THE SPECTRUM OF ULTRAPRODUCTS OF FINITE CARDINALS FOR AN ULTRAFILTER SH1026

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Abstract. We complete the characterization of the possible spectrum of regular ultrafilters D on a set I, where the spectrum is the set of ultraproducts of (finite) cardinals modulo D which are infinite.

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§ 0. Introduction

§ 0(A). **Background, questions and results.**

Ultraproducts were very central in model theory in the sixties, usually for regular ultrafilters. The question of ultraproducts of infinite cardinals had been resolved (see [CK73]): letting D be a regular ultrafilter on a set I, (for transparency we ignore the case of a filter)

$(*)_1$ if $\bar\lambda = \langle \lambda_s : s \in I\rangle$ and $\lambda_s \geq \aleph_0$ for $s \in I$ <u>then</u> $\prod\limits_{s\in I} \lambda_s/D = \mu^{|I|}$ when $\mu = \limsup_D(\bar\lambda) := \sup\{\chi$: the cardinal χ satisfies $\{s \in I : \lambda_s \geq \chi\} \in D\}$.

What about the ultraproducts of finite cardinals? Of course, under naive interpretation, if $\{\lambda_s : \lambda_s = 0\} \neq \emptyset$ the result is zero, so for notational simplicity we always assume $s \in I \Rightarrow \lambda_s \geq 1$. Also for every $n \geq 1$, letting $\lambda_s = n$ for $s \in I$ we have $\prod\limits_s \lambda_s/D = n$ so the question was

Question 0.1. Given an infinite set I

- (a) [the singleton problem] what infinite cardinals μ belong to $\mathscr{C}_I = \mathscr{C}_I^{\rm car}$, i.e. can be represented as $\{\prod\limits_{s\in I} \lambda_s/D : D$ a regular ultrafilter on $I, 1 \leq \lambda_s < \aleph_0\}\backslash\{\lambda : 1 \leq \lambda < \aleph_0\}$
- (b) [the spectrum problem] moreover what are the possible spectra, i.e. which sets of cardinals belong to $\mathbf{C}_I$ which is the family of sets $\mathscr{C}$ such that for some D, a regular ultrafilter on I we have $\mathscr{C} = {\rm upf}(D)$ where ${\rm upf}(D) = \{\prod\limits_{s\in I} \lambda_s/D : 1 \leq \lambda_s < \aleph_0$ for $s \in I\}\backslash\{\lambda : 1 \leq \lambda < \aleph_0\}$

Keisler [Kei67] asks and has started on 0.1: (assuming GCH was prevalent at the time as the situation was opaque otherwise)

$(*)_2$ assume GCH, a sufficient condition for $\mathscr{C} \in \mathbf{C}_I$ is:
- (a) $\mathscr{C}$ is a set of successor (infinite) cardinals
- (b) $\max(\mathscr{C}) = |I|^+$
- (c) if $\mu = \sup\{\chi < \mu : \chi \in \mathscr{C}\}$ then $\mu^+ \in \mathscr{C}$
- (d) if $\mu^+ \in \mathscr{C}$ then $\mu \cap \mathscr{C}$ has cardinality $< \mu$.

Keisler used products and D-sums of ultrafilters. Concerning the problem for singletons a conjecture of Keisler [Kei67, bottom of pg.49] was resolved in [She70]:

$(*)_3$ $\mu = \mu^{\aleph_0}$ <u>when</u> $\mu \in \mathscr{C}_I$, i.e. when $\mu = \prod\limits_{s\in I} \lambda_s/D$ is infinite, D an ultrafilter on I, each λ_s finite non-zero

The proof uses coding enough "set theory" on the n's and using the model theory of the ultra-product. This gives a necessary condition (for the singleton version), but is it sufficient? This problem was settled in [She78, Ch.V,§3] = [She90, Ch.VI,§3] proving that this is also a sufficient condition (+ the obvious condition $\mu \leq 2^{|I|}$), that is

$(*)_4$ $\mu \in \mathscr{C}_I := \cup\{\mathscr{C} : \mathscr{C} \in \mathbf{C}_I\}$ <u>iff</u> $\mu = \mu^{\aleph_0} \leq 2^{|I|}$.

The constructions in [She78, Ch.VI,§3] = [She90, Ch.VI,§3], use a family $\mathscr{F}$ of functions with domain I and a filter D on I such that $\mathscr{F}$ is independent over D (earlier Kunen used such family $\mathscr{F} \subseteq {}^\lambda\lambda$ for constructing a good ultrafilter on λ in ZFC; eliminating the use of an instance of GCH in the proof of Keisler; earlier Engelking-Karlowicz proved the existence of such $\mathscr{F}$). In particular in the construction in [She78, Ch.VI,§3] of maximal such filters and the Boolean Algebra $\mathbb{B} = \mathscr{P}(\lambda)/D$ are central. We decrease the family and increase D during the construction; specifically we construct $\mathscr{F}_\ell(\ell \le n)$ decreasing with ℓ, D_ℓ a filter on I increasing with ℓ, D_ℓ a maximal filter such that $\mathscr{F}_\ell$ is independent mod D_ℓ; so if $\mathscr{F}_n = \emptyset$ then D_0 is an ultrafilter and we have $\mathbb{B}_\ell = \mathscr{P}(I)/D_\ell$ is essentially $\lessdot$-decreasing and in the ultrapowers $\mathbb{N}^I/D_\ell$ the part which $\mathbb{B}_\ell$ induces for $\ell \le n$, is a sequence of initial segments of $\mathbb{N}^{\mathbb{B}}/D_0$ decreasing with ℓ.

In [She78, Ch.VI,Exercise 3.35] = [She90, pg.370] this is formalized:

$(*)_5$ if D_0 is a filter on $I, \mathbb{B}_0 = \mathscr{P}(I)/D_0, D_1 \supseteq D_0$ an ultrafilter, $D = \{A/D_0 : A \in D_1\}$ so $D \in \mathrm{uf}(\mathbb{B}_0)$ then $\mathbb{N}^{\mathbb{B}_0}/D_0^+$ is an initial segment of $\mathbb{N}^I/D$; (also $\mathbb{B}$ satisfies the c.c.c., but this is just to ensure $\mathbb{B}$ is complete, anyhow this holds in all relevant cases here).

It follows that we can replace $\mathscr{P}(I)$ by a Boolean Algebra $\mathbb{B}_1$ extending $\mathbb{B}_0$. The Boolean Algebra related to $\mathscr{F}$ is the completion of the Boolean Algebra generated by $\{x_{f,a} : f \in \mathscr{F}, a \in \mathrm{Rang}(f)\}$ freely except $x_{f,a} \cap x_{f,b} = 0$ for $a \ne b \in \mathrm{Rang}(f)$ and $f \in \mathscr{F}$. So if $\mathrm{Rang}(f)$ is countable for every $f \in \mathscr{F}$, the Boolean Algebra satisfies the $\aleph_1$-c.c. (in fact, is free), this was used there to deal with $\mathrm{lcf}(\kappa, D)$ for $\kappa = \aleph_0$ (for $\kappa > \aleph_0$ we need $\mathrm{Rang}(f) = \kappa$) and is continued lately in works of Malliaris-Shelah. But for $\mathrm{upf}(D)$ only the case of f's with countable range is used.

The problem of the spectrum (i.e. 0.1(b)) was not needed in [She78, Ch.VI,§3] for the model theoretic problems which were the aim of [She78, Ch.VI], still the case of finite spectrum was resolved there (also cofinality, i.e. $\mathrm{lcf}(\kappa, D)$ was addressed).

This was continued by Koppelberg [Kop80] using a possibly infinite $\lessdot$-increasing chains of complete Boolean Algebras; also she uses a system of projections instead of maximal filters but this is a reformulation as this is equivalent, see 0.11 below.

Koppelberg [Kop80] returns to the full spectrum problem proving:

$(*)_6$ $\mathscr{C} \in \mathbf{C}_I$ <u>when</u> $\mathscr{C}$ satisfies:

- (a) $\mathscr{C} \subseteq \mathrm{Card}$
- (b) $\max(\mathscr{C}) = 2^{|I|}$
- (c) $\mu = \mu^{\aleph_0}$ if $\mu \in \mathscr{C}$
- (d) if $\mu_n \in \mathscr{C}$ for $n < \omega$ then $\prod\limits_n \mu_n \in \mathscr{C}$.

Central in the proof is $(*)_5$ above ([She78, Ch.VI,Ex3.35,pg.370]). The result of Koppelberg is very strong, still the full characterization is not obtained; also Kanamori in his math review of her work asked about it.

Here we give a complete answer to the spectrum problem 0.1(b), that is, Theorem 1.20 gives a full ZFC answer to 0.1, that is.

Theorem 0.2. *For any infinite set $I, \mathscr{C} \in \mathbf{C}_I$ <u>iff</u> $\mathscr{C}$ is a set of cardinals such that $\mu \in \mathscr{C} \Rightarrow \mu = \mu^{\aleph_0} \le 2^{|I|}$ and $2^{|I|} \in \mathscr{C}$.*

We now comment on some further questions on ultra-powers.

The problem of cofinalities was central in [She78, Ch.VI,§3] in particular, $\mathrm{lcf}(\aleph_0, \lambda))$ (see 0.6 below). [Why? E.g. if $\mathrm{Th}(M)$, the complete first order theory of the model M is unstable then M^I/D is not $\mathrm{lcf}(\aleph_0, \lambda)^+$-saturated.] Another question was raised by the author [She72, pg.97] and independently by Eklof [Ekl73]:

Question 0.3. Assume $f_n \in {}^I\mathbb{N}, f_{n+1} <_D f_n$ and $\mu \leq \prod_{s \in I} f_n(s)/D$ for every n <u>then</u> is there $f \in {}^I\mathbb{N}$ such that $f <_D f_n$ for every n and $\mu \leq \prod_{s \in I} f(s)/D$?

The point in [She72, pg.75] was investigating saturation of ultrapowers (and ultraproducts) and Keisler order on first order theories. The point in [Ekl73] was ultraproduct of Abelian groups.

To explain the cofinalities problem, see 0.4. We can consider the following: for D a regular ultrafilter on I we consider $M = \mathbb{N}^\lambda/D$; for $a \in M$ let $\lambda_a = |\{b : b <_M a\}|$ and define $E_M = \{(a, b) : a, b \in M \text{ and } \lambda_a = \lambda_b \geq \aleph_0\}$. So E_M is a convex equivalence relation, and the equivalence classes are naturally linearly ordered and let $A_{D,\lambda} = \{a \in M : \lambda_a = \lambda\}$. So $\mathrm{upf}(D) = \{\lambda_a : A_{D,\lambda} \neq \emptyset\}$ and Question 0.3 asks: can the co-initiality of some $A_{D,\lambda}$ be $\aleph_0$. As M is $\aleph_1$-saturated, in this case the cofinality of $M{\restriction}\{c : \lambda_c < \lambda_a \text{ (hence } c <_M a)\}$ is $\mathrm{lcf}(\aleph_0, D)$ which is the co-initiality of $A_{D,\min(\mathrm{upf}(D))}$.

So a natural question is

Question 0.4. What are the possible $\mathrm{spec}_1(D) = \{(\lambda, \theta, \partial) : \lambda \in \mathrm{upf}(D), \partial$ the cofinality of $A_{D,\lambda}$ and θ the co-initiality of $A_{D,\lambda}\}$ for D a regular ultrafilter on I?

A further question is:

Question 0.5. Assume $\kappa = \mathrm{cf}(\kappa) < \lambda_1 = \lambda_1^{\aleph_0} < \lambda_2 = \lambda_2^{\aleph_0}, \lambda_1^{<\kappa>_{\mathrm{tr}}} \leq 2^\lambda$; see 0.7(4). Is there a regular ultrafilter D on λ such that for $n_i \in \mathbb{N}$ for $i < \lambda$ we have $\prod_i n_i/D = \lambda_1$ and $\prod_i 2^{n_i}/D = \lambda_2$?

This work was presented in the May 2013 Eilat Conference honoring Mati Rubin's retirement. In a work in preparation [S+], we try to build a counterexample to question 0.3.

§ 0(B). **Preliminaries.**

We define $\mathrm{lcf}(\kappa, D)$ and $M^{\mathbb{B}}/D$, when $\mathbb{B}$ is a Boolean Algebra and more.

Definition 0.6. For D an ultrafilter on I, κ a regular cardinal let $\mu = \mathrm{lcf}(\kappa, D)$ be the co-initiality of the linear order $(\kappa^I/D){\restriction}\{f/D : f \in {}^I\kappa$ is not D-bounded by any $\varepsilon < \kappa\}$.

Notation 0.7. 1) $\mathbb{B}$ denotes a Boolean Algebra, usually complete; let $\mathrm{comp}(\mathbb{B})$ be the completion of $\mathbb{B}$.
2) $\mathrm{uf}(\mathbb{B})$ is the set of ultrafilters on $\mathbb{B}$.
3) Let $\mathbb{B}^+ = \mathbb{B} \backslash \{0_{\mathbb{B}}\}$.
4) Let $\mathrm{cc}(\mathbb{B}) = \min\{\kappa : \mathbb{B}$ satisfies the κ-c.c.$\}$, necessarily a regular cardinal.
5) For $\lambda \geq \kappa = \mathrm{cf}(\kappa)$ let $\lambda^{\langle\kappa\rangle} = \mathrm{trp}_\kappa(\lambda) = \sup\{|\lim_\kappa(\mathscr{T})| : \mathscr{T} \subseteq {}^{\kappa>}\lambda$ is a subtree of cardinality $\leq \lambda\}$ where $\lim_\kappa(\mathscr{T}) = \{\eta \in {}^\kappa\mathrm{Ord} : \eta{\restriction}i \in \mathscr{T}$ for every $i < \kappa\}$.
6) For a Boolean algebra $\mathbb{B}$ let $\mathrm{comp}(\mathbb{B})$ be its completion.

Definition 0.8. For a Boolean Algebra $\mathbb{B}$ and a model or a set M.
1) Let $M^{\mathbb{B}}$ be the set of partial functions f from $\mathbb{B}^+$ into M such that for some maximal antichain $\langle a_i : i < i(*)\rangle$ of $\mathbb{B}, \mathrm{Dom}(f)$ includes $\{a_i : i < i(*)\}$ and is included in[1] $\{a \in \mathbb{B}^+ : (\exists i)(a \le a_i)\}$ and $f\restriction\{a \in \mathrm{Dom}(f) : a \le a_i\}$ is constant for each i.
1A) Naturally for $f_1, f_2 \in M^{\mathbb{B}}$ we say f_1, f_2 are D-equivalent, or $f_1 = f_2 \mod D$ <u>when</u> for some $b \in D$ we have $a_1 \in \mathrm{Dom}(f_1) \wedge a_2 \in \mathrm{Dom}(f_2) \wedge a_1 \cap a_2 \cap b > 0_{\mathbb{B}} \Rightarrow f_1(a_1) = f_2(a_2)$.
1B) Abusing notation, not only $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$ but $M^{\mathbb{B}_1}/D_1 \subseteq M^{\mathbb{B}_2}/D_2$ when $\mathbb{B}_1 \lessdot \mathbb{B}_2, D_\ell \in \mathrm{uf}(\mathbb{B}_\ell)$ for $\ell = 1, 2$ and $D_1 \subseteq D_2$, that is, for $f \in M^{\mathbb{B}_1}$ we identify f/D_1 and f/D_2.
2) For D an ultrafilter on the completion of the Boolean Algebra $\mathbb{B}$ we define $M^{\mathbb{B}}/D$ naturally, as well as $\mathrm{TV}(\varphi(f_0, \ldots, f_{n-1})) \in \mathrm{comp}(\mathbb{B})$ <u>when</u> $\varphi(x_0, \ldots, x_{n-1}) \in \mathbb{L}(\tau_M)$ and $f_0, \ldots, f_{n-1} \in M^{\mathbb{B}}$ where TV stands for truth value and $M^{\mathbb{B}}/D \models \varphi[f_0/D, \ldots, f_{n-1}/D]$ iff $\mathrm{TV}_M(\varphi(f_0, \ldots, f_{n-1})) \in D$.
3) We say $\langle a_n : n < \omega\rangle$ D-represents $f \in \mathbb{N}^{\mathbb{B}}$ <u>when</u> $\langle a_n : n < \omega\rangle$ is a maximal antichain of $\mathbb{B}$ (allowing $a_n = 0_{\mathbb{B}}$) and for some $f' \in \mathbb{N}^{\mathbb{B}}$ which is D-equivalent to f (see 0.8(1A)) we have $f'(a_n) = n$. We may omit D if $D = \{1_{\mathbb{B}}\}$ and say just $\langle a_n : n < \omega\rangle$ represents f.
4) We say $\langle (a_n, k_n) : n < \omega\rangle$ represents $f \in \mathbb{N}^{\mathbb{B}}$ <u>when</u>:

- (a) the k_n are natural numbers with no repetition
- (b) $\langle a_n : n < \omega\rangle$ is a maximal antichain of $\mathbb{B}$
- (c) $f(a_n) = k_n$.

The proofs in [She78, Ch.VI,§3] use downward induction on the cardinals.

Observation 0.9. *1) If $\mathbb{B}$ is a complete Boolean Algebra and $f \in \mathbb{N}^{\mathbb{B}}$ <u>then</u> some sequence $\langle a_n : n < \omega\rangle$ represents f.*
1A) If $\mathbb{B}$ is a c.c.c. Boolean Algebra and $f \in \mathbb{N}^{\mathbb{B}}$, <u>then</u> some sequence $\langle a_n, k_n : n < \omega\rangle$ represents f.
2) For a model M and Boolean Algebra $\mathbb{B}_1$ and ultrafilter D on its completion $\mathbb{B}_2$ we have $M^{\mathbb{B}_1}/D = M^{\mathbb{B}_2}/D$.

Fact 0.10. 1) If $\mathbb{B}_1 \subseteq \mathbb{B}_2$ are Boolean Algebras, $\mathbb{B}$ is a complete Boolean Algebra and π_1 is a homomorphism from $\mathbb{B}_1$ into $\mathbb{B}$ <u>then</u> there is a homomorphism π_2 from $\mathbb{B}_2$ into $\mathbb{B}$ extending π_1.
2) There is a homomorphism π_3 from $\mathbb{B}_3$ into $\mathbb{B}$ extending π_ℓ for $\ell = 0, 1, 2$ <u>when</u>:

- (a) $\mathbb{B}_0 \subseteq \mathbb{B}_\iota \subseteq \mathbb{B}_3$ are Boolean Algebras for $\iota = 1, 2$
- (b) $\mathbb{B}_1, \mathbb{B}_2$ are freely amalgamated over $\mathbb{B}_0$ inside $\mathbb{B}_3$
- (c) $\mathbb{B}$ is a complete Boolean Algebra
- (d) π_ℓ is a homomorphism from $\mathbb{B}_\ell$ into $\mathbb{B}$ for $\ell = 0, 1, 2$
- (e) $\pi_0 \subseteq \pi_1$ and $\pi_0 \subseteq \pi_2$.

Proof. 1) Well known.
2) Straightforward. $\square_{0.10}$

[1] for the $D_\ell \in \mathrm{uf}(\mathbb{B}_\ell)$ ultra-product, without loss of generality $\mathbb{B}$ is complete, then without loss of generality $f\restriction\{a_i : i < i(*)\}$ is one to one.

Observation 0.11. *Assume $\mathbb{B}_1 \lessdot \mathbb{B}_2$ are Boolean Algebras and $\mathbb{B}_1$ is complete.*
1) The following properties of D are equivalent:

- (a) *D is a maximal filter on $\mathbb{B}_2$ (among those) disjoint to $\mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$*
- (b) *there is a projection π of $\mathbb{B}_2$ onto $\mathbb{B}_1$ such that $D = \{a \in \mathbb{B}_2 : \pi(a) = 1_{\mathbb{B}_1}\}$.*

1A) Moreover D determines π uniquely and vice versa, in particular $\pi(c)$ is the unique $c' \in \mathbb{B}_1$ such that $c = c' \mod D$.
2) If D satisfies (1)(a) and D_1 is an ultrafilter of $\mathbb{B}_1$, then there is a one and only one ultrafilter $D_2 \in \mathrm{uf}(\mathbb{B}_2)$ extending $D_1 \cup D$.

Proof. 1) Clause (a) implies clause (b):

As D is a filter on $\mathbb{B}_2$ clearly for some Boolean Algebra $\mathbb{B}'_2$, there is a homomorphism $\mathbf{j}_0 : \mathbb{B}_2 \to \mathbb{B}'_2$ which is onto, such that $a \in \mathbb{B}_2 \Rightarrow (a \in D \leftrightarrow \mathbf{j}_0(a) = 1_{\mathbb{B}'_2})$. As $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ necessarily $\mathbf{j}_0 \restriction \mathbb{B}_1$ is one-to-one. Let $\mathbb{B}'_1 = \mathbf{j}_0(\mathbb{B}_1)$ so $\mathbf{j}_1 := (\mathbf{j}_0 \restriction \mathbb{B}_1)^{-1}$ is an isomorphism from $\mathbb{B}'_1$ onto $\mathbb{B}_1$ hence by 0.10(1) and the assumption that $\mathbb{B}_1$ is complete there is a homomorphism $\mathbf{j}_2$ from $\mathbb{B}'_2$ onto $\mathbb{B}_1$ extending $\mathbf{j}_1$. Hence $\mathbf{j}_3 = \mathbf{j}_2 \circ \mathbf{j}_0$ is a homomorphism from $\mathbb{B}_2$ onto $\mathbb{B}_1$ extending $\mathrm{id}_{\mathbb{B}_1}$, so it is a projection.

Lastly, $\mathbf{j}_3^{-1}\{1_{\mathbb{B}_1}\}$ is a filter extending D and disjoint to $\mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$. By the maximality of D we have equality.

An alternative proof is:

Let $\mathbb{B}'_2$ be the sub-algebra of $\mathbb{B}_2$ generated by $\mathbb{B}_1 \cup D$. Clearly every member of $\mathbb{B}'_2$ can be represented as $(a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$ with $a, a_m \in D$ for $m < n = \ell g(\bar{a})$ and $b \in \mathbb{B}_1, b_k \in \mathbb{B}_1$ for $k < \ell g(\bar{b}), \sigma$ a Boolean term such that $\bigwedge\limits_{k<n} a \le a_k$, equivalently $\bigwedge\limits_{k<n} a \cap (1 - a_k) = 0$. We try to define a function π from $\mathbb{B}'_2$ into $\mathbb{B}_1$ by:

$\oplus$ $\pi((a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))) = b$ for $a, \bar{a}, b, \bar{b}$ as above.

We have to prove that π is as promised.

$(*)_1$ π is a well defined (function from $\mathbb{B}'_2$ into $\mathbb{B}_1$).

Why? Obviously for every $c \in \mathbb{B}'_2$ there are $a, \bar{a}, b, \bar{b}, \sigma$ as above, so $\pi(c)$ has at least one definition, still we have to prove that any two such definitions agree. So assume $c = (a_\ell \cap b_\ell) \cup ((1-a_\ell) \cap \sigma_\ell(\bar{a}_\ell, \bar{b}_\ell))$ for $\ell = 1, 2$ as above so with $a_1, a_2, a_{1,k}, a_{2,m} \in D$ and $b_1, b_2, \bar{b}_1, \bar{b}_2 \in \mathbb{B}_1$ such that $a_1 \le a_{1,k}$ for every $k < \ell g(\bar{a}_1), a_2 \le a_{2,m}$ for every $m < \ell g(\bar{a}_2)$. We should prove that $b_1 = b_2$, if not without loss of generality $b_1 \nleq b_2$ hence $b := b_1 - b_2 > 0$. Clearly $a := a_1 \cap a_2 \in D$ and computing $c \cap b \cap a$ in two ways we get $a \cap b \cap b_1 = a \cap b \cap b_2$ hence $a \cap b = a \cap b \cap b_1 = a \cap b \cap b_2 = a \cap 0 = 0$ recalling $b = b_1 - b_2$, hence $a \le 1 - b$ so as $a \in D$ necessarily $1 - b \in D$. But $b \in \mathbb{B}_1^+$ so $1 - b \in \mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$, contradiction to the assumption on D.

$(*)_2$ π commutes with "$x \cap y$".

Why? Assume that for $\ell = 1, 2$ we have $c_\ell = (a_\ell \cap b_\ell) \cup ((1-a_\ell) \cap \sigma_\ell(\bar{a}_\ell, \bar{b}_\ell))$ with $a_\ell, b_\ell, \bar{a}_\ell, \bar{b}_\ell, \sigma_\ell$ as above.

So $\pi(c_\ell) = b_\ell$ and letting $a = a_1 \cap a_2 \in D$ we have $c := c_1 \cap c_2 = (a \cap (b_1 \cap b_2)) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$ where $\bar{a} = \bar{a}_1 {}^\frown \langle a_1 \rangle {}^\frown \bar{a}_2 {}^\frown \langle a_2 \rangle, \bar{b} = \bar{b}_1 {}^\frown \langle b_1 \rangle {}^\frown \bar{b}_2 {}^\frown \langle b_2 \rangle$ for some suitable term σ.

As $a \in D$, clearly $\pi(c) = b_1 \cap b_2 = \pi(c_1) \cap \pi(c_2)$, as required.

$(*)_3$ π commutes with "$1 - x$".

Why? Let $c = (a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$ hence $1-c = (a \cap (1-b)) \cup ((1-a) \cap (1-\sigma(\bar{a}, \bar{b}))$ hence $\pi(1-c) = 1-b = 1-\pi(c)$ so we are done.

$(*)_4$ π is a projection onto $\mathbb{B}_1$.

[Why? By $(*)_1, (*)_2, (*)_3$ clearly π is a homomorphism from $\mathbb{B}_2'$ into $\mathbb{B}_1$. So its range is $\subseteq \mathbb{B}_1$ and if $c \in \mathbb{B}_1$ let $b = c, a = 1_{\mathbb{B}_1}, \bar{a} = \langle\rangle = \bar{b}$ and $\sigma(\bar{a}, \bar{b}) = 0_{\mathbb{B}_1}$ so $c = (a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b})$ and $a, b, \bar{a}, \bar{b}, \sigma$ are as required so $\pi((a \cap b) \cap ((1-a) \cap \sigma(\bar{a}, \bar{b})) = b$ which means $\pi(c) = b = c$.]

Now we can finish: as $\mathbb{B}_1 \subseteq \mathbb{B}_2' \subseteq \mathbb{B}_2$ and π is a homomorphism from $\mathbb{B}_2'$ into $\mathbb{B}_1$ which is a complete Boolean Algebra, we can extend π to π^+, a homomorphism from $\mathbb{B}_2$ into $\mathbb{B}_1$, see 0.10. But π is a projection hence so is π^+. Clearly $(\pi^+)^{-1}\{1_{\mathbb{B}_1}\}$ includes D and equality holds by the assumption on the maximality of D and we have proved the implication.

<u>Clause (b) implies clause (a):</u>

First, clearly D is a filter of $\mathbb{B}_2$; also $a \in \mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\} \Rightarrow \pi(a) = a \neq 1_{\mathbb{B}_1} \Rightarrow a \notin D$.

Toward contradiction assume D_2 is a filter on $\mathbb{B}_2, D \subsetneqq D_2$ and $D_2 \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$. Choose $c_2 \in D_2 \backslash D$ and let $c_1 = \pi(c_2)$, consider the symmetric difference, $c_1 \Delta c_2$ it is mapped by π to $c_1 \Delta c_1 = 0_{\mathbb{B}_2}$ hence $\pi(1_{\mathbb{B}_2} - (c_1 \Delta c_2)) = 1_{\mathbb{B}_2} - \pi(c_1 \Delta c_2) = 1_{\mathbb{B}_2} - 0_{\mathbb{B}_2} = 1_{\mathbb{B}_2}$, so $1_{\mathbb{B}_2} - (c_1 \Delta c_2) \in D$ so $c_1 = c_2 \mod D$, hence (recalling $D_1 \subseteq D_2$) we have $c_1 = c_2 \mod D_2$ but $c_2 \in D_2$ hence $c_1 \in D_2$. But

- $c_1 \in \mathbb{B}_1$ being $\pi(c_2)$
- $c_1 \neq 1_{\mathbb{B}_1}$ as $\pi(c_2) = c_1$ and $c_2 \notin D$

and recall

- $c_1 \in D_2$

so c_1 contradicts $D_2 \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$. We comment that for this direction we do not use the completeness of $\mathbb{B}_1$.

1A) Now π determines D in the statement (b). Also D determines π because if π_1, π_2 are projections from $\mathbb{B}_2$ onto $\mathbb{B}_1$ such that $D = \{a \in \mathbb{B}_2 : \pi_\ell(a) = 1_{\mathbb{B}_1}\}$ for $\ell = 1, 2$ and $\pi_1 \neq \pi_2$ let $a \in \mathbb{B}_2$ be such that $\pi_1(a) \neq \pi_2(a)$; then as in $(b) \Rightarrow (a)$ in the proof of part (1), $\pi_\ell(a) = a \mod D$ for $\ell = 1, 2$ hence $\pi_1(a) = \pi_2(a) \mod D$, but $\pi_1(a), \pi_2(a) \in \mathbb{B}_1$ and $D_2 \cap \mathbb{B}_1 = 1_{\mathbb{B}_1}$ and $D \cap \mathbb{B}_1 = 1_{\mathbb{B}_1}$ hence $\pi_1(a) = \pi_2(a)$, contradiction.

2) Straightforward, but we elaborate; clearly $a \in D_1 \wedge b \in D \Rightarrow \pi(a \cap b) = \pi(a) \cap \pi(b) = a \cap 0_{\mathbb{B}_1} = a \geq 0_{\mathbb{B}_1}$ hence $a \in D_1 \wedge b \in D \Rightarrow a \cap b > 0_{\mathbb{B}_2}$, hence least one such $D_2 \in \text{uf}(\mathbb{B}_2)$. For uniqueness toward contradiction assume $\mathscr{E}_1, \mathscr{E}_2$ are from $\text{uf}(\mathbb{B}_2)$ and extend $D_1 \cup D$. So necessarily there is $a \in \mathscr{E}_1 \backslash \mathscr{E}_2$ so as above $a = \pi(a) \mod D$ but $D \subseteq \mathscr{E}_1 \cap \mathscr{E}_2$ hence for $\ell = 1, 2$ we have $a = \pi(a) \mod \mathscr{E}_\ell$ so $a \in \mathscr{E}_\ell \Leftrightarrow \pi(a) \in \mathscr{E}_\ell$. But $\mathscr{E}_1 \cap \mathbb{B}_1 = D_1 = \mathscr{E}_2 \cap \mathbb{B}_1$ and $\pi(a) \in \mathbb{B}_1$ has $\pi(a) \in \mathscr{E}_1 \Leftrightarrow \pi(a) \in \mathscr{E}_2$. By the last two sentences $a \in \mathscr{E}_1 \Leftrightarrow a \in \mathscr{E}_2$ contradicting the choice of a. $\square_{0.11}$

Fact 0.12. Assume $\mathbb{B}_1 \lessdot \mathbb{B}_2$ are complete Boolean Algebras, $D_\ell \in \text{uf}(\mathbb{B}_\ell)$ for $\ell = 1, 2$. If D is a maximal filter on $\mathbb{B}_2$ disjoint to $\mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$ and $D \cup D_1 \subseteq D_2$ <u>then</u> $\mathbb{N}^{\mathbb{B}_1}/D_1$ is an initial segment of $\mathbb{N}^{\mathbb{B}_2}/D_2$.

Remark 0.13. 1) This is [She78, Ch.VI,Ex3.35].
2) We can prove: if the homomorphism $\mathbf{j} : \mathbb{B}_2 \to_{\rm onto} \mathbb{B}_1$ maps $D_2 \in {\rm uf}(\mathbb{B}_2)$ onto $D_1 \in {\rm uf}(\mathbb{B}_1)$ then $\mathbb{N}^{\mathbb{B}_1}/D_1$ is canonically isomorphic to an initial segment of $\mathbb{N}^{\mathbb{B}_2}/D_2$ as in 0.12.

Proof. The desired conclusion will follow by $(*)_3$ below:

$(*)_1$ If $\mathscr{I}$ is a maximal antichain of $\mathbb{B}_1$ <u>then</u> $\{a/D : a \in \mathscr{I}\}$ is a maximal antichain of $\mathbb{B}_2/D$.

[Why? First,

- $a \in \mathscr{I} \Rightarrow a \in \mathbb{B}_1^+ \Rightarrow a/D \in (\mathbb{B}_2/D)^+$
- if $a \neq b \in \mathscr{I}$ then $\mathbb{B}_2 \models$ "$a \cap b = 0_{\mathbb{B}_1}$" hence $\mathbb{B}_2/D \models$ "$(a/D) \cap (b \cap D) = 0_{\mathbb{B}_2/D}$".

Hence, obviously $\mathscr{I}^* := \{a/D : a \in \mathscr{I}\}$ is an antichain of $\mathbb{B}_2/D$. Toward contradiction assume $\mathscr{I}^*$ is not maximal and let c/D witness it. By 0.11 there is $c' \in \mathbb{B}_1$ such that $c = c' \mod D$ and so without loss of generality $c \in \mathbb{B}_1$.

As $c/D \neq 0/D$ necessarily $c \in \mathbb{B}_1^+$ and if $b \in \mathscr{I}$ then $(b/D) \cap (c/D) = 0/D$ hence $b \cap c = 0 \mod D$ but $b, c \in \mathbb{B}_1$ hence $b \cap c = 0$, so c contradicts "$\mathscr{I}$ is a maximal antichain of $\mathbb{B}_1$".]

$(*)_2$ If $f \in \mathbb{N}^{\mathbb{B}_2}, c \in \mathbb{B}_1 \backslash D_1$ and ${\rm TV}(f > n) \cup c \in D$ for every n <u>then</u> $g \in \mathbb{N}^{\mathbb{B}_1} \Rightarrow g/D_2 < f/D_2$.

[Why? If g is a counter-example, then ${\rm TV}(f \le g)$ belongs to D_2 but $1 - c \in D_1 \subseteq D_2$ so ${\rm TV}(f \le g) - c$ belongs to D_2 hence to $D^+ := \{a \in \mathbb{B}_2 : 1 - a \notin D\}$ since $D \subseteq D_2$. Let $\langle b_n : n < \omega\rangle$ represent g as a member of $\mathbb{N}^{\mathbb{B}_1}$, then by $(*)_1$, $\langle b_n/D : n < \omega\rangle$ is a maximal anti-chain of $\mathbb{B}_2/D$ hence for some $n, {\rm TV}(f \le g) \cap b_n - c \in D^+$ but ${\rm TV}(f \le n) - c \ge {\rm TV}(f \le g) \cap b_n - c$ hence ${\rm TV}(f \le n) - c \in D^+$, contradiction to an assumption of $(*)_2$; so $(*)_2$ holds indeed.]

$(*)_3$ If $f \in \mathbb{N}^{\mathbb{B}_2}, g \in \mathbb{N}^{\mathbb{B}_1}$ and $f/D_2 \le g/D_2$ then for some $g' \in \mathbb{N}^{\mathbb{B}_1}/D$ we have $f/D_2 = g'/D_2$.

[Why? Let $\langle a_n : n < \omega\rangle$ represent f and let $a_{\ge n} = \bigcup\limits_{k \ge n} a_k \in \mathbb{B}_2$. If for some $b \in D_2$, we have $n < \omega \Rightarrow a_{\ge n} \cup (1 - b) \in D$ then there is $f' \in \mathbb{N}^{\mathbb{B}_2}$ such that $f'/D_2 = f/D_2$ and $n < \omega \Rightarrow {\rm TV}(f' \ge n) \in D$. Now we apply $(*)_2$ with $f', 0_{\mathbb{B}_1}$ here standing for f, c there and we get contradiction to "$f/D_2 \le g/D_2$". So we can assume there is no such b.

Let $a'_n \in \mathbb{B}_1$ be such that $a_n = a'_n \mod D$ so possibly $a'_n = 0_{\mathbb{B}_1}$, such a'_n exists by 0.11(1A). Clearly $\langle a'_n/D : n < \omega\rangle$ is an antichain of $\mathbb{B}_2/D$, so as $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ clearly $\langle a'_n : n < \omega\rangle$ is an antichain of $\mathbb{B}_1$.

<u>Case 1</u>: $c := \bigcup\limits_n a'_n \notin D_1$.

First, we argue that for every $n \in \omega, {\rm TV}(f > n) \cup c \in D$ as otherwise there is some $n \in \omega$ such that ${\rm TV}(f \le n) - c \in D^+$ hence for some $\ell \le n, a_\ell - c \in D^+$ hence (by the choice of a'_ℓ) we have $a'_\ell - c \in D^+$, contradiction to the choice of c. As $c \in \mathbb{B}_1$ we see by the assumption of this case that $c \in \mathbb{B}_1 \backslash D_1$, hence by $(*)_2$ we get a contradiction to the assumption "$f/D_2 \le g/D_2$" of $(*)_3$.

Case 2: $c := \bigcup_n a'_n \in D_1$ and $d = \bigcup_n (a_n \triangle a'_n) \notin D_2$.

As D_2 is an ultrafilter of $\mathbb{B}_2$ extending D_1, clearly $c' := c - d \in D_2$. We define $g' \in \mathbb{N}^{\mathbb{B}_1}$ as the function represented by $\langle a'_n : n < \omega \rangle$ and $g'' \in \mathbb{N}^{\mathbb{B}_2}$ as the function represented by $\langle a''_n : n < \omega \rangle$, where a''_n is $a'_n \cap c'$ if $n > 0$ and $a'_n \cup (1 - c')$ if $n = 0$. Easily $f/D_2 = g''/D_2$ because f, g'' "agree" on c' which belongs to D_2 and is disjoint to d and the choice of d; also $g''/D_2 = g'/D_2$ because $c' \in D_2$. Together we are done.

Case 3: $c := \bigcup_n a'_n \in D_1$ and $d = \bigcup_n (a_n \triangle a'_n) \in D_2$.

Let $d' \in \mathbb{B}_1$ be such that $d'/D = d/D$. Let $d_1 := \bigcup_n (a_n - a'_n)$ and $d_2 := \bigcup_n (a'_n - a_n)$ hence $d = d_1 \cup d_2$. Let $k < \omega$, now modulo D we have $d' \cap \bigcup_{n \le k} a'_n = d \cap \bigcup_{n \le k} a'_n = \bigcup_{\ell=1}^{2} (d_\ell \cap \bigcup_{n \le k} a'_n)$ and we shall deal separately with each term.

First, $d_1 \cap \bigcup_{n \le k} a'_n = \bigcup_{\ell \le k} ((a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n) \cup (\bigcup_{\ell > k} (a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n)$. Now the first term $\bigcup_{\ell \le k} ((a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n)$ is equal $\mod D$ to $(\bigcup_{n \le k} 0) \cap \bigcup_{n \le k} a'_k = 0_{\mathbb{B}_1}$, by the choice of the a'_ℓ. Next, the second term in the union, $(\bigcup_{\ell > k} (a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n)$ is modulo D again by the choice of the a'_ℓ, equal to $(\bigcup_{\ell > k} (a_\ell - a'_\ell)) \cap \bigcup_{n \le k} a_n$ which is zero as $\langle a_n : n < \omega \rangle$ is an antichain; together by the previous sentences $d_1 \cap \bigcup_{n \le k} a'_n = 0_{\mathbb{B}_2}$ mod D.

Similarly $d_2 \cap \bigcup_{n \le k} a'_n = 0_{\mathbb{B}_2} \mod D$ noting that $\langle a'_n : n < \omega \rangle$ is necessarily an antichain of $\mathbb{B}_1$. Hence $d' \cap \bigcup_{n \le k} a'_n = d \cap \bigcup_{n \le k} a'_n = \bigcup_{\ell=1}^{2} (d_\ell \cap \bigcup_{n \le k} a'_n)) = 0_{\mathbb{B}_2} \cup 0_{\mathbb{B}_2} = 0_{\mathbb{B}_2}$ mod D. But $d' \in \mathbb{B}_1$ and $a'_n \in \mathbb{B}_1$ for every n (and $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$, of course), hence $d' \cap \bigcup_{n \le k} a'_n = 0_{\mathbb{B}_1}$. However, as this holds for every k and the choice of c it follows that $d' \cap c = 0$, but by the case first assumption of the present case $c \in D_1 \subseteq D_2$ so $d' \notin D_2$, but by the case assumption $d'/D = d/D$ and $d \in D_2$ contradiction. $\square_{0.12}$

§ 1. Spectrum of the ultraproducts of finite cardinals

Definition 1.1. Assume D is an ultra-filter on I.
1) Let $\mathrm{upf}(D)$ be the spectrum of ultra-products mod D of finite cardinals, that is; $\{\prod\limits_{i \in I} n_i/D : n_i \in \mathbb{N}$ for $i \in I$ and $\prod\limits_{i \in I} n_i/D$ is infinite$\}$.
2) For $\lambda \in \mathrm{upf}(D)$ let $A_{D,\lambda} = \{a : a \in \mathbb{N}^I/D$ and the set $\{b \in \mathbb{N}^I/D : \mathbb{N}^I/D \models$ "$b < a$"$\}$ has cardinality $\lambda\}$; we consider it as a linearly ordered set by the order inherited from $\mathbb{N}^I/D$.
3) Let $\mathrm{spec}_1(D) = \{(\lambda, \theta, \partial) : \lambda \in \mathrm{upf}(D)$ and $A_{D,\lambda}$ has cofinality ∂ and co-initiality $\theta\}$.
4) Let $\mathrm{spec}_2(D)$ be the set of triples $(\lambda, \theta, \partial)$ such that:

- (a) $\lambda \in \mathrm{upf}(D)$
- (b) (α) if $\lambda < \max(\mathrm{upf}(D))$ then $A_{D,\lambda}$ has cofinality ∂
  - (β) if $\lambda = \max(\mathrm{upf}(D))$ then $\partial = 0$ (or $*$)
- (c) θ is the co-initiality of $A_{D,\lambda}$.

5) For D an ultrafilter on a complete Boolean Algebra $\mathbb{B}$ we define the above similarly considering $\mathbb{N}^{\mathbb{B}}/D$ instead $\mathbb{N}^J/D$ but in clause (b), ∂ is the cofinality of $A_{D,\lambda}$ in all cases.

Definition 1.2. Let K_α be the class of objects $\mathbf{k}$ consisting of:

- (a) $\mathbb{B}_\beta$ is a Boolean Algebra for $\beta \le \alpha$
- (b) $\langle \mathbb{B}_\beta : \beta \le \alpha \rangle$ is increasing
- (c) $\mathbb{B}_\beta$ is complete for $\beta < \alpha, \mathbb{B}_0$ is trivial
- (d) $\mathbb{B}_\beta \lessdot \mathbb{B}_\gamma$ if $\beta < \gamma \le \alpha$ and $\cup\{\mathbb{B}_{\beta_1} : \beta_1 < \gamma\} \lessdot \mathbb{B}_\gamma$ for limit $\gamma \le \alpha$
- (e) D_β is a filter on $\mathbb{B}_\alpha$ such that $\mathbb{B}_\beta \cap D_\beta = \{1_{\mathbb{B}_\beta}\}$
- (f) D_β is maximal under clause (e), so D_0 is an ultrafilter and $D_\alpha = \{1_{\mathbb{B}_\alpha}\}$
- (g) $\langle D_\beta : \beta \le \alpha \rangle$ is $\subseteq$-decreasing.

Definition 1.3. 1) Above let $\mathbb{B}[\mathbf{k}] = \mathbb{B}_{\mathbf{k}} = \mathbb{B}_\alpha, \mathbb{B}[\mathbf{k}, \beta] = \mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_\beta, \bar{\mathbb{B}}_{\mathbf{k}} = \langle \mathbb{B}_{\mathbf{k},\beta} : \beta \le \alpha \rangle, D_{\mathbf{k},\beta} = D_\beta, D_{\mathbf{k}} = D_{\mathbf{k},0}, \ell g(\mathbf{k}) = \alpha_{\mathbf{k}} = \alpha(\mathbf{k}) = \alpha$.
1A) Let K^{com}_α be the class of $\mathbf{k} \in K_\alpha$ such that $\mathbb{B}_{\mathbf{k}}$ is a complete Boolean Algebra.
2) Assume $\kappa > \aleph_0$ is regular. Let $K^{\mathrm{cc}(\kappa),1}_\alpha$ be the class of $\mathbf{k} \in K_\alpha$ such that $\mathbb{B}_\alpha$ satisfies the κ-c.c.
3) Let $K^{\mathrm{cc}(\kappa),2}_\alpha$ be the class of $\mathbf{k} \in K^{\mathrm{cc}(\kappa),1}_\alpha$ such that:

- $\mathbb{B}_{\mathbf{k}}$ is complete; recall that for every $\beta < \alpha, \mathbb{B}_\beta$ is complete
- if $\delta \le \alpha$ has cofinality $\ge \kappa$ <u>then</u> $\mathbb{B}_{\mathbf{k},\delta} = \bigcup\limits_{\beta<\delta} \mathbb{B}_{\mathbf{k},\beta}$
- if $\delta \le \alpha$ is limit of cofinality $< \kappa$, <u>then</u> $\mathbb{B}_{\mathbf{k},\delta}$ is the completion of $\bigcup\limits_{\beta<\delta} \mathbb{B}_{\mathbf{k},\beta}$.

3A) We may omit κ when $\kappa = \aleph_1$ so $K^{\mathrm{cc},\iota}_\alpha = K^{\mathrm{cc}(\aleph_1),\iota}_\alpha$; if we omit ι we mean 1.
4) Let $K = \bigcup\limits_\alpha K_\alpha$ and $K^{\mathrm{cc}(\kappa),\iota} = \cup\{K^{\mathrm{cc}(\kappa),\iota}_\alpha : \alpha$ is an ordinal$\}$ so $K^{\mathrm{cc}} = \bigcup\limits_\alpha K^{\mathrm{cc}}_\alpha$.
5) We say $\mathbb{B}$ is above $\bar{\mathbb{B}}_{\mathbf{k}}$ when $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}$ and $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$ for $\beta < \alpha_{\mathbf{k}}$.
6) $K^{\mathrm{fr}(\kappa)}_\alpha$ is the class of $\mathbf{f}$ consisting of:

(a) $\mathbf{k}_{\mathbf{f}} = (\bar{\mathbb{B}}, \bar{D})$ as in 1.2

(b) $\bar{\xi} = \langle \xi_\gamma : \gamma \le \alpha \rangle$ and $\bar{x} = \langle x_{\beta,\zeta,i} : i < \kappa, \beta < \alpha, \zeta < \xi_\beta \rangle, x_{\beta,\zeta,i} \in \mathbb{B}_{\mathbf{k}}$ are such that $\bar{x}$ is free except that $\beta < \xi_\alpha \wedge i < j < \kappa \Rightarrow x_{\beta,\zeta,i} \cap x_{\beta,\zeta,j} = 0$

(c) the sub-algebra which $\langle x_{\beta,\zeta,i} : \zeta < \xi_\gamma, i < \kappa \rangle$ generates is dense in $\mathbb{B}_{\mathbf{k},\gamma}$

(d) so $\bar{\xi}_{\mathbf{f}} = \bar{\xi}, \bar{x}_{\mathbf{f}} = \bar{x}, \bar{\mathbb{B}}_{\mathbf{f}} = \mathbb{B}_{\mathbf{k}}$, etc.

7) Let ${}_*K_\alpha$ be defined like K_α in 1.2 omitting clause (d) of 1.2, and define ${}_*K$, as above; not really needed here but we may comment.

Definition 1.4. 1) If $\beta \le \gamma$ and $\mathbf{m} \in K_\gamma$ then $\mathbf{k} = \mathbf{m} {\restriction} \beta$ is the unique $\mathbf{k} \in K_\beta$ such that $\mathbb{B}_{\mathbf{k}} = \mathbb{B}_{\mathbf{m},\beta}, \mathbb{B}_{\mathbf{k},\alpha} = \mathbb{B}_{\mathbf{m},\alpha}, D_{\mathbf{k},\alpha} = D_{\mathbf{m},\alpha} \cap \mathbb{B}_{\mathbf{k}}$ for $\alpha \le \beta$.
1A) If $\mathbf{k} \in K_\alpha$ and $\beta < \alpha$ <u>then</u> $\pi_{\mathbf{k},\beta}$ is the unique projection from $\mathbb{B}_{\mathbf{k}}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ such that $\pi^{-1}_{\mathbf{k},\beta}\{1_{\mathbb{B}_{\mathbf{k},\beta}}\} = D_{\mathbf{k},\beta}$ recalling 0.11; let $\pi_{\mathbf{k},\alpha} = \mathrm{id}_{\mathbb{B}_{\mathbf{k},\alpha}}$ and if $\gamma \le \beta \le \alpha$ then $\pi_{\mathbf{k},\beta,\gamma} = \pi_{\mathbf{k},\gamma} {\restriction} \mathbb{B}_{\mathbf{k},\beta}$.
2) We define the following two-place relations on K:

(A) $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{m}$ where at stands for atomic <u>iff</u>:
- (a) $\alpha_{\mathbf{k}} = \alpha_{\mathbf{m}}$
- (b) $\mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_{\mathbf{m},\beta}$ for $\beta < \alpha_{\mathbf{k}}$
- (c) $\mathbb{B}_{\mathbf{k},\alpha(\mathbf{k})} \lessdot \mathbb{B}_{\mathbf{m},\alpha(\mathbf{m})}$
- (d) $D_{\mathbf{k},\beta} \subseteq D_{\mathbf{m},\beta}$ for $\beta \le \alpha_{\mathbf{k}}$.

(B) $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{m}$, where ver stands for vertical <u>iff</u>
- (a) $\alpha_{\mathbf{k}} \le \alpha_{\mathbf{m}}$
- (b) $\mathbf{k} \le^{\mathrm{at}}_K (\mathbf{m} {\restriction} \alpha_{\mathbf{k}})$

(C) $\mathbf{k} \le^{\mathrm{hor}}_K \mathbf{m}$, where hor stands for horizontal <u>iff</u>
- (a) $\alpha_{\mathbf{k}} = \alpha_{\mathbf{m}} = \alpha$
- (b) $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}_{\mathbf{m},\beta}$ for $\beta \le \alpha$
- (c) $D_{\mathbf{k},\beta} \subseteq D_{\mathbf{m},\beta}$ for $\beta \le \alpha$

(D) (a) $\mathbf{f}_1 \le^{\mathrm{fr}(\kappa)}_K \mathbf{f}_2$ iff $\mathbf{f}_\ell \in K^{\mathrm{fr}(\kappa)}_{\alpha_\ell}$ and $\mathbf{k}_{\mathbf{f}_1} \le^{\mathrm{ver}}_K \mathbf{k}_{\mathbf{f}_2}$ and $\bar{x}_{\mathbf{f}_1} \trianglelefteq \bar{x}_{\mathbf{f}_2}$ which means:
- $\beta < \alpha_1 \Rightarrow \bar{x}_{\mathbf{f}_1,\beta} = \bar{x}_{\mathbf{f}_2,\beta}$ and $\beta = \alpha_1 \Rightarrow \xi_{\mathbf{f}_1,\beta} \le \xi_{\mathbf{f}_2,\beta} \wedge \bar{x}_{\mathbf{f}_1,\beta} = \bar{x}_{\mathbf{f}_2,\beta} {\restriction} \xi_{\mathbf{f}_2,\beta}$

(b) $\le^{\mathrm{at-fr}(\kappa)}_K$ is defined similarly

(E) $\mathbf{k} \le^{\mathrm{wa}}_K \mathbf{m}$ where wa stands for weakly atomic <u>iff</u>
- (a), (b), (d) as in Clause (A)
- (c) $\mathbb{B}_{\mathbf{k},\alpha(\mathbf{k})} \subseteq \mathbb{B}_{\mathbf{m},\alpha(\mathbf{m})}$.

Remark 1.5. Note that for the present work it is not a loss to use exclusively c.c.c. Boolean Algebras; moreover ones which have a dense subalgebra which is free. So using only free Boolean Algebras or their completion, i.e. $(K^{\mathrm{fr}(\aleph_1)}_\alpha, \le^{\mathrm{fr}(\kappa)}_K)$; so we are giving for $\mathbb{B}$ a set of generators (and the orders respect this).

Observation 1.6. *The relations $\le^{\mathrm{at}}_K, \le^{\mathrm{wa}}_K$ and $\le^{\mathrm{ver}}_K$ and $\le^{\mathrm{hor}}_K$ (the last one is not used) are partial orders on K.*

We need various claims on extending members of K, existence of upper bounds to an increasing sequence and amalgamation.

Claim 1.7. *Let δ be a limit ordinal.*
1) If $\langle \mathbf{k}_i : i < \delta\rangle$ is $\le^{\rm at}_K$-increasing <u>then</u> it has a $\le^{\rm at}_K$-lub $\mathbf{k}_\delta$, the union naturally defined so $|\mathbb{B}_{\mathbf{k}_\delta}| \le \Sigma\{|\mathbb{B}_{\mathbf{k}_i}| : i < \delta\}$.
1A) Like part (1) for $\le^{\rm wa}_K$.
2) If $\langle \mathbf{k}_i : i < \delta\rangle$ is a $\le^{\rm ver}_K$-increasing sequence, <u>then</u> it has a $\le^{\rm ver}_K$-upper bound $\mathbf{k} = \mathbf{k}_\delta$ which is the union which means:

(a) $\ell g(\mathbf{k}) = \cup\{\ell g(\mathbf{k}_i) : i < \delta\}$ *call it* α
(b) *if* $\beta < \alpha$ *then* $\mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_{\mathbf{k}_i,\beta}$ *for every large enough* i
(c) (α) *if* $\langle \ell g(\mathbf{k}_i) : i < \delta\rangle$ *is eventually constant (so* $\ell g(\mathbf{k}_i) = \alpha$ *for every* $i < \delta$ *large enough)* <u>*then*</u>
 - $\mathbb{B}_{\mathbf{k},\alpha} = \cup\{\mathbb{B}_{\mathbf{k}_i,\alpha} : i < \delta$ *is such that* $\ell g(\mathbf{k}_i) = \alpha\}$
 - $D_{\mathbf{k},\alpha} = \{1_{\mathbb{B}_{\mathbf{k},\alpha}}\}$, *redundant*

 (β) *if* $\langle \ell g(\mathbf{k}_i) : i < \delta\rangle$ *is not eventually constant* <u>*then*</u>
 - $\mathbb{B}_{\mathbf{k},\alpha} = \cup\{\mathbb{B}_{\mathbf{k}_i,\ell g(\mathbf{k}_i)} : i < \delta\}$
 - $D_{\mathbf{k},\alpha} = \{1_{\mathbb{B}_{\mathbf{k},\alpha}}\}$, *redundant*

(d) *if* $\beta < \alpha$ *then* $D_{\mathbf{k},\beta} = \cup\{D_{\mathbf{k}_i,\beta} : i < \delta$ *is such that* $\beta \le \ell g(\mathbf{k}_i)\}$.

3) In part (2) if $\mathbf{k}_i \in K^{{\rm cc}(\kappa),2}, {\rm cf}(\delta) \ge \kappa$ and the sequence $\langle \ell g(\mathbf{k}_i) : i < \delta\rangle$ is not eventually constant <u>then</u> $\mathbb{B}_{\mathbf{k}}$ is complete and ${\rm upf}(D_{\mathbf{k}}) = \cup\{{\rm upf}(D_{\mathbf{k}_i}) : i < \delta\}$.
4) Similarly for the ${}_*K$ version.

Proof. Straightforward (concerning part (3) note that recalling ${\rm cf}(\delta) \ge \kappa$ we have $\mathbb{N}^{\mathbb{B}(\mathbf{k})} = \cup\{\mathbb{N}^{\mathbb{B}(\mathbf{k}_i)} : i < \delta\}$. $\square_{1.7}$

Claim 1.8. *1) If $\mathbf{k} \in K_\alpha$ and $\mathbb{B}$ is above $\bar{\mathbb{B}}_{\mathbf{k}}$ (i.e. $\mathbb{B}_{\mathbf{k},\alpha} \subseteq \mathbb{B}$ and $\beta < \alpha \Rightarrow \mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$) <u>then</u> there is $\mathbf{m} \in K_\alpha$ such that $\mathbf{k} \le^{\rm wa}_K \mathbf{m}$ and $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$.*
2) If $\mathbf{k} \in K_\alpha$ and $\mathbb{B} \subseteq \mathbb{B}_{\mathbf{k}}$ and $\beta < \alpha \Rightarrow \mathbb{B}_{\mathbf{k},\beta} \subseteq \mathbb{B}$ (hence $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$) <u>then</u> there is $\mathbf{m} \in K_\alpha$ such that $\mathbf{m} \le^{\rm wa}_K \mathbf{k}$ and $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$.
3) If $\mathbf{k} \in K_\alpha, \mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}$ <u>then</u> for some $\mathbf{m} \in K_\alpha, \mathbf{k} \le^{\rm at}_K \mathbf{m}, \mathbb{B}_{\mathbf{m}} = \mathbb{B}$
4) In part (2) if $\mathbb{B} \lessdot \mathbb{B}_{\mathbf{k}}$ <u>then</u> we can add $\mathbf{m} \le^{\rm at}_K \mathbf{k}$.
5) If $\mathbf{k} \le^{\rm wa}_K \mathbf{m}$ and $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}_{\mathbf{m}}$ <u>then</u> $\mathbf{k} \le^{\rm at}_K \mathbf{m}$.

Proof. 1) For simplicity we concentrate on the main case: all the $\mathbb{B}_\beta$ and $\mathbb{B}$ satisfies the c.c.c. By 1.7(1A) without loss of generality $\mathbb{B}$ is generated by $\mathbb{B}_{\mathbf{k},\alpha} \cup \{a\}$ where $a \notin \mathbb{B}_{\mathbf{k},\alpha}$. So $\mathbb{B}_{\mathbf{m}}$ is uniquely defined and as required in Definition 1.2, but we have to define the $D_{\mathbf{m},\beta}$'s and, of course, let $D_{\mathbf{m},\alpha} = \{1_{\mathbb{B}_{\mathbf{m}}}\}$.

<u>Case 1</u>: $\alpha = 0$
This is trivial.

<u>Case 2</u>: $\alpha = \beta + 1$
As $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$ and $\pi_{\mathbf{k},\beta}$ is a projection from $\mathbb{B}_{\mathbf{k},\alpha}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ and $[B_{\mathbf{k},\alpha} \subseteq \mathbb{B}$ and $\mathbb{B}_{\mathbf{k},\beta}$ is complete by 0.10(1) there is a projection π from $\mathbb{B}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ extending $\pi_{\mathbf{k},\beta}$. Now for $\gamma < \alpha$ let $D_{\mathbf{m},\gamma}$ be the filter on $\mathbb{B}$ generated by $D_{\mathbf{k},\gamma} \cup \{(a \triangle \pi(a)\}$.

<u>Case 3</u>: α is a limit ordinal, $\mathbb{B}$ is c.c.c. (the main case) and α is of cofinality $> \aleph_0$
In this case, $\mathbb{B}' = \bigcup\limits_{\gamma<\alpha} \mathbb{B}_{\mathbf{k},\gamma}$ is complete and $\lessdot\mathbb{B}$ so we can continue as in Case 2 using $\mathbb{B}'$ instead $\mathbb{B}_{\mathbf{k},\beta}$.

<u>Case 4</u>: ${\rm cf}(\alpha) = \aleph_0$

Let $\alpha = \bigcup\limits_n \alpha_n, \alpha_n < \alpha_{n+1}$. For $\beta < \alpha$ let π_β be the projection of $\mathbb{B}_{\mathbf{k}}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ which maps $D_{\mathbf{k},\beta}$ onto $1_{\mathbb{B}_{\mathbf{k},\beta}}$. Let Π_β be the set of homomorphisms from $\mathbb{B}$ into $\mathbb{B}_{\mathbf{k},\beta}$ extending $\pi_\beta = \pi_{\mathbf{k},\beta}$, so not empty hence (recalling $\mathbb{B}_{\mathbf{k},\beta}$ is complete) there are $b_\beta \le c_\beta$ from $\mathbb{B}_{\mathbf{k},\beta}$ such that $\{\pi(a) : \pi \in \Pi_\beta\}$ is $\{a' \in \mathbb{B}_{\mathbf{k},\beta} : b_\beta \le a' \le c_\beta\}$. Clearly $\gamma < \beta < \alpha \Rightarrow b_\gamma \le b_\beta \le c_\beta \le c_\gamma$ in $\mathbb{B}_{\mathbf{k},\beta}$.

Now by induction on $\zeta < (\|\mathbb{B}\|^{|\alpha|})^+$ we define $\langle (b_{\beta,\zeta}, c_{\beta,\zeta}) : \beta < \alpha\rangle$ such that:

$(*)_\zeta$ (a) $\mathbb{B}_{\mathbf{k},\beta} \models$ "$b_{\beta,\zeta} \le c_{\beta,\zeta}$" for $\beta < \alpha$
(b) if $\gamma < \beta < \alpha$ then $\mathbb{B}_{\mathbf{k},\beta} \models$ "$b_{\gamma,\zeta} \le b_{\beta,\zeta} \le c_{\beta,\zeta} \le c_{\gamma,\zeta}$"
(c) if $\varepsilon < \zeta$ and $\beta < \alpha$ then $\mathbb{B}_{\mathbf{k},\beta} \models$ "$b_{\beta,\varepsilon} \le b_{\beta,\zeta} \le c_{\beta,\zeta} \le c_{\beta,\varepsilon}$".

Subcase 4A: For $\zeta = 0$ let $(b_{\beta,\zeta}, c_{\beta,\zeta}) = (b_\beta, c_\beta)$, so clause (a),(b) holds as said above and clause (c) is empty.

Subcase 4B: ζ is a limit ordinal
Let for $\beta < \alpha$

- $b_{\beta,\zeta} = \cup\{b_{\gamma,\varepsilon} : \varepsilon < \zeta\}$ in $\mathbb{B}_{\mathbf{k},\beta}$
- $c_{\beta,\zeta} = \cap\{b_{\gamma,\varepsilon} : \varepsilon < \zeta\}$.

They are well defined because $\mathbb{B}_{\mathbf{k},\beta}$ is a complete Boolean Algebra and it is easy to check that (a),(b),(c) hold.

Subcase 4C: $\zeta = \varepsilon + 1$
Let $b_{\beta,\zeta} = \cup\{\pi_{\mathbf{k},\gamma,\beta}(b_{\gamma,\varepsilon}) : \gamma \in (\beta,\alpha)\}, c_{\beta,\zeta} = \cap\{\pi_{\mathbf{k},\gamma,\beta}(c_{\gamma,\varepsilon}) : \gamma \in (\beta,\alpha)\}$.
Now check.

Having carried the induction, by $(*)_\zeta$ for $\zeta < (\|\mathbb{B}\|^{|\alpha|})^+$ for some $\zeta_* < (\|\mathbb{B}\|^{|\alpha|})^+$, $\langle (b_{\beta,\zeta}, c_{\beta,\zeta}) : \beta < \alpha\rangle$ is the same for all $\zeta \ge \zeta_*$ and let $a_\beta = b_{\beta,\zeta_*}$ for $\beta < \alpha$.

Easily $\gamma < \beta < \alpha \Rightarrow \pi_\beta(a_\beta) = a_\gamma$ and let $D_{\mathbf{m},\beta}$ be the filter of $\mathbb{B}$ generated by $D_{\mathbf{k},\beta} \cup \{(a \triangle a_\beta)\}$.
2) Easy.
3) By (1).
4),5) Should be easy. $\square_{1.8}$

Claim 1.9. *1) If $\mathbf{k} \in K_\alpha, \langle \beta_i : i \le i(*)\rangle$ is increasing with $\beta_{i(*)} = \alpha$ <u>then</u> there is one and only one $\mathbf{m} \in K_{i(*)}$ such that $(\mathbb{B}_{\mathbf{m},i}, D_{\mathbf{m},i}) = (\mathbb{B}_{\mathbf{k},\beta_i}, D_{\mathbf{m},\beta_i})$ for $i \le i(*)$.*
2) Above if $\mathbf{m} \le^{\rm at}_K \mathbf{m}_1$ <u>then</u> there is $\mathbf{k}_1$ such that $\mathbf{k} \le^{\rm at}_K \mathbf{k}_1$ and $\mathbb{B}_{\mathbf{k}_1} = \mathbb{B}_{\mathbf{m}_1}$ and $D_{\mathbf{k}_1,\beta_i} = D_{\mathbf{m}_1,i}$ for $i \le i(*)$.
2A) Similarly for $\le^{\rm wa}_K$.
3) Above if $\mathbf{m} \le^{\rm ver}_K \mathbf{m}_1 \in K_{i(*)+j(*)}$ <u>then</u> there is $\mathbf{k}_1 \in K_{\alpha+j(*)}$ such that $\mathbf{k} \le^{\rm ver}_K \mathbf{k}_1, \mathbb{B}_{\mathbf{k}_1} = \mathbb{B}_{\mathbf{m}_1}, \mathbb{B}_{\mathbf{k}_1,\alpha+j} = \mathbb{B}_{\mathbf{m}_1,i(*)+j}, D_{\mathbf{k}_1,\alpha+j} = D_{\mathbf{m}_1,i(*)+j}, D_{\mathbf{k}_1,\beta_i} = D_{\mathbf{m}_1,i}$ for $j < j(*), i \le i(*)$.
4) If $\mathbf{k} \in K_\alpha$ and $\beta_0 \le \beta_1 \le \beta_2 \le \alpha$ <u>then</u> $\pi_{\mathbf{k},\beta_2,\beta_0} = \pi_{\mathbf{k},\beta_1,\beta_0} \circ \pi_{\mathbf{k},\beta_2,\beta_1}$.

Proof. Straightforward. $\square_{1.9}$

Conclusion 1.10. *1) If $\mathbf{k} \in K_\alpha$ <u>then</u> there is $\mathbf{m}$ such that $\mathbf{k} \le^{\rm at}_K \mathbf{m}$ and $\mathbb{B}_{\mathbf{m},\alpha}$ is the completion of $\mathbb{B}_{\mathbf{k},\alpha}$ so $\mathbf{m} \in K^{\rm com}_\alpha$ and $\mathbf{k} \le^{\rm ver}_K \mathbf{m}$; so if $\mathbf{k} \in K^{{\rm cc}(\kappa)}_\alpha$ then $\mathbf{m} \in K^{{\rm cc}(\kappa)}_\alpha \cap K^{\rm com}$.*
2) If $\alpha < \beta, \mathbf{k} \in K_\alpha, \mathbf{n} \in K_\beta$ and $\mathbf{k} \le^{\rm ver}_K \mathbf{n}$, <u>then</u> for some $\mathbf{m}$ we have $\mathbf{k} \le^{\rm ver}_K \mathbf{m} \le^{\rm ver}_K \mathbf{n}$ and $\mathbb{B}_{\mathbf{m}}$ is the completion of $\mathbb{B}_{\mathbf{k}}$ inside $\mathbb{B}_{\mathbf{n}}$; so if $\mathbf{n} \in K^{{\rm cc}(\kappa)}$ then $\mathbf{m} \in K^{{\rm cc}(\kappa)}$.

Proof. 1) By 1.8.
2) Check the definitions. $\square_{1.10}$

Claim 1.11. *There is* $\mathbf{m}$ *such that* $\mathbf{k} \le^{\rm wa}_K \mathbf{m}, \mathbb{B}_{\mathbf{m}} = \mathbb{B}$ *and* $Y \subseteq D_{\mathbf{m}}$ *and* $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B} \Rightarrow \mathbf{k} \le^{\rm at}_K \mathbf{m}$ *when:*

- (a) $\mathbf{k} \in K_\alpha$
- (b) $\mathbb{B}$ *is a Boolean Algebra,* $Y \subseteq \mathbb{B}$
- (c) (α) $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}$
  - (β) $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$ *for* $\beta < \alpha_{\mathbf{k}}$
- (d) *if* $\beta < \alpha_{\mathbf{k}}$ *then for some* X_β *we have:*
  - (α) $X_\beta \subseteq Y$
  - (β) X_β *is a downward directed subset of* $\mathbb{B}$
  - (γ) *if* $x \in X_\beta$ *and* $b \in D_{\mathbf{k},\beta}$ *then* $x \cap b$ *is not disjoint to any* $a \in \mathbb{B}^+_{\mathbf{k},\beta}$
  - (δ) *if* $y \in Y$ *then for some* $b \in D_{\mathbf{k},\beta}$ *and* $x \in X_\beta$ *we*[2] *have* $b \cap x \le y$.

Proof. By 1.8 without loss of generality $\mathbb{B}$ is generated by $\mathbb{B}_{\mathbf{k}} \cup Y$ and let $\alpha = \alpha_{\mathbf{k}}$ and let X_β be as in clause (d) in the claim for $\beta < \alpha$ and define $\mathbf{m} \in K_\alpha$ as follows:

- $D_{\mathbf{m},\alpha} = \{1_{\mathbb{B}}\}, \mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$
- for $\beta < \alpha$ let $\mathbb{B}_{\mathbf{m},\beta} = \mathbb{B}_{\mathbf{k},\beta}$ and $D_{\mathbf{m},\beta}$ be the filter on $\mathbb{B}_{\mathbf{m},\alpha}$ generated by $D_{\mathbf{k},\beta} \cup X_\beta$.

The point is to check $\mathbf{m} \in K_\alpha$ as then $\mathbf{k} \le^{\rm at}_K \mathbf{m}$ and $Y \subseteq D_{\mathbf{m}}$ are obvious, also $\bar{\mathbb{B}}_{\mathbf{m}}$ is as required and $D_{\mathbf{m},\beta}$ a filter on $\mathbb{B}_{\mathbf{m},\alpha}$ including $D_{\mathbf{k},\beta}$ are obvious.

So proving $(*)_1, (*)_2, (*)_3$ below will suffice

$(*)_1$ if $\beta < \gamma < \alpha$ then $D_{\mathbf{m},\gamma} \subseteq D_{\mathbf{m},\beta}$.

[Why? If $a \in D_{\mathbf{m},\gamma}$ then by the choice of $D_{\mathbf{m},\gamma}$ (recalling $D_{\mathbf{k},\gamma}$ is downward directed being a filter and X_γ is downward directed by its choice (i.e. Clause $(d)(\beta)$ of the claim) for some $b \in D_{\mathbf{k},\gamma}$ and $x \in X_\gamma$ we have $b \cap x \le a$. So by $(d)(\alpha)$ applied to γ we have $x \in Y$ hence by $(d)(\delta)$ applied to β for some $b_1 \in D_{\mathbf{k},\beta}$ and $x_1 \in X_\beta$ we have $b_1 \cap x_1 \le x$ hence $(b \cap b_1) \cap x_1 = b \cap (b_1 \cap x_1) \le b \cap x \le a$ but $b \in D_{\mathbf{k},\gamma} \subseteq D_{\mathbf{k},\beta}, b_1 \in D_{\mathbf{k},\beta}$ hence $b \cap b_1 \in D_{\mathbf{k},\beta}$ and $x_1 \in X_\beta$ hence $a \in D_{\mathbf{m},\beta}$ by the choice of $D_{\mathbf{m},\beta}$.]

$(*)_2$ $D_{\mathbf{m},\beta}$ is a filter on $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$ disjoint to $\mathbb{B}_{\mathbf{m},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{k}}}\} = \mathbb{B}_{\mathbf{k},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{k}}}\}$.

[Why? By the definition of $D_{\mathbf{m},\beta}$ and clause $(d)(\gamma)$.]

$(*)_3$ if $\beta < \alpha$ then $D_{\mathbf{m},\beta}$ is a maximal filter of $\mathbb{B}_{\mathbf{m}}$ disjoint to $\mathbb{B}_{\mathbf{k},\beta} \backslash \{1_{\mathbf{m},\beta}\} = \mathbb{B}_{\mathbf{m},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{m},\beta}}\}$.

Why? If $b \in \mathbb{B} = \mathbb{B}_{\mathbf{m},\alpha}$ then for some Boolean terms $\sigma(y_0, y_1, \dots, z_0, z_1, \dots)$ and $a_0, a_1, \dots, \in \mathbb{B}_{\mathbf{k}}$ and $x_0, x_1, \dots \in Y$ we have $b = \sigma(a_0, a_1, \dots, x_0, x_1, \dots)$ hence modulo the filter $D_{\mathbf{m},\beta}, b$ is equal to $\sigma(a_0, a_1, \dots, 1_{\mathbb{B}_{\mathbf{k},\beta}}, 1_{\mathbb{B}_{\mathbf{k},\beta}}, \dots)$. But for each a_ℓ there is $c_\ell \in \mathbb{B}_{\mathbf{k},\beta}$ such that $a_\ell = c_\ell \mod D_{\mathbf{k},\beta}$ hence b is equal to $\sigma(c_0, c_1, \dots, 1_{\mathbb{B}_{\mathbf{k},\beta}}, 1_{\mathbb{B}_{\mathbf{k},\beta}}, \dots)$ which belongs to $\mathbb{B}_{\mathbf{k},\beta}$.

[2] Does this contradict $(d)(\gamma)$? No, as $\mathbf{D}_{\mathbf{k},\beta}$ is disjoint to $\mathbb{B}_{\mathbf{k},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{k},\beta}}\}$.

As this holds for any $b \in \mathbb{B}$ we are easily done. $\square_{1.11}$

Definition 1.12. 1) We say $\mathbf{k}$ is reasonable in α when $\alpha + 1 \le \alpha_{\mathbf{k}}$ (so $\mathbb{B}_{\mathbf{k},\alpha}$ is complete) and there is a maximal antichain of $\mathbb{B}_{\alpha+1}$ included in $\{a \in \mathbb{B}_{\mathbf{k},\alpha+1} : \pi_{\mathbf{k},\alpha+1,\alpha}(a) = 0_{\mathbb{B}_{\mathbf{k},\alpha}}\}$.
2) We say $\mathbf{k}$ is reasonable when it is reasonable in α whenever $\alpha + 1 \le \alpha_{\mathbf{k}}$.
3) Let

- $A^1_{\mathbf{k},\alpha} = \{f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]} : f \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\alpha]}$ and if $\beta < \alpha$ then $f/D_{\mathbf{k}} \notin \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}/D_{\mathbf{k}}\}$
- $A^2_{\mathbf{k},\alpha} = \{f/D_{\mathbf{k}} : f \in A^1_{\mathbf{k},\alpha}\}$
- $A^1_{\mathbf{k},<\alpha} = \bigcup_{\beta<\alpha} A^1_{\mathbf{k},\beta}$ and $A^2_{\mathbf{k},<\alpha} = \bigcup_{\beta<\alpha} A^2_{\mathbf{k},\beta}$, etc.

4) We say f is reasonable[3] in $(\mathbf{k}, \alpha, \beta)$ when $\alpha < \beta < \alpha_{\mathbf{k}}$ and $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ and for some $f' \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$, we have $f'/D_{\mathbf{k}} = f/D_{\mathbf{k}}$ and f' is represented by $\langle a_n : n < \omega\rangle$ and $\pi_{\mathbf{k},\alpha}(a_n) = 0$ for every n large enough and $a_n \notin D_{\mathbf{k}}$ for every n. If $\beta = \alpha + 1$ we may omit it.
5) We say f is reasonable in $(\mathbf{k}, < \alpha)$ when it is reasonable in $(\mathbf{k}, \beta, \gamma)$ for some $\beta + 1 = \gamma < \alpha$.

Observation 1.13. *If $\beta < \alpha_{\mathbf{k}}, \mathbf{k} \in K^{\rm com}$ and $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]}$ is represented by $\langle a_n : n < \omega\rangle$, then $f \in A^1_{\mathbf{k},\le\beta}$ iff $\bigcup_n (a_n \triangle \pi_{\mathbf{k},\beta}(a_n)) \notin D_{\mathbf{k}}$.*

Proof. By the proof of 0.12. $\square_{1.13}$

Claim 1.14. *1) If $\mathbf{k} \in K^{\rm cc}_{\alpha+1}$ then there is $\mathbf{m} \in K^{\rm cc}_{\alpha+1}$ such that $\mathbf{k} \le^{\rm at}_K \mathbf{m}, \mathbb{B}_{\mathbf{m}}$ is complete and $\mathbf{m}$ is reasonable in α and $\|\mathbb{B}_{\mathbf{m}}\| = \|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0}$.*
2) If $\mathbf{k} \in K^{\rm cc}_{\alpha+1}$ is reasonable in α and $\mathbf{k} \le^{\rm at}_K \mathbf{m}$ or $\mathbf{k} \le^{\rm ver}_K \mathbf{m}$ then $\mathbf{m}$ is reasonable in α.
3) If $\langle \mathbf{k}_i : i < \delta\rangle$ is $\le^{\rm ver}_K$-increasing in $K^{\rm cc}$ and each $\mathbf{k}_i$ is reasonable then there is a $\le^{\rm ver}_K$-upper bound $\mathbf{k}$ of cardinality $(\sum_i \|\mathbb{B}_{\mathbf{k}_i}\|)^{\aleph_0}$ which is reasonable.
4) If f is reasonable in $(\mathbf{k}, \alpha)$ then it is reasonable in $(\mathbf{k}, < \alpha + 1)$.
5) If $f \in A^1_{\mathbf{k},\alpha}$ then f is reasonable in $(\mathbf{k}, \alpha)$.
6) In 1.7,(2),(3) if $\mathbf{k}_i$ is reasonable for every $i < \delta$ then so is $\mathbf{k}$.

Proof. Straightforward; e.g.:
2) Because $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}_{\mathbf{m},\alpha(\mathbf{k})}$, see Definition 1.4(2) and read Definition 1.12(1).
5) Let $\langle a_n : n < \omega\rangle$ represent f.

Let $a'_n = \pi_{\mathbf{k},\alpha+1}(a_n)$, so $\pi_{\mathbf{k},\alpha+1,\alpha}(a_n - a'_n) = \pi_{\mathbf{k},\alpha+1,\alpha}(a_n) - \pi_{\mathbf{k},\alpha+1,\alpha}(a'_n) = a'_n - a'_n = 0$. Now $\langle a'_n : n < \omega\rangle$ is an antichain (using $\pi_{\mathbf{k},\alpha}$) and let $g \in \mathbb{N}^{\mathbb{B}_\alpha}$ be such that $g(a'_n) = n$ and $g(1_{\mathbb{B}_{\mathbf{k},\alpha}} - \bigcup_n a'_n) = 0$. So $g \in \mathbb{N}^{\mathbb{B}_{\mathbf{k}}}$ and by "$f \in A^1_{\mathbf{k},\alpha}$", necessarily ${\rm TV}(f \ne g) \in D_{\mathbf{k}}$ and clearly ${\rm TV}(f \ne g) \le \bigcup_n (a_n - a'_n) \cup {\rm TV}(f = 0)$ but ${\rm TV}(f \ne 0) \in D_{\mathbf{k}}$, i.e. $a_0 \notin D_{\mathbf{k}}$ by the assumption on f, hence necessarily $b := \bigcup_n (a_n - a'_n) \in D_{\mathbf{k}}$. Now define $f'' \in \mathbb{N}^{\mathbb{B}_k}$ represented by $\langle b_n : n < \omega\rangle$ where $b_n = a_n - a'_n$ for $n \ge 1$ and $b_n = 1 - \bigcup_{m\ge1} b_m$ for $n = 0$. So $\pi_{\mathbf{k},\alpha}(b_n) = 0_{\mathbb{B}_{\mathbf{k},n}}$ for $n \ge 1$ and $b_n \notin D_{\mathbf{k}}$ for $n \ge 0$. Clearly f'' is reasonable in $(\mathbf{k}, \alpha)$. $\square_{1.14}$

[3]Note that "some f is reasonable in $(\mathbf{k}, \alpha)$" is close to but not equivalent to "$\mathbf{k}$ is reasonable in $\mathbf{h}$".

Claim 1.15. *If (A) then (B) where:*

(A) (a) $\mathbf{k} \in K^{\rm cc}_\alpha$
 (b) $\beta_n < \beta_{n+1} < \beta = \bigcup\limits_k \beta_k \le \alpha$
 (c) $\mathbf{k}$ *is reasonable in* β_n *for every* $n \in \omega$

(B) *if* $f_1 \in A^1_{\mathbf{k},\beta}$, *i.e.* $f_1 \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$, $f_1/D_{\mathbf{k}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\gamma]}/D_{\mathbf{k}} : \gamma < \beta\}$ *then there is* f_2 *such that:*
 (a) $f_2 \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$
 (b) $f_2/D_{\mathbf{k}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\gamma]}/D_{\mathbf{k}} : \gamma < \beta\}$
 (c) $f_2/D_{\mathbf{k}} < f_1/D_{\mathbf{k}}$
 (d) *there is* $\langle (a_i, k_i) : i < \omega\rangle$ *representing* f_2 *such that:*
  (α) *for each* i, *letting* $k(i) = k_i$ *we have* $a_i \in \mathbb{B}_{\mathbf{k},\beta_{k(i)+1}}$ *and* $\pi_{\mathbf{k},\beta_{k(i)+1},\beta_{k(i)}}(a_i) = 0$
  (β) *for each* ℓ *the set* $\{i : k_i < \ell\}$ *is finite.*

Proof. For each n let $\langle a_{n,\ell} : \ell < \omega\rangle$ be a maximal antichain of $\mathbb{B}_{\beta_n+1}$ such that $\pi_{\mathbf{k},\beta_n+1,\beta_n}(a_{n,\ell}) = 0$ for $\ell < \omega$, exists as $\mathbf{k}$ is reasonable in β_n for every $n \in \omega$, see Definition 1.12(2).

Let

$(*)_0$ (a) $\mathscr{T}_n = \{\eta : \eta \in {}^n\omega \text{ and } \bigcap\limits_{k<n} a_{k,\eta(k)} > 0\}$
 (b) $\mathscr{T} = \bigcup\limits_n \mathscr{T}_n$.

Hence

$(*)_1$ (a) $\langle a_\eta : \eta \in \mathscr{T}_n\rangle$ is a maximal antichain of $\mathbb{B}_{\beta_{n+1}}$ on which $\pi_{\mathbf{k},\beta_n}$ is zero
 (b) $\mathscr{T}$ is a subtree of ${}^{\omega>}\omega$.

Now choose a sequence $\bar{k}$ of natural numbers such that:

$(*)_2$ (a) $\bar{k} = \langle k_s : s \in \mathscr{T}\rangle$
 (b) if $\nu \triangleleft \eta$ then $k_\nu < k_\eta$
 (c) if $k_\eta = k_\nu$ then $\eta = \nu$

$(*)_3$ let $g_n \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta(n)+1]}$ be represented by $\langle (a_\eta, k_\eta) : \eta \in \mathscr{T}_n\rangle$, see Definition 0.8(4).

[Why? Well defined by $(*)_1(a)$.]

$(*)_4$ let $\mathbf{M}_{\mathbf{k}} = \{\bar{c} : \bar{c} = \langle c_\ell : \ell < \omega\rangle$ is a maximal antichain of $\mathbb{B}_{\beta_0+1}$ disjoint to $D_{\mathbf{k}}\}$.

What is the point of $\mathbf{M}_{\mathbf{k}}$? $g_n \in A^1_{\mathbf{k},\beta(n)}$ hence $\langle g_n/D_{\mathbf{k}} : n < \omega\rangle$ is increasing and cofinal in $\cup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\beta(n)]}/D : n < \omega\}$ hence if in $\mathbb{N}^{\mathbb{B}[\mathbf{k}]}/D_{\mathbf{k}}$ we have a definable sequence, the n-th try being g_n/D, in "non-standard places" we have the $g_{\bar{c}}$'s defined below members of $A^2_{\mathbf{k},\alpha}$ and those are co-initial in it.

$(*)_5$ for each $\bar{c} \in \mathbf{M}_{\mathbf{k}}$ let
 (a) $S_{\bar{c}} = \{(\ell, \eta) : \ell < \omega, \eta \in \mathscr{T}_\ell \text{ and } c_\ell \cap a_\eta > 0\}$

(*b*) for $(\ell,\eta) \in S_{\bar c}$ let $a_{(\ell,\eta)} = c_\ell \cap a_\eta$

(*c*) $g_{\bar c} \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ be represented by $\langle (a_{(\ell,\eta)}, k_\eta) : (\ell,\eta) \in S_{\bar c}\rangle$

$(*)_6$ g_n is $(\mathbf{k},\beta_n)$-reasonable.

[Why? By the Definition 1.12.]

$(*)_7$ $g_n \in A^1_{\mathbf{k},\beta_n+1}$.

[Why? Follows from the definition of g_n in $(*)_3$ and the choice of the a_η's and k_ν's in $(*)_1$ and $(*)_2$.]

$(*)_8$ $g_{\bar c} \in A^1_{\mathbf{k},\beta}$ for $\bar c \in \mathbf{M}_\mathbf{k}$.

[Why? As $\bar k$ is with no repetition and the definition.]

$(*)_9$ there is $\bar c \in \mathbf{M}_\mathbf{k}$ such that $g_{\bar c}/D_\mathbf{k} < f_1/D_\mathbf{k}$.

[Why? See explanation after $(*)_4$ and 1.14(5).] $\square_{1.15}$

Claim 1.16. *If (A) then (B) <u>where</u>:*

(A) (a) $\mathbf{k} \in K^{\rm cc}_\alpha$ *is* $\mathbb{B}_{\mathbf{k},\alpha}$ *infinite*

(b) $\beta_n = \beta(n) < \alpha$ *is increasing with limit* α

(c) $\mathbf{k}$ *is reasonable in* β_n

(d) $f \in A^1_{\mathbf{k},\alpha}$

(B) *there are* $\mathbf{m}, g$ *such that:*

(a) $\mathbf{k} \le^{\rm at}_K \mathbf{m}$ *and* $\|\mathbb{B}_\mathbf{m}\| = (\|\mathbb{B}_\mathbf{k}\|^{\aleph_0})^+$

(b) $g \in A^1_{\mathbf{m},\alpha}$

(c) $g/D_\mathbf{m} < f/D_\mathbf{m}$

(d) $g/D_\mathbf{m} \notin \mathbb{N}^{\mathbb{B}[\mathbf{k}]}/D_\mathbf{m}$.

Proof. Without loss of generality $\mathbb{B}_\mathbf{k}$ is complete of cardinality λ.

Let $f_2, \langle (a_n,k_n) : n < \omega\rangle$ be as in 1.15 for $f_1 = f$ and let $u_n = \{\ell : a_\ell \in \mathbb{B}_{\mathbf{k},\beta_n}\}$, by 1.15 clearly u_n is finite.

Let $\mathbb{B}^0$ be the Boolean algebra extending $\mathbb{B}_\mathbf{k}$ generated by $\mathbb{B}_\mathbf{k} \cup \{x_{\varepsilon,n,\ell} : \ell \le n$ and $\varepsilon < \lambda^+\}$ freely except the equation $x_{\varepsilon,n,\ell} \le a_n, x_{\varepsilon,n,\ell_1} \cap x_{\varepsilon,n,\ell_2} = 0$ and $\bigcup\limits_{\ell \le n} x_{\varepsilon,n,\ell} = a_n$ for $\varepsilon < \lambda^+, \ell \le n, \ell_1 < \ell_2 \le n$ and let $\mathbb{B}$ be the completion on $\mathbb{B}^0$. Let $g_\varepsilon \in \mathbb{N}^\mathbb{B}$ be represented by $\langle x_{g_\varepsilon,\ell} := \cup\{x_{\varepsilon,n,\ell} : n \text{ satisfies } \ell \le n\} : \ell < \omega\rangle$, clearly

$(*)_1$ $g_\varepsilon/D \le f_2/D$ for any $D \in {\rm uf}(\mathbb{B})$.

For $\varepsilon \ne \zeta < \lambda^+$ let $c_{\varepsilon,\zeta} = \bigcup\limits_\ell (x_{g_\varepsilon,\ell} \triangle x_{g_\zeta,\ell})$.

Now

$(*)_2$ $c_{\varepsilon,\zeta} = \bigcup\limits_n \bigcup\limits_{\ell \le n} (x_{\varepsilon,n,\ell} \triangle x_{\zeta,n,\ell})$ and $c_{\varepsilon,\zeta} = c_{\zeta,\varepsilon}$.

[Why? As $\langle x_{\varepsilon,n,\ell} : \ell \le n\rangle$ is a partition of a_n and $\langle a_n : n < \omega\rangle$ is an antichain of $\mathbb{B}$.]

Let $\mathbb{B}'$ be the sub-algebra of $\mathbb{B}$ generated by $\mathbb{B}_\mathbf{k} \cup Y$ where $Y := \{c_{\varepsilon,\zeta} : \varepsilon < \zeta < \lambda^+\}$

$(*)_3$ we define $\pi^1_n : \mathbb{B}_{\mathbf{k}} \cup Y \to \mathbb{B}_{\beta_n}$ by:

- $\pi^1_n {\restriction} \mathbb{B}_{\mathbf{k}} = \pi_{\mathbf{k},\alpha(\mathbf{k}),\beta(n)}$
- $\pi^1_n(c_{\varepsilon,\zeta}) = 1_{\mathbb{B}_{\beta(n)}}$ for $\varepsilon \ne \zeta < \lambda^+$

$(*)_4$ π^1_n has an extension $\pi^2_n \in \mathrm{Hom}(\mathbb{B}', \mathbb{B}_{\beta(n)})$, necessarily unique

[Why? It is enough to show that if $d_0,\dots,d_{m-1} \in \mathbb{B}_{\mathbf{k}}$ and $\varepsilon_\ell < \zeta_\ell < \lambda^+$ for $\ell < k$ and $\sigma(y_0,\dots,y_{m-1},x_0,\dots,x_{k-1})$ is a Boolean term and

$$\mathbb{B} \models \sigma(d_0,\dots,d_{m-1},c_{\varepsilon_0,\zeta_0},\dots,c_{\varepsilon_{k-1},\zeta_{k-1}}) = 0$$

then $\mathbb{B}_{\mathbf{k},\beta(n)} \models \sigma(\pi^1_n(d_0),\dots,\pi^1_n(d_{m-1}),\pi^1_n(c_{\varepsilon_0,\zeta_0},\dots)) = 0$. As $d_0,\dots,d_{m-1} \in \mathbb{B}_{\mathbf{k}}$ and $\pi^1_n(c_{\varepsilon_\ell,\zeta_\ell}) = 1_{\mathbb{B}_{\mathbf{k},\beta(n)}}$ it is sufficient to prove: if $d \in \mathbb{B}_{\mathbf{k}}$ and $\mathbb{B} \models$ "$d \cap \bigcap\limits_{\ell<k} c_{\varepsilon_\ell,\zeta_\ell} = 0$" then $\mathbb{B}_{\mathbf{k},\beta(n)} \models$ "$(\pi_{\mathbf{k},\alpha(\mathbf{k}),\beta(k)}(d)) = 0$".

Now if for some $\ell \ge 1, \ell \notin u_n, d \cap a_\ell > 0$ the assumption does not hold and otherwise, necessarily $d \le \bigcup\limits_{\ell \in u_n} a_\ell$ hence the conclusion holds. So indeed π^2_n is well defined.]

$(*)_5$ $\pi^2_n = \pi_{\mathbf{k},\beta(n+1),\beta(n)} \circ \pi^2_{n+1}$.

[Why? See the definition of π^1_m recalling 1.9(4).]

$(*)_6$ there is $\mathbf{n}$ such that $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{n}$ and $\mathbb{B}_{\mathbf{n}} = \mathbb{B}', \pi_{\mathbf{n},\alpha(\mathbf{k}),\beta(n)} = \pi^2_n$ hence $D_{\mathbf{n}} \supseteq Y$.

[Why? Check the definitions.]

$(*)_7$ there is $\mathbf{m}$ such that $\mathbf{n} \le^{\mathrm{wa}}_K \mathbf{m}, \mathbb{B}_{\mathbf{m}} = \mathbb{B}$ hence $\mathbf{k} \le^{\mathrm{wa}}_K \mathbf{m}$

[Why? By claim 1.8(1),(3), the "hence" by 1.8(5) recalling that $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}$ by the choice of $\mathbb{B}$ and $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}' \subseteq \mathbb{B}$ by the choice of $\mathbb{B}'$.]

$(*)_8$ there is $\varepsilon < \lambda^+$ such that $g_\varepsilon \in A^1_{\mathbf{n},\alpha(\mathbf{k})}$.

[Why? As $A^1_{\mathbf{n},<\alpha(\mathbf{k})}$ has cardinality $\le \lambda$.]

$(*)_9$ $\mathbf{m}$ is as required.

[By $(*)_7 + (*)_8$ and 1.8(2).] $\square_{1.16}$

Observation 1.17. *In claim 1.16 we can demand* $\|\mathbb{B}_{\mathbf{m}}\| = \|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0}$.

Proof. By the Löwenheim-Skolem-Tarski argument. $\square_{1.17}$

Claim 1.18. *Assume* $\mathbf{k} \in K^{\mathrm{cc}}_\alpha$ *and* $\mathrm{cf}(\alpha) > \aleph_0$.

If $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]}$ *and* $f/D \notin \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}/D_{\mathbf{k}}$ *for* $\beta < \alpha$, *<u>then</u> for some* $\mathbf{m}$ *and* g *we have:*

$(*)$
- (a) $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{m}$
- (b) $g \in \mathbb{N}^{\mathbb{B}[\mathbf{m}]}$
- (c) $g/D_{\mathbf{m}} < f/D_{\mathbf{m}}$
- (d) $h/D_{\mathbf{m}} < g/D_{\mathbf{m}}$ <u>*when*</u> $h \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ *for some* $\beta < \alpha$
- (e) $|\mathbb{B}_{\mathbf{m}}| \le |\mathbb{B}_{\mathbf{k}}|$.

Proof. Like the proof of 1.16 only simpler and shorter. Let $\bar{a} = \langle a_n : n < \omega \rangle$ represent $f, \lambda = \|\mathbb{B}_{\mathbf{k}}\|$. By 1.14(4), without loss of generality f is reasonable in $(\mathbf{k}, \alpha)$; let $\{x_{\varepsilon,n,\ell} : \varepsilon < \lambda^+, \ell \leq n\}, \mathbb{B}^0, \mathbb{B}, Y, \mathbb{B}'$ be as there and define $\pi_1 : \mathbb{B}_{\mathbf{k}} \cup Y \to \mathbb{B}_{\mathbf{k},\alpha}$ as there. $\pi_1 {\restriction} \mathbb{B}_{\mathbf{k},\alpha} = \pi_{\mathbf{k},\alpha+1,\alpha}, \pi_1(x_{\varepsilon,\zeta}) = 1_{\mathbb{B}_{\mathbf{k},\alpha}}$ for $\varepsilon < \zeta < \lambda^+$.

The rest is as there. $\square_{1.18}$

Claim 1.19. *If* $\mathbf{k} \in K^{\rm com}_\alpha, \lambda \geq \|\mathbb{B}_{\mathbf{k}}\| + 2^{\aleph_0}$ *and* $p(x)$ *is a type in the model* $\mathbb{N}^{\mathbb{B}[\mathbf{k}]}/D_{\mathbf{k}}$ *then for some* $\mathbf{m} \in K_{\alpha+1}$ *we have* $\mathbf{k} \leq^{\rm ver}_K \mathbf{m}$ *and* $p(x)$ *is realized in* $\mathbb{N}^{\mathbb{B}[\mathbf{m}]}/D_{\mathbf{m}}$.

Proof. Easy. $\square_{1.19}$

Having established all these statements, we can prove now the main result of this paper:

Theorem 1.20. *For any infinite cardinal* λ*, for some regular ultrafilter* D *on* λ *we have* $\mathrm{upf}(D) = \mathscr{C}$ *iff:*

(∗) (a) $\mathscr{C}$ *is a set of cardinals* $\leq 2^\lambda$
(b) $\mu = \mu^{\aleph_0}$ *whenever* $\mu \in \mathscr{C}$
(c) $2^\lambda \in \mathscr{C}$ *hence is the maximal member of* $\mathscr{C}$.

Proof. The implication $\Rightarrow$, we already know, so we shall deal with the $\Leftarrow$ implication; the proof relies on earlier definitions and claims so the reader can return to a second reading.

Let $\langle \lambda_\alpha : \alpha \leq \alpha(*) \rangle$ list $\mathscr{C}$ in increasing order. Let $S = \{\alpha : \alpha \leq \alpha(*) + 1$ and $\mathrm{cf}(\alpha) \neq \aleph_0\}$. We choose $\mathbf{k}_\alpha$ by induction on $\alpha \in S \cap (\alpha(*) + 2)$ such that:

⊞ (a) $\mathbf{k}_\alpha \in K^{\rm com}_\alpha \cap K^{\rm cc}_\alpha$, see Definition 1.2, 1.3(1A)
(b) $\mathbf{k}_\beta \leq^{\rm ver}_K \mathbf{k}_\alpha$ for $\beta \in \alpha \cap S$, see 1.4(2)(B)
(c) if $f \in A^1_{\mathbf{k},\beta}, \beta < \alpha$ then $\lambda_\beta = |\{g/D_{\mathbf{k}} : g \in \mathbb{N}^{\mathbb{B}}, g/D < f(D)\}|$
(d) if $\mathrm{cf}(\alpha) > \aleph_0$ then $\mathbb{B}_{\mathbf{k}_\alpha} = \cup\{\mathbb{B}_{\mathbf{k}_\beta} : \beta < \alpha\}$
(e) $\mathbf{k}_\alpha$ is reasonable (see Definition 1.12)
(f) the set $A^1_{\mathbf{k},\beta}$ has cardinality λ_β.

Case 1: For $\alpha = 0$

$\mathbb{B}_{\mathbf{k}_0}$ is the trivial Boolean Algebra, so really there is nothing to prove.

Case 2: $\mathrm{cf}(\alpha) > \aleph_0$

Use 1.7(2),(3) to find $\mathbf{k}_\alpha$ satisfying clauses (a),(b),(c),(d). Now $\mathbf{k}_\alpha$ satisfies clause (e) by 1.14(6).

Case 3: $\alpha = \beta + 1$

We choose $\mathbf{k}_{\beta,i}$ by induction for $i \leq \lambda_\beta$ such that

(∗) (a) (α) if $\beta \in S$ then $\mathbf{k}_\beta \leq^{\rm ver}_K \mathbf{k}_{\beta,i} \in K^{\rm com}_\alpha \cap K^{\rm cc}_\alpha$
(β) if $\beta \notin S$ then $\gamma \in \beta \cap S \Rightarrow \mathbf{k}_\gamma \leq^{\rm ver}_K \mathbf{k}_{\beta,i} \in K^{\rm com}_\alpha \cap K^{\rm cc}_\alpha$
(γ) if $i = 0$ then there is $g \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$ such that $g/D_{\mathbf{k}_{\beta,i}} \notin \{f/D_{\mathbf{k}_{\beta,i}} : f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_\gamma]}$ for some $\gamma \in \alpha \cap S\}$
(δ) $\mathbb{B}_{\mathbf{k}_{\beta,i}}$ is infinite
(b) $\langle \mathbf{k}_{\beta,j} : j \leq i \rangle$ is $\leq^{\rm at}_K$-increasing continuous
(c) $\mathbb{B}_{\mathbf{k}_{\beta,i}}$ has cardinality $\leq \lambda_\beta$

(d) if $i = j+1$:

(α) bookkeeping gives us $g_{\beta,j} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$ such that $g_{\beta,j}/D_{\mathbf{k}_{\beta,j}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\gamma,\mathbf{k}_\beta]} : \gamma \in \alpha \cap S\}$

(β) there is $f_{\beta,j} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$ such that $f_{\beta,j}/D_{\mathbf{k}_{\beta,i}} < g_{\beta,j}/D_{\mathbf{k}_{\beta,i}}$ and $f_{\beta,j}/D_{\mathbf{k}_{\beta,i}} \notin \cup\{\mathbb{N}^{\mathbb{B}[\gamma,\mathbf{k}_\beta]} : \gamma \in \alpha \cap S\}$

(e) if $i < \lambda_\beta$ and g satisfies $(d)(\alpha)$ then for some $i_1 \in [i, \lambda_\beta], g_{\beta,i_1} = g$

(f) if $i = j+1$ then $\mathbb{B}_{\mathbf{k}_{\beta,i}}$ is complete and reasonable.

Note that by 1.14(1) we can take care of clause (f), so we shall ignore it.

For $i = 0$ we use 1.14(1) if $\beta \in S$ and we use 1.7 if $\beta \notin S$.

For i limit use 1.7(1).

For $i = j+1, \mathrm{cf}(j) > \aleph_0$ use the claim 1.18.

If $i = j+1, \mathrm{cf}(j) = \aleph_0$ use the claim 1.16.

For $i = j+1, \mathrm{cf}(j) = 1$ we use 1.14(1).

Having carried the induction on $i \le \lambda_\beta$ let $\mathbf{k}_\alpha = \mathbf{k}_{\beta,\lambda_\beta}$. In particular $\mathbb{B}_{\mathbf{k}_{\beta,\lambda_\beta}}$ is complete as $\lambda_\beta = \sup\{i < \lambda_\beta : \mathbb{B}_{\mathbf{k},i}$ is complete$\}$ by clause (f) and $\mathrm{cf}(\lambda_\beta) > \aleph_0$ as $\lambda_\beta = \lambda_\beta^{\aleph_0}$.

Having carried the induction on $\alpha \le \alpha(*)+1$ clearly the pair $(\mathbb{B}_{\mathbf{k}_{\alpha(*)+1}}, D_{\mathbf{k}_{\alpha(*)+1}})$ is almost as required. That is, (see [She90, Ch.VI,§3]) we know that for some regular filter D_* on $\mathscr{P}(I)$, there is a homomorphism $\mathbf{j}$ from the Boolean Algebra $\mathscr{P}(I)$ onto $\mathbb{B}_{\mathbf{k}_{\alpha(*)+1}}$ and let $D = \{A \subseteq \lambda : \mathbf{j}(A) \in D_{\mathbf{k}_{\alpha(*)+1}}\}$. $\square_{1.20}$

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