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COMPACTNESS OF CHROMATIC NUMBER II SH1018

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Abstract. We try to look again at results of the form. There is a graph with chromatic number $> \aleph_0$ but every subgraph of cardinality $< \mu$ has chromatic number $\leq \aleph_0$.

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§ 0. Introduction

This continues [She13a] but does not rely on it.

In [She13a] we prove that if there is $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ of cardinality μ, λ-free not free then we can get a failure of λ-compactness for the chromatic number being κ. This gives (using [She94, Ch.II]) that if μ is strong limit singular of cofinality κ and $2^\mu > \mu^+$ then we get the above for $\lambda = \mu^+$ (and more).

Our original objective is to answer a problem of Magidor: $\aleph_\omega$-compactness fails for being $\aleph_0$-chromatic, however lately Magidor prove the consistency. An earlier version was wrong and a new proof will be presented in a version under preparation.

We thank Komjath and Kojman for pointing out a terminal error in a previous attempt. Komjath also asked on the case $\mu = \lambda > \chi = \aleph_0$ when λ singular. Definition 0.2 tries to have a more general frame.

We intend to continue in [S$^+$a].

Another problem on incompactness is about the existence of λ-free Abelian groups G which with no non-trivial homomorphism to $\mathbb{Z}$, in [She07], for $\lambda = \aleph_n$ using $n - BB$. In [She13b] we get more λ's, almost in ZFC by 1-BB (black box). This proof suffices here (but not in ZFC). This is continued in [S$^+$b] which originally we use here, but presently is not connected.

Definition 0.1. 1) Assume $\mu \geq \lambda = \mathrm{cf}(\lambda) \geq \chi$. We say "we have (μ, λ)-incompactness for the $(< \chi)$-chromatic number" or $\mathrm{INC}_{\mathrm{chr}}(\mu, \lambda, < \chi)$ <u>when</u> there is an increasing continuous sequence $\langle G_i : i \leq \lambda \rangle$ of graphs each with $\leq \mu$ nodes, G_i an induced subgraph of G_λ with $\mathrm{ch}(G_\lambda) \geq \chi$ but $i < \lambda \Rightarrow \mathrm{ch}(G_i) < \chi$.
2) Replacing (in part (1)) χ by $\bar{\chi} = (< \chi_0, \chi_1)$ means $\mathrm{ch}(G_\lambda)) \geq \chi_1$ and $i < \lambda \rightarrow \mathrm{ch}(G_i) < \chi_0$; similarly in parts 3),4) below.
3) We say we have incompactness for length λ for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number <u>when</u> we fail to have (μ, λ)-compactness for $(< \chi)$-chromatic (or $\bar{\chi}$-chromatic) number for some μ.
4) We say we have $[\mu, \lambda]$-incompactness for $(< \chi)$-chromatic number or $\mathrm{INC}_{\mathrm{chr}}[\mu, \lambda, < \chi]$ <u>when</u> there is a graph G with μ nodes, $\mathrm{ch}(G) \geq \chi$ but $G^1 \subseteq G \wedge |G^1| < \lambda \Rightarrow \mathrm{ch}(G^1) < \chi$.
5) Let $\mathrm{INC}^+_{\mathrm{chr}}(\mu, \lambda, < \chi)$ be as in part (1) but we add that there is a partition $\langle A_{1,\varepsilon} : \varepsilon < \kappa \rangle$ of the set of nodes of G_i such that $c\ell(G_i \restriction A_{i,\varepsilon})$, the colouring number of $G_i \restriction A_{i,\varepsilon}$ is $< \chi$ for $i < \lambda$, see below.
6) Let $\mathrm{INC}^+_{\mathrm{chr}}[\mu, \lambda, < \chi]$ be as in part (4) but we add: if $G^1 \subseteq G$ and $|G^1| < \lambda$ <u>then</u> there is a partition $\langle A_\varepsilon : \varepsilon < \varepsilon_* \rangle$ of the nodes of G^1 to $\varepsilon_* < \chi$ sets such that $\varepsilon < \varepsilon_* \Rightarrow c\ell(G^1 \restriction A_\varepsilon) < \chi$.
7) If $\chi = \kappa^+$ we may write κ instead of "$< \chi$".
8) Let $\mathrm{INC}(\lambda, < \chi)$ means $\mathrm{INC}(\lambda, \lambda, < \chi)$, and similarly in the other cases.

Definition 0.2. In Definition 0.1 we allow λ similar when we replace $\bar{G}$ by $(G, \bar{A}), G$ a graph with $\leq \mu$ nodes, $\bar{A} = \langle A_i : i < \lambda \rangle$ a partition of the set of nodes, $\mathrm{Ch}(G_u) \leq \chi$ for $u \in [\lambda]^{<\lambda}$ where $G_\eta = G \restriction \bigcup\limits_{i \in u} A_i$.

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§ 1. A sufficient criterion and relations to transversals

Definition 1.1. 1) Let $\mathrm{Inc}[\mu,\lambda,\kappa]$ mean that we can find $\mathbf{a} = (\mathscr{A}, \bar{R})$ witnessing it which means that:

- (a) $|\mathscr{A}| = \mu$
- (b) $\bar{R} = \langle R_\varepsilon : \varepsilon < \kappa\rangle$
- (c) R_ε is a two-place relation on $\mathscr{A}$, so we may write $\nu R_\varepsilon \eta$
- (d) $\mathscr{A}$ is not free (for $\mathbf{a}$), see $(*)_1$ below or just not strongly free, see $(*)_2$ below
- (e) $\mathbf{a} = (\mathscr{A}, \bar{R})$ is λ-free which means $\mathscr{B} \subseteq \mathscr{A} \wedge |\mathscr{B}| < \lambda \Rightarrow \mathscr{B}$ is $\mathbf{a}$-free

<u>where</u>

$(*)_1$ if $\mathscr{B} \subseteq \mathscr{A}$ <u>then</u> $\mathscr{B}$ is $\mathbf{a}$-free means that there is a witness $(h, <_*)$ which means

- (α) $<_*$ a well ordering of $\mathscr{B}$
- (β) h is a function from $\mathscr{B}$ to κ
- (γ) if $h(\eta) = h(\nu)$ and $\nu R_\zeta \eta$ for some ζ <u>then</u> $\nu <_* \eta$ (so really only $<_* \restriction \{\eta \in \mathscr{B} : h(\eta) = \varepsilon\}$ for $\varepsilon < \kappa$ count); so it is reasonable to assume each R_ε is irreflexive
- (δ) for any $\eta \in \mathscr{B}$ the set[1] $\exp(\eta, h, <_*)$ has cardinality $< \kappa$ where (recall that $\mathscr{B} = \mathrm{Dom}(h)$)
  - $\exp(\eta, h, <_*) = \exp(\eta, h, <_*, \mathbf{a}) = \{\zeta < \kappa$: there is $\nu <_* \eta$ such that $\nu R_\zeta \eta$ and $h(\nu) = h(\eta)\}$

$(*)_2$ if $\mathscr{B} \subseteq \mathscr{A}$ <u>then</u> $\mathscr{B}$ is strongly $\mathbf{a}$-free means that for every well ordering $<_*$ of $\mathscr{B}$ there is a function $h : \mathscr{B} \to \kappa$ such that $(h, <_* \restriction \mathscr{B})$ witness $\mathscr{B}$ is $\mathbf{a}$-free

$(*)_3$ if $\mathscr{B} \subseteq \mathscr{A}$ <u>then</u> $\mathscr{B}$ is weakly free means that there is a witness h which means

- (α) h is a function from $\mathscr{B}$ to κ
- (β) for every $\eta \in \mathscr{B}$ the set $\exp(\eta, h)$ has cardinality $< \kappa$ where
  - $\exp(\eta, h) = \exp(\eta, h, \mathbf{a}) = \{\zeta < \kappa$: there is $\nu \in \mathscr{B}$ such that $\nu R_\zeta \eta$ and $h(\nu) = h(\eta)\}$.

2) Let $\mathrm{Inc}(\mu,\lambda,\kappa)$ mean that we can find $(\mathscr{A}, \bar{\mathscr{A}}, \bar{R})$ witnessing it which means that:

- $(a) - (d)$ as above
- $(e)'$ $\bar{\mathscr{A}} = \langle \mathscr{A}_\alpha : \alpha < \lambda\rangle$ is a partition[2] of union $\mathscr{A}$ such that for each $u \in [\lambda]^{<\lambda}$ the set $\cup\{\mathscr{A}_\alpha : \alpha \in u\}$ is free (i.e. for $(\mathscr{A}, \bar{R})$).

3) We call $\mathbf{a} = (\mathscr{A}, \bar{R})$ a pre-witness for $[\mu,\lambda,\kappa]$ or $[\mu,\kappa)$ <u>when</u> it satisfies clauses (a),(b),(c) of part (1). For such $\mathbf{a}$ let $G_{\mathbf{a}}$ be the graph with set of notes $\mathscr{A}$ and set of edges $\{\{\eta,\nu\} : \eta R_\varepsilon \nu$ for some $\varepsilon < \kappa\}$.

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[1] exp stands for exceptional

[2] If λ is regular we can use $\langle \bigcup_{\alpha<\beta} \mathscr{A}_\alpha : \beta < \lambda\rangle$, so an increasing sequence of length λ with union $\mathscr{A}$ each set is free.

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Claim 1.2. *We have* $\mathrm{INC}_{\mathrm{chr}}(\mu,\lambda,\kappa)$ *or* $\mathrm{INC}_{\mathrm{chr}}[\mu,\lambda,\kappa]$, *see Definition 0.1(4)* *when*:

$\boxplus$ (a) $\mathrm{Inc}(\chi,\lambda,\kappa)$ *or* $\mathrm{Inc}[\chi,\lambda,\kappa]$ *respectively*

(b) $\chi \le \mu = \mu^\kappa$.

Proof. Fix $\mathbf{a} = (\mathscr{A},\bar{\mathscr{A}},\bar R)$ or $\mathbf{a} = (\mathscr{A},\bar R)$ witnessing $\mathrm{Inc}(\mu,\lambda,\kappa)$ or $\mathrm{Inc}[\mu,\lambda,\kappa]$ respectively. Now we define $\tau_{\mathscr{A}}$ as the vocabulary $\{P_\eta : \eta \in \mathscr{A}\} \cup \{F_\varepsilon : \varepsilon < \kappa\}$ where P_η is a unary predicate, F_ε a unary function (but it may be interpreted as a partial function).

We further let $K_{\mathbf{a}}$ be the class of structures M such that:

$\boxplus_1$ (a) $M = (|M|, F^M_\varepsilon, P^M_\eta)_{\varepsilon<\kappa,\eta\in\mathscr{A}}$

(b) $\langle P^M_\eta : \eta \in \mathscr{A}\rangle$ is a partition of $|M|$, so for $a \in M$ let $\eta[a]$ $= \eta^M_a$ be the unique $\eta \in \mathscr{A}$ such that $a \in P^M_\eta$

(c) if $a_\ell \in P^M_{\eta_\ell}$ for $\ell = 1,2$ and $F^M_\zeta(a_2) = a_1$ then $\eta_1 R_\zeta \eta_2$.

Let $K^*_{\mathbf{a}}$ be the class of M such that:

$\boxplus_2$ (a) $M \in K_{\mathbf{a}}$

(b) $\|M\| = \mu$

(c) if $\eta \in \mathscr{A}, u \subseteq \kappa$ and for $\zeta \in u$ we have $\nu_\zeta \in \mathscr{A}, \nu_\zeta R_\zeta \eta$ and $a_\zeta \in P^M_{\nu_\zeta}$ then for some $a \in P^M_\eta$ we have $\zeta \in u \Rightarrow F^M_\zeta(a) = a_\zeta$ and $\zeta \in \kappa\backslash u \Rightarrow F^M_\zeta(a)$ not defined.

Clearly

$\boxplus_3$ there is $M \in K^*_{\mathbf{a}}$.

[Why? Obvious as we are assuming $|\mathscr{A}| = \chi \le \mu = \mu^\kappa$.]

$\boxplus_4$ for $M \in K_{\mathbf{a}}$ let G_M be the graph with:

- set of nodes $|M|$
- set of edges $\{\{a, F^M_\varepsilon(a)\} : a \in |M|, \varepsilon < \kappa$ when $F^M_\varepsilon(a)$ is defined$\}$.

We shall show that the graph G_M is as required in Definition 0.1(1) or 0.1(4) (recalling κ^+ here stands for χ there, see 0.1(7)). Clearly G_M is a graph with μ nodes so recalling Definition 1.1(2) or 1.1(1) it suffices to prove $\boxplus_5$ and $\boxplus_7$ below.

$\boxplus_5$ if $\mathscr{B} \subseteq \mathscr{A}$ is free, and $M \in K_{\mathbf{a}}$ then $G_{M,\mathscr{B}} := G_M{\restriction}(\cup\{P^M_\eta : \eta \in \mathscr{B}\})$ has chromatic number $\le \kappa$.

[Why? Let the pair $(h,<_*)$ witness that $\mathscr{B}$ is free (for $\mathbf{a} = (\mathscr{A},\bar R)$, see $1.1(1)(*)_1$) so $h : \mathscr{B} \to \kappa$ and let $\mathscr{B}_\varepsilon = \{\eta \in \mathscr{B} : h(\eta) = \varepsilon\}$ for $\varepsilon < \kappa$.

Clearly

$\boxplus_{5.1}$ it suffices for each $\varepsilon < \kappa$ to prove that $G_{M,\mathscr{B}_\varepsilon}$ has chromatic number $\le \kappa$.

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Let $\langle \eta_\alpha : \alpha < \alpha(*)\rangle$ list $\mathscr{B}$ in $<_*$-increasing order. We define $\mathbf{c}_\varepsilon : G_{M,\mathscr{B}_\varepsilon} \to \kappa$ by defining a colouring $\mathbf{c}_{\varepsilon,\alpha} : G_{M,\{\eta_\beta:\beta<\alpha\}\cap\mathscr{B}_\varepsilon} \to \kappa$ by induction on $\alpha \le \alpha(*)$ such that $\mathbf{c}_{\varepsilon,\alpha}$ is increasing continuous with α. For $\alpha = 0$, let $\mathbf{c}_{\varepsilon,\alpha} = \emptyset$, and for α limit take union. If $\alpha = \beta + 1$ and $\eta_\beta \notin \mathscr{B}_\varepsilon$ then we let $\mathbf{c}_\alpha = \mathbf{c}_\beta$.

Lastly, assume $\alpha = \beta + 1, \eta_\beta \in \mathscr{B}_\varepsilon$ then note that the set $u_{\varepsilon,\beta} = \{\zeta < \kappa$: there is $\nu <_* \eta_\beta$ such that $\nu \in \mathscr{B}_\varepsilon$ and $\nu R_\zeta \eta\}$ has cardinality $< \kappa$ because the pair $(<_*, h)$ witness "$\mathscr{B}$ is free". Hence, recalling $M \in K_\mathbf{a}$, for each $a \in P^M_{\eta_\beta}$, the set $u_{\varepsilon,\beta,a} := \{\zeta < \kappa_\varepsilon : F^M_\zeta(a) \in \{P^M_\nu : \nu <_* \eta_\beta$ and $\nu \in \mathscr{B}_\varepsilon\}\}$ is $\subseteq u_{\varepsilon,\beta}$ hence has cardinality $\le |u_{\varepsilon,\beta}| < \kappa$. But by $(*)_1(\gamma)$ of 1.1 and the definition of $K_\mathbf{a}$, $A_a := \{b \in G_{M,\{\eta_\gamma:\gamma<\beta\cap\mathscr{B}_\varepsilon\}} : \{b,a\}$ is an edge of $G_M\}$ is $\subseteq \{F^M_\zeta(a) : \zeta \in u_{\varepsilon,\beta,a}\}$ hence the set A_a has cardinality $\le |u_{\varepsilon,\beta,a}| < \kappa$. So define $\mathbf{c}_{\varepsilon,\alpha}$ extending $\mathbf{c}_{\varepsilon,\beta}$ by, for $a \in P^M_{\eta_\beta}$ letting $\mathbf{c}_{\varepsilon,\alpha}(a) = \min(\kappa\backslash\{\mathbf{c}_{\varepsilon,\beta}(b) : b \in P^M_\nu$ for some $\nu <_* \eta_\beta$ from $\mathscr{B}_\varepsilon$ and $\{b,a\}$ is an edge of $G_M\})$. Recalling there is no edge $\subseteq P_{\eta_\beta}$ this is a colouring.

So we can carry the induction. So indeed $\boxplus_5$ holds.]

> $\boxplus_6$ if $\mathscr{B} \subseteq \mathscr{A}$ is free and $M \in K_\mathbf{a}$ then $G_{M,\mathscr{B}}$ is the union of $\le \kappa$ sets each with colouring number $\le \kappa$ hence also chromatic number $\le \kappa$.

[Why? By the proof of $\boxplus_5$.]

> $\boxplus_7$ $\mathrm{chr}(G_M) > \kappa$ if $M \in K^*_\mathbf{a}$.

Why? Toward contradiction assume $\mathbf{c} : G_M \to \kappa$ is a colouring and let $<_*$ be a well ordering of $\mathscr{A}$. For each $\eta \in \mathscr{A}$ and $\varepsilon, \zeta < \kappa$ let $\Lambda_{\eta,\varepsilon,\zeta} = \{\nu : \nu \in \mathscr{A}, \nu <_* \eta, \nu R_\zeta \eta$ and $\varepsilon \in \mathscr{H}_\nu\}$ where for $\nu \in \mathscr{A}$ we define $\mathscr{H}_\nu = \{\varepsilon$: for some $a \in P^M_\nu$ we have $\mathbf{c}(a) = \varepsilon\}$.

<u>Case 1</u>: There is $\eta \in \mathscr{A}$ such that $(\forall \varepsilon \in \mathscr{H}_\eta)(\exists^\kappa \zeta < \kappa)[\Lambda_{\eta,\varepsilon,\zeta} \neq \emptyset]$.

So we can find a one-to-one function $g : \mathscr{H}_\eta \to \kappa$ such that $\Lambda_{\eta,\varepsilon,g(\varepsilon)} \neq \emptyset$ for every $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$. For each $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$ choose $\nu_\varepsilon \in \Lambda_{\eta,\varepsilon,g(\varepsilon)}$; possible as $\Lambda_{\eta,\varepsilon,g(\varepsilon)} \neq \emptyset$ by the choice of the function g. By the definition of "$\nu_\varepsilon \in \Lambda_{\eta,\varepsilon,g(\varepsilon)}$" there is $a_\varepsilon \in P^M_{\nu_\varepsilon}$ such that $\mathbf{c}(\nu_\varepsilon) = \varepsilon$; recalling $\nu_\varepsilon \in \Lambda_{\eta,\varepsilon,\zeta}$ we have $\nu_\varepsilon R_\zeta \eta$ holds. So as $M \in K^*_\mathbf{a}$ there is $a \in P^M_\eta$ such that $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa \Rightarrow F^M_{g(\varepsilon)}(a) = a_\varepsilon$, but then $\{a, a_\varepsilon\} \in \mathrm{edge}(G_M)$ hence $\mathbf{c}(a) \neq \mathbf{c}(a_\varepsilon) = \varepsilon$ for every $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$, contradiction to the definition of $\mathscr{H}_\eta$.

<u>Case 2</u>: Not Case 1

So for every $\eta \in \mathscr{A}$ there is $\varepsilon \in \mathscr{H}_\eta \subseteq \kappa$ such that there are $< \kappa$ ordinals $\zeta < \kappa$ such that $\Lambda_{\eta,\varepsilon,\zeta} \neq \emptyset$. This means that there is $h : \mathscr{A} \to \kappa$ such that:

- $\bullet_1$ $\eta \in \mathscr{A} \Rightarrow h(\eta) \in \mathscr{H}_\eta$ and
- $\bullet_2$ $\eta \in \mathscr{A} \Rightarrow \kappa > |\{\zeta < \kappa : \Lambda_{\eta,h(\eta),\zeta} \neq \emptyset\}|$.

This implies that:

- $\bullet_3$ $\eta \in \mathscr{A} \Rightarrow \kappa > |\exp(\eta, h, \mathbf{a}, <_*)|$

because (we have $\bullet_2$ and):

- $\bullet_4$ if $\eta \in \mathscr{A}$ and $\varepsilon = h(\eta)$ <u>then</u> $\exp(\eta, h, <_*, \mathbf{a}) \subseteq \{\zeta < \kappa : \Lambda_{\eta,\varepsilon,\zeta} \neq \emptyset\}$.

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[Why? As $h : \mathscr{A} \to \kappa$ and if $\zeta \in \exp(\eta, h, <_*, \mathbf{a})$ let ν exemplify this, that is, $\nu <_* \eta, \nu R_\zeta \eta$ and $h(\nu) = h(\eta) = \varepsilon$ and recall $h(\nu) = \varepsilon$ implies $\varepsilon \in \mathscr{H}_\nu$ by $\bullet_1$. But this means that $\nu \in \Lambda_{\eta,\varepsilon,\zeta}$ hence $\Lambda_{\eta,\varepsilon,\eta} \neq \emptyset$ as required.]

As $<_*$ was any well ordering of $\mathscr{A}$, this means, see 1.1$(*)_2$ holds, that $\mathscr{A}$ is strongly free, contradiction to 1.1(d). $\square_{1.2}$

We can now reprove a result from [She13a].

Conclusion 1.3. *1) We have* $\mathrm{Inc}(\mu, \lambda, \kappa)$ *<u>when</u>*

$(*)$ *for some* $\mathscr{F}$ *and natural number* $\mathbf{k} > 0$ *we have*

- (a) $\mathscr{F} \subseteq {}^\kappa\mu$ *has cardinality* μ *and is tree like (i.e.* $f_1(\bar{d}) = f_2(j) \wedge \{f_1, f_2\} \subseteq \mathscr{F} \Rightarrow f_1{\restriction}i = f_2{\restriction}j$
- (b) $\mathscr{F}$ *is not free where*
  - $\mathscr{F}' \subseteq \mathscr{F}$ *is free means:*
  - *there is a sequence* $\langle \mathscr{F}'_i : i < \kappa \rangle$ *such that* $\mathscr{F}' = \cup\{\mathscr{F}'_i : i < \kappa\}$ *and for each* $i, \mathscr{F}'_i$ *has a transversal which means that* $\{\mathrm{Rang}(\eta) : \eta \in \mathscr{F}'_i\}$ *has a transversal* *(= one-to-one choice function)*
- (c) $\mathscr{F}$ *is the increasing union of* $\langle \mathscr{F}_\alpha : \alpha < \lambda \rangle$ *such that each* $\mathscr{F}_\alpha$ *is free.*

2) We have $\mathrm{Inc}[\mu, \lambda, \kappa]$ *<u>when</u>*

$(*)$ *as above but replacing clause (c) by:*

$(c)'$ *every* $\mathscr{F}' \subseteq \mathscr{F}$ *of cardinality* $< \lambda$ *has a transversal.*

Proof. 1), 2) We define $\mathbf{a}$ by choosing (for our $\mathscr{F}$):

- $\mathscr{A}_\mathbf{a} = \mathscr{F}$
- $<_\mathscr{A}$ any well ordering of $\mathscr{F}$; not part of $\mathbf{a}$
- R_ε is defined by: $f R_\varepsilon g$ iff $f <_\mathscr{A} g \wedge f(\varepsilon) = g(\varepsilon)$
- for part (1) let $\bar{\mathscr{A}}$ be a sequence witnessing clause (c).

So it suffices to prove $\mathrm{Inc}(\mu, \lambda, \kappa)$ or $\mathrm{Inc}[\mu, \lambda, \kappa]$; hence it suffices to prove that $\mathbf{a}$ witness it.

Now in Definition 1.1, clauses (a),(b),(c) are obvious. For clause (e), assume $\mathscr{F}_2 \subseteq \mathscr{F}$ is free in the sense of 1.3(1)(b), and we shall prove that $\mathscr{F}_2$ is $\mathbf{a}$-free, this suffices for clause (e). By the assumption on $\mathscr{F}_2$, clearly $\mathscr{F}_2$ is the union of $\langle \mathscr{F}_{2,\zeta} : \zeta < \kappa \rangle, \mathscr{F}_{2,\zeta}$ has a transversal $\mathbf{h}_\zeta$. Now we define $h : \mathscr{F}_2 \to \kappa$ by: $h(f) = \mathrm{pr}(\zeta, \varepsilon)$ where $\zeta = \min\{\xi : f \in \mathscr{F}_{2,\xi}\}$ and ε is minimal such that $\mathbf{h}_\zeta(\mathrm{Rang}(f)) = f(\varepsilon)$, now the pairs $(h, <_\mathscr{A} {\restriction}\mathscr{F}_2)$ witness that $\mathscr{F}_2$ is free (for $\mathbf{a}$).

For clause (d) toward contradiction assume that $h : \mathscr{F} \to \kappa$ and well ordering $<_*$ of $\mathscr{A}$ witness $\mathscr{F}$ is free for $\mathbf{a}$, hence $\bar{\mathscr{B}} = \langle \mathscr{B}_\varepsilon : \varepsilon < \kappa \rangle$ is a partition of $\mathscr{F}$ when we let $\mathscr{B}_\varepsilon = \{f \in \mathscr{F} : h(f) = \varepsilon\}$.

By Definition 1.1, for each $\varepsilon < \kappa$ and $f \in \mathscr{B}_\varepsilon$ the set $u_f = \{\zeta < \kappa$: for some $g \in \mathscr{B}_\varepsilon$ we have $g R_\zeta f\}$ has cardinality $< \kappa$ and let $\zeta_f \in \kappa \backslash u_f$. For $\varepsilon, \zeta < \kappa$ let $\mathscr{B}_{\varepsilon,\zeta} = \{f \in \mathscr{B}_\varepsilon : \zeta_f = \zeta\}$ so $\langle \mathscr{B}_{\varepsilon,\zeta} : \varepsilon, \zeta < \kappa \rangle$ is a partition of $\mathscr{A}$. Now for each $\varepsilon, \zeta < \kappa$, if $f \neq g \in \mathscr{B}_{\varepsilon,\zeta}$ <u>then</u> $f(\zeta) \neq g(\zeta)$. Why? By symmetry we can assume $g <_\mathscr{A} f$ now $\zeta = \zeta_f \in \kappa \backslash u_f$, so g cannot witness $\zeta \in u_f$. So $\langle \mathscr{B}_{\varepsilon,\zeta} : \varepsilon, \zeta < \kappa \rangle$ contradicts clause (b) of the claim's assumption. $\square_{1.3}$

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Claim 1.4. *If* $\mathrm{INC}[\mu,\lambda,\kappa]$ *or* $\mathrm{INC}(\mu,\lambda,\kappa)$ *then* $\mathrm{Inc}[\mu,\lambda,\kappa]$ *or* $\mathrm{Inc}(\mu,\lambda,\kappa)$ *respectively.*

Proof. As the two cases are similar we do the $\mathrm{INC}(\mu,\lambda,\kappa)$ case, so let $G, \langle G_i : i < \lambda\rangle$ witness it.

Let $<_*$ be a well ordering of the set of nodes of G. Define $\mathbf{a} = (\mathscr{A}, \bar{\mathscr{A}}, \bar{R})$ by:

- $\mathscr{A}$ is the set of nodes of G
- $\bar{\mathscr{A}} = \langle \mathscr{A}_i : i < \lambda\rangle$ with $\mathscr{A}_i$ the set of nodes of G_i
- $R_\varepsilon = \{(\nu,\eta) : \{\nu,\eta\}$ an edge of G and $\nu <_* \eta\}$.

Now check, noting when checking, that e.g. in $(*)_1$ of Definition 1.1, $\exp(\eta,\alpha,<_*)$ is equal to κ or to $\emptyset$ as $\bigwedge_\varepsilon R_\varepsilon = R_0$. $\square_{1.4}$

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