

ON PARTIAL ORDERINGS HAVING PRECALIBRE- \aleph_1 AND FRAGMENTS OF MARTIN'S AXIOM

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ABSTRACT. We define a countable antichain condition (ccc) property for partial orderings, weaker than precalibre- \aleph_1 , and show that Martin's axiom restricted to the class of partial orderings that have the property does not imply Martin's axiom for σ -linked partial orderings. This yields a new solution to an old question of the first author about the relative strength of Martin's axiom for σ -centered partial orderings together with the assertion that every Aronszajn tree is special. We also answer a question of J. Steprans and S. Watson (1988) by showing that, by a forcing that preserves cardinals, one can destroy the precalibre- \aleph_1 property of a partial ordering while preserving its ccc-ness.

A question asked in [1] is if $MA(\sigma\text{-centered})$ plus "Every Aronszajn tree is special" implies $MA(\sigma\text{-linked})$. The interest in this question originated in the result of Harrington-Shelah [5] showing that if \aleph_1 is accessible to reals, i.e., there exists a real number x such that the cardinal \aleph_1 in the model $L[x]$ is equal to the real \aleph_1 , then MA implies that there exists a $\Delta_3^1(x)$ set of real numbers that does not have the Baire property. The hypothesis that \aleph_1 is accessible to reals is necessary, for if \aleph_1 is inaccessible to reals and MA holds, then \aleph_1 is actually weakly-compact in L ([5]), and K. Kunen showed that starting from a weakly compact cardinal one can get a model where MA holds and every projective set of reals has the Baire property. In [1], using Todorćević's ρ -functions ([12]), it was shown that $MA(\sigma\text{-centered})$ plus "Every Aronszajn tree is special" is sufficient to produce a $\Delta_3^1(x)$ of real numbers without the Baire property, assuming $\aleph_1 = \aleph_1^{L[x]}$. Thus, it was natural to ask how weak is $MA(\sigma\text{-centered})$ plus "Every Aronszajn tree is special" as compared to the full MA , and in particular if it implies $MA(\sigma\text{-linked})$. The answer is negative, as it has been observed by D. Chodounsky and J. Zapletal that a finite-support iteration of σ -centered posets combined with the forcing that specializes Aronszajn trees has the Y-c.c. property, and therefore does not add random reals (see [2]). In the first part of the paper we give a new and stronger negative answer to the question, namely we show that a fragment of MA that includes $MA(\sigma\text{-centered})$, and even

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$MA(3\text{-Knaster})$, and implies “Every Aronszajn tree is special”, does not imply $MA(\sigma\text{-linked})$. A partial ordering with the precalibre- \aleph_1 property plays the key role in the construction of the model.

In the second part of the paper we answer a question of Steprans-Watson [9]. They ask if it is possible to destroy the precalibre- \aleph_1 property of a partial ordering, while preserving its ccc-ness, in a forcing extension of the set-theoretic universe V that preserves cardinals. This is a natural question considering that, as shown in [9], on the one hand, assuming MA plus the Covering Lemma, every precalibre- \aleph_1 partial ordering has precalibre- \aleph_1 in every forcing extension of V that preserves cardinals; and on the other hand the ccc property of a partial ordering having precalibre- \aleph_1 can always be destroyed while preserving \aleph_1 , and consistently even preserving all cardinals.

We answer the Steprans-Watson question positively, and in a very strong sense. Namely, we show that it is consistent, modulo ZFC, that the Continuum Hypothesis holds and there exist a forcing notion T of cardinality \aleph_1 that preserves \aleph_1 (and therefore it preserves all cardinals, cofinalities, and the cardinal arithmetic), and two precalibre- \aleph_1 partial orderings, such that forcing with T preserves their ccc-ness, but it also forces that their product is not ccc and therefore they don’t have precalibre- \aleph_1 .

1. PRELIMINARIES

Recall that a partially ordered set (or poset) \mathbb{P} is *ccc* if every antichain of \mathbb{P} is countable; it is *productive-ccc* if the product of \mathbb{P} with any ccc poset is also ccc; it is *Knaster* (or has *property- \mathcal{K}*) if every uncountable subset of \mathbb{P} contains an uncountable subset consisting of pairwise compatible elements. More generally, for $k \geq 2$, \mathbb{P} is *k -Knaster* if every uncountable subset of \mathbb{P} contains an uncountable subset such that any k -many of its elements have a common lower bound. Thus, Knaster is the same as 2-Knaster. \mathbb{P} has *precalibre- \aleph_1* if every uncountable subset of \mathbb{P} has an uncountable subset such that any finite set of its elements has a common lower bound; it is *σ -linked* (or *σ -2-linked*) if it can be partitioned into countably-many pieces so that each piece is pairwise compatible. More generally, for $k \geq 2$, \mathbb{P} is *σ - k -linked* if it can be partitioned into countably-many pieces so that any k -many elements in the same piece have a common lower bound. Finally, \mathbb{P} is *σ -centered* if it can be partitioned into countably-many pieces so that any finite number of elements in the same piece have a common lower bound. We have the following implications, for every $k \geq 2$:

$$\sigma\text{-centered} \Rightarrow \sigma\text{-}k\text{-linked} \Rightarrow k\text{-Knaster} \Rightarrow \text{productive-ccc} \Rightarrow \text{ccc},$$

and

$$\sigma\text{-centered} \Rightarrow \text{precalibre-}\aleph_1 \Rightarrow k\text{-Knaster}.$$

These are the only implications that can be proved in ZFC.

For any property Γ of posets that implies the ccc, and an infinite cardinal κ , *Martin’s Axiom for Γ and for families of κ -many dense open sets*, denoted by $MA_\kappa(\Gamma)$, asserts: for every \mathbb{P} that satisfies the property Γ and every family $\{D_\alpha : \alpha < \kappa\}$ of dense open subsets of \mathbb{P} , there exists a filter $G \subseteq \mathbb{P}$ that is *generic* for the family, that is, $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$.

When $\kappa = \aleph_1$ we omit the subscript and write $MA(\Gamma)$ for $MA_{\aleph_1}(\Gamma)$. Also, for an infinite cardinal θ , the notation $MA_{<\theta}(\Gamma)$ means: $MA_\kappa(\Gamma)$ for all $\kappa < \theta$. The axiom $MA_{\aleph_0}(\Gamma)$ is provable in ZFC; and it is consistent, modulo ZFC, that the Continuum Hypothesis fails and $MA_{<2^{\aleph_0}}(ccc)$ holds (see [7], or [6]). *Martin's axiom*, denoted by MA , is $MA(ccc)$.

Thus, we have the following implications, for every $k \geq 2$:

$$\begin{aligned} MA_\kappa(ccc) &\Rightarrow MA_\kappa(\text{productive-ccc}) \Rightarrow \\ &\Rightarrow MA_\kappa(k\text{-Knaster}) \Rightarrow MA_\kappa(\sigma\text{-}k\text{-linked}) \Rightarrow MA_\kappa(\sigma\text{-centered}), \end{aligned}$$

and

$$MA_\kappa(k\text{-Knaster}) \Rightarrow MA_\kappa(\text{precalibre-}\aleph_1) \Rightarrow MA_\kappa(\sigma\text{-centered}).$$

Again, the arrows cannot be reversed (see [13], [10] for even finer distinctions, and also [11] for Borel examples).

For all the facts mentioned in the rest of the paper without a proof, as well as for all undefined notions and notations, see [6].

2. THE PROPERTY Pr_k

Let us consider the following property of partial orderings, weaker than the k -Knaster property.

Definition 1. For $k \geq 2$, let $Pr_k(\mathbb{Q})$ mean that \mathbb{Q} is a forcing notion such that if $p_\varepsilon \in \mathbb{Q}$, for all $\varepsilon < \aleph_1$, then we can find \bar{u} such that:

- (a) $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$.
- (b) u_ξ is a finite subset of \aleph_1 .
- (c) $u_{\xi_0} \cap u_{\xi_1} = \emptyset$, whenever $\xi_0 \neq \xi_1$.
- (d) If $\xi_0 < \dots < \xi_{k-1}$, then we can find $\varepsilon_l \in u_{\xi_l}$, for $l < k$, such that $\{p_{\varepsilon_l} : l < k\}$ have a common lower bound.

Notice that $Pr_k(\mathbb{Q})$ implies that \mathbb{Q} is ccc, and that $Pr_{k+1}(\mathbb{Q})$ implies $Pr_k(\mathbb{Q})$. Also note that if \mathbb{Q} is k -Knaster, then $Pr_k(\mathbb{Q})$. For given a subset $\{p_\varepsilon : \varepsilon < \aleph_1\}$ of \mathbb{Q} , there exists an uncountable $X \subseteq \aleph_1$ such that $\{p_{\varepsilon_l} : l < k\}$ has a common lower bound, for every $\varepsilon_0 < \dots < \varepsilon_{k-1}$ in X , so we can take u_ξ to be the singleton that contains the ξ -th element of X . Finally, observe that if \mathbb{Q} has precalibre- \aleph_1 , then $Pr_k(\mathbb{Q})$ holds for every $k \geq 2$.

Recall that if T is an Aronszajn tree on ω_1 , then the forcing that specializes T consists of finite functions p from ω_1 into ω such that if $\alpha \neq \beta$ are in the domain of p and are comparable in the tree ordering, then $p(\alpha) \neq p(\beta)$. The ordering is the reversed inclusion. It is consistent, modulo ZFC, that the specializing forcing is not productive-ccc, an example being the case when T is a Suslin tree. However, we have the following:

Lemma 2. If T is an Aronszajn tree and $\mathbb{Q} = \mathbb{Q}_T$ is the forcing that specializes T with finite conditions, then $Pr_k(\mathbb{Q})$ holds, for every $k \geq 2$.

Proof. Without loss of generality, $T = (\omega_1, <_T)$. Let $p_\alpha \in \mathbb{Q}$, for $\alpha < \aleph_1$. By a Δ -system argument we may assume that $\{dom(p_\alpha) : \alpha < \aleph_1\}$ forms a Δ -system, with root r . Moreover, we may assume that for some fixed n , $|dom(p_\alpha) \setminus r| = n$, for all $\alpha < \omega_1$. Let $\langle \alpha_1, \dots, \alpha_n \rangle$ be an enumeration of $dom(p_\alpha) \setminus r$. We may also assume that if $\alpha < \beta$, then the highest level of T

that contains some α_i ($1 \leq i \leq n$) is strictly lower than the lowest level of T that contains some β_j ($1 \leq j \leq n$).

Fix a uniform ultrafilter D over ω_1 . For each $\alpha < \omega_1$ and $1 \leq i, j \leq n$, let

$$D_{\alpha,i,j} := \{\beta > \alpha : \alpha_i <_T \beta_j\}$$

and let

$$D_{\alpha,i,0} := \{\beta > \alpha : \alpha_i \not<_T \beta_j, \text{ all } j\}.$$

For every α and every i , there exists $j_{\alpha,i} \leq n$ such that $D_{\alpha,i,j_{\alpha,i}} \in D$. Moreover, for every $1 \leq i \leq n$, there exists $E_i \in D$ such that $j_{\alpha,i}$ is fixed, say with value j_i , for all $\alpha \in E_i$. We claim that $j_i = 0$, for all $1 \leq i \leq n$. For suppose i is so that $j_i \neq 0$. Pick $\alpha < \beta < \gamma$ in $E_i \cap D_{\alpha,i,j_i} \cap D_{\beta,i,j_i}$. Then $\alpha_i, \beta_i <_T \gamma_{j_i}$, hence $\alpha_i <_T \beta_i$. This yields an ω_1 -chain in T , which is impossible. Now let $E := \bigcap_{1 \leq i \leq n} E_i \in D$.

We claim that for every m and every α we can find $u \in [\omega_1 \setminus \alpha]^m$ such that if $\beta < \gamma$ are in u , then $\beta_i \not<_T \gamma_j$, for every $1 \leq i, j \leq n$. Indeed, given m and α , choose any $\beta^0 \in E \setminus \alpha$. Now given β^0, \dots, β^l , all in E , let $\beta^{l+1} \in E \cap \bigcap_{1 \leq i \leq n} \bigcap_{l' \leq l} D_{\beta^{l'},i,0}$. Then the set $u := \{\beta^0, \dots, \beta^{m-1}\}$ is as required.

We can now choose $\langle u_\xi : \xi < \aleph_1 \rangle$ pairwise-disjoint, with $|u_\alpha| > k \cdot n$, so that if $\xi_1 < \xi_2$, then $\sup(u_{\xi_1}) < \min(u_{\xi_2})$, and each u_ξ is as above, i.e., if $\beta < \gamma$ are in u_ξ , then $\beta_i \not<_T \gamma_j$, for every $1 \leq i, j \leq n$. We claim that $\langle u_\xi : \xi < \aleph_1 \rangle$ is as required. So, suppose $\xi_0 < \dots < \xi_{k-1}$. We choose $\alpha^\ell \in u_{\xi_\ell}$ by downward induction on $\ell \in \{0, \dots, k-1\}$ so that $\{p_{\alpha^\ell} : \ell < k\}$ has a common lower bound. Let α^{k-1} be any element of $u_{\xi_{k-1}}$. Now suppose $\alpha^{\ell+1}, \dots, \alpha^{k-1}$ have been already chosen and we shall choose α^ℓ . We may assume that for each $\beta \in u_{\xi_\ell}$, p_β is incompatible with $p_{\alpha^{\ell'}}$, some $\ell' \in \{\ell+1, \dots, k-1\}$, for otherwise we could take as our α^ℓ any $\beta \in u_{\xi_\ell}$ with p_β compatible with all $p_{\alpha^{\ell'}}$, $\ell' \in \{\ell+1, \dots, k-1\}$. Thus, for each $\beta \in u_{\xi_\ell}$ there exist $\ell' \in \{\ell+1, \dots, k-1\}$ and $1 \leq i, j \leq n$ such that $\beta_i <_T \alpha_j^{\ell'}$. So, since $|u_{\xi_\ell}| > k \cdot n$, there must exist $\beta, \beta' \in u_{\xi_\ell}$ and ℓ' such that $\beta_i, \beta_{i'} <_T \alpha_j^{\ell'}$, for some $1 \leq i, i', j \leq n$ with $\beta_i \neq \beta_{i'}$. But this implies that β_i and $\beta_{i'}$ are $<_T$ -comparable, contradicting our choice of u_{ξ_ℓ} . \square

We show next that the property Pr_k for forcing notions is preserved under iterations with finite support, of any length.

Lemma 3. *For any $k \geq 2$, the property Pr_k is preserved under finite-support forcing iterations. That is, if*

$$\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta; \alpha \leq \lambda, \beta < \lambda \rangle$$

is a finite-support iteration of forcing notions such that $Pr_k(\mathbb{P}_0)$ and $\Vdash_{\mathbb{P}_\beta}$ “ $Pr_k(\mathbb{Q}_\beta)$ ”, for every $\beta < \lambda$, then $Pr_k(\mathbb{P}_\lambda)$.

Proof. By induction on $\alpha \leq \lambda$. For $\alpha = 0$ it is trivial. If α is a limit ordinal with $cf(\alpha) \neq \aleph_1$, and $p_\varepsilon \in \mathbb{P}_\alpha$, for all $\varepsilon < \aleph_1$, then either uncountably many p_ε have the same support (in the case $cf(\alpha) = \omega$) or the support of all p_ε is bounded by some $\alpha' < \alpha$. In either case $Pr_k(\mathbb{P}_\alpha)$ follows easily from the induction hypothesis.

If $cf(\alpha) = \aleph_1$, then we may use a Δ -system argument, as in the usual proof of the preservation of the ccc.

So, suppose $\alpha = \beta + 1$. Let $p_\varepsilon \in \mathbb{P}_\alpha$, for all $\varepsilon < \aleph_1$. Without loss of generality, we may assume that $\beta \in \text{dom}(p_\varepsilon)$, for all $\varepsilon < \aleph_1$.

Since \mathbb{P}_β is ccc, there is $q \in \mathbb{P}_\beta$ such that

$$q \Vdash_{\mathbb{P}_\beta} "|\{\varepsilon : p_\varepsilon \restriction \beta \in \mathcal{G}_\beta\}| = \aleph_1".$$

Let $G \subseteq \mathbb{P}_\beta$ be generic over V and with $q \in G$. In $V[G]$ we have that $p_\varepsilon(\beta)[G] \in \mathbb{Q}_\beta[G]$, and $Pr_k(\mathbb{Q}_\beta[G])$ holds. So, there is $\langle u_\xi^0 : \xi < \aleph_1 \rangle$ as in Definition 1 for the sequence $\langle p_\varepsilon(\beta)[G] : p_\varepsilon \restriction \beta \in G \rangle$. So,

$$q \Vdash_{\mathbb{P}_\beta} "\langle u_\xi^0 : \xi < \aleph_1 \rangle \text{ is as in Definition 1 for } \langle p_\varepsilon(\beta) : p_\varepsilon \restriction \beta \in \mathcal{G}_\beta \rangle".$$

For each ξ , let (q_ξ, u_ξ^1) be such that

$$q_\xi \in \mathbb{P}_\beta \text{ and } q_\xi \leq q.$$

$$q_\xi \Vdash_{\mathbb{P}_\beta} "u_\xi^0 = u_\xi^1", \text{ so } u_\xi^1 \text{ is finite.}$$

$$q_\xi \leq p_\varepsilon \restriction \beta, \text{ for every } \varepsilon \in u_\xi^1. \text{ (This can be ensured because if } \varepsilon \in u_\xi^1, \text{ then } q_\xi \Vdash_{\mathbb{P}_\beta} "p_\varepsilon \restriction \beta \in \mathcal{G}_\beta", \text{ so we may as well take } q_\xi \leq p_\varepsilon \restriction \beta.)$$

Now apply the induction hypothesis for \mathbb{P}_β to obtain $\langle u_\zeta^2 : \zeta < \aleph_1 \rangle$ as in the definition of Pr_k for the sequence $\langle q_\xi : \xi < \aleph_1 \rangle$. We may assume, by refining the sequence if necessary, that $\max(u_\zeta^2) < \min(u_{\zeta'}^2)$ whenever $\zeta < \zeta'$.

Let $u_\zeta^* := \bigcup \{u_\xi^1 : \xi \in u_\zeta^2\}$. We claim that $\bar{u}^* = \langle u_\zeta^* : \zeta < \aleph_1 \rangle$ is as in the definition, for the sequence $\langle p_\varepsilon : \varepsilon < \aleph_1 \rangle$. Clearly, the u_ζ^* are finite and pairwise-disjoint. Moreover, given $\zeta_0 < \dots < \zeta_{k-1}$, we can find $\xi_0 \in u_{\zeta_0}^2, \dots, \xi_{k-1} \in u_{\zeta_{k-1}}^2$ such that in \mathbb{P}_β there is a common lower bound q_* to $\{q_{\xi_0}, \dots, q_{\xi_{k-1}}\}$. Since $q_* \leq q_{\xi_0}, \dots, q_{\xi_{k-1}} \leq q$, there are some $q_{**} \leq q_*$ and $\varepsilon_l \in u_{\xi_l}^1$, for each $l < k$, such that for some \mathbb{P}_β -name p ,

$$q_{**} \Vdash_{\mathbb{P}_\beta} "p \leq_{\mathbb{Q}_\beta} p_{\varepsilon_0}(\beta), \dots, p_{\varepsilon_{k-1}}(\beta)".$$

Then the condition $q_{**} * p$ is a common lower bound for the conditions $p_{\varepsilon_0}, \dots, p_{\varepsilon_{k-1}}$. \square

3. ON FRAGMENTS OF MA

We shall now prove that $MA(Pr_{k+1})$ does not imply $MA(\sigma$ - k -linked), which yields a negative answer to the first question stated in the Introduction. The following is the main lemma.

Lemma 4. *For $k \geq 2$, there is a forcing notion $\mathbb{P}_* = \mathbb{P}_*^k$ and \mathbb{P}_* -names \mathcal{A} and $\mathbb{Q}_{\mathcal{A}} = \mathbb{Q}_{\mathcal{A}}^k$ such that*

- (1) \mathbb{P}_* has precalibre- \aleph_1 and is of cardinality \aleph_1 .
- (2) $\mathbb{P}_* \Vdash_{\mathbb{P}_*} "\mathcal{A} \subseteq [\aleph_1]^{k+1}"$
- (3) $\mathbb{P}_* \Vdash_{\mathbb{P}_*} "\mathbb{Q}_{\mathcal{A}} = \{v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \mathcal{A} = \emptyset\}$, ordered by \supseteq , is σ - k -linked."
- (4) $\mathbb{P}_* \Vdash_{\mathbb{P}_*} "I_\alpha := \{v \in \mathbb{Q}_{\mathcal{A}} : v \not\subseteq \alpha\}$ is dense, all $\alpha < \aleph_1$."

- (5) $\Vdash_{\mathbb{P}_*}$ “If $v_\alpha \in \mathbb{Q}_{\mathcal{A}}$ is such that $v_\alpha \not\subseteq \alpha$, for $\alpha < \aleph_1$; and $u_\xi \in [\aleph_1]^{<\aleph_0}$, for $\xi < \aleph_1$, are non-empty and pairwise disjoint, then there exist $\xi_0 < \dots < \xi_k$ such that for every $\langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_{\xi_\ell}$ the set $\bigcup_{\ell \leq k} v_{\alpha_\ell}$ does not belong to $\mathbb{Q}_{\mathcal{A}}$.”

Proof. We define \mathbb{P}_* by: $p \in \mathbb{P}_*$ if and only if p has the form $(u, A, h) = (u_p, A_p, h_p)$, where

- (a) $u \in [\aleph_1]^{<\aleph_0}$,
- (b) $A \subseteq [u]^{k+1}$, and
- (c) $h : \wp_p \rightarrow \omega$, where $\wp_p := \{v \subseteq u : [v]^{k+1} \cap A = \emptyset\}$, is such that if $w_0, \dots, w_{k-1} \in \wp_p$ and h is constant on $\{w_0, \dots, w_{k-1}\}$, then $w_0 \cup \dots \cup w_{k-1} \in \wp_p$.

The order is given by: $p \leq q$ if and only if $u_q \subseteq u_p$, $A_q = A_p \cap [u_q]^{k+1}$, and $h_q \subseteq h_p$ (hence $\wp_q = \wp_p \cap \mathcal{P}(u_q)$ and $h_p \upharpoonright \wp_q = h_q$).

(1): Clearly, \mathbb{P}_* has cardinality \aleph_1 , so let us show that it has precalibre- \aleph_1 . Given $\{q_\xi = (u_\xi, A_\xi, h_\xi) : \xi < \aleph_1\} \subseteq \mathbb{P}_*$, and writing \wp_ξ instead of the more cumbersome \wp_{q_ξ} , we can find an uncountable $W \subseteq \aleph_1$ such that:

- (i) The set $\{u_\xi : \xi \in W\}$ forms a Δ -system with heart u_* .
- (ii) The sets $[u_*]^{k+1} \cap A_\xi$, for $\xi \in W$, are all the same. Hence the sets $\wp_\xi \cap \mathcal{P}(u_*)$, for $\xi \in W$, are also all the same.
- (iii) The functions $h_\xi \upharpoonright (\wp_\xi \cap \mathcal{P}(u_*))$, for $\xi \in W$, are all the same.
- (iv) The ranges of h_ξ , for $\xi \in W$, are all the same, say R . So, R is finite.
- (v) For each $i \in R$, the sets $\{w \cap u_* : h_\xi(w) = i\}$, for $\xi \in W$, are the same.

We will show that every finite subset of $\{q_\xi : \xi \in W\}$ has a common lower bound. Given $\xi_0, \dots, \xi_m \in W$, let $q = (u_q, A_q, h_q)$ be such that

- $u_q = \bigcup_{\ell \leq m} u_{\xi_\ell}$
- $A_q = \bigcup_{\ell \leq m} A_{\xi_\ell}$. Note that this implies that the \wp_{ξ_ℓ} are contained in $\wp_q = \{v \subseteq u_q : [v]^{k+1} \cap A_q = \emptyset\}$. Indeed, if, say, $w \in \wp_{\xi_\ell}$, then $[w]^{k+1} \cap A_{\xi_\ell} = \emptyset$, and we claim that also $[w]^{k+1} \cap A_{\xi_j} = \emptyset$, for $j \leq m$. For if $v \in [w]^{k+1} \cap A_{\xi_j}$, with $j \neq \ell$, then $v \subseteq u_*$, and therefore $v \in [u_*]^{k+1} \cap A_{\xi_j} = [u_*]^{k+1} \cap A_{\xi_\ell}$. Hence, $v \in [w]^{k+1} \cap A_{\xi_\ell}$, which is impossible because $[w]^{k+1} \cap A_{\xi_\ell}$ is empty.
- $h_q : \wp_q \rightarrow \omega$ is such that $h_q(v) = h_{\xi_\ell}(v)$ for all $v \in \wp_{\xi_\ell}$, and the $h_q(v)$ are all distinct and greater than $\sup\{h_q(v) : v \in \bigcup_{\ell \leq m} \wp_{\xi_\ell}\}$, for $v \notin \bigcup_{\ell \leq m} \wp_{\xi_\ell}$. Notice that h_q is well-defined because the restrictions $h_{\xi_\ell} \upharpoonright (\wp_{\xi_\ell} \cap \mathcal{P}(u_*))$, for $\ell \leq m$, are all the same.

We claim that $q \in \mathbb{P}_*$. For this, we only need to show that if $\{w_0, \dots, w_{k-1}\} \subseteq \wp_q$ and h_q is constant on $\{w_0, \dots, w_{k-1}\}$, then $[\bigcup_{j < k} w_j]^{k+1} \cap A_q = \emptyset$. So fix a set $\{w_0, \dots, w_{k-1}\} \subseteq \wp_q$ and suppose h_q is constant on it, say with constant value i . By definition of h_q we must have $\{w_0, \dots, w_{k-1}\} \subseteq \bigcup_{\ell \leq m} \wp_{\xi_\ell}$. Now suppose, towards a contradiction, that $v \in [\bigcup_{j < k} w_j]^{k+1} \cap A_{\xi_\ell}$, some $\ell \leq m$. Let $s = \{w_j : j < k\} \cap \wp_{\xi_\ell}$, and let $t = \{w_j : j < k\} \setminus s$. Thus, $v \subseteq \bigcup s \cup (\bigcup t \cap u_*)$, for if $\alpha \in v \setminus \bigcup s$, then $\alpha \in \bigcup t$ and $\alpha \in \bigcup \wp_{\xi_{\ell'}}$, for some $\ell' \neq \ell$, hence $\alpha \in u_\xi \cap u_{\xi'} = u_*$.

By (v),

$$\{w \cap u_* : h_{\xi_\ell}(w) = i\} = \{w \cap u_* : h_{\xi_{\ell'}}(w) = i\}$$

for every $\ell' \leq m$. So, for every $w_j \in t$, there exists $w'_j \in \wp_{\xi_\ell}$ such that $w_j \cap u_* = w'_j \cap u_*$ and $h_{\xi_\ell}(w'_j) = i$. Let $t' = s \cup \{w'_j : w_j \in t\}$. Note that $t' \subseteq \wp_{\xi_\ell}$ and $t' \subseteq \{w : h_{\xi_\ell}(w) = i\}$. So,

$$v \subseteq \bigcup t' \subseteq \bigcup \{w : h_{\xi_\ell}(w) = i\}.$$

Thus, $v \in [\bigcup \{w : h_{\xi_\ell}(w) = i\}]^{k+1} \cap A_{\xi_\ell}$. But this is impossible because $\bigcup \{w : h_{\xi_\ell}(w) = i\} \in \wp_{\xi_\ell}$ (since h_{ξ_ℓ} satisfies property (c) above) and therefore

$$[\bigcup \{w : h_{\xi_\ell}(w) = i\}]^{k+1} \cap A_{\xi_\ell} = \emptyset.$$

Now one can easily check that $q \leq q_{\xi_0}, \dots, q_{\xi_m}$. And this shows that the set $\{q_\xi : \xi \in W\}$ is finite-wise compatible.

(2): Let

$$\mathcal{A} = \{(\check{v}, p) : v \in A_p, p \in \mathbb{P}_*\}.$$

Thus, \mathcal{A} is a name for the set $\bigcup \{A_p : p \in G\}$, where G is the \mathbb{P}_* -generic filter. Clearly, (2) holds.

(3): Let

$$\mathbb{Q}_{\mathcal{A}} = \{(\check{v}, p) : v \in \wp_p, p \in \mathbb{P}_*\}.$$

Thus, $\mathbb{Q}_{\mathcal{A}}$ is a name for the set $\bigcup \{\wp_p : p \in G\}$, where G is the \mathbb{P}_* -generic filter. Clearly, $\Vdash_{\mathbb{P}_*} \text{“}\mathbb{Q}_{\mathcal{A}} = \{v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \mathcal{A} = \emptyset\}$ ”. Moreover, if G is \mathbb{P}_* -generic over V , then, by (c), the function $\bigcup \{h_p : p \in G\}$ witnesses that the interpretation $i_G(\mathbb{Q}_{\mathcal{A}})$, ordered by \supseteq , is σ - k -linked.

(4): Clear.

(5): Suppose that $p \in \mathbb{P}_*$ forces $\dot{v}_\alpha \in \mathbb{Q}_{\mathcal{A}}$ is such that $\dot{v}_\alpha \not\subseteq \alpha$, all $\alpha < \aleph_1$; and it also forces $\dot{u}_\xi \in [\aleph_1]^{<\aleph_0}$, all $\xi < \aleph_1$, are non-empty and pairwise disjoint.

For each $\xi < \aleph_1$, let $q_\xi = (u_\xi, A_\xi, h_\xi) \leq p$ and let $u_\xi^* \in [\aleph_1]^{<\aleph_0}$ and $\bar{v}_\xi^* = \langle v_{\xi, \alpha}^* : \alpha \in u_\xi^* \rangle$, with $v_{\xi, \alpha}^* \in [\aleph_1]^{<\aleph_0}$, be such that

$$q_\xi \Vdash_{\mathbb{P}_*} \text{“}\dot{u}_\xi = u_\xi^* \text{ and } \dot{v}_\alpha = v_{\xi, \alpha}^*, \text{ for } \alpha \in u_\xi^* \text{.”}$$

We may assume, by extending q_ξ if necessary, that $u_\xi^* \cup \bigcup_{\alpha \in u_\xi^*} v_{\xi, \alpha}^* \subseteq u_\xi$.

As in (1), we can find an uncountable $W \subseteq \aleph_1$ such that (i)-(v) hold for the set of conditions $\{q_\xi : \xi \in W\}$. Hence $\{q_\xi : \xi \in W\}$ is pairwise compatible (in fact, finite-wise compatible), from which it follows that the set $\{u_\xi^* : \xi \in W\}$ is pairwise disjoint. Now choose $\xi_0 < \dots < \xi_k$ from W so that

- The heart u_* of the Δ -system $\{u_\xi : \xi \in W\}$ is an initial segment of u_{ξ_ℓ} , all $\ell \leq k$,
- $\sup(u_{\xi_\ell}) < \inf(u_{\xi_{\ell+1}} \setminus u_*)$, for all $\ell < k$, and
- $u_{\xi_\ell}^* \subseteq u_{\xi_\ell} \setminus u_*$, for all $\ell \leq k$.

For each $\sigma = \langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_{\xi_\ell}^*$, pick $w_\sigma \in [\bigcup_{\ell \leq k} v_{\xi_\ell, \alpha_\ell}^*]^{k+1}$ such that $|w_\sigma \cap (v_{\xi_\ell, \alpha_\ell}^* \setminus \alpha_\ell)| = 1$, for all $\ell \leq k$. This is possible because $v_{\xi_\ell, \alpha_\ell}^* \not\subseteq \alpha_\ell$.

Claim 5. $w_\sigma \not\subseteq u_{\xi_\ell}$, hence $w_\sigma \not\subseteq A_{\xi_\ell}$, for all $\sigma \in \prod_{\ell \leq k} u_{\xi_\ell}^*$ and all $\ell \leq k$.

Proof of Claim. Fix $\sigma = \langle \alpha_\ell : \ell \leq k \rangle$ and $\ell \leq k$, and suppose, for a contradiction, that $w_\sigma \subseteq u_{\xi_\ell}$. Then $w_\sigma \subseteq u_{\xi_\ell} \setminus u_*$. If $\ell < k$, then since $\sup(u_{\xi_\ell}) < \inf(u_{\xi_{\ell+1}} \setminus u_*) \leq \inf(u_{\xi_{\ell+1}}^*) \leq \alpha_{\ell+1}$, we would have $w_\sigma \setminus \alpha_{\ell+1} = \emptyset$, which contradicts our choice of w_σ . But if $\ell = k$, then since $\sup(v_{\xi_{k-1}, \alpha_{k-1}}^*) \leq \sup(u_{\xi_{k-1}}) < \inf(u_{\xi_k} \setminus u_*)$, we would have $w_\sigma \cap v_{\xi_{k-1}, \alpha_{k-1}}^* = \emptyset$, which contradicts again our choice of w_σ . \square

Now define $q = (u_q, A_q, h_q)$ as follows:

- $u_q = \bigcup_{\ell \leq k} u_{\xi_\ell}$
- $A_q = (\bigcup_{\ell \leq k} A_{\xi_\ell}) \cup \{w_\sigma : \sigma \in \prod_{\ell \leq k} u_{\xi_\ell}^*\}$. Note that since $w_\sigma \not\subseteq u_{\xi_\ell}$ (Claim 5), we have that $w_\sigma \not\subseteq \wp_{\xi_\ell}$, for all $\sigma \in \prod_{\ell \leq k} u_{\xi_\ell}^*$ and $\ell \leq k$. Hence, $\wp_{\xi_\ell} \subseteq \wp_q$, all $\ell \leq k$.
- $h_q : \wp_q \rightarrow \omega$ is such that $h_q(v) = h_{\xi_\ell}(v)$ for $v \in \wp_{\xi_\ell}$, for all $\ell \leq k$, and the $h_q(v)$ are all distinct and greater than $\sup\{h_q(v) : v \in \bigcup_{\ell \leq k} \wp_{\xi_\ell}\}$, for $v \notin \bigcup_{\ell \leq k} \wp_{\xi_\ell}$.

As in (1), we can now check that $q \in \mathbb{P}_*$. Moreover, by Claim 5, $A_{\xi_\ell} = A_q \cap [u_{\xi_\ell}]^{k+1}$. Hence, $q \leq q_{\xi_\ell}$, all $\ell \leq k$, and so

$$q \Vdash_{\mathbb{P}_*} \text{“} \dot{u}_{\xi_\ell} = u_{\xi_\ell}^* \text{ and } \dot{v}_\alpha = v_{\xi_\ell, \alpha}^*, \text{ for } \alpha \in u_{\xi_\ell}^* \text{.”}$$

And since $w_\sigma \in [\bigcup_{\ell \leq k} v_{\alpha_\ell}^*]^{k+1} \cap A_q$, for every $\sigma \in \prod_{\ell \leq k} u_{\xi_\ell}^*$, we have that

$$q \Vdash_{\mathbb{P}_*} \text{“} \bigcup_{\ell \leq k} \dot{v}_{\alpha_\ell} \notin \mathbb{Q}_{\mathcal{A}}, \text{ for all } \langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} \dot{u}_{\xi_\ell} \text{.”}$$

\square

Lemma 6. Let $k \geq 2$ and let \mathbb{P}_* be as in Lemma 4. Suppose \mathbb{Q} is a \mathbb{P}_* -name for a forcing notion that satisfies Pr_{k+1} . Then,

$$\Vdash_{\mathbb{P}_* * \mathbb{Q}} \text{“There is no directed } G \subseteq \mathbb{Q}_{\mathcal{A}} \text{ such that } \dot{I}_\alpha \cap G \neq \emptyset, \text{ all } \alpha < \aleph_1 \text{.”}$$

where \dot{I}_α is a name for the dense open set $\{v \in \mathbb{Q}_{\mathcal{A}} : v \not\subseteq \alpha\}$.

Proof. Suppose, for a contradiction, that $p * \dot{q} \in \mathbb{P}_* * \mathbb{Q}$ and

$$p * \dot{q} \Vdash_{\mathbb{P}_* * \mathbb{Q}} \text{“There exists } G \subseteq \mathbb{Q}_{\mathcal{A}} \text{ directed, with } \dot{I}_\alpha \cap G \neq \emptyset, \text{ all } \alpha < \aleph_1 \text{.”}$$

Suppose $G_0 \subseteq \mathbb{P}_*$ is a filter generic over V , with $p \in G_0$. So, in $V[G_0]$, letting $q = i_{G_0}(\dot{q})$ and $\mathbb{Q} = i_{G_0}(\mathbb{Q})$, we have that for some \mathbb{Q} -name \dot{G} ,

$$q \Vdash_{\mathbb{Q}} \text{“} \dot{G} \subseteq \mathbb{Q}_{\mathcal{A}} \text{ is directed and } I_\alpha \cap \dot{G} \neq \emptyset, \text{ all } \alpha < \aleph_1 \text{.”}$$

For each $\alpha < \aleph_1$, let $q_\alpha \leq q$, and let $v_\alpha \in [\aleph_1]^{< \aleph_0}$ be such that

$$q_\alpha \Vdash_{\mathbb{Q}} \text{“} \check{v}_\alpha \in I_\alpha \cap \dot{G} \text{”}.$$

Thus, $v_\alpha \not\subseteq \alpha$, for all $\alpha < \aleph_1$.

Since \mathbb{Q} satisfies Pr_{k+1} , there exists $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$ such that

- (a) u_ξ is a finite subset of \aleph_1 , all $\xi < \aleph_1$,

- (b) $u_{\xi_0} \cap u_{\xi_1} = \emptyset$ whenever $\xi_0 \neq \xi_1$, and
- (c) if $\xi_0 < \dots < \xi_k$, then we can find $\alpha_\ell \in u_{\xi_\ell}$, for $\ell \leq k$, such that $\{q_{\alpha_\ell} : \ell \leq k\}$ have a common lower bound.

By Lemma 4, we can find $\xi_0 < \dots < \xi_k$ such that for every $\langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_{\xi_\ell}$ the set $\bigcup_{\ell \leq k} v_{\alpha_\ell}$ does not belong to \mathbb{Q}_A .

By (c), let $\alpha_\ell \in u_{\xi_\ell}$, for $\ell \leq k$, be such that $\{q_{\alpha_\ell} : \ell \leq k\}$ have a common lower bound, call it r . Then r forces that $\{\check{v}_{\alpha_\ell} : \ell \leq k\} \subseteq \check{G}$. And since r forces that \check{G} is directed, it also forces that $\bigcup_{\ell \leq k} v_{\alpha_\ell} \in \mathbb{Q}_A$. A contradiction. \square

All elements are now in place to prove the main result of this section.

Theorem 7. *Let $k \geq 2$. Assume $\lambda = \lambda^{<\theta}$, where $\theta = cf(\theta) > \aleph_1$. Then there is a finite-support iteration*

$$\bar{\mathbb{P}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta; \alpha \leq \lambda, \beta < \lambda \rangle$$

where

- (1) \mathbb{P}_0 is the forcing \mathbb{P}_* from Lemma 4.
- (2) $\Vdash_{\mathbb{P}_\beta}$ “ $Pr_{k+1}(\mathbb{Q}_\beta)$ ”, for every $0 < \beta < \lambda$.
- (3) In $V^{\mathbb{P}^\lambda}$ the axiom $MA_{<\theta}(Pr_{k+1})$ holds, hence in particular (Lemma 2) every Aronszajn tree on ω_1 is special.
- (4) \mathbb{Q}_A witnesses that $MA(\sigma$ - k -linked) fails in $V^{\mathbb{P}^\lambda}$.

Proof. To obtain (3), we proceed in the standard way as in all iterations forcing (some fragment of) MA , that is, we iterate all posets with the Pr_{k+1} property and having cardinality $< \theta$, which are given by some fixed book-keeping function (see [6] or [7] for details).

Since after forcing with \mathbb{P}_0 the rest of the iteration $\bar{\mathbb{P}}$ has the property Pr_{k+1} (Lemma 3), (4) follows immediately from Lemma 6. \square

Corollary 8. *For every $k \geq 2$, ZFC plus $MA(Pr_{k+1})$ does not imply $MA(\sigma$ - k -linked).*

Thus, since $MA(Pr_{k+1})$ implies both $MA(\sigma$ -centered) and “Every Aronszajn tree is special”, the corollary answers in the negative and in a strong way the question from [1]: Does $MA(\sigma$ -centered) plus “Every Aronszajn tree is special” imply $MA(\sigma$ -linked)?

4. ON DESTROYING PRECALIBRE- \aleph_1 WHILE PRESERVING THE CCC

We turn now to the second question stated in the Introduction (Steprans-Watson [9]): Is it consistent that there exists a precalibre- \aleph_1 poset which is ccc but does not have precalibre- \aleph_1 in some forcing extension that preserves cardinals?

Note that the forcing extension cannot be ccc, since ccc forcing preserves the precalibre- \aleph_1 property. Also, as shown in [9], assuming MA plus the Covering Lemma, every forcing that preserves cardinals also preserves the precalibre- \aleph_1 property. Moreover, the examples provided in [9] of cardinal-preserving forcing notions that destroy the precalibre- \aleph_1 they do so by actually destroying the ccc property.

A positive answer to Question 1 is provided by the following theorem. But first, let us recall a strong form of Jensen's diamond principle, *diamond-star relativized to a stationary set S* , which is also due to Jensen. For S a stationary subset of ω_1 , let

\diamond_S^* : There exists a sequence $\langle \mathcal{S}_\alpha : \alpha \in S \rangle$, where \mathcal{S}_α is a countable set of subsets of α , such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$, for every $\alpha \in C \cap S$.

The principle \diamond_S^* holds in the constructible universe L , for every stationary $S \subseteq \omega_1$ (see [3], 3.5, for a proof in the case $S = \omega_1$, which can be easily adapted to any stationary S). Also, \diamond_S^* can be forced by a σ -closed forcing notion (see [7], Chapter VII, Exercises H18 and H20, where it is shown how to force the even stronger form of diamond known as \diamond_S^+).

Theorem 9. *It is consistent, modulo ZFC, that the CH holds and there exist*

- (1) *A forcing notion T of cardinality \aleph_1 that preserves cardinals.*
- (2) *Two posets \mathbb{P}_0 and \mathbb{P}_1 of cardinality \aleph_1 that have precalibre- \aleph_1 and such that*

$$\Vdash_T \text{“}\mathbb{P}_0, \mathbb{P}_1 \text{ are ccc, but } \mathbb{P}_0 \times \mathbb{P}_1 \text{ is not ccc.”}$$

Hence \Vdash_T “ \mathbb{P}_0 and \mathbb{P}_1 don't have precalibre- \aleph_1 ”.

Proof. Let $\{S_1, S_2\}$ be a partition of $\Omega := \{\delta < \omega_1 : \delta \text{ a limit}\}$ into two stationary sets. By a preliminary forcing, we may assume that $\diamond_{S_1}^*$ holds. So, there exists $\langle \mathcal{S}_\alpha : \alpha \in S_1 \rangle$, where \mathcal{S}_α is a countable set of subsets of α , such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$, for every $\alpha \in C \cap S_1$. In particular, the CH holds. Using $\diamond_{S_1}^*$, we can build an S_1 -oracle, i.e., an \subset -increasing sequence $\bar{M} = \langle M_\delta : \delta \in S_1 \rangle$, with M_δ countable and transitive, $\delta \in M_\delta$, $M_\delta \models \text{“}ZFC^- + \delta \text{ is countable”}$, and such that for every $A \subseteq \omega_1$ there is a club $C_A \subseteq \omega_1$ such that $A \cap \delta \in M_\delta$, for every $\delta \in C_A \cap S_1$. (For the latter, one simply needs to require that $\mathcal{S}_\delta \subseteq M_\delta$, for all $\delta \in S_1$.) Moreover, we can build \bar{M} so that it has the following additional property:

- (*) For every regular uncountable cardinal χ and a well ordering $<_\chi^*$ of $H(\chi)$, the set of all (universes of) countable $N \preceq \langle H(\chi), \in, <_\chi^* \rangle$ such that the Mostowski collapse of N belongs to M_δ , where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$.

The property (*) will be needed to prove that the tree partial ordering T (defined below) has many branches, and also to prove that the product partial ordering $\mathbb{Q} \times T$ (defined below) is S_1 -proper (Claim 10), and so it does not collapse \aleph_1 .

To ensure (*), take a big-enough regular cardinal λ and define the sequence \bar{M} so that, for every $\delta \in S_1$, M_δ is the Mostowski collapse of a countable elementary substructure X of $H(\lambda)$ that contains $\bar{M} \upharpoonright \delta$, all ordinals $\leq \delta$, and all elements of \mathcal{S}_δ . To see that (*) holds, fix a regular uncountable cardinal χ , a well ordering $<_\chi^*$ of $H(\chi)$, and a club $E \subseteq [H(\chi)]^{\aleph_0}$. Let $\bar{N} = \langle N_\alpha : \alpha < \aleph_1 \rangle$ be an \subset -increasing and \in -increasing continuous chain of elementary substructures of $\langle H(\chi), \in, <_\chi^* \rangle$ with the universe of N_α in E ,

for all $\alpha < \aleph_1$. We shall find $\delta \in S_1$ such that the transitive collapse of N_δ belongs to M_δ , where $\delta = N_\delta \cap \omega_1$.

Fix a bijection $h : \aleph_1 \rightarrow \bigcup_{\alpha < \aleph_1} N_\alpha$, and let $\Gamma : \aleph_1 \times \aleph_1 \rightarrow \aleph_1$ be the standard pairing function (cf. [6], 3). Observe that the set

$$D := \{\delta < \aleph_1 : \delta \text{ is closed under } \Gamma \text{ and } h \text{ maps } \delta \text{ onto } N_\delta\}$$

is a club. Now let

$$\begin{aligned} X_1 &:= \{\Gamma(i, j) : h(i) \in h(j)\} \\ X_2 &:= \{\Gamma(\alpha, i) : h(i) \in N_\alpha\} \\ X_3 &:= \{\Gamma(i, j) : h(i) <^*_\chi h(j)\} \\ X &:= \{3j + i : j \in X_i \text{ and } i \in \{1, 2, 3\}\}. \end{aligned}$$

The set $S'_1 := \{\delta \in S_1 : X \cap \delta \in M_\delta\}$ is stationary. Thus, since the set $C := \{\delta < \aleph_1 : \delta = N_\delta \cap \omega_1\}$ is a club, we can pick $\delta \in C \cap D \cap S'_1$. Since $\delta \in D$, the structure

$$Y := \langle X_2 \cap \delta, \{\langle i, j \rangle : \Gamma(i, j) \in X_1 \cap \delta\}, \{\langle i, j \rangle : \Gamma(i, j) \in X_3 \cap \delta\} \rangle$$

is isomorphic to N_δ , and therefore Y and N_δ have the same transitive collapse. And Y belongs to M_δ , because $\delta \in S'_1$. Hence, since $M_\delta \models ZFC^-$, the transitive collapse of Y belongs to M_δ . Finally, since $\delta \in C$, $\delta = N_\delta \cap \omega_1$.

We shall define now the forcing T . Let us write $\aleph_1^{<\aleph_1}$ for the set of all countable sequences of countable ordinals. Let

$$T := \{\eta \in \aleph_1^{<\aleph_1} : \text{Range}(\eta) \subset S_1, \eta \text{ is increasing and continuous, of successor length, and if } \varepsilon < lh(\eta), \text{ then } \eta \upharpoonright \varepsilon \in M_{\eta(\varepsilon)}\}.$$

Let \leq_T be the partial order on T given by end-extension. Thus, (T, \leq_T) is a tree. Note that, since $\delta \in M_\delta$ for every $\delta \in S_1$, if $\eta \in T$, then $\eta \in M_{\text{supRange}(\eta)}$. Also notice that if $\eta \in T$, then $\eta \frown \langle \delta \rangle \in T$, for every $\delta \in S_1$ greater than $\text{supRange}(\eta)$. In particular, every node of T of finite length has \aleph_1 -many extensions of any bigger finite length. Now suppose $\alpha < \omega_1$ is a limit, and suppose, inductively, that for every successor $\beta < \alpha$, every node of T of length β has \aleph_1 -many extensions of every higher successor length below α . We claim that every $\eta \in T$ of length less than α has \aleph_1 -many extensions in T of length $\alpha + 1$ (and in fact, the set of their suprema is stationary). For every $\delta < \omega_1$, let $T_\delta := \{\eta \in T : \text{supRange}(\eta) < \delta\}$. Notice that T_δ is countable: otherwise, uncountably-many $\eta \in T_\delta$ would have the same $\text{supRange}(\eta)$, and therefore they would all belong to the model $M_{\text{supRange}(\eta)}$, which is impossible because it is countable. Now fix a node $\eta \in T$ of length less than α , and let $B := \{b_\gamma : \gamma < \omega_1\}$ be an enumeration of all the *branches* (i.e., linearly-ordered subsets of T closed under predecessors) b of T that contain η and have length α (i.e., $\bigcup\{\text{dom}(\eta') : \eta' \in b\} = \alpha$). For a club C of δ the set $\{b_\gamma : \gamma < \delta\}$ belongs to M_δ . We shall next build a sequence $B^* := \langle b_\xi^* : \xi < \omega_1 \rangle$ of branches from B so that the set $\text{sup}B^* := \langle \text{supRange}(\bigcup b_\xi^*) : \xi < \omega_1 \rangle$ is the increasing enumeration of a club. To this end, start by fixing an increasing sequence $\langle \alpha_n : n < \omega \rangle$ of successor ordinals converging to α , with α_0 greater than the length of η . Then let $b_0^* := b_0$. Given b_ξ^* , let γ be the least ordinal such that $\bigcup b_\gamma(\alpha_0) > \text{supRange}(\bigcup b_\xi^*)$, and let $b_{\xi+1}^* := b_\gamma$. Finally, given b_ξ^* for all $\xi < \delta$, where $\delta < \omega_1$ is a limit ordinal, pick an increasing sequence $\langle \xi_n : n < \omega \rangle$ converging to δ .

By construction, the sequence $\langle \text{supRange}(\bigcup b_{\xi_n}^*) : n < \omega \rangle$ is increasing. Now let $f : \alpha \rightarrow \aleph_1$ be such that $f \upharpoonright [0, \alpha_0] = \bigcup b_{\xi_0}^* \upharpoonright [0, \alpha_0]$, and $f \upharpoonright (\alpha_n, \alpha_{n+1}] = \bigcup b_{\xi_{n+1}}^* \upharpoonright (\alpha_n, \alpha_{n+1}]$, for all $n < \omega$. Then set $b_\delta^* := \{f \upharpoonright \beta : \beta < \alpha \text{ is a successor}\}$. One can easily check that b_δ^* is a branch of T of length α with $\text{supRange}(\bigcup b_\delta^*) = \text{sup}\{\text{supRange}(\bigcup b_\xi^*) : \xi < \zeta\}$. Finally, notice that if $\delta \in S_1 \cap C$ is greater than α and belongs to the club enumerated by $\text{sup}B^*$, then since $M_\delta \models \text{“}\delta \text{ is countable”}$, we can pick the sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle \xi_n : n < \omega \rangle$ in M_δ . Then the sequence $\langle b_{\xi_n}^* : n < \omega \rangle$ belongs to M_δ , and therefore $(\bigcup b_\delta^*) \frown \delta \in T$.

By (*) the set of all countable $N \preceq \langle H(\aleph_2), \in, <_{\aleph_2}^* \rangle$ that contain B^* and $\langle \alpha_n : n < \omega \rangle$, with $\alpha \subseteq N$, and such that the Mostowski collapse of N belongs to M_δ , where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$. So, since the set $\text{Lim}(\text{sup}B^*)$ of limit points of $\text{sup}B^*$ is a club, there is such an N with $\delta := N \cap \omega_1 \in \text{Lim}(\text{sup}B^*)$. If \bar{N} is the transitive collapse of N , we have that $B^* \upharpoonright \delta \in \bar{N} \in M_\delta$, and so in M_δ we can build, as above, the branch b_δ^* . Therefore, since $\delta = \text{supRange}(\bigcup b_\delta^*)$, we have that $\bigcup b_\delta^* \cup \{\langle \alpha, \delta \rangle\} \in T$ and extends η . We have thus shown that η has \aleph_1 -many extensions in T of length $\alpha + 1$. Even more, the set $\{\text{supRange}(\bigcup b) : b \text{ is a branch of length } \alpha + 1 \text{ that extends } \eta\}$ is stationary.

Note however that since the complement of S_1 is stationary, T has no branch of length ω_1 , because the range of such a branch would be a club contained in S_1 . But since every $\eta \in T$ has extensions of length $\alpha + 1$, for every α greater than or equal to the length of η , forcing with (T, \geq_T) yields a branch of T of length ω_1 .

In order to obtain the forcing notions \mathbb{P}_0 and \mathbb{P}_1 claimed by the theorem, we need first to force with the forcing \mathbb{Q} , which we define as follows. For u a subset of T , let $[u]_T^2$ be the set of all pairs $\{\eta, \nu\} \subseteq u$ such that $\eta \neq \nu$ and η and ν are $<_T$ -comparable. Let

$$\mathbb{Q} := \{p : [u]_T^2 \rightarrow \{0, 1\} : u \text{ is a finite subset of } T\},$$

ordered by reversed inclusion.

It is easily seen that \mathbb{Q} is ccc, and it has cardinality \aleph_1 , so forcing with \mathbb{Q} does not collapse cardinals, does not change cofinalities, and preserves cardinal arithmetic. (In fact, \mathbb{Q} is equivalent, as a forcing notion, to the poset for adding \aleph_1 Cohen reals, which is σ -centered, but we shall not make use of this fact.)

Notice that if $G \subseteq \mathbb{Q}$ is a generic filter over V , then $\bigcup G : [T]_T^2 \rightarrow \{0, 1\}$.

Recall that, for $S \subseteq \aleph_1$ stationary, a forcing notion \mathbb{P} is called S -proper if for all (some) large-enough regular cardinals χ and all (stationary-many) countable $\langle N, \in \rangle \preceq \langle H(\chi), \in \rangle$ that contain \mathbb{P} and such that $N \cap \aleph_1 \in S$, and all $p \in \mathbb{P} \cap N$, there is a condition $q \leq p$ that is (N, \mathbb{P}) -generic. If \mathbb{P} is S -proper, then it does not collapse \aleph_1 . (See [8], or [4] for details.)

Claim 10. *The forcing $\mathbb{Q} \times T$ is S_1 -proper, hence it does not collapse \aleph_1 .*

Proof of the claim. Let χ be a large-enough regular cardinal, and let $<_\chi^*$ be a well-ordering of $H(\chi)$. Let $N \preceq \langle H(\chi), \in, <_\chi^* \rangle$ be countable and such that $\mathbb{Q} \times T$ belongs to N , $\delta := N \cap \aleph_1 \in S_1$, and the Mostowski collapse of N

belongs to M_δ . Fix $(q_0, \eta_0) \in (\mathbb{Q} \times T) \cap N$. It will be sufficient to find a condition $\eta_* \in T$ such that $\eta_0 \leq_T \eta_*$ and (q_0, η_*) is $(N, \mathbb{Q} \times T)$ -generic.

Let

$$\mathbb{Q}_\delta := \{p \in \mathbb{Q} : \text{if } \{\eta, \nu\} \in \text{dom}(p), \text{ then } \eta, \nu \in T_\delta\}.$$

Thus, \mathbb{Q}_δ is countable. Moreover, notice that $T_\delta = T \cap N$, and therefore $\mathbb{Q}_\delta = \mathbb{Q} \cap N$. Hence, T_δ and \mathbb{Q}_δ are the Mostowski collapses of T and \mathbb{Q} , respectively, and so they belong to M_δ .

In M_δ , let $\langle (p_n, D_n) : n < \omega \rangle$ list all pairs (p, D) such that $p \in \mathbb{Q}_\delta$, and D is a dense open subset of $\mathbb{Q}_\delta \times T_\delta$ that belongs to the Mostowski collapse of N . That is, D is the Mostowski collapse of a dense open subset of $\mathbb{Q} \times T$ that belongs to N .

Also in M_δ , fix an increasing sequence $\langle \delta_n : n < \omega \rangle$ converging to δ , and let

$$D'_n := \{(p, \nu) \in D_n : lh(\nu) > \delta_n\}.$$

Clearly, D'_n is dense open.

Note that, as the Mostowski collapse of N belongs to M_δ , we have that $\langle \chi^* \upharpoonright (\mathbb{Q}_\delta \times T_\delta) = (\langle \chi^* \upharpoonright (\mathbb{Q} \times T)) \cap N \in M_\delta$.

Now, still in M_δ , and starting with (q_0, η_0) , we inductively choose a sequence $\langle (q_n, \eta_n) : n < \omega \rangle$, with $q_n \in \mathbb{Q}_\delta$ and $\eta_n \in T_\delta$, and such that if $n = m + 1$, then:

- (a) $p_n \geq q_n$ and $\eta_m <_T \eta_n$.
- (b) $(q_n, \eta_n) \in D'_n$.
- (c) (q_n, η_n) is the $\langle \chi^*$ -least such that (a) and (b) hold.

Then, $\eta_* := (\bigcup_n \eta_n) \cup \{\delta, \delta\} \in T$, and $\eta^* \in M_\delta$, hence $(q_0, \eta_*) \in \mathbb{Q} \times T$. Clearly, $(q_0, \eta_*) \leq (q_0, \eta_0)$. So, we only need to check that (q_0, η_*) is $(N, \mathbb{Q} \times T)$ -generic.

Fix an open dense $E \subseteq \mathbb{Q} \times T$ that belongs to N . We need to see that $E \cap N$ is predense below (q_0, η_*) . So, fix $(r, \nu) \leq (q_0, \eta_*)$. Since \mathbb{Q} is ccc, q_0 is (N, \mathbb{Q}) -generic, so we can find $r' \in \{p : (p, \eta) \in E, \text{ some } \eta\} \cap N$ that is compatible with r . Let n be such that $p_n = r'$ and D_n is the Mostowski collapse of E . Then (p_n, η_n) belongs to the transitive collapse of E , hence to $E \cap N$, and is compatible with (r, ν) , as $(p_n, \eta_*) \leq (p_n, \eta_n)$. \square

We thus conclude that if $G \subseteq \mathbb{Q}$ is a filter generic over V , then in $V[G]$ the forcing T does not collapse \aleph_1 , and therefore, being of cardinality \aleph_1 , it preserves cardinals, cofinalities, and the cardinal arithmetic.

We shall now define the \mathbb{Q} -names for the forcing notions \mathbb{P}_ℓ , for $\ell \in \{0, 1\}$, as follows: in $V^\mathbb{Q}$, let $\tilde{b} = \bigcup \tilde{G}$, where \tilde{G} is the standard \mathbb{Q} -name for the \mathbb{Q} -generic filter over V . Then let

$$\mathbb{P}_\ell := \{(w, c) : w \subseteq T \text{ is finite, } c \text{ is a function from } w \text{ into } \omega \text{ such that if } \{\eta, \nu\} \in [w]_T^2 \text{ and } \tilde{b}(\{\eta, \nu\}) = \ell, \text{ then } c(\eta) \neq c(\nu)\}.$$

A condition (w, c) is stronger than a condition (v, d) if and only if $w \supseteq v$ and $c \supseteq d$.

We shall show that if G is \mathbb{Q} -generic over V , then in the extension $V[G]$, the partial orderings $\mathbb{P}_\ell = \mathbb{P}_\ell[G]$, for $\ell \in \{0, 1\}$, and T are as required.

Claim 11. *In $V[G]$, \mathbb{P}_ℓ has precalibre- \aleph_1 .*

Proof of the claim. Assume $p_\alpha = (w_\alpha, c_\alpha) \in \mathbb{P}_\ell$, for $\alpha < \omega_1$. We shall find an uncountable $S \subseteq \aleph_1$ such that $\{p_\alpha : \alpha \in S\}$ is finite-wise compatible. For each $\delta \in S_2$, let

$$s_\delta := \{\eta \upharpoonright (\gamma+1) : \eta \in w_\delta, \text{ and } \gamma \text{ is maximal such that } \gamma < lh(\eta) \wedge \eta(\gamma) < \delta\}.$$

As η is an increasing and continuous sequence of ordinals from S_1 , hence disjoint from S_2 , the set s_δ is well-defined. Notice that s_δ is a finite subset of $T_\delta := \{\eta \in T : supRange(\eta) < \delta\}$, which is countable.

Let $s_\delta^1 := w_\delta \cap T_\delta$. Note that $s_\delta^1 \subseteq s_\delta$.

Let $f : S_2 \rightarrow \omega_1$ be given by $f(\delta) = \max\{supRange(\eta) : \eta \in s_\delta\}$. Thus, f is regressive, hence constant on a stationary $S_3 \subseteq S_2$. Let δ_0 be the constant value of f on S_3 . Then, $s_\delta \subseteq T_{\delta_0}$, for every $\delta \in S_3$. So, since T_{δ_0} is countable, there exist $S_4 \subseteq S_3$ stationary and s_* such that $s_\delta = s_*$, for every $\delta \in S_4$. Further, there is a stationary $S_5 \subseteq S_4$ and s_*^1 and c_* such that for all $\delta \in S_5$,

$$s_\delta^1 = s_*^1, \quad c_\delta \upharpoonright s_*^1 = c_*, \quad \text{and } \forall \alpha < \delta (w_\alpha \subseteq T_\delta).$$

Hence, if $\delta_1 < \delta_2$ are from S_5 , then not only $w_{\delta_1} \cap w_{\delta_2} = s_*^1$, but also if $\eta_1 \in w_{\delta_1} - s_*^1$ and $\eta_2 \in w_{\delta_2} - s_*^1$, then η_1 and η_2 are $<_T$ -incomparable: for suppose otherwise, say $\eta_1 <_T \eta_2$. If $\gamma + 1 = lh(\eta_1)$, then $\eta_2 \upharpoonright (\gamma + 1) = \eta_1 <_T \eta_2$, and $\eta_2(\gamma) = \eta_1(\gamma) < \delta_2$, by choice of S_5 . Hence, by the definition of s_{δ_2} , $\eta_2 \upharpoonright (\gamma + 1) = \eta_1$ is an initial segment of some member of $s_{\delta_2} = s_*$, and so it belongs to T_{δ_1} , hence $\eta_1 \in s_*^1$, contradicting the assumption that $\eta_1 \notin s_*^1$.

So, $\{p_\delta : \delta \in S_5\}$ is as required. \square

It only remains to show that forcing with T over $V[G]$ preserves the ccc-ness of \mathbb{P}_0 and \mathbb{P}_1 , but makes their product not ccc.

Claim 12. *If G_T is T -generic over $V[G]$, then in the generic extension $V[G][G_T]$, the forcing \mathbb{P}_ℓ is ccc.*

Proof of the claim. First notice that, by the Product Lemma (see [6], 15.9), G is \mathbb{Q} -generic over $V[G_T]$, and $V[G][G_T] = V[G_T][G]$. Now suppose $\tilde{A} = \{(\tilde{w}_\alpha, \tilde{c}_\alpha) : \alpha < \omega_1\} \in V[G_T]$ is a \mathbb{Q} -name for an uncountable subset of \mathbb{P}_ℓ . For each $\alpha < \omega_1$, let $p_\alpha \in \mathbb{Q}$ and (w_α, c_α) be such that $p_\alpha \Vdash "(\tilde{w}_\alpha, \tilde{c}_\alpha) = (w_\alpha, c_\alpha)"$. Let u_α be such that $dom(p_\alpha) = [u_\alpha]_T^2$. By extending p_α , if necessary, we may assume that $w_\alpha \subseteq u_\alpha$, for all $\alpha < \omega_1$. We shall find $\alpha \neq \beta$ and a condition p that extends both p_α and p_β and forces that (w_α, c_α) and (w_β, c_β) are compatible. For this, first extend (w_α, c_α) to (u_α, d_α) by letting d_α give different values in $\omega \setminus Range(c_\alpha)$ to all $\eta \in u_\alpha \setminus w_\alpha$. We may assume that the set $\{u_\alpha : \alpha < \omega_1\}$ forms a Δ -system with root r . Moreover, we may assume that p_α restricted to $[r]_T^2$ is the same for all $\alpha < \omega_1$, and also that d_α restricted to r is the same for all $\alpha < \omega_1$. Now pick $\alpha \neq \beta$ and let $p : [u_\alpha \cup u_\beta]_T^2 \rightarrow \{0, 1\}$ be such that $p \upharpoonright [u_\alpha]_T^2 = p_\alpha$, $p \upharpoonright [u_\beta]_T^2 = p_\beta$, and $p(\{\eta, \nu\}) \neq \ell$, for all other pairs in $[u_\alpha \cup u_\beta]_T^2$. Then, p extends both p_α and p_β , and forces that (u_α, d_α) and (u_β, d_β) are compatible, hence it forces that (w_α, c_α) and (w_β, c_β) are compatible. \square

But in $V[G][G_T]$, the product $\mathbb{P}_0 \times \mathbb{P}_1$ is not ccc. For let $\eta^* = \bigcup G_T$. For every $\alpha < \omega_1$, let $p_\alpha^\ell := (\{\eta^* \upharpoonright (\alpha + 1)\}, c_\alpha^\ell) \in \mathbb{P}_\ell$, where $c_\alpha^\ell(\eta^* \upharpoonright (\alpha + 1)) = 0$. Then the set $\{(p_\alpha^0, p_\alpha^1) : \alpha < \omega_1\}$ is an uncountable antichain. \square

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