $\text{tp}_{\text{bs}}$ .

(c) if  $H_1 \subseteq G_1 \subseteq G_2, H_2 \subseteq G_2$  and  $\bar{c}$  realizes  $q_s(\bar{a}, G_1)$  in  $G_2$  and  $\bar{c}' \in {}^n(G_2)$  realizes  $q_t(\bar{a}, G_2)$  then  $\text{tp}_{\text{at}}(\bar{c}', G_1, G_2) = \cap \{\text{tp}_{\text{at}}(\pi(\bar{c}), G_1, G_2) : \pi \in K\}$ .

*Remark 3.21.* Toward this in §(4B) we deal with finding such amalgamations and  $\mathfrak{s}$ 's.

*Proof.* Straightforward.

□3.20

## § 4. FOR FIXING A DISTINGUISHED SUBGROUP

In the construction of complete members of  $\mathbf{K}_{\text{exlf}}$  (and related aims) we fix large enough  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$  and build a  $\subseteq$ -increasing continuous sequence  $\langle G_\alpha : \alpha < \lambda \rangle$ ,  $|G_\alpha| < \lambda$ ; normally we demand for  $\alpha < \beta < \lambda$  that “usually”  $G_\alpha \leq_{\mathfrak{S}} G_\beta$  (i.e. except for  $\delta \in S$ , where  $S \subseteq S_{\aleph_0}^\lambda$ ). But at some moment for  $\alpha = \delta + n$ , we like to use  $p = \text{tp}_{\text{bs}}(c, G_\alpha, G_{\alpha+1})$  which extends some  $r \in \mathbf{S}_{\text{bs}}(K)$ ,  $K \subseteq G_\alpha$  finite but such that  $c$  commutes with  $G_\delta$ . Also toward this in §(4A) we deal with a relative  $\text{NF}^3$  of  $\text{NF}_f$ , in which we demand  $\mathbf{C}_{G_1}(G_3)$  is large, this continues §2 concentrating on the case  $G_0$  is with trivial center. In §(4B) we use this to define some schemes from  $\Omega[\mathbf{K}]$ , see e.g. 4.10.

Another problem is that given  $G_1$  instead of extending  $G_1$  to  $G_2$  such that  $q_t(G_1)$  is realized by  $\bar{c} \in \omega^{>}(G_2)$  for some  $t \in \text{def}_{\mathfrak{S}}(G_1)$ , we like to have an infinite  $\bar{c} = (\dots \hat{c}_i \dots)_{i \in I}$ , with  $\text{tp}(\bar{c} \upharpoonright u, G_1, G_2) \in q_{t_u}(G_1)$  for every finite  $u \subseteq I$ ; used in stage D of the proof of Theorem 5.1. This is done in §4(C).

## § 4(A). Preserving Commutation.

**Claim 4.1.** *The subgroups  $H'_1, H'_2$  of  $G_3$  commute when:*

- (\*) (a)  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ ;
- (b)  $G_\ell = G_{\mathbf{x}, \ell}, G'_\ell = \mathbf{j}_{\mathbf{x}, \ell}(G_\ell)$  for  $\ell = 0, 1, 2$ ;
- (c)  $G_3 = G_{\mathbf{x}}$ ;
- (d)  $H_1 \subseteq G_1$  and  $H'_1 = \mathbf{j}_{\mathbf{x}, 1}(H_1)$ ;
- (e)  $H_1 = \cup \{b(H_1 \cap G_0) : b \in \mathbf{I}_1\}$  where  $\mathbf{I}_1 = \mathbf{I}_{\mathbf{x}, 1} \cap H_1$ ;
- (f) if  $g \in \mathbf{I}_{\mathbf{x}, 1}$  and<sup>17</sup>  $b \in \mathbf{I}_1$  then  $gb \in \mathbf{I}_{\mathbf{x}, 1}$ ;
- (g) the subgroups  $G_0, H_1$  of  $G_1$  commute;
- (h)  $H_2 \subseteq G_2$  commutes with  $G_0 \cap H_1$  and  $H'_2 = \mathbf{j}_{\mathbf{x}, 2}(H_2)$ ;
- (i)  $H_2 = \cup \{a(G_0 \cap H_2) : a \in \mathbf{I}_2\}$  where  $\mathbf{I}_2 = \mathbf{I}_{\mathbf{x}, 2} \cap H_2$ .

*Remark 4.2.* 1) Really here it suffices to deal with the case  $G_0 \cap H_1 = \{e\}$ .

2) A natural case is  $\mathbf{Z}(G_0) = \{e_{G_0}\}, H_1 = \mathbf{C}_{G_1}(G_0), H_2 = G_2$ .

3) See the proof of 5.1.

*Notation 4.3.* Let  $\mathbf{X}_{\text{lf}}^3 = \mathbf{X}_{\mathbf{K}_{\text{lf}}}^3$  be the class of tuple  $(\mathbf{x}, H_1, H_2)$  which satisfies (\*) of Claim 4.1.

*Proof.* Let  $a \in H_2, b \in H_1, f_a = \mathbf{j}_{\mathbf{x}, 2}(a), f_b = \mathbf{j}_{\mathbf{x}, 1}(b)$ , so by (\*) (d), (h) we just have to prove that  $f_b f_a((g_0, g_1, g_2)) = f_a f_b((g_0, g_1, g_2))$  for any  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$ .

Clearly

- if  $a \in G_0$  or  $b \in G_0$  this holds.

[Why? First, if  $a \in G_0$  then  $f_a = \mathbf{j}_{\mathbf{x}, 2}(a) = \mathbf{j}_{\mathbf{x}, 0}(a) = \mathbf{j}_{\mathbf{x}, 1}(a) \in \mathbf{j}_{\mathbf{x}, 1}(G_1) = G'_1 \subseteq G_{\mathbf{x}}$  and as  $b \in H_1 \subseteq G_{\mathbf{x}}$ , by (\*) (g) we have  $G_1 \models$  “ $a, b$  commute” hence  $G_{\mathbf{x}} \models$  “ $\mathbf{j}_{\mathbf{x}, 2}(a), \mathbf{j}_{\mathbf{x}, 1}(b)$  commute” and so  $G_{\mathbf{x}} \models$  “ $f_a, f_b$  commute”. Second, if  $b \in G_0$  then  $b \in G_0 \cap H_1 \subseteq G_0 \subseteq G_2$  and  $a \in H_2 \subseteq G_2$ , so by clause (\*) (h) clearly  $G_2 \models$  “ $a, b$  commute” and we finish as above.]

<sup>17</sup>As  $G_1$  is locally finite, necessarily  $\mathbf{I}_1$  is a subgroup of  $H_1$ .

Moreover, as  $H_1 = \langle (G_0 \cap H_1) \cup \mathbf{I}_1 \rangle_{G_1}$  by clause  $(*) (e)$ , recalling  $\bullet$  above, without loss of generality

$$\boxplus_1 \quad b \in \mathbf{I}_1 \subseteq \mathbf{I}_{\mathbf{x},1}.$$

Similarly as  $H_2 = \langle (G_0 \cap H_2) \cup \mathbf{I}_1 \rangle$ . By clause  $(*) (i)$ , recalling  $\bullet$  above without loss of generality:

$$\boxplus_2 \quad a \in \mathbf{I}_2 \subseteq \mathbf{I}_{\mathbf{x},2}.$$

Let<sup>18</sup>  $f_x((g_0, g_1, g_2)) = (g_0^x, g_1^x, g_2^x)$  and  $f_y f_x((g_0, g_1, g_2)) = (g_0^{x,y}, g_1^{x,y}, g_2^{x,y})$  for  $x \in \{a, b\}$  and  $y \in \{a, b\} \setminus \{x\}$ .

We shall prove that  $g_\ell^{a,b} = g_\ell^{b,a}$  for  $\ell = 0, 1, 2$ ; this suffices.

Clearly,

- <sub>1</sub>  $g_1^a = g_1$  and  $g_2 g_0 a = g_2^a g_0^a$ ;
- <sub>2</sub>  $g_2^{a,b} = g_2^a$  and  $g_1^a g_0^a b = g_1^{a,b} g_0^{a,b}$ ;
- <sub>3</sub>  $g_2^b = g_2$  and  $g_1 g_0 b = g_1^b g_0^b$ ;
- <sub>4</sub>  $g_1^{b,a} = g_1^b$  and  $g_2^b g_0^b a = g_2^{b,a} g_0^{b,a}$ .

Now

$$\boxplus_3 \quad g_1^{a,b} G_0 = g_1^{a,b} g_0^{a,b} G_0 = g_1^a g_0^a b G_0 = (g_1^a b)(g_0^a G_0) = (g_1^a b) G_0.$$

[Why? As  $g_0^{a,b} \in G_0$ , by the second statement of  $\bullet_2$ , noting that  $b, g_0^a$  commute by  $(*) (g)$ , and as  $g_0^a \in G_0$ , respectively.]

But  $g_1^a \in \mathbf{I}_{\mathbf{x},1}$  (as  $(g_0^a, g_1^a, g_2^a) \in \mathcal{U}_{\mathbf{x}}$ ), and  $b \in \mathbf{I}_1 \subseteq \mathbf{I}_{\mathbf{x},1}$  by  $\boxplus_1$ , hence by  $(*) (f)$  we have  $g_1^a b \in \mathbf{I}_{\mathbf{x},1}$  and also  $g_1^{a,b} \in \mathbf{I}_{\mathbf{x},1}$  (as  $(g_0^{a,b}, g_1^{a,b}, g_2^{a,b}) \in \mathcal{U}_{\mathbf{x}}$ ). Now by  $\boxplus_3$ ,  $g_1^{a,b} G_0 = (g_1^a b) G_0$  and by the last sentence  $g_1^{a,b}, g_1^a \in \mathbf{I}_{\mathbf{x},1}$  and thus

$$\bullet_5 \quad g_1^{a,b} = g_1^a b.$$

So by  $\bullet_5$  and the second equation in  $\bullet_2$  we have  $g_1^a b g_0^{a,b} = g_1^{a,b} g_0^{a,b} = g_1^a g_0^a b = g_1^a b g_0^a$ , the last equality by recalling  $b, g_0^a$  commute by  $(*) (g)$ , hence we have:

$$\bullet_6 \quad g_0^{a,b} = g_0^a.$$

Similarly to  $\boxplus_3$  we have

$$\boxplus_4 \quad g_1^b G_0 = g_1^b g_0^b G_0 = g_1 g_0 b G_0 = (g_1 b)(g_0 G_0) = (g_1 b) G_0.$$

[Why? As  $g_0^b \in G_0$ , by  $\bullet_3$  second statement, as  $b, g_0$  commute by  $(*) (g)$ , and as  $g_0 \in G_0$  respectively.]

Also  $g_1 \in \mathbf{I}_{\mathbf{x},1}$  as  $(g_0, g_1, g_2) \in \mathcal{U}_{\mathbf{x}}$  and  $b \in \mathbf{I}_1$  by  $\boxplus_1$  so recalling  $(*) (f)$  we deduce  $g_1, g_1 b \in \mathbf{I}_{\mathbf{x},1}$  thus from  $\boxplus_4$  we deduce:

$$\bullet_7 \quad g_1^b = g_1 b.$$

Hence by  $\bullet_7$  and  $\bullet_3$  second statement we have  $g_1 b g_0^b = g_1^b g_0^b = g_1 g_0 b = g_1 b g_0$ , the last equation recalling  $b, g_0$  commute (by  $(*) (g)$ ), hence we have:

$$\bullet_8 \quad g_0^b = g_0.$$

<sup>18</sup>Note that  $g_\ell^x$  is not conjugation by  $x$ .

So by  $\bullet_4, \bullet_7, \bullet_1, \bullet_6$ ,  $b$  commuting with  $G_0$  and  $\bullet_2$  second statement respectively, we have

$$\boxplus_5 \quad g_1^{b,a} = g_1^b = (g_1 b) = (g_1^a b) = (g_1^a b)(g_0^a (g_0^{a,b})^{-1}) = (g_1^a g_0^a b)(g_0^{a,b})^{-1} = g_1^{a,b},$$

and thus

$$\bullet_9 \quad g_1^{b,a} = g_1^{a,b}.$$

Also by  $\bullet_4, \bullet_3, \bullet_8, \bullet_1, \bullet_6, \bullet_2$  we have

$$\boxplus_6 \quad g_2^{b,a} g_0^{b,a} = g_2^b g_0^b a = g_2 g_0^b a = g_2 g_0 a = g_2^a g_0^a = g_2^a g_0^{a,b} = g_2^{a,b} g_0^{a,b}.$$

So

$$\bullet_{10} \quad g_2^{b,a} g_0^{b,a} = g_2^{a,b} g_0^{a,b}$$

but  $g_0^{b,a}, g_0^{a,b} \in G_0$  and  $g_2^{b,a}, g_2^{a,b} \in \mathbf{I}_{\mathbf{x},2}$  hence recalling  $(g_0^{a,b}, g_1^{a,b}, g_2^{a,b}), (g_0^{b,a}, g_1^{b,a}, g_2^{b,a}) \in \mathcal{U}_{\mathbf{x}}$  we have:

$$\bullet_{11} \quad g_2^{b,a} = g_2^{a,b} \text{ and } g_0^{b,a} = g_0^{a,b}.$$

But  $\bullet_{11} + \bullet_9$  imply that we are done. □<sub>4.1</sub>

The following claim is like Definition 2.5, but now we preserve a large  $\mathbf{C}_{G_1}(G_0)$  using 4.1.

**Definition 4.4.** Let  $\text{NF}^3(\bar{G}, H_1, L, H_2)$  mean:

- (A) (a)  $\bar{G} = \langle G_\ell : \ell \leq 3 \rangle$  are from  $\mathbf{K}_{\text{lf}}$ ;
- (b)  $G_0 \subseteq G_\ell$  for  $\ell = 1, 2$ ;
- (c)  $G_0$  is finite;
- (d)  $H_1 \subseteq \mathbf{C}_{G_1}(G_0), L \subseteq H_1, L \cap G_0 = \{e_{G_0}\}, H_1 = \langle L, G_0 \cap H_1 \rangle_{G_1}$ ;
- (e)  $G_1 \cap G_2 = G_0$ ;
- (f)  $H_2 \subseteq \mathbf{C}_{G_2}(H_1 \cap G_0)$ ;
- (B) (a)  $G_\ell \subseteq G_3$  for  $\ell = 1, 2$ ;
- (b) for  $\sigma(\bar{x}, \bar{y})$  a group-term,  $\bar{a} \in {}^{\ell g(\bar{x})}(G_1)$  and  $\bar{b} \in {}^{\ell g(\bar{y})}(G_2)$  the following conditions are equivalent:
  - $G_3 \models \text{“}\sigma(\bar{a}, \bar{b}) = e_{G_3}\text{”}$ ,
  - if  $(\mathbf{x}, H_1, H_2) \in \mathbf{X}_{\text{lf}}^3$ , see 4.3,  $G_\ell = G_{\mathbf{x},\ell}$  for  $\ell = 0, 1, 2$  and  $\bar{a}' = \mathbf{j}_{\mathbf{x},1}(\bar{a})$  and<sup>19</sup>  $\bar{b}' = \mathbf{j}_{\mathbf{x},2}(\bar{b})$  then  $G_{\mathbf{x}} \models \text{“}\sigma(\bar{a}', \bar{b}') = e_{G_{\mathbf{x}}}\text{”}$ .

**Convention 4.5.** In 4.4, if  $H_1 = L$  we may in addition omit  $L$ . We may omit  $L, H_2$  when  $L = H_1, H_2 = \mathbf{C}_{G_2}(H_1 \cap G_0)$ . Lastly, if  $\mathbf{Z}(G_0) = \{e_{G_0}\}, L = \mathbf{C}_{G_1}(G_0)$  and  $H_1 = L$  and  $H_2 = G_2$ , then we may omit  $H_1, L$  and  $H_2$ ; see 4.6(3) below.

**Claim 4.6.** Assume  $\bar{G} = \langle G_\ell : \ell < 3 \rangle, H_1, L, H_2$  are as in 4.4(A).

- 1) We can find  $\mathbf{x}$  such that  $(\mathbf{x}, H_1, H_2) \in \mathbf{X}_{\text{lf}}^3$ .
- 2) There is  $G_3 \in \mathbf{K}$  such that  $\text{NF}^3(\langle G_0, G_1, G_2, G_3 \rangle, H_1, L, H_2)$  and  $G_3$  is unique up to isomorphism over  $G_1 \cup G_2$ .
- 3) If  $\bar{G}$  satisfies (A)(a),(b),(c) of Definition 4.4,  $\mathbf{Z}(G_0) = \{e_{G_0}\}, H_1 = L = \mathbf{C}_{G_1}(G_0)$  and  $H_2 = G_2$ , then  $(\bar{G}, H_1, L, H_2)$  satisfies 4.4(A).

<sup>19</sup>We may add  $\mathbf{I}_1 = L$ .

4) The relation  $\text{NF}^3(\bar{G}), \bar{G} = \langle G_\ell : \ell \leq 3 \rangle$  satisfies the parallel of 2.10 omitting symmetry, so having uniqueness, monotonicity and both sides definability, i.e.  $G_1 \leq_{\Omega[\mathbf{K}]} G_3, G_2 \leq_{\Omega[\mathbf{K}]} G_3$ .

*Proof.* 1) It suffices to prove we can choose  $\mathbf{I}_1^*, \mathbf{I}_2^*$  satisfying the demands on  $\mathbf{I}_{\mathbf{x},1}, \mathbf{I}_{\mathbf{x},2}$  in 4.1.

Why can we do it? For  $\mathbf{I}_2^*$  the demands are just clauses (b),(c) from 2.2(1) and  $(*) (i)$  of 4.1 so just choose  $\mathbf{I}_2 \subseteq H_2$  such that  $e_{G_0} \in \mathbf{I}_2$  and  $\langle g(G_0 \cap H_2) : g \in \mathbf{I}_2 \rangle$  is a partition of  $H_2$  and then let  $\mathbf{I}_2^*$  be such that  $\mathbf{I}_2 \subseteq \mathbf{I}_2^* \subseteq G_2$  and  $\langle gG_0 : g \in \mathbf{I}_2^* \rangle$  is a partition of  $G_2$ .

For  $\mathbf{I}_1^*$  we have to take care of clauses (b),(c) from 2.2(1), of  $(*) (e)$  (the parallel of  $(*) (i)$ ) and of  $(*) (f)$  from 4.1. For this let  $H_1^+ := \langle G_0, H_1 \rangle_{G_1}$ . First, choose  $\mathbf{I}'_1 = L$  so clearly  $e_{G_0} \in \mathbf{I}'_1$  and thus  $\langle gG_0 : g \in \mathbf{I}'_1 \rangle$  is a partition of  $H_1^+$ . Why? Recalling that  $L \subseteq H_1 \subseteq G_1, L \cap G_0 = \{e_{G_0}\}$  and  $H_1 = \langle L, G_0 \cap H_1 \rangle_{G_1}$  and  $H_1$  commute with  $G_0$  in  $G_1$ ; by clause (A)(d) we know that this is satisfied. Also let  $\mathbf{J}_1 \subseteq G_1$  be such that  $e_{G_0} = e_{G_1} \in \mathbf{J}_1$  and  $\langle gH_1^+ : g \in \mathbf{J}_1 \rangle$  is a partition of  $G_1$ . Now let  $\mathbf{I}_1^* = \{gb : g \in \mathbf{J}_1 \text{ and } b \in \mathbf{I}'_1\}$ .

Clearly  $\langle gG_0 : g \in \mathbf{I}_1^* \rangle = \langle g(bG_0) : b \in \mathbf{I}'_1, g \in \mathbf{J}_1 \rangle$  is a partition of  $G_1$  (refining  $\langle gH_1^+ : g \in \mathbf{J}_1 \rangle$ ), so clause 2.2(1)(b) holds. Furthermore,  $\mathbf{I}_1^* \cap H_2^+ = L = \mathbf{I}'_1$  so clause 4.1(e) holds.

Next as  $e_{G_0} \in \mathbf{J}_1$  and  $e_{G_0} \in \mathbf{I}'_1$  clearly  $e_{G_0} \in \mathbf{I}_1^*$ , so  $\mathbf{I}_1^*$  satisfies clause 2.2(1)(c). Also if  $g \in \mathbf{I}_1^* \wedge b \in \mathbf{I}'_1$  then for some  $g_1 \in \mathbf{J}_1, b_1 \in \mathbf{I}'_1$  we have  $G_1 \models "g = g_1 b_1"$  hence  $G_1 \models "gb = (g_1 b_1)b = g_1(b_1 b)"$  and recall  $g_1 \in \mathbf{J}_1$  and  $b_1 b \in \mathbf{I}'_1$  as  $\mathbf{I}'_1 = L$  is closed under products. Thus together  $gb \in \mathbf{I}_1^*$ , hence clause 4.1(1)(f) is satisfied. So  $\mathbf{I}_1^*, \mathbf{I}_2^*$  are as required in 2.2(1) and 4.1. Hence there is  $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$  such that  $G_{\mathbf{x},\ell} = G_\ell$  for  $\ell = 0, 1, 2$  and  $\mathbf{I}_{\mathbf{x},\ell} = \mathbf{I}_\ell^*$  for  $\ell = 1, 2$ .

2) Consider clause (B) of 4.4, the "if  $\mathbf{x} \in \dots$ " is not empty so  $G_3$  is a well defined group. Easily  $G_1 \subseteq G_3$  and  $G_2 \subseteq G_3$  but is  $G_3$  locally finite? This follows from the results in §2, in particular 2.10. That is, as there if  $G'_\ell$  is finite,  $G_0 \subseteq G'_\ell \subseteq G_\ell$  for  $\ell = 1, 2$  then we have finitely many possible choices of  $(\mathbf{I}_{\mathbf{x},1} \cap x_1 G'_1, \mathbf{I}_{\mathbf{x},2} \cap x_2 G'_2)$  for  $x_1 \in G_1, x_2 \in G_2$  hence the group  $G_3$  that we get is locally finite. Probably better this is  $G'_3$  such that  $\text{NF}_f(G_0, G_1, G_2, G'_3)$ , by the definition there is a homomorphism from  $G'_3$  onto  $G_3$  over  $G_1 \cup G_2$ . Now as  $G'_3$  is lf so is  $G_3$ .

3),4) Should be clear. □4.6

#### § 4(B). Schemes and derived sets.

**Definition 4.7.** 1) Let  $\mathbf{X}_0$  be the set of  $\mathbf{x}$  such that:

- (a)  $\mathbf{x}$  has the form  $(K_1, K_2, \bar{a}_2, \bar{a}_1) = (K_1[\mathbf{x}], K_2[\mathbf{x}], \bar{a}_2[\mathbf{x}], \bar{a}_1[\mathbf{x}])$ ;
- (b)  $K_1 \subseteq K_2$  are finite groups;
- (c)  $\bar{a}_1$  is a finite sequence generating  $K_1$ ;
- (d)  $\bar{a}_2$  is a finite sequence from  $K_2$  such that  $\bar{a}_2 \hat{\ } \bar{a}_1$  generates  $K_2$  (if  $\bar{a}_2 = \langle a_2 \rangle$  we may write just  $a_2$ );
- (e)  $K_1$  has trivial center.

2) Let  $\mathbf{X}_1$  be the set of  $\mathbf{x}$  such that:

- (a)  $\mathbf{x} = (K, \bar{a}) = (K[\mathbf{x}], \bar{a}[\mathbf{x}])$ ;
- (b)  $K \in \mathbf{K}$  is finite;
- (c)  $\bar{a}$  is a finite sequence from  $K$  generating  $K$ ,  $\ell g(\bar{a}) \geq 1$ ; let  $a_* = a_*[\mathbf{x}] = a_0$ , the first element of  $\bar{a}$ .

3) Let  $\mathbf{X}_2$  be the set of  $\mathbf{x} \in \mathbf{X}_1$  such that:

- (\*)  $K$  has trivial center.

4) Let  $\mathbf{X}_3$  be the set of  $\mathbf{x} \in \mathbf{X}_1$  such that<sup>20</sup>:

- (\*) if  $f$  is a non-trivial automorphism of  $K$  then for some conjugate  $b$  of  $a_* = a_*[\mathbf{x}] = a_0[\mathbf{x}]$  we have  $f(b) \notin \langle a_* \rangle_K$ ; equivalently, for some conjugate  $b$  of  $a_*$ ,  $\langle b \rangle_K \neq \langle a \rangle_K$ .

**Observation 4.8.** If  $m \in \{2, 3, \dots\}$  then for some  $\mathbf{x} \in \mathbf{X}_3$  the element  $a_*[\mathbf{x}] \in K[\mathbf{x}]$  has order  $m$ .

**Claim 4.9.** If  $\mathbf{x} \in \mathbf{X}_0$ , then there is one and only one  $\mathfrak{s}$ , call it  $\mathfrak{s}_{\text{cm}} = \mathfrak{s}_{\text{cm}}[\mathbf{x}]$  such that:

- (a)  $\mathfrak{s} \in \Omega[\mathbf{K}_{\text{lf}}]$ ;
- (b)  $k_{\mathfrak{s}} = \ell g(\bar{a}_1[\mathbf{x}])$  and  $n_{\mathfrak{s}} = \ell g(\bar{a}_2[\mathbf{x}])$ ;
- (c)  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = \text{tp}_{\text{bs}}(\bar{a}_1[\mathbf{x}], \emptyset, K[\mathbf{x}])$ ;
- (d) if  $G_1 \subseteq G_3 \in \mathbf{K}_{\text{lf}}$  and  $\text{tp}_{\text{bs}}(\bar{a}, \emptyset, G_1) = \text{tp}_{\text{bs}}(\bar{a}_1[\mathbf{x}], \emptyset, K[\mathbf{x}])$  and  $\bar{c}$  realizes  $q_{\mathfrak{s}}(\bar{a}, G_1)$  in  $G_3$  then  $\text{NF}^3(\langle \bar{a} \rangle_{G_1}, G_1, \langle \bar{a} \hat{\ } \bar{c} \rangle_{G_3}, G_3)$ .

*Proof.* As in §2 using §(4A). Let  $K_\ell = K_\ell[\mathbf{x}]$  for  $\ell = 1, 2$ ; and let  $G_0 = K_1$  and  $G_1 \in \mathbf{K}$  be existentially closed, extend  $K_1$  and be such that  $K_2 \cap G_1 = K_0$ . Let  $L = \mathbf{C}_{G_0}(G_1)$ , so as  $G_0 = K_1$  has trivial center (by 4.7(1)(e)), clearly we have  $L \cap G_0 = \{e_{G_0}\}$  and let  $H_1 = \text{cl}(G_0 \cup L, G_1)$ ,  $H_0 = \{e_{K_1}\}$  and let  $H_2 = G_2 := K_2$ . Now we apply Claim 4.6(2), so there is  $G_3$  such that  $\text{NF}^3(G_0, G_1, G_2, G_3)$  see Definition 4.4. By it, the type  $\text{tp}_{\text{bs}}(\bar{a}_2[\mathbf{x}], G_2, G_{\mathbf{x}})$  does not split over  $G_0 = K_1$ . From this it is easy to define  $\mathfrak{s}$  and to prove it is as required.  $\square_{4.9}$

**Definition/Claim 4.10.** For  $\mathbf{x} \in \mathbf{X}_1$  let  $\mathfrak{s} = \mathfrak{s}_{\text{ab}}[\mathbf{x}]$  be such that:

- (a)  $\mathfrak{s} \in \Omega[\mathbf{K}_{\text{lf}}]$ ;
- (b)  $k_{\mathfrak{s}} = 0$ ;
- (c) if  $\bar{c}$  realizes  $q_2(\langle \cdot \rangle, G_1)$  in  $G_2$  so  $G_1 \subseteq G_2$  then  $\bar{c}$  realizes  $\text{tp}_{\text{bs}}(\bar{a}[\mathbf{x}], \emptyset, K[\mathbf{x}])$  and commutes with  $G_1$ , and  $\langle \bar{c} \rangle_{G_2} \cap G_1 = \{e\}$ .

*Proof.* Easy.  $\square_{4.11}$

**Definition/Claim 4.11.** For  $\mathbf{x} \in \mathbf{X}_2$  we define  $\mathfrak{s} = \mathfrak{s}_{\text{gm}}[\mathbf{x}]$  such that:

- (a)  $\mathfrak{s} \in \Omega[\mathbf{K}_{\text{lf}}]$ ;
- (b)  $k_{\mathfrak{s}} = 2\ell g(\bar{a}[\mathbf{x}])$  and  $n_{\mathfrak{s}} = 1$ ;
- (c) if  $G_1 \subseteq G_2 \in \mathbf{K}_{\text{lf}}$  and  $\text{tp}_{\text{bs}}(\bar{a}_\ell, \emptyset, G_1) = \text{tp}_{\text{bs}}(\bar{a}[\mathbf{x}], \emptyset, K[\mathbf{x}])$  for  $\ell = 1, 2$  and  $\langle \bar{a}_1 \rangle_{G_1}, \langle \bar{a}_2 \rangle_{G_1}$  commute in  $G_1$  and<sup>21</sup> have intersection  $\{e_G\}$  then  $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = \text{tp}_{\text{bs}}(\bar{a}_1 \hat{\ } \bar{a}_2, \emptyset, G_1)$ ;

<sup>20</sup>So  $\mathbf{x}_3 \supseteq \mathbf{x}_2$ .

<sup>21</sup>In fact this follows.



(d) moreover, in clause (c), if  $c \in G_2$  realizes  $q_{\mathfrak{s}}(\bar{a}_1 \hat{\ } \bar{a}_2, G_1)$  in  $G_2$  then conjugation by  $c$  interchanges  $\bar{a}_1, \bar{a}_2$  and is the identity on  $\mathbf{C}_{G_1}(\bar{a}_1 \hat{\ } \bar{a}_2)$ .

*Proof.* Let  $G_2 \in \mathbf{K}_{\text{exlf}}$  be an extension of  $K[\mathbf{x}]$  in which some  $c$  realizes  $q_{\mathfrak{s}_{\text{cg}}}(K_{\mathbf{x}})$ ; let  $\bar{a}_1 = \bar{a}[\mathbf{x}], \bar{a}_2 = c^{-1}\bar{a}_1c := \langle c^{-1}a_{1,\ell}c : \ell < \ell g(\bar{a}_1) \rangle$  in  $G_2$ .

Note that, by inspection,  $G_0 = \langle \bar{a}_1 \hat{\ } \bar{a}_2 \rangle_{G_1}$  is finite with trivial center and let  $G_0 \subseteq G_1 \in \mathbf{K}_{\text{lf}}$ . Now use 4.1 with  $G_0, G_1, \text{cl}(\bar{a}_1 \hat{\ } \bar{a}_2 \hat{\ } \langle c \rangle, G_2), \mathbf{C}_{G_1}(G_0), \mathbf{C}_{G_1}(G_0), \text{cl}(\bar{a}_1 \hat{\ } \bar{a}_2 \hat{\ } \langle c \rangle, G_2)$  here standing for  $G_0, G_1, G_2, G_1, H_1, L, H_2$  there.  $\square_{4.10}$

**Definition 4.12.** 1) For  $\mathfrak{s} \in \Omega[\mathbf{K}]$  and  $G_1 \subseteq G_2$  let  $\text{cp}_{\mathfrak{s}}(G_1, G_2) = \{c_0 : \bar{c} \in {}^{n(\mathfrak{s})}(G_2)$  realizes  $q_t(G_1)$  where  $t \in \text{def}(G_1)$  satisfies  $\mathfrak{s}_t = \mathfrak{s}\}$ .

2) For  $\mathbf{x} \in \mathbf{X}_1$  and  $G_1 \subseteq G_2$  let  $\text{cp}_{\mathbf{x}}(G_1, G_2) = \text{cp}_{\mathfrak{s}_{\text{ab}}[\mathbf{x}]}(G_1, G_2)$ .

3) For  $G_1 \subseteq G_2 \in \mathbf{K}_{\text{lf}}$  and  $\ell \in \{1, 2, 3\}$  let  $\text{cp}_{\ell}(G_1, G_2) = \cup\{\text{cp}_{\mathfrak{s}_{\text{ab}}[\mathbf{x}]}(G_1, G_2) : \mathbf{x} \in \mathbf{X}_{\ell}\}$ ; if  $\ell = 2$  we may omit it.

### § 4(C). Larger Definable Types.

**Definition 4.13.** 1) For  $G \in \mathbf{K}, \mathfrak{S} \subseteq \Omega[\mathbf{K}]$  and set  $I$  let  $\text{Def}_{I, < \kappa}(G, \mathfrak{S})$  be the set of  $t$  such that:

- (a)  $t = \langle t_u : u \subseteq I \text{ finite} \rangle$ ;
- (b)  $t_u \in \text{def}_{\mathfrak{S}}(G)$  with  $\bar{x}_{t_u} = \langle x_i : i \in u \rangle$  and  $\bar{a}_{t_u} = \bar{a}_t$  or pedantically  $\bar{a}_{t_u} = \bar{a}_t \upharpoonright w_u$  where  $w_u \subseteq \ell g(\bar{a}_t)$  is finite;
- (c)  $\ell g(\bar{a}_t) := I$  has cardinality  $< \kappa$  and  $\text{Rang}(\bar{a}_t) \subseteq G$ ;
- (d) if  $G \subseteq H \subseteq L \in \mathbf{K}_{\text{lf}}$  and  $u \subseteq v \subseteq I$  are finite and  $\bar{b} \in {}^v L$  realizes  $q_{t_v}(H)$  then  $\bar{b} \upharpoonright u$  realizes  $q_{t_u}(H)$ .

2) We define  $\Omega_{I, < \kappa}[\mathbf{K}, \mathfrak{S}]$  parallelly and if  $\mathfrak{S} = \Omega[\mathbf{K}]$  then we may omit it.

3) If  $t \in \text{Def}_{I, < \kappa}(G, \mathfrak{S})$  then  $q_t(G) \in \mathbf{S}_{\text{bs}}^I(G)$  is defined by  $\cup\{q_{t_u}(\langle x_i : i \in u \rangle) : u \subseteq I \text{ finite}\}$ .

4) Omitting  $\kappa$  means  $\aleph_0$ . We may replace “ $< \kappa^+$ ” by  $\kappa$  and even a set  $I_1$ . We may replace  $I$  by “ $< \mu$ ” meaning “some  $\chi < \mu$ ”. Similarly for “ $\leq \mu$ ”.

5) For  $n < \omega$  and  $\mathfrak{s}_0, \dots, \mathfrak{s}_{n-1} \in \Omega_{< \mu, < \kappa}[\mathbf{K}]$  we define  $\mathfrak{s}_0 \oplus \dots \oplus \mathfrak{s}_{n-1}$  and  $\mathfrak{s}_0 \otimes \dots \otimes \mathfrak{s}_{n-1}$  naturally.

**Claim 4.14.** 1) If  $G \in \mathbf{K}, \mathfrak{S} \subseteq \Omega[\mathbf{K}]$  and  $t \in \text{Def}_I(G, \mathfrak{S})$  then for some pair  $(\bar{c}, H)$  we have  $G \subseteq H \in \mathbf{K}_{\text{lf}}, \bar{c} \in {}^I H, H = \langle G \cup \bar{c} \rangle_H$  and  $\text{tp}_{\text{bs}}(\bar{c}, G, H) = q_t(G)$ .

2) If  $\mathfrak{S}$  is closed then above  $G \leq_{\mathfrak{S}} H$ .

**Definition 4.15.** Assume  $\bar{H} = \langle H_i : i < \delta \rangle$  is  $\subseteq$ -increasing in  $\mathbf{K}$  and  $H_{\delta} = \cup\{H_i : i < \delta\}$ . We say  $\mathcal{A}$  is a one step  $(< \mu, < \kappa, \delta, \mathfrak{S}) - \otimes$ -construction (if  $\delta = \omega$  we may omit it) when: as in 3.16 except that

- (c)'  $t_{\alpha, i} \in \text{Def}_{I_{\alpha, i, < \kappa}}(H_i, \mathfrak{S})$  for some set  $I_{\alpha, i}$  of cardinality  $< \mu$ .

The case we shall actually use in §5 is:

**Claim 4.16.** Assume  $K \subseteq L \in \mathbf{K}_{\text{lf}}, K$  is finite and  $f$  embeds  $K$  into  $G_1 \in \mathbf{K}_{\text{lf}}$  and  $\langle c_i : i < \mu \rangle$  list the members of  $L$  and  $\{c_{\ell} : \ell < n\}$  is the set of elements of  $K$ . Then there is  $t \in \text{Def}_{\leq \mu}(G_1, \mathfrak{S}[\mathbf{K}])$  such that: if  $\bar{c}^* = \langle c_i^* : i < \mu \rangle \in {}^{\mu}(G_2)$  realizes  $q_t(G_1)$  in  $G_2$ , so  $G_1 \subseteq G_2$ , then  $c_i \mapsto c_i^*$  (for  $i < \mu$ ) is an embedding of  $L$  into  $G_2$  extending  $f$ .

*Proof.* Straightforward by §2.

□4.16

**Discussion 4.17.** Those definable types are still locally definable over finite sets.

§ 5. CONSTRUCTING COMPLETE EXISTENTIALLY CLOSED  $G$ 

**Theorem 5.1.** *Assume if  $G \in \mathbf{K}_{\text{lf}}$  and  $|G| \leq \mu = \mu^{\aleph_0}$ .*

- 1) *There is a complete  $G' \in \mathbf{K}_{\text{lf}}$  which extend  $G$  such that  $|G'| = \mu^+$  and  $G'$  is existentially closed.*
- 2) *Moreover  $G \leq_{\Omega[\mathbf{K}_{\text{lf}}]} G'$  and  $G'$  is full.*
- 3) *There is  $G'$  such that  $G \leq_{\mathfrak{S}} G'$  and  $G' \in \mathbf{K}_{\mu^+}^{\text{exlf}}$  is complete and  $\mathfrak{S}$ -full provided that  $\mathfrak{S}$  satisfies:*

- (\*) (α)  $\mathfrak{S} \subseteq \Omega[\mathbf{K}_{\text{lf}}]$
- (β)  $\mathfrak{S}$  is dense and  $\otimes$ -closed (for  $\mathbf{K}_{\text{lf}}$ )
- (γ) *some schemes introduced earlier belongs to  $\mathfrak{S}$ , specifically:*
  - $\mathfrak{s}_{\text{ab}(2)}$  from Definition 2.20, used in the paragraph before  $\boxplus_3$
  - $\mathfrak{s}_{\text{cm}}$  from Definition 4.9, used in  $(*)_{4.3}$
  - $\mathfrak{s}_{\text{cg}}$ , from Definition 2.17(1), 2.18(2) used after  $\boxplus_7$  Stage E
  - $\mathfrak{s}_{\text{gl}}$  from Definition 2.17(2), 2.18(3)
  - $\mathfrak{s}_{\text{gm}}$  from Definition 4.10, see  $(*)_{5.1(f)}$ .

*Proof. Proof of 5.1*

We let  $\mathfrak{S} = \Omega[\mathbf{K}_{\text{lf}}]$  for parts (1),(2) and fix  $\mathfrak{S}$  for part (3) as there.

Stage A: Without loss of generality the universe of  $G$  is an ordinal  $\leq \mu$  and let  $\lambda = \mu^+$ .

Let  $S \subseteq S_{\aleph_0}^\lambda := \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$  be a stationary subset of  $\lambda$  such that also  $S_{\aleph_0}^\lambda \setminus S$  is stationary in  $\lambda$  and  $\alpha \in S \Rightarrow (\mu \text{ divides } \alpha)$ . Let  $\langle S_\zeta : \zeta < \lambda \rangle$  be a partition of  $S$  to stationary sets. Let  $S_* \subseteq \lambda \setminus S$  be stationary and a set of limit ordinals.

Let  $C_\delta$  be an unbounded subset of  $\delta$  of order type  $\omega$  for  $\delta \in S$  such that  $\bar{C}_\zeta = \langle C_\delta : \delta \in S_\zeta \rangle$  guess clubs for each  $\zeta < \lambda$ , this means that for every club  $E$  of  $\lambda$  the set  $\{\delta \in S_\zeta : C_\delta \subseteq E\}$  is a stationary subset of  $\lambda$ ; such  $\langle C_\delta : \delta \in S_\zeta \rangle$  exists by [She94a, Ch.III] = [She94b].

Let  $\alpha_\delta(n)$  be the  $n$ -th member of  $C_\delta$ .

Let  $\bar{\tau}$  be such that:

- $\bar{\tau} = \langle \tau_\zeta : \zeta < \lambda \rangle$
- $\tau_\zeta \subseteq \mathcal{H}(\aleph_0)$  is a countable vocabulary
- if  $\tau \subseteq \mathcal{H}(\aleph_0)$  is a countable vocabulary then  $\{\zeta : \tau_\zeta = \tau\}$  has cardinality  $\lambda$ .

By [Sheb, 3.26(3)=L6.11A,pg.31] there is  $\mathbf{b}_\zeta$ , a BB, black box for  $(S_\zeta, \bar{C}_\zeta)$  say  $\mathbf{b}_\zeta = \langle N_i^\delta : i \in \mathcal{I}_\delta, \delta \in S_\zeta \rangle$ , that is:

- $\boxplus_{0,\zeta}$  (a)  $N_i^\delta$  is a model of cardinality  $\aleph_0$  with universe  $\subseteq \delta = \sup(N_i^\delta)$  and vocabulary  $\tau_\zeta \subseteq \mathcal{H}(\aleph_0)$
- (b) if  $N$  is a  $\tau_\zeta$ -model with universe  $\lambda$  then for stationarily many  $\delta \in E_N \cap S_\zeta$  for some  $i \in \mathcal{I}_\delta$  we have  $C_\delta \subseteq E_N \setminus S$  where  $E_N := \{\alpha : N \upharpoonright \alpha \prec N\}$  and  $N_i^\delta \prec N$ ; moreover
- (b)<sup>+</sup> if  $\tau = \tau_\zeta, \bar{N} = \langle N_\eta : \eta \in \mathcal{I} \rangle, \mathcal{I}$  a non-empty subtree of  $\omega^{>\lambda}$  such that  $\tau(N_\eta) = \tau_\zeta, \eta \triangleleft \nu \Rightarrow N_\eta \prec N_\nu$  and  $|N_\eta| \in [\aleph_0, \aleph_1]$  and  $E$  a club of  $\lambda, \eta \in \mathcal{I} \Rightarrow (\exists^\lambda \alpha)(\eta \hat{\ } \langle \alpha \rangle \in \mathcal{I})$  and  $\eta \triangleleft \nu \in \mathcal{I} \Rightarrow \sup(N_\eta) < \sup(N_\nu)$

- then for some  $\delta \in S_\zeta \cap E$  we have  $C_\delta \subseteq E, i \in \mathcal{T}_\delta$  and  $\eta \in \lim_\omega(\mathcal{T})$   
we have  $N_i^\delta = \cup\{N_{\eta \upharpoonright n} : n < \omega\}$ ;
- (c) if  $i \neq j \in \mathcal{T}_\delta$  then  $N_i^\delta \cap N_j^\delta$  is bounded in  $\delta$  (used just after  $(*)_{5.5}$ ),  
moreover:
- (c)<sup>+</sup> if  $i \neq j \in \mathcal{T}_\delta$  then the set  $\{\beta < \delta : \beta \text{ a limit ordinal such that } \sup(N_i^\delta \cap \beta) = \beta = \sup(N_j^\delta \cap \beta)\}$  is bounded in  $\delta$ ;
- (d)  $N_i^\delta \cap (\alpha_\delta(n), \alpha_\delta(n+1)) \neq \emptyset$  and  $N_i^\delta \upharpoonright \alpha_\delta(n) \prec N_i^\delta$  for  $n < \omega, \delta \in S, i \in \mathcal{T}_\delta$ ;
- (e) for notational simplicity we assume  $\mathcal{T}_\delta \subseteq \mu$ .

Stage B: By induction on  $\gamma < \lambda$  we shall choose the following:

- $\boxplus_1$  (a)  $G_\gamma \in \mathbf{K}_{\text{If}}$  of cardinality  $\mu$  and the universe of  $G_\gamma$  is an ordinal  $< \lambda$ ;
- (b)  $G_0 = G$ ;
- (c)  $\langle G_\beta : \beta \leq \gamma \rangle$  is increasing continuous;
- (d) if  $\beta \in \gamma \setminus S$  then  $G_\beta \leq_{\mathfrak{S}} G_\gamma$
- (e) if  $\gamma = \beta + 1, \beta \notin S$ , then:
- ( $\alpha$ )  $G_\gamma$  is generated by  $\{\bar{c}_{\beta,i} : i \in \mathcal{T}_\beta\} \cup G_\beta$ , where  $\mathcal{T}_\beta$  is a set of  
cardinality  $\leq \mu$  (to be chosen),
- ( $\beta$ )  $t_{\beta,i} \in \text{Def}_{\leq \mu}(G_\beta, \mathfrak{S})$ , non-trivial (see Definition 4.13(5)) for  $i \in \mathcal{T}_\beta$ ,
- ( $\gamma$ )  $\text{tp}_{\text{bs}}(\bar{c}_{\beta,i}, G_\beta, G_\gamma) = q_{t_{\beta,i}}(G_\beta)$  for  $i \in \mathcal{T}_\beta$ ,
- ( $\delta$ ) if  $n < \omega$  and  $i(0), \dots, i(n-1) \in \mathcal{T}_\beta$  are pairwise distinct, then  
 $\text{tp}_{\text{bs}}(\bar{c}_{\beta,i(0)} \hat{\ } \dots \hat{\ } \bar{c}_{\beta,i(n-1)}, G_\beta, G_\gamma) = q_t(G_\beta)$ , where  $t = t_{\beta,i(0)} \otimes \dots \otimes t_{\beta,i(n-1)}$ ,
- ( $\varepsilon$ ) if  $t = (\mathfrak{s}, \bar{a}) \in \text{def}_{\mathfrak{S}}(G_\beta)$  is non-trivial then for some  $i \in \mathcal{T}_\beta$  we  
have  $t_{\beta,i} = t$ ;
- (f) if  $\gamma = \delta + 1, \delta \in S$  then:
- ( $\alpha$ )  $G_\gamma$  is generated by  $\{\bar{c}_{\delta,i} : i \in \mathcal{T}_\delta\} \cup G_\delta$ ,
- ( $\beta$ )  $\mathcal{A}_\gamma = (G_{\delta+1}, G_\delta, \langle \bar{c}_{\delta,i}, t_{\delta,i,n} : i \in \mathcal{T}_\delta \rangle)$  is a one step ( $< \aleph_0, < \aleph_0, \mathfrak{S}$ )  $\otimes$ -construction over  $\langle G_{\alpha_\delta(n)} : n < \omega \rangle$ , see 3.16; used in  
 $(*)_{5.2}$ 's proof<sup>22</sup>,
- (g)  $t_{\beta,i} = (\mathfrak{s}_{\beta,i}, \bar{a}_{\beta,i})$  for  $\beta \in \gamma \setminus S$ .

First we shall show:

$\boxplus_2$  we can carry the induction.

Why? For  $\gamma = 0$  we have nothing to do by clause (b).

For  $\gamma$  limit we let  $G_\gamma = \cup\{G_\beta : \beta < \gamma\}$ .

For  $\gamma = \beta + 1, \beta \notin S$  we have some freedom, as we have  $t_{\beta,i} \in \text{Def}_{\leq \mu}(G_\beta, \mathfrak{S})$   
not just  $\text{def}(G_\beta, \mathfrak{S})$ . So let  $\mathcal{T}_\beta = \mu, \{t_{\beta,i} : i \in \mathcal{T}_\beta\} \subseteq \text{Def}_{\leq \mu}(G_\beta, \mathfrak{S})$  be of cardinality  
 $\mu$  and including  $\text{def}(G_\beta, \mathfrak{S})$  and so  $\langle t_{\beta,i} = (\mathfrak{s}_{\beta,i}, \bar{a}_{\beta,i}) : i < \mu \rangle$ , possibly with  
repetitions. Clearly  $\boxplus_1(e)(\varepsilon)$  holds.

Now as in Claim 3.13 we can find  $G_\gamma, \langle \bar{c}_{\beta,i} : i < \mu \rangle$  such that:

<sup>22</sup>Actually can use a one step ( $\leq \mu, < \aleph_0, \mathfrak{S}$ )  $\otimes$ -construction.

- $G_\beta \leq_{\mathfrak{S}} G_\gamma$ ;
- $G_\gamma = \langle \{\bar{c}_{\beta,i} : i < \mu\} \cup G_\beta \rangle_{G_\gamma}$ ;
- if  $n < \omega$  and  $i_k < \mu$  for  $k < n$  and  $\langle i_k : \ell < n \rangle$  is with no repetitions then  
 $\bar{c}_{\beta,i_0} \hat{\ } \dots \hat{\ } \bar{c}_{\beta,i_{n-1}}$  realizes  $q_t(G_\beta)$  where  $t = t_{\beta,i_0} \otimes \dots \otimes t_{\beta,i_{n-1}}$ .

If  $\gamma = \delta + 1, \delta \in S$  we can let  $\mathfrak{s}_{\delta,i} = \mathfrak{s}_{\text{ab}(2)}$ , clearly we satisfy clause (f); but we may act differently. Clearly, as in the previous case, there is some freedom left: what we do for  $\gamma = \delta + 1, \delta \in S$  and this will depend on the  $\langle N_i^\delta : i \in \mathcal{T}_\delta \rangle$  from  $\boxplus_0$ . During the rest of the proof we shall use (some of the freedom left) to guarantee that  $G_*$  (see below) is as required.

Of course, we let:

$$\boxplus_3 G_* = G_\lambda = \cup \{G_\alpha : \alpha < \lambda\}.$$

We now point out some useful properties of the construction:

- (\*)<sub>3.1</sub> there is a model  $N_*$  expanding  $G_*$ , so with universe  $\lambda$ , and a countable vocabulary such that for any  $N \subseteq N_*$  we have:
  - (a)  $G_* \upharpoonright N$  is a subgroup of  $G_*$ ;
  - (b)  $\beta \in N$  iff  $N \cap G_{\beta+1} \setminus G_\beta \neq \emptyset$  iff  $\beta + 1 \in N$ ;
  - (c) if  $\gamma = \beta + 1, \gamma \in N$  then  $N \cap G_\gamma = \langle \cup \{\bar{c}_{\beta,i} : i \in N \cap \mathcal{T}_\beta\} \cup (N \cap G_\beta) \rangle_{G_\gamma}$ ;
  - (d) if  $i \in N \cap \mathcal{T}_\beta$  and  $\beta \in N$ , then  $|\text{lg}(\bar{c}_{\beta,i})| \leq \omega \Rightarrow \bar{c}_{\beta,i} \subseteq N \cap G_{\beta+1}$  and  $|\text{lg}(\bar{a}_{t_{\beta,i}})| \leq \omega \Rightarrow \bar{a}_{t_{\beta,i}} \subseteq N \cap G_\beta$ ;
  - (e)  $\tau(N_*) \subseteq \mathcal{H}(\aleph_0)$ , but  $\mathcal{H}(\aleph_0) \setminus \tau(G_*)$  is infinite;
  - (f) if  $\delta \in N \cap S$  then  $C_\delta \subseteq N$ .

Now note

- (\*)<sub>3.2</sub> if  $\alpha < \lambda$  is a limit ordinal, then  $G_\alpha \in \mathbf{K}_{\text{exlf}}$ .

[Why? Recall clause (e)( $\varepsilon$ ) of  $\boxplus_1$  noting that  $S$  is a set of limit ordinals, hence  $\alpha = \sup(\alpha \setminus S)$ .]

We now assume:

$$\boxplus_4 \mathbf{h} \text{ is an automorphism of } G_*.$$

We shall eventually prove that (if we suitably use the freedom left in  $\boxplus_1$ , then)  $\mathbf{h}$  is an inner automorphism, i.e.  $b \in G_* \Rightarrow \mathbf{h}(b) = a^{-1}ba$  for some  $a \in G_*$ , this clearly suffices noting that  $G_*$  has trivial center as  $\mathfrak{s}_{\text{cg}} \in \mathfrak{S}$ .

We shall often use

- (\*)<sub>4.1</sub> for limit  $\beta \in \lambda \setminus S$  let  $L_\beta^* = \text{cp}(G_\beta, G_{\beta+\omega})$  (see Definition 4.12(3)), i.e.  $c \in L_\beta^*$  iff for some finite  $K \subseteq \mathbf{C}_{G_{\beta+\omega}}(G_\beta)$  with trivial center we have  $c \in K$  and  $K \cap G_\beta = \{e_{G_\beta}\}$ .

Note that

- (\*)<sub>4.2</sub> the last demand in (\*)<sub>4.1</sub>, “ $K \cap G_\beta = \{e_{G_*}\}$ ”, is redundant.

[Why? Recall  $\beta$  is a limit ordinal hence by (\*)<sub>3.2</sub>,  $G_\beta$  has trivial center.]

Note:

(\*)<sub>4.3</sub> if  $a \in L_\beta^*$  and  $K$  witnesses it, then  $K \subseteq L_\beta^*, K \cap G_\beta = \{e\}$ , and moreover there is  $L \in \mathbf{K}_{\text{exlf}}$  included in  $L_\beta^*$  and including  $K$ .

[Why? We can choose  $\bar{K} = \langle K_n : n < \omega \rangle$  such that  $K_0 = K, K_n$  is a finite group with trivial center,  $K_n \subseteq K_{n+1}$  and  $\bigcup K_n \in \mathbf{K}_{\text{exlf}}$ . We now choose by induction on  $n$  an embedding  $f_n$  of  $K_n$  into  $G_{\beta+\omega}^n$  such that  $f_0 = \text{id}_K, f_n \subseteq f_{n+1}$  and  $\text{Rang}(f_n) \subseteq L_\beta^*$ ; the induction step is possible by 4.9. Now  $\bigcup_n f_n(K_n)$  is as required.]

We shall use:

(\*)<sub>4.4</sub> let  $E_{\mathbf{h}} = \{\delta : \delta \text{ is a limit ordinal and } \mathbf{h} \text{ maps } G_\delta \text{ onto } G_\delta \text{ and } (N^* \upharpoonright \delta, \mathbf{h} \upharpoonright \delta) \prec (N^*, \mathbf{h})\}$ .

Now

(\*)<sub>4.5</sub>  $E_{\mathbf{h}}$  is a club of  $\lambda$ .

[Why? Just look at (\*)<sub>4.4</sub>.]

Stage C: We shall prove

$\boxplus_5$  for some  $\alpha(*) < \lambda$ , for every  $\beta \in S_* \cap E_{\mathbf{h}} \setminus \alpha(*)$  and  $c \in L_\beta^*$  we have  $\mathbf{h}(c) \in \text{cl}(G_{\alpha(*)} \cup \{c\}, G_*)$ .

Why? If not, for every  $\alpha < \lambda$  there are  $\beta_\alpha \in S_* \cap E_{\mathbf{h}} \setminus \alpha, m(\alpha) = m_\alpha \in \{2, 3, \dots\}$  and  $c_\alpha \in L_{\beta_\alpha}^*$  of order  $m_\alpha$  such that  $\mathbf{h}(c_\alpha) \notin \text{cl}(G_\alpha \cup \{c_\alpha\}, G_*)$ . Now let  $\bar{c}_\alpha$  witness that  $c_\alpha \in L_{\beta_\alpha}^*$  with  $c_{\alpha,0} = c_\alpha$ , i.e.  $\bar{c}_\alpha$  list the members of a finite subgroup of  $G_{\beta_\alpha+\omega}$  commuting with  $G_{\beta_\alpha}$  with trivial center and so included in  $L_{\beta_\alpha}^*$ .

(\*)<sub>5.0</sub> without loss of generality  $\mathbf{h}(c_\alpha) \in \text{cl}(G_\alpha \cup \bar{c}_\alpha)$ .

[Why? Let  $K_0$  be the subgroup of  $G_{\beta_\alpha+\omega}$  with universe  $\bar{c}_\alpha$ ; we can find  $K_1, K_2, K_3$  such that  $K_3$  is a finite group and for  $\ell = 0, 1, 2$  and  $K_1 \cap K_2 = \langle c_0 \rangle_{K_0}$  without loss of generality  $K_3 \subseteq G_{\beta_\alpha+\omega}$ , so we can replace  $K_0$  by  $K_1$  or by  $K_2$ .]

Let  $\mathbf{x}_\alpha \in X_2$  be such that  $\bar{c}_\alpha$  realizes  $q_{\mathfrak{s}_{\text{ab}}[\mathbf{x}_\alpha]}(\langle \rangle, G_{\beta_\alpha})$ , see 4.7(2) + 4.11. But if  $\alpha_1 < \alpha_2$  then  $(\beta_{\alpha_2}, c_{\alpha_2}, m_{\alpha_2})$  can serve as  $(\beta_{\alpha_1}, c_{\alpha_1}, m_{\alpha_1})$ , hence, without loss of generality,  $\mathbf{x}_\alpha = \mathbf{x}, m_\alpha = m_*$  for every  $\alpha$ .

- (\*)<sub>5.1</sub> (a) Let  $\bar{b}_{\alpha,1} = \bar{c}_\alpha$ ; let  $k_{\alpha,1} < \omega$  be such that  $\bar{b}_{\alpha,1} \subseteq G_{\beta_\alpha+k_{\alpha,1}+1}, \bar{b}_{\alpha,1} \not\subseteq G_{\beta_\alpha+k_{\alpha,1}}$ ;  
 (b) let  $k_{\alpha,*} \in (k_{\alpha,1} + 1, \omega)$  be such that:  $\text{tp}_{\text{bs}}(\mathbf{h}(\bar{b}_{\alpha,1}), G_{\beta_\alpha+\omega}, G_*) = q_{\mathfrak{s}_\alpha}(\bar{a}_\alpha^\bullet, G_{\beta_\alpha+\omega})$  for some  $\mathfrak{s}_\alpha \in \mathfrak{S}$  with  $\bar{a}_\alpha^\bullet \subseteq G_{\beta_\alpha+k_{\alpha,*}}$ ;  
 (c) let  $\bar{b}_{\alpha,2} \subseteq G_{\beta_\alpha+\omega}$  realize  $q_{\mathfrak{s}_{\text{ab}}[\mathbf{x}]}(\langle \rangle, G_{\beta_\alpha+k_{\alpha,*}})$ ;  
 (d) let  $k_{\alpha,2} < \omega$  be such that  $\bar{b}_{\alpha,2} \subseteq G_{\beta_\alpha+k_{\alpha,2}+1}, \bar{b}_{\alpha,2} \not\subseteq G_{\beta_\alpha+k_{\alpha,2}}$ , so actually without loss of generality  $k_{\alpha,2} = k_{\alpha,*} + 1$ ;  
 (e) note that  $\bar{b}_{\alpha,1} \hat{\ } \bar{b}_{\alpha,2}$  realizes  $p_{\mathfrak{s}_{\text{gm}}}(\bar{x})$ , see 4.11;  
 (f) let  $k_{\alpha,3} < \omega$  be  $> k_{\alpha,1}, k_{\alpha,2}$  and let  $b_{\alpha,3} \in G_{\beta_\alpha+k_{\alpha,3}+1}$  realizes  $q_{\mathfrak{s}_{\text{gm}}[\mathbf{x}]}(\bar{b}_{\alpha,1} \hat{\ } \bar{b}_{\alpha,2}, G_{\beta_\alpha+k_{\alpha,3}}, G_{\beta_\alpha})$ , (see Definition 4.11); so it commutes with  $\mathbf{C}_{G_{\beta_\alpha+k_{\alpha,2}+1}}(\bar{b}_{\alpha,1} \hat{\ } \bar{b}_{\alpha,2})$ , hence with  $G_{\beta_\alpha}$  and conjugating by it interchange  $\bar{b}_{\alpha,1}, \bar{b}_{\alpha,2}$ ;

- (g) without loss of generality  $\mathfrak{s}_\alpha = \mathfrak{s}_*$  and  $(\ell g(\bar{b}_{\alpha,1}), k_{\alpha,2}, \ell g(\bar{b}_{\alpha,2}))$  does not depend on  $\alpha$ .

Our intention (in this stage) is to find  $\alpha_n < \lambda$  increasing with  $n$  satisfying  $\beta_{\alpha_n} < \alpha_{n+1}$  and element  $d$  such that, on the one hand, conjugating with  $d$  maps  $c_{\alpha_n} = b_{\alpha_n,1,0}$  to  $b_{\alpha_n,2,0}$  for each  $n$ , and on the other hand,  $\text{tp}_{\text{bs}}(d, G_{\beta_n+\omega}, G_\lambda)$  does not split over  $G_{\beta_n} + b_{\alpha_n,3}$ , a contradiction.

Let  $N$  be such that:

- (\*)<sub>5,2</sub> (a)  $N$  is a model with universe  $\lambda$ ;  
 (b)  $N$  is with countable vocabulary;  
 (c)  $N$  expands  $N_*$  from (\*)<sub>3,1</sub>;  
 (d)
  - $F_0^N = \mathbf{h}$ , so  $F_0$  is a unary function symbol,
  - $F_{1,\ell}^N(\alpha) = b_{\alpha,\ell}$  for  $\ell = 1, 2$  and  $\ell < \ell g(\bar{b}_{\alpha,\ell})$ , (if  $\ell = 0$  we may omit it),
  - $F_{1,3}^N(\alpha) = b_{\alpha,3}$ ,
  - $F_2^N(\alpha) = \beta_\alpha$ ,
  - $F_{2,\ell}(\alpha) = \beta_\alpha + k_{\alpha,\ell}$  for  $\ell = 1, 2, 3$ ,
  - $F_3^N(\alpha) = \beta_\alpha + \omega$ ,
 (e)  $F_{4,n}^N$  is an  $(n+1)$ -place function such that: if  $\alpha_0 < \dots < \alpha_n, c_{\alpha_\ell} \in G_{\alpha_\ell+1}$ , each  $\alpha_\ell$  is a limit ordinal then  $F_{4,n}^N(\alpha_0, \dots, \alpha_n)$  is the product of  $a_0 a_1 \dots a_n$  where  $a_k = F_{1,1,0}(\alpha_k)$ ;  
 (f)  $P^N = \{(\alpha, c) : \alpha < \lambda \text{ and } c \in G_\alpha\}$ .

Without loss of generality  $\tau_N \subseteq \mathcal{H}(\aleph_0)$ , choose  $\zeta(1) < \lambda$  such that  $\tau_{\zeta(1)} = \tau_N$  and for each  $\delta \in S_{\zeta(1)}$  we use the amount of freedom we are left with (see before  $\boxplus_3$ ), choosing  $G_{\delta+1}$  such that:

- (\*)<sub>5,3</sub> if  $\delta \in S_{\zeta(1)}, i \in \mathcal{T}_\delta$ , letting  $\alpha_{\delta,i,n} := \min(N_i^\delta \setminus \alpha_\delta(n))$  then (a)  $\Rightarrow$  (b) where:

- (a)
  - $\beta_{\delta,i,n} := F_2^{N_i^\delta}(\alpha_{\delta,i,n})$  is  $\geq \alpha_{\delta,i,n}$  but  $< \alpha_\delta(n+1)$ ,
  - $F_3^{N_i^\delta}(\alpha_{\delta,i,n}) = \beta_{\delta,i,n} + \omega$ ,
  - $b_{\delta,i,n,\ell} = F_{1,\ell}^N(\alpha_{\delta,i,n})$  for  $\ell = 1, 2$  and  $\ell = 0$ ,
  - $k_{\delta,i,n,\ell} = F_{2,\ell}^{N_i^\delta}(\alpha_{\delta,i,n}) - \alpha_{\delta,i,n}$  for  $\ell = 1, 2, 3$ ,
  - $b_{\delta,i,n,3} = F_3^{N_i^\delta}(\beta_{\delta,i,n})$ ,
  - $b_{\delta,i,n,\ell} \in G_{\beta_{\delta,i,n} + k_{\delta,i,n,\ell} + 1}$  commute with  $G_{\beta_{\delta,i,n}}$  and conjugating by  $b_{\delta,i,n,3}$  interchange  $b_{\delta,i,n,1,\ell}, b_{\delta,i,n,2,\ell}$ ,
  - $\delta$  is the set of elements of  $G_\delta$ , similarly  $\alpha_{\delta,i,n}$  (as they  $\in E_{\mathbf{h}}$ ),
  - for every  $\beta < \delta$  we have  $(G_{\beta+1} \setminus G_\beta) \cap N_i^\delta \neq \emptyset \Leftrightarrow \beta \in N_{\delta,i}$ ,
  - if  $\beta \in N_i^\delta \setminus S$  and  $\bar{c} \in \omega^>(N_i^\delta)$ , so  $\bar{c} \in \omega^>(G_\delta)$ , then  $\text{tp}_{\text{bs}}(\bar{c}, G_\beta, G_\delta) \in q_t(G_\beta)$  for some  $t \in \text{def}(G_\beta)$  satisfying  $\bar{a}_t \in \omega^>(N_i^\delta \cap G_\beta)$ ,
 (b)  $\bar{c}_{\delta,i} = \langle c_{\delta,i} \rangle$  and  $\text{tp}_{\text{bs}}(c_{\delta,i}, G_\delta, G_{\delta+1})$  is as in claim 2.19 with  $G_{\alpha_\delta(n)}(n < \omega), G_\delta, b_{\beta_{\delta,i,n,3}} \in N_i^\delta(n < \omega)$  here standing for  $G_n(n < \omega), G_\omega, a_n^t(n < \omega)$  with  $I = \{t\}$  there;

- (\*)<sub>5.4</sub> let  $\mathcal{T}'_\delta = \{i \in \mathcal{T}_\delta : \text{clause (a) of } (* )_{5.3} \text{ holds}\}$ ;  
 (\*)<sub>5.5</sub> let  $\beta_{\delta,i,n} = \alpha_{\delta,i,n} + \omega$  and  $b_{\delta,i,n} = b_{\delta,i,n,\iota} \in G_{\beta_{\delta,i,n}+1}$  for  $\iota = 1, 2, 3$  realizes  $q_{\mathfrak{S}_{\text{ab}(2)}}(\langle \cdot \rangle, G_{\beta_{\delta,i,n}})$  when the assumption of clause (a) fails.

Why can we fulfill (\* )<sub>5.3</sub>? Let  $\langle i_\ell : \ell < \ell(*) \rangle$  be a finite sequence of members of  $\mathcal{T}_\delta$ . For  $\ell < \ell(*)$  and  $n < \omega$  let  $d_{\ell,n} = b_{\delta,i_\ell,n,3}$ .

Now

- (\*)<sub>5.6</sub>  $\langle d_{\ell,n} : n < \omega \rangle$  pairwise commute if  $i(\ell) \in \mathcal{T}'_\delta$  for each  $\ell < \ell(*)$ .

[Why? As  $b_{\delta,i(\ell),n,3} \in \mathbf{C}(G_{\beta_{\delta,i(\ell),n}}, G_{\beta_{\delta,i(\ell),n}+\omega})$  for  $n < \omega$  and  $\beta_{\delta,i(\ell),n} + \omega < \alpha_\delta(n+1) \leq \alpha_{\delta,i(\ell),n+1} \leq \beta_{\delta,i(\ell),n+1}$ , recalling  $N_\delta^i \upharpoonright \alpha_\delta(n+1) \prec N_\delta^i$  and  $N_\delta^i \cap (\alpha_\delta(n), \alpha_\delta(n+1)) \neq \emptyset$ .]

- (\*)<sub>5.7</sub>  $\langle d_{\ell,n} : n < \omega \rangle$  pairwise commute when  $i(\ell) \notin \mathcal{T}'_\delta$ .

[Why? Even easier.]

- (\*)<sub>5.8</sub> if  $\ell(1) \neq \ell(2)$  then for every  $n(1) < n(2)$  the elements  $b_{\delta,i(\ell(1)),n(1),3}, b_{\delta,i(\ell(2)),n(2),3}$  commute.

[Why? Recall that  $b_{\delta,i(\ell(1)),n(1),3} \in G_{\alpha_\delta(n(2))} \subseteq G_{\beta_{\delta,i(\ell(2)),n(2)}}$ ; note that  $b_{\delta,i,n,\iota} \in G_{\beta_{\delta,i,n,\iota}+\omega}$  commute with  $G_{\beta_{\delta,i,n}}$  rather than with  $G_{\alpha_{\delta,i,n}}$  but not used.]

- (\*)<sub>5.9</sub> if  $\ell(1), \ell(2) < \ell(*)$ , then for  $n$  large enough, for every  $n(1), n(2) \in (n, \omega)$  the elements  $d_{\ell(1),n(1)}, d_{\ell(2),n(2)}$  of  $G_\delta$  commute.

[Why? Similarly, as  $N_{i_{\ell(1)}}^\delta \cap N_{i_{\ell(2)}}^\delta$  is bounded in  $\delta$ , but not used.]

- (\*)<sub>5.10</sub> The conditions in 2.19 hold hence we can fulfill (\* )<sub>5.3</sub>, (\* )<sub>5.4</sub>, i.e. we can carry the induction in  $\boxplus_1$ .

[Why? Think.]

Next let

- (\*)<sub>5.11</sub>  $E = \{\delta < \lambda : \delta \text{ a limit ordinal is the universe of } G_\delta \text{ and } N \upharpoonright \delta \prec N, \text{ hence } \mathbf{h} \text{ maps } G_\delta \text{ onto itself}\}$ .

Clearly  $E$  is a club of  $\lambda$ , hence by  $\boxplus_{0,\zeta(1)}$  from stage A, there is a pair  $(\delta, i_*) = (\delta, i(*))$  such that

- (\*)<sub>5.12</sub>  $\delta \in E \cap S_{\zeta(1)}$  and  $i_* \in \mathcal{T}_\delta$  and  $N_{i_*}^\delta \prec N$ .

Let  $d = \mathbf{h}(c_{\delta,i_*}) \in G_*$ , so:

- (\*)<sub>5.13</sub> (a) the pair  $(\delta, i_*)$  satisfies the demands in (\* )<sub>5.3</sub>(a);  
 (b) for some finite set  $u_* \subseteq \mathcal{T}_\delta$  and  $\bar{b}_* \in {}^\omega(G_\delta)$ , the type  $\text{tp}_{\text{bs}}(d, G_{\delta+1}, G_*)$  does not split over  $\{c_{\delta,i} : i \in u_*\} \cup \bar{b}_*$ ;  
 (c) without loss of generality  $i_* \in u_*$ .

[Why? For clause (a), as  $\delta \in E$  and  $N_{i_*}^\delta \prec N$ , recalling the choice of  $N$  (including  $\mathbf{h} = F_0^N$ ). For clause (b), apply properties of the construction in  $\boxplus_1$ , i.e.  $G_{\delta+1} \leq_{\mathfrak{E}} G_*$ .]

- (\*)<sub>5.14</sub> conjugating by  $d$  in  $G_*$  interchange  $b_{\delta,i(*),n,1}$  with  $b_{\delta,i(*),n,2}$  for  $n < \omega$ .



[Why? Should be clear as for  $m \in \omega \setminus \{n\}$  and  $\iota(1), \iota(2) \in \{1, 2, 3\}$ , the element  $b_{\delta, i(*), m, \iota(1)}$  commutes with  $b_{\delta, i(*), m, \iota(2)}$ .]

Recalling  $\boxplus_0(c)$  there is  $n(*) < \omega$  large enough such that:

(\*)<sub>5.15</sub>  $\bar{b}_* \subseteq G_{\beta_{\delta, i(*), n(*)}}$  and  $j_1 \neq j_2 \in u_* \Rightarrow N_{j_1} \cap N_{j_2} \subseteq G_{\alpha_{\delta}(n(*))}$  and  $j_1, j_2$  are like  $i, j$  as in  $\boxplus_{0, \zeta(1)}(c)^+$ .

Clearly for some  $\beta(*) < \lambda$  we have  $\mathbf{h}(b_{\delta, i(*), n(*), 1}) \in G_{\beta(*)+1} \setminus G_{\beta(*)}$ . As  $\alpha_{\delta, i(*), n(*)} = \min(N_{i_*}^\delta \setminus \alpha_{\delta}(n(*)) \in N_{i_*}^\delta$ , clause (d) of  $\boxplus_0$  and  $N_{i_*}^\delta \prec N$ , clearly:

(\*)<sub>5.16</sub> (a)  $\mathbf{h}$  maps  $G_{\alpha_{\delta, i(*), n(*)}} \cap N_{i_*}^\delta$  onto itself and so  $\beta(*) \in N_{i_*}^\delta \setminus \alpha_{\delta, i(*), n(*)}$   
 (b)  $\mathbf{h}$  maps  $G_{\alpha_{\delta, i(*), n(*)+1}}$  onto itself hence  $\beta(*) \in N_{i_*}^\delta \cap \alpha_{\delta, i(*), n(*)+1} \setminus \alpha_{\delta, i(*), n(*)}$ .

Also,

(\*)<sub>5.17</sub> if  $\beta(*) < \beta_{\delta, i(*), n(*)} + \omega$  then  $\beta(*) \leq \beta_{\delta, i(*), n(*)} + k_{\delta, i(*), n(*)}, 2$ .

[Why? By (\*)<sub>5.1</sub>.]

Now,

(\*)<sub>5.18</sub> there is  $\beta \in N_{i_*}^\delta \cap (\beta(*) + 1) \setminus \alpha_{\delta}(n(*)) \setminus S$  such that  $[\beta, \beta(*) + \omega] \cap N_{i_*}^\delta$  is disjoint from  $N_j^\delta$  if  $j \in u_*$  but  $j \neq i_*$ .

[Why? First assume  $\beta(*) \notin S$ , let  $\beta = \beta(*)$ , so clearly  $\beta \in N_{i_*}^\delta$  by (\*)<sub>5.14</sub>,  $\beta \in (\beta(*) + 1)$ , also  $\beta \notin \alpha_{\delta}(n(*))$  as by (\*)<sub>5.6</sub> and the fact that  $\beta \notin S$  by its choice. Also  $[\beta, \beta(*) + \omega] = [\beta(*), \beta(*) + \omega] \subseteq N_{i_*}^\delta$  as  $N$  is closed under  $\alpha \mapsto \alpha + 1$  by (\*)<sub>3.1(b)</sub>. If  $j \in u_*$  but  $j \neq i_*$  then  $N_j^\delta \cap N_{i_*}^\delta \subseteq \alpha_{\delta}(n(*)) \leq \beta$ , hence  $[\beta, \beta(*) + \omega] \cap N_j^\delta = \emptyset$ , so we are done.

Second, assume  $\beta(*) \in S$ , hence  $\text{cf}(\delta) = \aleph_0$ , and by (\*)<sub>3.1(f)</sub>,  $\{\alpha_{\beta(*)}(n) : n < \omega\} \subseteq N_{i_*}^\delta$ . But by  $\boxplus_0(c)^+$  we have  $j \in u_* \wedge j \neq i_* \Rightarrow \sup(N_j^\delta \cap \beta(*)) < \beta(*)$ . As  $u_*$  is finite there is  $\beta \in \{\alpha_{\beta(*)}(n) : n < \omega\}$  such that  $(\beta, \beta(*)) \cap N_j^\delta = \emptyset$ ; hence as before also  $(\beta, \beta(*) + \omega) \cap N_j^\delta = \emptyset$ , whenever  $j \in u_* \wedge j \neq i_*$ . So (\*)<sub>5.16</sub> holds indeed.]

We finish the proof of  $\boxplus_5$  by getting a contradiction as follows.

Case 1:  $\beta(*) \geq \beta_{\delta, i(*), n(*)} + \omega$ .

So by the choice of  $\beta$  and the proof of (\*)<sub>5.3</sub> the type  $\text{tp}_{\text{bs}}(d, G_{\beta(*)+\omega}, G_*)$  does not split over  $G_\beta$ , and even over some finite subset of it.

Now by  $\boxplus_1(e)$  in  $G_{\beta(*)+\omega}$  there is  $d' \neq \mathbf{h}(b_{\beta_{\delta, i(*), n(*)}, 1})$  realizing

$$\text{tp}_{\text{bs}}(\mathbf{h}(b_{\beta_{\delta, i(*), n(*)}, 1}), G_\beta, G_{\beta(*)+\omega}) \text{ so } \mathbf{h}(b_{\beta_{\delta, i(*), n(*)}, 3}) \notin \text{cl}(G_\beta \cup \{d\}).$$

However,  $G_* \models d^{-1} \mathbf{h}(c_{\beta_{\delta, i(*), n(*)}, 1})d = \mathbf{h}(c_{\beta_{\delta, i(*), n(*)}, 2})$ , contradiction.

Case 2:  $\beta(*) < \beta_{\delta, i(*), n(*)} + \omega$ .

Hence  $\beta(*) \leq \beta_{\delta, i(*), n(*)} + k_{\delta, i(*), n(*)}, 2$  and so  $\{\text{tp}_{\text{bs}}(d, G_{\beta_{\delta, i(*), n(*)}+\omega}, G_*)\}$  does not split over  $G_{\beta_{\delta, i(*), n(*)}} \cup \{b_{\delta, i(*), n(*)}, 3\}$  but  $\text{tp}(b_{\delta, i(*), n(*)}, 3, G_{\beta_{\delta, i(*), n(*)}+k_{\delta, i(*), n(*)}, 3}, G_*)$  does not split over  $G_{\beta_{\delta, i(*), n(*)}} \cup \text{Rang}(\bar{b}_{\delta, i(*), n(*)}, 1) \cup \text{Rang}(\bar{b}_{\delta, i(*), n(*)}, 2)$ .

It follows that  $\text{tp}_{\text{bs}}(d, G_{\beta_{\delta, i(*), n(*)}+k_{\delta, i(*), n(*)}, 2}, G_*)$  does not split over  $G_{\beta_{\delta, i(*), n(*)}} \cup \bar{b}_{\delta, i(*), n(*)}, 1$  and recall  $\mathbf{h}(b_{\beta_{\delta, i(*), n(*)}, 1}) \subseteq G_{\beta_{\delta, i(*), n(*)}+k_{\delta, i(*), n(*)}, 2}$ , contradiction by (\*)<sub>5.0</sub>.

So we have finished proving  $\boxplus_5$ .

Stage D:

- $\boxplus_6$  (a) for some stationary  $S_1^* \subseteq S_* (\subseteq \lambda \setminus S)$  for every  $\beta \in S_1^* \setminus \alpha(*)$  if  $b \in L_\beta^*$  then  $\mathbf{h}(b) = \sigma^{G_*}(b, \bar{a})$  for some  $\bar{a} \in \omega^{>}(G_\beta)$  and group-term  $\sigma(x, \bar{y})$   
 (b) moreover  $\mathbf{h}(b) = \sigma^{G_*}(b)$  if  $b \in L_\beta^*$ .

Why?

- (\*)<sub>6.1</sub> clause (a) of  $\boxplus_6$  holds even for every  $\beta \in S_2^* := S_* \cap E_{\mathbf{h}} \setminus \alpha(*)$ .

[Why? By  $\boxplus_5$ .]

- (\*)<sub>6.2</sub> Without loss of generality if  $\beta \in S_2^*$  and  $b \in L_\beta^*$ , then  $\mathbf{h}(b) = \sigma_b(b)a_b$  for some  $a_b \in G_\beta$ .

[Why? This by (\*)<sub>6.1</sub> because  $\mathbf{h}$  maps  $G_\beta$  onto itself,  $b$  commutes with  $G_\beta$  whereas  $\bar{a}_b \in \omega^{>}(G_\beta)$ .]

- (\*)<sub>6.3</sub> (a)  $b \mapsto \sigma_b(b)$  is a homomorphism from the set  $L_\beta^*$  into  $L_\beta^*$  (but we did not claim  $L_\beta^*$  is a subgroup);  
 (b)  $b \mapsto a_b$  induces a homomorphism from the set  $L_\beta^*$  into the group  $G_\beta$ , that is if  $\sigma(x_0, \dots, x_{n-1})$  is a group term and  $b_0, \dots, b_{n-1} \in L_\beta^*$  and  $G_{\beta+\omega} \models \sigma(b_0, \dots, b_{n-1}) = e$  then  $G_\beta \models \sigma(a_{b_0}, \dots, a_{b_{n-1}}) = e$ .

[Why? As  $\mathbf{h}$  is an automorphism of  $G_*$  and as  $a_{b_1}, \sigma_{b_2}(b_2)$  commute for  $b_1, b_2 \in L_\beta^*$ .]

We try to get rid of the homomorphism from (\*)<sub>6.3</sub>(b) in order to prove  $\boxplus_6(b)$ .

Toward contradiction assume (for the rest of this stage):

- (\*)<sub>6.4</sub>  $\gamma \in S_2^* \subseteq \lambda \setminus S_1^*$  is a limit ordinal and  $b_* \in L_\gamma^*$  and  $a_{b_*} \neq e$ .

Now as  $\gamma \in S_* \subseteq \lambda \setminus S$  we can find a sequence  $\bar{f}^\gamma = \langle f_\eta^\gamma : \eta \in \omega^\mu \rangle$  satisfying  $f_\eta^\gamma$  is a function from  $\{\eta \upharpoonright n : n < \omega\}$  into  $G_\gamma$  such that for every  $f : \omega^{>}\mu \rightarrow G_\gamma$  for some  $\eta \in \omega^\mu$  we have  $f_\eta^\gamma \subseteq f$ ; i.e. a simple black box, see [Sheb, Fact 1.5=L4.5A], it exists as  $\mu = \mu^{\aleph_0}$ . Now generally for  $\gamma \in \lambda \setminus S$  let  $\mathscr{W}_\gamma = \{\eta \in \omega^\mu : \text{for some } c \in G_\gamma \text{ of order 2 we have } n < \omega \Rightarrow c^{-1}f_\eta^*(\eta \upharpoonright (2n))c = f_\eta^*(\eta \upharpoonright (2n+1))\}$ .

Let  $K_*$  be the group of permutations of  $I = \omega^{>}\mu \times \{0, 1\}$  with finite support, i.e.  $\{f \in \text{Sym}(I) : (\exists \langle \aleph_0 t \in I \rangle)(f(t) \neq t)\}$ . For  $\eta \in \omega^{>}\mu$  let  $h_\eta \in K_*$  be such that  $h_\eta((\eta, \iota)) \equiv (\eta, 1 - \iota)$ , for  $\iota = 0, 1$ , and is the identity otherwise. Let  $K_\gamma$  be the group of permutations of  $I = \omega^{>}\mu \times \{0, 1\}$  generated by  $K_* \cup \{y_\eta : \eta \in \omega^{>}\mu\}$ , where:

- (\*)<sub>6.5</sub> (a) if  $\eta \in \mathscr{W}_\gamma$  then  $y_\eta$  interchanges  $(\eta \upharpoonright (2n+1), \iota), (\eta \upharpoonright (2n+2), \iota)$  for  $n < \omega, \iota = 0, 1$  and otherwise is the identity;  
 (b) if  $\eta \in \omega^\mu \setminus \mathscr{W}_\gamma$  then  $y_\eta$  interchanges  $(\eta \upharpoonright (2n), \iota)$  and  $(\eta \upharpoonright (2n+1), \iota)$  for  $n < \omega, \iota = 0, 1$ , and is the identity otherwise.

Let

- $d$  be the permutation of  $I$  interchanging  $(\langle \rangle, 0), (\langle \rangle, 1)$  and being the identity otherwise.

Now we shall use some of the amount of freedom left, clearly:

- (\*)<sub>6.6</sub> (a) there is  $K \subseteq \mathbf{C}_{G_{\gamma+\omega}}(G_\beta)$  finite with trivial center such that  $b_* \in K$ ;

- (b) there is  $\bar{b}$  which lists the member of  $K$  such that  $d_0 = b_*$ ;
  - (c) there is  $\bar{d}$ , a finite sequence from  $K_\gamma$  realizing  $\text{tp}(\bar{b}, \emptyset, G_*)$ ;
  - (d) there is  $n(*)$  such that  $K \subseteq G_{\gamma+n(*)}$ .
- (\*)<sub>6.7</sub> There is an embedding  $g_\gamma$  of  $K_\gamma$  into  $\mathbf{C}_{G_{\gamma+n(*)+1}}(G_\gamma)$  mapping  $\bar{d}$  to  $\bar{b}$  hence  $d_0$  to  $b_*$ ;
- (\*)<sub>6.8</sub>  $b \mapsto a_b$  (for  $b \in g_\gamma(K_\gamma)$ ) is a homomorphism from  $g_\gamma(K_\gamma)$  into  $G_\gamma$ ;
- (\*)<sub>6.9</sub> let  $f : \omega^\lambda \rightarrow G_\beta$  be defined by  $f(\eta) = a_{g_\gamma(h_\eta)}$ .

By the choice of  $\langle f_\eta : \eta \in \omega^\lambda \rangle$  for some  $\eta \in \omega^\lambda$  we have  $n < \omega \Rightarrow f_\eta^\gamma(\eta \upharpoonright n) = f(\eta \upharpoonright n)$ .

Now does  $\eta \in \mathcal{W}_\gamma$ ? First, assume  $\eta \notin \mathcal{W}_\gamma$ , then (by the choice of  $K_\gamma$ )  $(g_\gamma(y_\eta) \in G_\gamma$  and) conjugating by  $g_\gamma(y_\eta)$  for each  $n$ , interchanges  $g_\gamma(h_{\eta \upharpoonright (2n)}), g_\gamma(h_{\eta \upharpoonright (2n+1)})$  which means that in  $K_\gamma$ , conjugating by  $h_\eta$  interchanges  $f_\eta^\gamma(\eta \upharpoonright (2n)), f_\eta^\gamma(\eta \upharpoonright (2n+1))$ , but by the choice of  $\mathcal{W}_\gamma$  this means  $\eta \in \mathcal{W}_\gamma$ .

Second, assume  $\eta \in \mathcal{W}_\gamma$ , by the definition of  $\mathcal{W}_\gamma$  there is  $c \in G_\gamma$  of order 2 such that conjugating by  $c$  for each  $n$  interchanges  $g_\gamma(h_{\eta \upharpoonright (2n)}), g_\gamma(h_{\eta \upharpoonright (2n+1)})$ . But conjugating by  $g_\gamma(y_\eta)$  for  $n$  interchange  $g_\gamma(h_{\eta \upharpoonright (2n+1)}), g_\gamma(h_{\eta \upharpoonright (2n+2)})$ . So in  $G_*$ , the subgroup generated by  $\{c, g_\gamma(y_\eta), g_\gamma(h_{\eta \upharpoonright 1})\}$  includes  $g_\gamma(h_{\eta \upharpoonright n})$  for  $\eta = 1, 2, \dots$ ; why? just prove it by induction on  $n$ . But  $\{g_\gamma(h_{\eta \upharpoonright n}) : n = 1, 2, \dots\} \subseteq G_*$  is infinite, contradiction.

Stage E:

- $\boxplus_7$  there is a finite sequence  $\bar{a}_*$  such that for every  $b \in G_*$  we have  $\mathbf{h}(b) \in \text{cl}(\bar{a}_* \cup \{(b, G_*)\})$ .

[Why? For  $\beta \in S_1^*$  let  $d_\beta \in G_{\beta+1}$  realize  $\mathfrak{s}_{\text{cg}}(\langle \cdot \rangle, G_\beta)$  in  $G_{\beta+1}$ . So for every  $a \in G_\beta$  of order  $m$  as  $G_\beta$  is existentially closed there is a finite  $K_a \subseteq G_\beta$  with trivial center to which  $a$  belongs. Hence the element  $d_\beta a d_\beta^{-1}$  commute with  $G_\beta$  and belongs to  $G_{\beta+1}$  and moreover to  $L_\beta^*$ . Hence, by  $\boxplus_6(b)$ , for some  $k(a) < m$  we have:

$$(*)_{7.1} \quad \mathbf{h}(d_\beta^{-1} a d_\beta) = (d_\beta^{-1} a d_\beta)^{k(a)}.$$

Hence

$$(*)_{7.2} \quad \mathbf{h}(a) = \mathbf{h}(d_\beta^{-1}) \mathbf{h}(d_\beta^{-1} a d_\beta) \mathbf{h}(d_\beta^{-1}) = \mathbf{h}(d_\beta^{-1}) (d_\beta^{-1} a d_\beta)^{k(a)} \mathbf{h}(d_\beta).$$

Also, as  $\beta \notin S$ , there is a finite  $K_\beta \subseteq G_\beta$  such that  $\text{tp}_{\text{bs}}(\langle \mathbf{h}(d_\beta), d_\beta \rangle, G_\beta, G_*; \mathbf{K}_{\text{lf}})$  does not split over  $K_\beta$ . By  $(*)_{7.2}$ ,  $\text{tp}_{\text{bs}}(\mathbf{h}(a), G_\beta, G_*; \mathbf{K}_{\text{lf}})$  does not split over  $K_\beta \cup \{d\}$ , but  $\mathbf{h}(a) \in G_\beta$  hence  $\mathbf{h}(a) \in \langle K \cup \{d\} \rangle_{G_*}$ . By Fodor's lemma this is enough for  $\boxplus_7$ .

Clearly we are done by 2.23. □<sub>5.1</sub>

\* \* \*

- Question 5.2.* 1) In 5.1 we can easily get  $2^\lambda$  pairwise non-isomorphic groups  $G'$ . But can they be pairwise far? (i.e. no  $G \in \mathbf{K}_\lambda$ , can be embedded in two of them)?
- 2) Even more basically can we demand  $G_*$  has no uncountable Abelian subgroup (when  $G$  does not)? Or at least no Abelian group of cardinality  $\lambda$ ?
- 3) Can we prove 5.1 for every  $\lambda > \aleph_0$ ? or at least  $\lambda \geq \beth_\omega$ ?

**Discussion 5.3.** 1) Concerning 5.2(1), the problem with our approach is using  $p \in \mathbf{S}_{\mathfrak{S}}(G)$ , so as  $\lambda$  is regular we will get subgroups generated by indiscernible sequences, but let us elaborate. Assume  $G_* \in \mathbf{K}_\lambda, G_* = \cup\{G_\alpha : \alpha < \lambda\}, G_\alpha$  increases with  $\alpha$  and  $|G_\alpha| < \lambda$ . Further, assume  $\mathfrak{s} \in \Omega[\mathbf{K}]$  and  $\bar{a} \in {}^{n(\mathfrak{s})}G_*$  and  $S = \{\alpha < \lambda : \bar{a} \subseteq G_\alpha \text{ and the type } q_{\mathfrak{s}}(\bar{a}, G_\alpha) \text{ is realized in } G_*\}$  is unbounded in  $\lambda$  and thus it is an end segment. Let  $\bar{c}_\alpha \in {}^{k(\mathfrak{s})}G_*$  realize  $q_{\mathfrak{s}}(\bar{a}, G_\alpha)$  and so for some club  $E$  of  $\lambda, \alpha \in S \cap E \Rightarrow \bar{c}_\alpha \in G_{\min(E \setminus (\alpha+1))}$ . Now  $\bar{\mathfrak{c}} = \langle \bar{c}_\alpha : \alpha \in S \cap E \rangle$  satisfies: if  $h$  is a partial increasing finite function from  $S \cap E$  to  $S \cap E$ , then it induces a partial automorphism of  $G_* : \bar{c}_\alpha \mapsto \bar{c}_{h(\alpha)}$ . This is a case of indiscernible sequences. Hence the isomorphism type of  $\text{cl}(\cup\{\bar{c}_\alpha : \alpha \in S \cap E\}, G_*)$  depends only on  $\mathfrak{s}$  (and  $\text{tp}_{\text{bs}}(\bar{a}, \emptyset, G_*)$ ). Hence the number of pairwise far such  $G_*$ 's is  $\leq |\mathfrak{S}| + \aleph_0$ .

2) Concerning 5.2(2), the problem with our approach is that we use  $\mathfrak{s} = \mathfrak{s}_{\text{ab}(k)}$  and more generally  $\mathfrak{s} \in \Omega[\mathbf{K}]$  such that if  $q_{\mathfrak{s}}(\bar{a}, G) = \text{tp}_{\text{bs}}(\bar{c}, G, H)$  then some  $c \in H \setminus G$  commute with every (or simply many) members of  $G$ . Hence in the construction above,  $G_*$  has Abelian subgroups of cardinality  $\lambda$ .

3) What about considering the class of  $(G, F_h)_{h \in H}, F_h \in \text{aut}(G), G \in \mathbf{K}_{\text{lf}}, h \mapsto F_h$  a homomorphism? We intend to deal with it in [Shee].

**Discussion 5.4.** 1) Naturally the construction in the proof of 5.1 is not unique, the class has many complicated models. In the construction in the proof of 5.1 we choose one where we realize many definable types.

2) We may like in  $\boxplus_5$  of Stage C in the proof of 5.1 to consider  $c \in G_\lambda$ , not necessarily from  $G_{\beta+\omega}$ ; (so later the role of  $\mathfrak{s}_{\text{cg}}$  in translating knowledge on  $\mathbf{h}|G_{\beta+\omega}$  to knowledge on  $G_\beta +$  use of Fodor is not necessary). Presently the way we combine  $\langle b_{\delta, i(\ell), n, 3} : n < \omega, \ell < \ell(*) \rangle$  to one  $n$ -type in  $\mathbf{S}_{\text{bs}}(G_\delta)$  works using 2.19.

Concerning the existence of complete groups in  $\mathbf{K}_\lambda^{\text{lf}}$  extending any  $G \in \mathbf{K}_\lambda^{\text{lf}}$  there are some restrictions.

**Claim 5.5.** *Assume  $\lambda > \text{cf}(\lambda) = \aleph_0, \chi = \lambda^{\aleph_0}$ .*

- 1) *If  $G \in \mathbf{K}_\lambda^{\text{lf}}$  is full, then its outer automorphism group has cardinality  $\geq \chi$ .*
- 2)  *$G$  has  $\geq \chi$  outer automorphisms when  $G \in \mathbf{K}_\lambda^{\text{lf}}$  and for some sequence  $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$  listing the elements of  $G$ , letting  $G_\alpha = \text{cl}(\{a_\beta : \beta < \alpha\}, G)$  we have:*

- (a) *for every  $\alpha < \lambda$  for  $\lambda$  ordinals  $\beta < \lambda, a_\beta$  commutes with  $G_\alpha$*
- (b) *for every  $a \in G \setminus \{e_G\}$  some element  $b \in G, a$  does not commute with  $b$ .*

3) *Like (2) but  $G_\alpha$  has center of cardinality  $< \lambda$ .*

4) *Instead of (a),(b) we can use:*

- (a)' *for every  $\alpha < \lambda$  we have  $\lambda = |\{a/\text{Cent}(G) : a \in G \text{ commute with } G_\alpha\}|$ .*

*Proof.* 1) We reduce it to part (2). Let  $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$  witness fullness (so  $\lambda \geq 2^{\aleph_0}$ ). Now using the schemes  $\mathfrak{s} = \mathfrak{s}_{\text{ab}(2)}$ , the pair  $(G, \bar{a})$  satisfies clause (a) of part (2). Using, e.g. the scheme  $\mathfrak{s} = \mathfrak{s}_{\text{cg}}$  and the claim on non-commuting, 2.21, also clause (b) there holds.

2) Let  $\lambda = \sum_n \lambda_n, \lambda_n < \lambda_{n+1}$ . For each  $n$ , by clause (a) we have  $|S_n^1| = \lambda$  where  $S_n^1 := \{\alpha : a_\alpha \text{ commute with } \text{cl}(\{a_\beta : \beta < \lambda_n\}, G)\}$ . Hence for some  $k_n > n$  we have  $S_n^3 = \{\alpha < \lambda_{k_n} : \alpha \in S_n^1\}$  has cardinality  $> \lambda_n$ .

Replacing  $\langle \lambda_n : n < \omega \rangle$  by a subsequence without loss of generality  $\bigwedge_n k_{2n} = 2n + 1$ . Let  $\langle \alpha_{n,i} : i < \lambda_n \rangle$  be a sequence of pairwise distinct members of  $S_{2n}^3 \setminus \lambda_{2n}$ .

Now for each  $\eta \in \prod_{\ell < n} \lambda_{2\ell}$  let  $b_\eta = a_{\eta(0)}a_{\eta(1)} \dots a_{\eta(n-1)} \in G$  and so  $h_\eta := \square_{b_\eta}$ , conjugation by  $b_\eta$ , is an inner automorphism of  $H$ . Also  $\nu \triangleleft \eta \in \prod_{\ell < n} \lambda_{2\ell} \Rightarrow \square_{b_\eta}, \square_{b_\nu}$  agree on  $\{a_\beta : \beta < \lambda_{2\ell g(\nu)}\}$ .

Hence if  $\eta \in \prod_n \lambda_{2n}$  then  $\langle h_{\eta \upharpoonright n} : n < \omega \rangle$  converge, i.e. for every  $a \in G$ , the sequence  $\langle h_{\eta \upharpoonright n}(a) : n < \omega \rangle$  is eventually constant and called the eventual value  $h_\eta(a)$ .

So  $h_\eta$  is an automorphism of  $G$  (for each  $\eta \in \prod_n \lambda_{2n}$ ). Now if  $\eta_1, \eta_2 \in \prod_n \lambda_{2n}$ ,  $\eta_1(k) \neq \eta_2(k)$ ,  $\eta_1 \upharpoonright k = \eta_2 \upharpoonright k$  and for some  $\alpha < \lambda_{2k}$ ,  $a_\alpha$  does not commute with  $a_{\eta_1(k)}a_{\eta_2(k)}^{-1}$  then  $h_{\eta_1} \neq h_{\eta_2}$ . Hence we can easily find  $2^{\aleph_0}$  pairwise distinct  $h_\eta$ 's. So if  $\lambda < 2^{\aleph_0}$  we are done; otherwise, let  $\mu = \min\{\mu : \mu^{\aleph_0} \geq \lambda \text{ equivalently } \mu^{\aleph_0} = \lambda^{\aleph_0}\}$ , so  $2^{\aleph_0} < \mu < \lambda$  and  $\alpha < \mu \Rightarrow |\alpha|^{\aleph_0} < \mu$ .

Choose  $\bar{\mu} = \langle \mu_n : n < \omega \rangle$  such that  $\sum_n \mu_n = \mu$ ,  $\mu_n < \mu_{n+1}$ ; moreover each  $\mu_n$  regular and  $\alpha < \mu_n \Rightarrow |\alpha|^{\aleph_0} < \mu_n$ . Now for  $n < k$  let  $E_{n,k} = \{(i, j) : i, j < \mu_n \text{ and the conjugation } \square_{a_{\alpha_n, i}}, \square_{a_{\alpha_n, j}} \text{ agree on } \{a_\beta : \beta < \lambda_{2k}\}\}$ , an equivalence relation. By clause (b) in the assumption,  $\bigcap_{k > n} E_{n,k}$  is the equality on  $\mu_n$ , hence for some  $k(n) > n$ ,  $\mu_n/E_{n,k}$  has  $\mu_n$  equivalence class. The rest should be clear.  
3),4) Similarly. □<sub>5.5</sub>

## § 6. OTHER CLASSES

Note that

**Theorem 6.1.** *The results of §1 holds for any universal class  $\mathbf{K}$  - see [Sheg].*

However, we cannot in general prove the existence of dense  $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ , in fact, possibly  $\Omega[\mathbf{K}] = \emptyset$ . We refer the reader to §0 before 0.13, and to 0.17, 2.1. We may expand an lf group by choosing representations for left cosets  $bK$ , for  $K$  a finite subgroup of  $G, b \in G$ . Then the density of  $\Omega[\mathbf{K}]$  is easy.

**Definition 6.2.** 1) Let  $\mathbf{K}_{\text{clf}}$  be the class of structures  $M$  such that  $M$  is an expansion of an lf group  $G = G_M$  by  $F_n = F_n^M$  for  $n \geq 1$  such that:

- (a)  $F_n^M$  is a partial  $(n+1)$ -place function from  $G$  to  $G$ ;
- (b) if  $(a_0, \dots, a_n) \in \text{Dom}(F_n^M)$  then  $(a_0, \dots, a_{n-1})$  list without repetitions the elements of a subgroup of  $G_M$  and  $a_n \in G_M$ , of course;
- (c) if  $F_n^M(a_0, \dots, a_n) = b$  then  $b \in \{a_n a_\ell : \ell < n\}$ ;
- (d) if  $K$  is a finite subgroup of  $G_M$  with  $n$  elements and for some  $(a_0, \dots, a_{n-1})$  listing its elements with no repetitions and  $b$  we have  $(a_0, \dots, a_{n-1}, b) \in \text{Dom}(F_n^M)$ , then for every  $(a'_0, \dots, a'_{n-1})$  listing the members of  $K$  and  $b' \in bK \subseteq G_M$  we have  $(a'_0, \dots, a'_{n-1}, b') \in \text{Dom}(F_n^M)$  and  $b'K = bK \Rightarrow F_n^M(a_0, \dots, a_{n-1}, b') = F_n^M(a_0, \dots, a_{n-1}, b)$ ;
- (e) if  $K_1, K_2$  are as in clause (d) then also  $K_1 \cap K_2$  is;
- (f) if  $A \subseteq G_M$  is finite then there is a minimal  $K$  as in clause (d) which contains  $A$  and if  $A$  is empty then  $K = \{e_{G_M}\}$ .

**Definition 6.3.** Let  $\mathbf{K}_{\text{plf}}$  be the class of structures  $M$  such that:  $M$  expands a lf group  $G$  by  $P_n^M$  for  $n < \omega$  and  $F_n^M$  for  $n < \omega$  (actually definable from the rest) such that:

- (a)  $P_n^M$  is an  $(n+3)$ -place relation;
- (b) if  $\bar{a} = (a_0, \dots, a_{n+2}) \in P_n^M$  then  $\{a_0, \dots, a_{n-1}\}$  list with no repetitions the elements of a finite subgroup of  $G_M$ ;
- (c) if  $\{a_0, \dots, a_{n-1}\} = \{a'_0, \dots, a'_{n-1}\}$  are as above and moreover  $b, b' \in M$  and  $\{ba_0, \dots, ba_{n-1}\} = \{b'a'_0, \dots, b'a'_{n-1}\}$  then  $M \models "P_n(a_0, \dots, a_{n-1}, b, c, d) = P_n(a'_0, \dots, a'_{n-1}, b', c, d)"$  for every  $c, d \in M$ ;
- (d) if  $(a_0, \dots, a_{n-1})$  list the members of a finite subgroup  $K$  of  $G$  with no repetitions and  $b \in G$  then  $\{(c, d) : (a_0, \dots, a_{n-1}, b, c, d) \in P_n^M\}$  is a linear order on the right coset  $bK$ , which we denote by  $<_{K,b}^M$ ;
- (e) if the sequence  $(a_0, \dots, a_{n-1})$  is as above and  $b \in G$  then  $F_n^M(a_0, \dots, a_{n-1}, b)$  is the first element by the order there in  $\{ba_0, \dots, ba_{n-1}\}$ .

**Definition 6.4.** 1) For  $M \in \mathbf{K}_{\text{clf}}$  let  $\text{fsb}(M)$  be the set of finite subgroups  $K$  of  $G_M$  such that for some  $a_0, \dots, a_{n-1}$  listing with no repetitions the elements of  $K$  and for some  $b \in G_M$  we have  $(a_0, \dots, a_{n-1}, b) \in \text{Dom}(F_n^M)$ , i.e. they are as in clause (d) of Definition 6.2.

2) In this case we may write  $F_K^M(b) = F_n^M(a_0, \dots, a_{n-1}, b)$ .

3) For  $M, N \in \mathbf{K}_{\text{clf}}$  let  $M \leq_{\text{elf}} N$  or  $M \subseteq N$  mean that  $G_M \subseteq G_N$  and  $F_n^M = F_n^N \upharpoonright M$  hence  $K \in \text{sfb}(N) \wedge K \subseteq M \Rightarrow K \in \text{fsb}(M)$ . We define similarly  $\leq_{\text{plf}}, \leq_{\text{olf}}$ , see Definition 6.3, 0.15. We may write  $M \leq_{\mathbf{K}} N$  for the appropriate  $\mathbf{K}$ , etc.

- 4) “ $M \in \mathbf{K}_{\text{clf}}$  is (existentially closed)” is defined as in 0.13(2).
- 5) Let clf-group mean a member of  $\mathbf{K}_{\text{clf}}$  and similarly an olf-group.
- 6) Similarly for “olf-groups” and “plf-groups”.

**Convention 6.5.** 1) Let  $\mathbf{K}$  denote one of the classes defined above, but let it be  $\mathbf{K}_{\text{clf}}$  if not said otherwise.

**Definition/Claim 6.6.** 1) For  $M \in \mathbf{K}_{\text{olf}}$  let  $M^{[\text{clf}]}$  be the unique  $N \in \mathbf{K}_{\text{clf}}$  such that:  $G_N = G_M$  and  $\text{fsb}(N) = \{K : K \subseteq G_M \text{ is finite}\}$  and  $F_K^M(b)$  is the  $<_M$ -first member of  $\text{bK} \subseteq G$  (well defined as  $\text{bK}$  is finite non-empty).

- 1A) For  $M \in \mathbf{K}_{\text{olf}}$  we define  $M^{[\text{plf}]}$  and for  $M \in \mathbf{K}_{\text{plf}}$  we define  $M^{[\text{clf}]}$  parallelly.
- 2) For  $M \in \mathbf{K}_{\text{clf}}$  and  $A \subseteq M$ , there is  $N \subseteq M$  from  $\mathbf{K}_{\text{clf}}$  with universe  $A$  iff for every finite  $A \subseteq B$  there is  $K \in \text{fsb}(M)$  such that  $A \subseteq K \subseteq B$ .
- 2A) So if  $M \in \mathbf{K}_{\text{clf}}$  and  $K \in \text{fsb}(M)$  then  $M \setminus K \in \mathbf{K}_{\text{clf}}$  and is finite.
- 3) For  $A \subseteq M \in \mathbf{K}$  let  $\text{cl}(A, M)$  be the minimal  $N \subseteq M$  such that  $A \subseteq N$ , equivalently  $\cup\{K : K \in \text{fsb}(M) \text{ and there is no } L \in \text{fsb}(M) \text{ such that } A \cap K \subseteq L \subset K\}$ .
- 4) For  $A \subseteq M \in \mathbf{K}$  let  $\text{cl}_{\text{gr}}(A, M)$  be the closure of  $A$  under the group operations.
- 5) We call  $M \in \mathbf{K}_{\text{clf}}$  full when  $\text{fsb}(M)$  is the set of finite  $K \subseteq G_M$ .

**Claim 6.7.** 1) The objects in 6.6 are well defined (in the right class).

- 2) If  $M \in \mathbf{K}_{\text{olf}}$  or  $M \in \mathbf{K}_{\text{plf}}$  then  $M^{[\text{clf}]} \in \mathbf{K}_{\text{clf}}$  is full.
- 3)  $\mathfrak{S}(\mathbf{K}_{\text{olf}})$  is dense.
- 4)  $\mathfrak{S}(\mathbf{K}_{\text{clf}})$  is dense.

*Proof.* 1) Straightforward, e.g. in part (3) for  $\mathbf{K}_{\text{clf}}$  the closure is well defined because  $\text{fsb}(M)$  is closed under intersections.

2) Easy, too.

3),4) As in §2. □6.7

*Remark 6.8.* Call  $M \in \mathbf{K}_{\text{clf}}$  invariant when for every finite  $K \subseteq G_M$  there is a function  $F_K^M : G \rightarrow G$  such that  $F_K^M(g) \in gK$  and is equal to  $F_n^M(a_0, \dots, a_{n-1})$  when  $a_0, \dots, a_{n-1}$  list the members of  $K$  with no repetitions. Restricting ourselves to such  $M$  seems to cause problems in amalgamations, whereas for  $\mathbf{K}_{\text{plf}}$  this is not so.

**Definition 6.9.** For  $M \in \mathbf{K}$  and  $n < \omega$  let  $\mathbf{S}_{\text{gd}}^n(M)$  be the set of good  $n$ -types  $p(\bar{x}) \in \mathbf{S}_{\text{bs}}^n(M)$  which means:  $p = \text{tp}(\bar{a}, M, N)$  where  $M \subseteq N \in \mathbf{K}$  and  $\bar{a} \in {}^n N$  and  $\text{cl}_{\text{gr}}(\bar{a} + M, N) = \text{cl}(\bar{a} + M, N)$ .

**Claim 6.10.** The classes  $\mathbf{K} = \mathbf{K}_{\text{clf}}, \mathbf{K}_{\text{plf}}, \mathbf{K}_{\text{olf}}$  have dense closed  $\mathfrak{S} \subseteq \Omega[K]$ .

*Proof.* Straightforward. □6.10

\* \* \*

**Definition 6.11.** 1) Let  $\mathbf{K}_{\text{sl}}$  be the class of locally finite semi-groups, i.e.  $G$ , it has only one operation, binary which is associative.

2) Let  $\mathbf{K}_{\text{usl}}$  be defined similarly with an individual constant  $e$  such that  $G \models ge_G = g = e_G g$  for every  $g \in G \in \mathbf{K}_{\text{usl}}$ .

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