CLOSED SETS WHICH CONSISTENTLY HAVE FEW TRANSLATIONS COVERING THE LINE

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ABSTRACT. We characterize the compact subsets K of 2^ω for which one can force the existence of a set X of cardinality less than the continuum such that $K + X = 2^\omega$.

1. Introduction

In this this note we answer a variant of the following well-known question: For which compact subsets K of the real line can one force that the real line is covered by fewer than continuum many translations of K (as reinterpreted in the forcing extension)? This question has been considered by several authors and the following are known.

- The real line is not covered by fewer than $2^{\aleph_0}$ many translations of the ordinary Cantor set. (Gruenhage)
- If C has packing dimension less than 1 then $\mathbb{R}$ is not covered by fewer than $2^{\aleph_0}$ translations of C. (Darji-Keleti, [3]),
- There is a compact set K of measure zero such that $\mathbb{R}$ is covered by $\mathsf{cof}(\mathcal{N})$ (which is consistently $< 2^{\aleph_0}$) many translations of K (Elekes-Steprāns, [4]). The same holds in any locally compact Abelian Polish group. (Elekes-Toth, [5]).

Instead of the real line, we will work in the space 2^ω, with addition as coordinate-wise addition modulo 2. For all sets $X, K \subseteq 2^\omega$, and any $z \in 2^\omega$, $X \subseteq K + z$ if and only if $z \notin (2^\omega \setminus K) + X$ (this observation uses the fact that $-z = z$ for all $z \in 2^\omega$). Replacing K with its complement, this says that 2^ω is covered by the set of translations of K by elements of X if and only if X is not covered by a single translation of $2^\omega \setminus K$. The following lemma shows then that we can restrict our attention to compact sets K which are nowhere dense and have measure zero with respect to the standard product measure on 2^ω.

Lemma 1. *Let K be a closed subset of 2^ω.*

(1) If K is somewhere dense then 2^ω is covered by finitely many translations of K.
(2) If K has positive measure and $\mathsf{non}(\mathcal{N}) < 2^{\aleph_0}$ then 2^ω is covered by fewer than $2^{\aleph_0}$ many translations of K.

Proof. For the first part, if K is somewhere dense then it contains a basic open set, which implies that a finite set of translations of K covers 2^ω. For the second, if $\mathsf{non}(\mathcal{N}) < 2^{\aleph_0}$

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there there exists a set $X \subseteq 2^\omega$ of cardinality less than $2^{\aleph_0}$ and outer measure 1. Suppose that K has positive measure. Since $X \not\subseteq (2^\omega \setminus K) + z$ for any $z \in 2^\omega$ it follows that $K + X = 2^\omega$. □

Known proofs that fewer than $2^{\aleph_0}$ many translations of a given set K do not cover 2^ω are based on the following property.

Definition 2. *Let K be a subset of 2^ω. We say that K is* small *if there exists a perfect set $P \subseteq 2^\omega$ such that for every $z \in 2^\omega$,*

$$(K + z) \cap P \text{ is countable.}$$

If K is small then we need $2^{\aleph_0}$ translations of K to cover 2^ω since we need that many translations to cover P. Furthermore, for closed K, the property "K is small" is Σ^1_2 in a parameter for K, hence absolute. To see this, note that K is small if and only if there exists P such that

(1) P is closed and uncountable, and
(2) $\forall z\ (K + z) \cap P$ is countable.

The first clause is $\mathbf{\Sigma}^1_1$ and the second is $\mathbf{\Pi}^1_1$, by the well-known fact that

$$\{W \in \mathcal{K}(2^\omega) : W \text{ is countable}\}$$

is a $\mathbf{\Pi}^1_1$ set, where $\mathcal{K}(2^\omega)$ is the hyperspace of compact subsets of 2^ω (see Section 33.B of [7]).

The notion of being small can be generalized as follows:

Definition 3. *Suppose that K is a subset of 2^ω, Y is a subset of 2^ω and $\mathcal{J}$ is an ideal on Y. We say that K is $\mathcal{J}$-small if for every $z \in 2^\omega$, $(K + z) \cap Y \in \mathcal{J}$.*

In particular, K is small if it is $\mathcal{J}$-small for $\mathcal{J}$ the ideal of countable subsets of some fixed perfect set. The following lemma connects the previous definition with the topic of this paper.

Lemma 4. *Suppose that X, Y and K are subsets of 2^ω, and that $\mathcal{J}$ is an ideal on Y such that K is $\mathcal{J}$-small. If $X + K = 2^\omega$, then $|X| \geq \mathsf{cov}(\mathcal{J})$.*

A compact subset K of 2^ω, being closed, can be viewed as the set of paths through the tree $T_K = \{x \restriction n \mid x \in K, n \in \omega\}$. This tree gives rise to a natural reinterpretation of K in any forcing extension, using the set of paths through T_K in that extension.

Theorem 5 is the main result of this paper. Theorem 38, in conjunction with Theorem 48, gives a more precise formulation.

Theorem 5. *Suppose that K is a compact set in 2^ω. Then exactly one of the following holds.*

(1) In some forcing extension, 2^ω is covered by fewer than continuum many translations of the reinterpretation of K.

(2) There exist a set $Y \subseteq 2^\omega$ of size $2^{\aleph_0}$ and an ideal $\mathcal{J}$ on Y such that

(a) K is $\mathcal{J}$-small, and

(b) $\mathsf{cov}(\mathcal{J}) = 2^{\aleph_0}$.

The theorem easily gives that if the second case holds, then it holds in all forcing extensions. In fact, our characterization of the dichotomy is absolute between models of set theory with the same ordinals (this is not a corollary of Theorem 5 as stated, but see Remark 32).

The paper is organized as follows. In Section 2 we give a simple criterion which implies that in a c.c.c. forcing extension fewer than $2^{\aleph_0}$ translations of K cover 2^ω. In Section 3 we give examples of sets that satisfy this criterion. Section 4 reviews basic information about Sacks forcing, and Section 5 introduces a rank function on Sacks names for reals. In Sections 6 and 7 we prove the two parts of the main result, and give necessary and sufficient conditions for a compact set K to cover 2^ω with fewer than $2^{\aleph_0}$ many translations.

2. Special cases and simple tests

In this section we introduce a property of a set $K \subseteq 2^\omega$ which implies that in some c.c.c. forcing extension fewer than $2^{\aleph_0}$ translations of the corresponding reinterpretation of K cover 2^ω.

Definition 6. *Let s be a sequence in $2^{<\omega}$, let $n \in \omega$ be an integer and let $K \subseteq 2^\omega$ be a perfect set. An integer $j \in \omega$* witnesses that K is big in $[s]$ for n-tuples *if for all $X \in [2^\omega]^{\leq n}$ and $x \in 2^\omega$, if*

(1) $[s] \not\subseteq (2^\omega \setminus K) + X$ and

(2) $x \restriction j \in X \restriction j = \{y \restriction j : y \in X\}$,

then

$$[s] \not\subseteq (2^\omega \setminus K) + (X \cup \{x\}).$$

We say that K is big *if for each $n \in \omega$ there exists a $j_n \in \omega$ which witnesses that K is big in 2^ω for n-tuples. We say that K is* big* *if there exists an unbounded set $\{j_n : n \in \omega\} \subseteq \omega$ such that for each $n \in \omega$, $j_n \in \omega$ witnesses, for each $s \in 2^\omega$ with $|s| \leq j_n$ and $K \cap [s] \neq \emptyset$, that K is big in $[s]$ for n-tuples.*

If K is big* then the collection of finite sets covered by translations of K resembles an ideal, in the following sense: if $X_0, X_1 \subseteq 2^\omega$ are sets of size n, $(2^\omega \setminus K) + X_0 \neq 2^\omega$ and $X_0 \restriction j_{2n} = X_1 \restriction j_{2n}$, then $(2^\omega \setminus K) + (X_0 \cup X_1) \neq 2^\omega$. The assertions that K is big and big* are each Π^1_2 in a code for K, and therefore absolute.

Lemma 7. *If K is big then K is not small.*

Proof. Suppose that $P \subseteq 2^\omega$ is a perfect set. Build recursively a sequence

$$\{x_n : n \in \omega\} \subseteq P$$

such that

(1) $Q = \mathsf{cl}(\{x_n : n \in \omega\})$ is perfect,
(2) for all $n \in \omega$, $(2^\omega \setminus K) + \{x_m : m < n\} \neq 2^\omega$.

Given $\{x_m : m < n\}$ satisfying (2), choose $x_n \in P$ such that $x_n \restriction j_n = x_m \restriction j_n$ for some $m < n$. This will guarantee that (2) continues to hold. Since P is perfect, condition (1) can be arranged by careful bookkeeping, ensuring that each x_n is in the closure of $\{x_m : m \in \omega \setminus \{n\}\}$.

By (2), $L_n = \{z \in 2^\omega : \{x_m : m < n\} \subseteq K + z\}$ is a nonempty compact set, for each $n \in \omega$. For $z \in \bigcap_n L_n$, we have $\{x_n : n \in \omega\} \subseteq K + z$, and thus $Q \subseteq K + z$. □

The following theorem is essentially proved in [5].

Theorem 8. *If K is big* then there exists $X \subseteq 2^\omega$ such that $|X| \leq \mathsf{cof}(\mathcal{N})$ and*

$$X + K = 2^\omega.$$

We prove an alternate version of this theorem, as follows.

Theorem 9. *If K is big*, then there is a c.c.c. forcing extension in which 2^ω is covered by fewer than continuum many translations of the reinterpretation of K.*

Let $\mathbb{Q} = \{q \in 2^\omega : \forall^\infty n\ q(n) = 0\}$. Before beginning the proof, we prove the following lemma.

Lemma 10. *Suppose that $K \subseteq 2^\omega$ is big*. There exists a c.c.c. forcing notion $\mathbb{P}_K$ which adds real $z_K \in 2^\omega$ such that*

$$\Vdash_{\mathbb{P}_K} \forall x \in 2^\omega \cap \mathbf{V}\ \exists q \in \mathbb{Q}\ x \in K + z_K + q.$$

Proof. Suppose that $K \subseteq 2^\omega$ is big*. Let $\mathbb{P}_K$ be the collection of pairs (t, X) such that

(1) $t \in 2^{<\omega}$ and X is a finite subset of 2^ω,
(2) $K \cap [t] \neq \emptyset$,
(3) $[t] \not\subseteq ((2^\omega \setminus K) + X)$.

For $(t_0, X_0), (t_1, X_1) \in \mathbb{P}_K$, we put $(t_1, X_1) \geq (t_0, X_0)$ if $t_0 \subseteq t_1$ and $X_0 \subseteq X_1$. We will show that $\mathbb{P}_K$ has the required properties.

To see that $\mathbb{P}_K$ is c.c.c., suppose that $\{(t_\alpha, X_\alpha) : \alpha < \omega_1\}$ is a subset of $\mathbb{P}_K$. Without loss of generality we can assume that there exist $t \in 2^{<\omega}$ and $n \in \omega$ such that $t_\alpha = t$ and $|X_\alpha| = n$ for all $\alpha < \omega_1$. Let $\{j_m : m \in \omega\}$ witness that K is big*. Letting $m \in \omega \setminus 2n$ be such that $j_m \geq |t|$, we can assume that $X_\alpha \restriction j_m = X_\beta \restriction j_m$ for all $\alpha, \beta < \omega_1$. It follows then that $(t, X_\alpha \cup X_\beta) \in \mathbb{P}_K$ is a condition extending both (t_α, X_α) and (t_β, X_β).

Let $z_K = \bigcup\{t : (t, X) \in G\}$, where G is the generic filter. To see that z_K is as desired, fix a $\mathbb{P}_K$-condition (t, X), and let $x \in (2^\omega)^V$ be given. Let $n = |X|$. Find $q \in \mathbb{Q}$ such that $q + x\restriction j_n \in X\restriction j_n$. Since K is big*, it follows that $(t, X \cup \{x + q\}) \in \mathbb{P}_K$. Furthermore,

$$(t, X \cup \{x + q\}) \Vdash_{\mathbb{P}_K} x \in K + z_K + q,$$

since it forces that the generic real z_K will be in the closure of $x + q + K$.

In particular,

$$\mathbf{V}^{\mathbb{P}_K} \models 2^\omega \cap \mathbf{V} \subseteq K + z_K + \mathbb{Q},$$

which finishes the proof. □

Proof of Theorem 9. Let $\mathbf{V}[g]$ be a c.c.c. extension of the universe satisfying $\neg\,\mathsf{CH}$ and let $\mathbb{P}_{\omega_1}$ be the finite support iteration of $\mathbb{P}_K$ of length $\aleph_1$ defined in $\mathbf{V}[g]$. Let H be $\mathbf{V}[g]$-generic for $\mathbb{P}_{\omega_1}$. For each $\alpha < \omega_1$, let H_α denote the restriction of H to the first α many stages of $\mathbb{P}_{\omega_1}$, and let z_α be the generic real added at the αth stage. Let

$$X = \{z_\alpha + q : \alpha < \omega_1, q \in \mathbb{Q}\}.$$

For each $x \in 2^\omega \cap \mathbf{V}[g, H]$ there is an $\alpha < \omega_1$ such that $x \in \mathbf{V}[g, H_\alpha]$, and it follows that for some $q \in \mathbb{Q}$, $x \in K + z_\alpha + q$. Thus in $\mathbf{V}[g, H]$, $2^\omega \subseteq X + K$ and $|X| < 2^{\aleph_0}$. □

3. Examples of big sets and small sets

In this section we will provide some examples of small sets and big* sets. Fix a partition $\{I_n : n \in \omega\}$ of ω into finite sets of increasing size, and let K_n be a subset of 2^{I_n}, for each $n \in \omega$. Consider a set of the form $K = \prod_n K_n$. This is a typical compact set in 2^ω whose combinatorial properties are hereditary with respect to all full subtrees, i.e. subtrees of form $K \cap [s]$, where $K \cap [s] \neq \emptyset$ and $s \in 2^{<\omega}$. In particular if such set is big it is also big*.

Theorem 11. *If* $\lim_{n\to\infty} \dfrac{|K_n|}{|2^{I_n}|} = 1$ *then* K *is big**.

We use the following lemma.

Lemma 12 ([5]). *Suppose that* $I \subseteq \omega$ *is finite, and* $n \in \omega$ *and* $C \subseteq 2^I$ *are such that*

$$\frac{|C|}{2^{|I|}} \geq 1 - \frac{1}{n+1}.$$

For any $X \subseteq 2^I$ *of size* $\leq n$ *there exists* $t \in 2^I$ *such that* $t + X \subseteq C$.

Proof. For any $s \in X$,

$$\frac{|\{t \in 2^I : t + s \notin C\}|}{|2^I|} \leq \frac{1}{n+1}.$$

Thus

$$\frac{|\{t \in 2^I : \exists s \in X\ t + s \notin C\}|}{|2^I|} \leq \frac{n}{n+1} < 1.$$

□

Proof of Theorem 11. For each $n \in \omega$, let $j_n = \sum_{m \leq k} |I_m|$, where k is such that

$$|K_j|/2^{|I_j|} \geq 1 - \frac{1}{n+2}$$

for all $j \geq k$. Then for any $n \in \omega$, $s \in 2^{<\omega}$ of length at most j_n, $X \subseteq 2^\omega$ of size at most n and $x \in 2^\omega$, repeated application of Lemma 12 will produce a translation as desired (the initial segment of the translation up to j_n being given by the assumption that some translation in $[s]$ already covers X). □

If the sets I_n are large enough then we can choose sets K_n $(n \in \omega)$ so that

$$1 - \frac{1}{n+1} \leq \frac{|K_n|}{|2^{I_n}|} \leq 1 - \frac{1}{2n+1}$$

holds for all $n \in \omega$. Then $K = \prod_{n \in \omega} K_n$ has measure zero since $\prod_{n \in \omega} \dfrac{1}{2n+1} = 0$.

The next two lemmas show that if $\lim_{n \to \infty} \dfrac{|K_n|}{|2^{I_n}|} < 1$ then K may be small or big*, depending on the choice of K_n's. In the following lemma, the sets K_n can be chosen so that the ratios $\dfrac{|K_n|}{|2^{I_n}|}$ are eventually any given dyadic rational value in the interval $[0, \dfrac{1}{2}]$.

Lemma 13. *For each $n \in \omega$, let J_n be a nonempty proper subset of I_n, and let K_n be the set of $s \in 2^{I_n}$ such that $s(i) = 0$ for all $i \in J_n$. Then $K = \prod_{n \in \omega} K_n$ is small.*

Proof. Put $J = \bigcup_n J_n$ and let $P = \{x \in 2^\omega : \forall n \notin J \ x(n) = 0\}$. For each $z \in 2^\omega$, $(K + z) \cap P$ has at most one element. □

Lemma 14. *Fix a sequence of positive reals $\{\varepsilon_n : n \in \omega\}$. There exists a sequence $K_n \subseteq 2^{I_n}$ such that for each n, $|K_n|/2^{|I_n|} \leq \varepsilon_n$ and $K = \prod_{n \in \omega} K_n$ is big*.*

Lemma 14 can be proved in the same way as Lemma 12, with the following theorem (which is Theorem 3.3 of [1], with $1 - \varepsilon$ in place of ε) used instead of Lemma 11.

Theorem 15 ([1])**.** *Suppose that $m \in \omega$ and $0 < \delta < 1 - \epsilon < 1$. There exists $n \in \omega$ such that for every finite set $I \subseteq \omega$ of size at least n, there exists a set $C \subseteq 2^I$ such that $\varepsilon + \delta \geq |C| \cdot 2^{-|I|} \geq \varepsilon - \delta$ and for every set $X \subseteq 2^I$, $|X| \leq m$*

$$\left| \frac{|\bigcap_{s \in X}(C + s)|}{2^{|I|}} - \varepsilon^{|X|} \right| < \delta.$$

Theorem 15 says that we can choose C is such a way that for all sequences $s_1, \ldots, s_m$ in 2^I the sets $s_1 + C, \ldots, s_m + C$ are probabilistically independent with error δ.

Proof of Lemma 14. Thus, if we choose δ to be much smaller than ε^m, then if $|X| < m$ it follows that $\bigcap_{s \in X}(C + s) \neq \emptyset$. In particular, if $t \in \bigcap_{s \in X}(C + s)$ then $t + X \subseteq C$.

The rest of the argument is just like in Theorem 11. □

4. The Sacks model

In the following section we will describe necessary and sufficient conditions for a compact set K to (consistently) cover 2^ω by fewer than $2^{\aleph_0}$ translations. This characterization is intrinsically connected to the Sacks model.

The Sacks model, obtained by a length ω_2 countable support iteration of perfect set forcing, is a natural candidate to witness that $\aleph_1$ translations of a compact set K covers 2^ω. This follows from Zapletal's work on tame cardinal invariants in [13]. More specifically, we have the following:

Definition 16. *A tame cardinal invariant is defined as*

$$\min\{|A| : A \subseteq \mathbb{R} \;\&\; \phi(A) \& \; \psi(A)\}$$

where $\phi(A)$ is a statement of the model $\langle TC(A), \in, A\rangle$ and $\psi(A)$ is a statement of form "$\forall x \in \mathbb{R}\ \exists y \in A\ \theta(x, y)$", where θ is a formula whose quantifiers range over reals and ω only.

If $K \subseteq 2^\omega$ is a compact set than

$$\min\{|A| : A \subseteq 2^\omega \quad \forall x \in 2^\omega\ \exists y \in A\ x + y \in K\}$$

is a tame cardinal invariant.

Theorem 17 (Zapletal [13])**.** *Assuming the existence of a proper class of inaccessible cardinals δ which are limits of Woodin cardinals and of $<\delta$-strong cardinals, if $\mathfrak{x}$ is a tame cardinal invariant, and $\mathfrak{x} < 2^{\aleph_0}$ holds in a set forcing extension, then $\mathfrak{x} < 2^{\aleph_0}$ holds in the iterated Sacks extension.*

A natural attempt would be to show that if K is not small then in the Sacks model $\mathbf{V}^{\mathbb{S}_{\omega_2}}$,

$$\forall x \in 2^\omega\ \exists z \in \mathbf{V} \cap 2^\omega\ x \in K + z.$$

Translating to the Sacks model it would suffice that the following statement holds:

Proposition 18 (false)**.** *Suppose that $p \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in 2^\omega$. Then there exists $p' \geq p$ and a perfect set $P \subseteq 2^\omega$ such that for every perfect set $Q \subseteq P$ there exists $q \geq p'$ such that $q \Vdash \dot{x} \in Q$.*

Indeed, suppose that K is not small and let $p \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in 2^\omega$. If there is $p' \geq p$ and $x \in \mathbf{V} \cap 2^\omega$ such that $p' \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} = x$ then any $z \in (K + x) \cap \mathbf{V}$ will be as required. Otherwise, find $p' \geq p$ and P as in Proposition 18. Since K is not small there is $z \in 2^\omega$ such that $P \cap (K + z)$ is uncountable. Let $Q \subseteq P \cap (K + z)$ be a perfect set. It follows that there is $q \geq p'$ such that $q \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in Q \subseteq K + z$. Since $\dot{x}$ was arbitrary, this finishes the proof.

Proposition 18 is true for a single Sacks forcing but fails for an iteration of two or more Sacks reals. To see this note that if $(p, \dot{q}) \Vdash_{\mathbb{S}\star\mathbb{S}} \dot{x} \in 2^\omega$, then $(p, \dot{q})$ can be represented as a closed subset $\bar{p} \subseteq 2^\omega \times 2^\omega$, where $p = \{x : (\bar{p})_x \neq \emptyset\}$, and $(\bar{p})_x \in \mathbb{S}$ whenever $(\bar{p})_x \neq \emptyset$. Furthermore, we can find a one-to-one continuous function $f : \bar{p} \longrightarrow 2^\omega$ such that $\bar{p} \Vdash_{\mathbb{S}\star\mathbb{S}} \dot{x} = f(s_0, s_1)$, where s_0, s_1 are first and second Sacks reals. Let $x_0 \in p$ be a real that is not Sacks-generic (for example a real that is in $\mathbf{V}$) , and put

$$Q = \{z : \exists y \in (\bar{p})_{x_0}\ z = f(x_0, y)\}.$$

Clearly Q is a perfect set (since $(\bar{p})_{x_0}$ is and f is one-to-one) and $\bar{p} \Vdash \dot{x} \notin Q$ (since x_0 is not Sacks-generic).

In spite of this counterexample, the basic idea in the Proposition 18 is sound and in the sequel we will look for a largeness condition on Q such that Proposition 18 is true for the iteration as well. Then we will require that K is such that for some $z \in 2^\omega$, $P \cap (K + z)$ satisfies this condition.

We begin with a review of well known properties of Sacks forcing and its iterations.

Sacks forcing $\mathbb{S}$ is defined as the collection of perfect subtrees of $2^{<\omega}$ ordered by inclusion (we write $T \geq T'$ to indicate that $T \subseteq T'$). We will often identify a tree T with the corresponding (perfect) set $[T]$ consisting of its branches, and use letters p, q, etc. to refer to these perfect sets. Given a closed set $p \subseteq 2^\omega$, we let $\mathsf{split}(p)$ be the set of $s \in 2^{<\omega}$ such that $s^\frown\langle 0\rangle$ and $s^\frown\langle 1\rangle$ are both initial segments of members of p. For each $n \in \omega$, we let $\mathsf{split}_n(p)$ be the set of $s \in \mathsf{split}(p)$ having exactly n proper initial segments in $\mathsf{split}(p)$.

For $T, T' \in \mathbb{S}$ and $n \in \omega$ define

$$T \geq_n T' \iff T \geq T' \ \&\ T{\restriction}n = T'{\restriction}n.$$

Lemmas 19-23 are taken from [2] (which in turn is modeled after [8]). Lemmas 19 and 20 are well known (see, for instance, pages 244-245 of [6]).

Lemma 19. *Suppose that $p \in \mathbb{S}$ and $p \Vdash_{\mathbb{S}} \dot{x} \in 2^\omega$. For every $n \in \omega$ there exist $q \geq_n p$ and a continuous function $F : [q] \longrightarrow 2^\omega$ such that $q \Vdash_{\mathbb{S}} \dot{x} = F(\dot{g})$, where $\dot{g}$ is the canonical name for the generic real. Moreover, we can require that for every $v \in \mathsf{split}_n(q)$ and any $x_1, x_2 \in [q_v]$, $F(x_1){\restriction}n = F(x_2){\restriction}n$.*

Lemma 20. *Suppose that $p \in \mathbb{S}$, $n \in \omega$ and $p \Vdash_{\mathbb{S}} \dot{x} \in 2^\omega$. Let $F : [p] \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{S}} \dot{x} = F(\dot{g})$.*

There exists $q \geq p$ such that $F{\restriction}[q]$ is constant, or there exists $q \geq_n p$ such that $F{\restriction}[q]$ is one-to-one. In particular, the generic real is minimal.

For each ordinal $\gamma \leq \omega_2$, we let $\mathbb{S}_\gamma$ denote the countable support iteration of $\mathbb{S}$ of length γ. So $\mathbb{S}_\gamma$ is the set of functions p such that

(1) $\mathsf{dom}(p) = \gamma$,
(2) $\mathsf{supp}(p) = \{\beta : p(\beta) \neq \emptyset\}$ is countable,
(3) $\forall \beta < \gamma\ p{\restriction}\beta \Vdash_{\mathbb{S}_\beta} p(\beta) \in \mathbb{S}$.

For $F \in [\gamma]^{<\omega}$, $n \in \omega$, and $p, q \in \mathbb{S}_\gamma$ define

$$q \geq_{F,n} p \iff q \geq p \ \& \ \forall \beta \in F \ q{\restriction}\beta \Vdash_{\mathbb{S}_\beta} q(\beta) \geq_n p(\beta).$$

For $p \in \mathbb{S}_\gamma$ let $\mathsf{cl}(p)$ be the smallest set $w \subseteq \gamma$ such that p can be evaluated using the generic reals $\langle \dot{g}_\beta : \beta \in w \rangle$. In other words, $\mathsf{cl}(p)$ consists of those $\beta < \gamma$ such that the transitive closure of p (as a set) contains a $\mathbb{S}_\beta$-name for an element of $\mathbb{S}$. It is well-known [12] that $\{p \in \mathbb{S}_\gamma : \mathsf{cl}(p) \in [\gamma]^{\leq\omega}\}$ is dense in $\mathbb{S}_\gamma$.

Suppose that $p \in \mathbb{S}_\gamma$, $w = \mathsf{cl}(p)$ is countable and $\gamma_p = \mathsf{ot}\big(\mathsf{cl}(p)\big)$. Let $\mathbb{S}_w$ be the countable support iteration of $\mathbb{S}$ with domain w. In other words, consider the countable support iteration $\langle \mathcal{P}_\beta, \dot{\mathcal{Q}}_\beta : \beta < \sup(w) \rangle$ such that

$$\forall \beta < \sup(w) \ \Vdash_{\mathcal{P}_\beta} \dot{\mathcal{Q}}_\beta \simeq \begin{cases} \mathbb{S} & \text{if } \beta \in w \\ \emptyset & \text{if } \beta \notin w \end{cases}.$$

It is clear that $\mathbb{S}_w$ is forcing-equivalent to $\mathbb{S}_{\gamma_p}$. Moreover, we can view the condition p as a member of $\mathbb{S}_w$.

Let γ be a countable ordinal and $p \in \mathbb{S}_\gamma$. Define $\overline{p} \subseteq (2^\omega)^\gamma$ as follows:

$$\langle x_\beta : \beta < \gamma \rangle \in \overline{p}$$

if for every $\beta < \gamma$,

$$x_\beta \in \Big[p(\beta)\big[\langle x_\gamma : \gamma < \beta \rangle\big]\Big].$$

Note that $p(\beta)[\langle x_\gamma : \gamma < \beta \rangle]$ is the interpretation of $p(\beta)$ using reals $\langle x_\gamma : \gamma < \beta \rangle$ so it may be undefined if these reals are not sufficiently generic.

For a set $G \subseteq (2^\omega)^\gamma$, $u \subseteq \gamma$, and $x \in (2^\omega)^u$ let

$$(G)_x = \{y \in (2^\omega)^{\gamma \setminus u} : \exists z \in G \ z{\restriction}u = x \ \& \ z{\restriction}(\gamma \setminus u) = y\},$$

and for $\beta \in \gamma$ let $(G)_\beta = \{x(\beta) : x \in G\}$.

We say that $p \in \mathbb{S}_\gamma$ is *good* if

(1) $\overline{p}$ is compact,
(2) for every $\beta < \gamma$ and $x \in \overline{p{\restriction}\beta}$, $\overline{p[x]} = (\overline{p})_x$ and $\overline{p(\beta)[x]} = ((\overline{p})_x)_\beta$.
(3) $\overline{p}$ is homeomorphic to $(2^\omega)^\gamma$ via a homeomorphism h such that for every $\beta < \gamma$ and $x \in \overline{p{\restriction}\beta}$, $h{\restriction}((\overline{p})_x)_\beta$ is a homeomorphism between $((\overline{p})_x)_\beta$ and 2^ω.

Lemma 21. *$\{p \in \mathbb{S}_\gamma : \overline{p}$ is good$\}$ is dense in $\mathbb{S}_\gamma$.*

From now on we will always work with conditions p such that $\overline{p}$ is good.

As in the lemma 19 we show that:

Lemma 22. *Suppose that $p \in \mathbb{S}_\gamma$ and $p \Vdash_{\mathbb{S}_\gamma} \dot{x} \in 2^\omega$. Then there exists $q \geq p$ and a continuous function $F : \overline{p} \longrightarrow 2^\omega$ such that $q \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \gamma \rangle$ is the sequence of generic reals.*

The following lemma is an analogue of Lemma 20.

Lemma 23. *Suppose that $p \in \mathbb{S}_\gamma$, $n \in \omega$ and $p \Vdash_{\mathbb{S}_\gamma} \dot{x} \in 2^\omega$. Let $F : \overline{p} \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \gamma \rangle$ is the sequence of generic reals. There exists $q \geq p$ such that exactly one of the following conditions hold:*

(1) $F{\restriction}\overline{q}$ is constant,

(2) $F{\restriction}\overline{q}$ is one-to-one.

5. A rank function

In this section we will work towards formulating a correct version of Proposition 18. Let K be a perfect subset of 2^ω and fix a tree $\widetilde{T}$ such that $K = [\widetilde{T}]$.

Our main objective is to find property of K which will lead to the following dichotomy:

Let K be a compact subset of 2^ω and let $\mathbb{P}$ be a partial order forcing CH. Either

$$\mathbf{V}^{\mathbb{P}*\mathbb{S}_{\omega_2}} \models K + (\mathbf{V}^{\mathbb{P}} \cap 2^\omega) = 2^\omega$$

or, in all outer models of ZFC,

$$\forall X \subseteq 2^\omega \ (|X| < 2^{\aleph_0} \to K + X \neq 2^\omega).$$

We need to look only at iterations of Sacks forcing of countable length over models of CH.

Lemma 24. *The following are equivalent for a model $\mathbf{V} \models \mathsf{CH}$:*

(1) $\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega$,

(2) for every $\gamma < \omega_1$, $\mathbf{V}^{\mathbb{S}_\gamma} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega$.

Proof. Implication (1) $\to$ (2) is obvious. To show that (2) $\to$ (1) observe that every real in $\mathbf{V}^{\mathbb{S}_{\omega_2}}$ depends only on countably many Sacks reals. If $\mathbf{G} \subseteq \mathbb{S}_{\omega_2}$ is a generic filter over $\mathbf{V}$ and $x \in \mathbf{V}[\mathbf{G}] \cap 2^\omega$ then there exists a countable ordinal γ and a $\mathbf{H} \subseteq \mathbb{S}_\gamma$ generic filter over $\mathbf{V}$ which belongs to $\mathbf{V}[\mathbf{G}]$ such that $x \in \mathbf{V}[\mathbf{H}]$. It follows that

$$\mathbf{V}[\mathbf{H}] \models \exists z \in 2^\omega \cap \mathbf{V} \ x \in K + z.$$

By absoluteness, the same holds in $\mathbf{V}[\mathbf{G}]$. □

Definition 25. *For $\gamma < \omega_1$ let $\mathbf{Q}_\gamma$ be the collection of triples $\vec{p} = (p, F, T)$ where $p \in \mathbb{S}_\gamma$ is good and $F \colon \bar{p} \to [T]$ is a homeomorphism.*

Elements of $\mathbf{Q}_\gamma$ represent $\mathbb{S}_\gamma$-names for elements of 2^ω. By Lemma 22, when

$$p \Vdash_{\mathbb{S}_\gamma} \dot{x} \in 2^\omega$$

we can find a homeomorphism $F \colon \bar{p} \to P$ such that $p \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, possibly after passing to a stronger condition. In place of F, we typically use a function $F' = F \circ h$, for some homeomorphism $h \colon (2^\omega)^\gamma \to \bar{p}$. Since F is a homeomorphism, every branch of T reconstructs the entire generic sequence of γ Sacks reals.

Definition 26. *Suppose that $(p, F, T) \in \mathbf{Q}_\gamma$. Given $u \in \mathsf{split}(T)$ and $\alpha < \gamma$, let $\mathsf{proj}_\alpha(u)$ be the portion of α-th Sacks real computed by u.*

The notation $\mathsf{proj}_\alpha(u)$ suppresses the parameter (p, F, T), which will be clear in context. Since $[u]$ is a clopen set, $\mathsf{proj}_\alpha(u) \neq \emptyset$ only for finitely many $\alpha < \gamma$. More precisely, we have the following:

Lemma 27. *For every $u \in \mathsf{split}(T)$ there is $A_u \in [\gamma]^{<\omega}$ such that*

$$F^{-1}([u]) = \{x \in \bar{p} : \forall \alpha \in A_u \ \mathsf{proj}_\alpha(u) \subseteq x(\alpha)\}.$$

The following lemma is central to our argument.

Lemma 28. *Suppose that $(p, F, T) \in \mathbf{Q}_\gamma$. For every $v \in \mathsf{split}(T)$, and any $\delta \in \gamma$ there are nodes $t_0, t_1 \in \mathsf{split}(T)$ such that*

(1) $v \subseteq t_0, t_1$,

(2) $\mathsf{proj}_\delta(t_0)$, $\mathsf{proj}_\delta(t_1)$ are incompatible,

(3) $\mathsf{proj}_\alpha(t_0) = \mathsf{proj}_\alpha(t_1)$ for $\alpha < \delta$.

Proof. Let $A_v \in [\gamma]^{<\omega}$ be such that $F^{-1}[v] = \{x \in \bar{p} : \forall \alpha \in A_v \ \mathsf{proj}_\alpha(u) \subseteq x(\alpha)\}$. Choose two branches $x_0, x_1 \in F^{-1}[v]$ such that $x_0(\alpha) = x_1(\alpha)$ for all $\alpha < \delta$ and $x_0(\delta) \neq x_1(\delta)$. Recall that we assumed that $\dot{x}$ depends on all Sacks reals so this is always possible. Now $F(x_0)$ and $F(x_1)$ are two branches extending v. Let $n \in \omega$ be so large that $\mathsf{proj}_\delta(F(x_0)\restriction n)$, $\mathsf{proj}_\delta(F(x_1)\restriction n)$ are incompatible.

Now let $t_0 = F(x_0)\restriction n$ and $t_1 = F(x_1)\restriction n$. Since $x_0(\alpha) = x_1(\alpha)$ for $\alpha < \delta$, it follows that $\mathsf{proj}_\alpha(t_0) = \mathsf{proj}_\alpha(t_1)$ for all $\alpha < \delta$. □

In the proof above, n may have to be quite large to determine that $x_0(\delta)\restriction n \neq x_1(\delta)\restriction n$, and its value depends on F and T. To illustrate this point suppose that we are dealing with just two Sacks reals and $\dot{x}$ is a name for the sum of them. Even if we know that the first digit of $\dot{x}$ is 0 we only know that the first digits of both Sacks reals are the same. It depends on the tree T how far we have to extend v to determine the value of the first digit of either Sacks real.

Definition 29. *Given a tree $T \subseteq 2^{<\omega}$ we let $\mathsf{obj}(T)$ be the collection of triples*

$$x = (n_x, t_x, s_x)$$

such that

(1) $n_x \in \omega$,

(2) $t_x \subseteq T$ is a finite tree whose all maximal nodes have length n_x,

(3) $s_x \in 2^{n_x}$.

For $x = (n_x, t_x, s_x)$ and $y = (n_y, t_y, s_y)$ we say that $x \geq y$ if

(1) $n_x \geq n_y$,

(2) $t_x \cap 2^{n_y} = t_y$,

(3) $s_y \subseteq s_x$.

Let $\mathbf{0}$ be $(0, \emptyset, \emptyset)$, the smallest element in $\mathsf{obj}(T)$.

The following definition is modeled after Lemma 28.

Definition 30. *Given $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$ and $x = (n_x, t_x, s_x)$ in $\mathsf{obj}(T)$, a $(\vec{p}, x)$-*challenge *is a pair (v, ξ) such that v is a maximal node of t_x and $\xi < \gamma$. We say that y is a* response *to (v, ξ) if*

(1) $y \geq x$,

(2) there are maximal nodes $t_0, t_1 \in t_y$ such that

  (a) $v \subseteq t_0, t_1$,

  (b) $\mathsf{proj}_\xi(t_0)$, $\mathsf{proj}_\xi(t_1)$ are incompatible and

  (c) $\forall \zeta < \xi$ $\mathsf{proj}_\zeta(t_0) = \mathsf{proj}_\zeta(t_1)$.

Definition 31. *Suppose that $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$ and $K = [\widetilde{T}]$ is a fixed compact set. The rank function $\mathsf{rk}_{\vec{p}} : \mathsf{obj}(T) \longrightarrow \omega_1 \cup \{\infty\}$ is defined as follows.*

(1) $\mathsf{rk}_{\vec{p}}(x) = 0$ if $t_x + s_x \not\subseteq \widetilde{T} \cap 2^{n_x}$,

(2) $\mathsf{rk}_{\vec{p}}(x) \geq \alpha > 0$ if for every $\beta < \alpha$, and every $(\vec{p}, x)$-challenge (v, ξ) there exists a response $y \in \mathsf{obj}(T)$ with $\mathsf{rk}_{\vec{p}}(y) \geq \beta$.

Let $\mathsf{rk}_{\vec{p}}(x) = \infty$ if $\mathsf{rk}_{\vec{p}}(x) \geq \alpha$ for all α.

By Lemma 28, if x is in $\mathsf{obj}(T)$ and $t_x + s_x \subseteq \widetilde{T} \cap 2^{n_x}$ then $\mathsf{rk}_{\vec{p}}(x)$ is equal to

$$\min_{\xi < \gamma} \min_{v \in t_x \cap 2^{n_x}} \sup\{\mathsf{rk}_{\vec{p}}(y) + 1 : y \geq x,\ y \text{ responds to } (\vec{p}, x)\text{-challenge } (v, \xi)\}.$$

Remark 32. *For $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$, the members of $\mathsf{obj}(T)$ are hereditarily finite, and the function $\mathsf{rk}_{\vec{p}}$ depends only on $\mathsf{obj}(T)$ and $\vec{p}$. It follows that $\mathsf{rk}_{\vec{p}}$ takes the same values in every wellfounded model of ZFC containing $\vec{p}$. Similarly, the existence of a countable ordinal γ and $\vec{p} \in \mathbf{Q}_\gamma$ such that the corresponding rank function $\mathsf{rk}_{\vec{p}}$ takes value γ at $\mathbf{0}$ is a Σ^1_2 statement, so absolute to models of ZFC containing ω_1.*

Lemma 33. *If $x \leq y$ then $\mathsf{rk}_{\vec{p}}(x) \geq \mathsf{rk}_{\vec{p}}(y)$.*

Proof. If (v, ξ) is a $(\vec{p}, y)$-challenge then $(v \restriction n_x, \xi)$ is a $(\vec{p}, x)$-challenge. □

Lemma 34. *Suppose that $x \in \mathsf{obj}(T)$ and $y \geq x$ is a response to $(\vec{p}, x)$-challenge (v, ξ). Then there exists a minimal $x \leq y' \leq y$ which responds to (v, ξ).*

Proof. Suppose that $x = (n_x, t_x, s_x)$ and $y = (n_y, t_y, s_y)$. First find $n_x \leq n_{y'} \leq n_y$ such that $t_0 \restriction n_{y'}, t_1 \restriction n_{y'}$ are still responses to (v, ξ). Let $t_{y'}$ consist of these two nodes plus one extension of length $n_{y'}$ for each maximal node of t_x. □

Observe that in the definition of rank we can limit ourselves to extensions that are minimal in the above sense.

Lemma 35. *Suppose that $\mathsf{rk}_{\vec{p}}(x) = \infty$ and $\xi < \gamma$. Then there exists $y \geq x$ such that*

(1) $\mathsf{rk}_{\vec{p}}(y) = \infty$,

(2) for every maximal node $v \in t_x$*, y responds to the* $(\vec{p}, x)$*-challenge* (v, ξ).

Proof. Let $v_1, \dots, v_k$ be a list of maximal nodes of t_x. Let $x_0 = x$ and define by recursion a sequence $x_1, \dots, x_k = y$ such that for every $i < k$,

(1) $x_{i+1} \geq x_i$,

(2) $\mathsf{rk}_{\vec{p}}(x_i) = \infty$,

(3) for every $j > i$, v_j has a unique maximal extension $v_j^\star$ in t_{x_i},

(4) x_{i+1} is a response to the $(\vec{p}, x_i)$-challenge $(v_{i+1}^\star, \xi)$.

If x_i is already constructed then by the induction hypothesis v_i has a unique extension $v_i^\star$ in x_i. Let x_{i+1} be any maximal extension of x_i responding to $(v_i^\star, \xi)$ with

$$\mathsf{rk}_{\vec{p}}(x_{i+1}) = \infty.$$

It is easy to see that $y = x_k$ has required properties. □

The definition of rank depends on the set K. The following examples relate it to the concepts from previous sections. Lemma 36 below follows from the general theorem which we are aiming to prove (Theorem 38), but here we will provide a direct argument. The weakening of the Lemma 37 with "big*" in place of "big" is an immediate corollary of Theorems 9 and 38 (and the remarks on absoluteness in Remark 32 and after Definition 6).

Lemma 36. *Suppose that* $\vec{p} = (p, F, T)$ *and* $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$*. Then there exists* $z \in 2^\omega$ *such that* $K \cap (z + [T])$ *is uncountable. In particular, if* $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ *then K is not small.*

Proof. Suppose that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$. Recursively construct a sequence $\langle x_k : k \in \omega \rangle$ such that for every k,

(1) $x_k = \langle n_{x_k}, t_{x_k}, s_{x_k} \rangle$,

(2) $x_{k+1} \geq x_k$,

(3) $\mathsf{rk}_{\vec{p}}(x_k) = \infty$,

(4) x_{k+1} responds to all $(\vec{p}, x_k)$-challenges $(v, 1)$ for each maximal node $v \in t_{x_k}$.

For the step (4) we use Lemma 35 with $\xi = 1$.

Let $\bar{T} = \bigcup_k t_{x_k}$ and $z = \bigcup_k s_{x_k}$. It follows that $\bar{T}$ is a perfect tree and

$$[\bar{T}] \subseteq [\widetilde{T}] + z.$$

□

Lemma 37. *Suppose that K is big. Then* $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ *for all* $\gamma < \omega_1$ *and all* $\vec{p} \in \mathbf{Q}_\gamma$.

Proof. Fix $\gamma < \omega_1$ and let $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$. It suffices to to find a tree $T' \subseteq T$ and a real $z \in 2^\omega$ such that

(1) $[T'] \subseteq [\widetilde{T}] + z$,

(2) for all $v \in \mathsf{split}(T')$ and all $\delta < \gamma$ there are nodes t_0, t_1 such that
  (a) $v \subseteq t_0, t_1$,
  (b) $\mathsf{proj}_\delta(t_0)$, $\mathsf{proj}_\delta(t_1)$ are incompatible,
  (c) $\forall \eta < \delta\ \mathsf{proj}_\eta(t_0) = \mathsf{proj}_\eta(t_1)$.

If we succeed in finding such T' and z then for every $x = (n_x, t_x, s_x) \in \mathsf{obj}(T)$ satisfying

(1) $t_x \subseteq T' \cap 2^{n_x}$
(2) $s_x \subseteq z$

we have $\mathsf{rk}_{\vec{p}}(x) > 0$. Working by induction, one can show then that $\mathsf{rk}_{\vec{p}}(x) = \infty$ for all such x.

We refine the argument from Lemma 7. We build recursively a sequence

$$\{y_n : n \in \omega\} \subseteq [T]$$

such that

(1) $\mathsf{cl}(\{y_n : n \in \omega\})$ is a perfect set,
(2) for all $n \in \omega$, $(2^\omega \setminus K) + \{y_m : m < n\} \neq 2^\omega$,
(3) for every $t \in \{y_n{\restriction}i : i, n \in \omega\} \cap \mathsf{split}(T)$ and every $\eta < \gamma$, there exist $m, p < \omega$ and $j \in \omega$ such that
  (a) $t \subseteq y_m, y_p$,
  (b) $\mathsf{proj}_\eta(y_m{\restriction}j)$, $\mathsf{proj}_\eta(y_p{\restriction}j)$ are incompatible,
  (c) $\forall \gamma < \eta\ \mathsf{proj}_\gamma(y_m{\restriction}j) = \mathsf{proj}_\gamma(y_p{\restriction}j)$.

The first two conditions can be obtained as in the proof of Lemma 7. The third can be obtained using Lemma 28, along with the assumption that K is big.

Arguing as in the last paragraph of Lemma 7, we find $z \in 2^\omega$ such that

$$\{y_s : s < f\} \subseteq K + z.$$

Let T' be a tree such that $[T']$ is the closure of $\{y_s : s < f\}$. Observe that T' has the required properties. □

The following theorem is a refinement of Theorem 5. It characterizes the compact sets K that require continuum many translations to cover 2^ω in all forcing extensions. The second half of the theorem is proved in Section 6. The first half is proved in Section 8.

Theorem 38. *Suppose that K is a compact subset of 2^ω. If for some $\gamma < \omega_1$ there exists a $\vec{p} \in \mathbf{Q}_\gamma$ such that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) < \omega_1$, then 2^ω is not covered by fewer than $2^{\aleph_0}$ many translations of K. If CH holds and $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ for all $\gamma < \omega_1$ and all $\vec{p} \in \mathbf{Q}_\gamma$, then*

$$\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega.$$

6. The consistency result

In this section we will show the second part of Theorem 38.

Theorem 39. *Suppose that* CH *holds. If for every $\gamma < \omega_1$ and every $p \in \mathbf{Q}_\gamma$, $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ then for*

$$\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega.$$

The proof of this theorem will occupy the rest of this section. As we observed in Lemma 24, it suffices to show that for every $\gamma < \omega_1$,

$$\mathbf{V}^{\mathbb{S}_\gamma} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega.$$

Fix $\gamma < \omega_1$. We have to show that for every real $x \in \mathbf{V}^{\mathbb{S}_\gamma} \cap 2^\omega$ there exists $z \in \mathbf{V} \cap 2^\omega$ such that $x \in K + z$.

Suppose that $x \in \mathbf{V}^{\mathbb{S}_\gamma} \cap 2^\omega$. Without loss of generality, x depends on all the generic reals, i.e., γ is minimal. We can find $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$ such that

$$p \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}}),$$

where $\mathbf{g} = \langle g_\beta : \beta < \gamma \rangle$ is the sequence of Sacks reals. We need to find $q \in \mathbb{S}_\gamma$ and $z \in \mathbf{V} \cap 2^\omega$ such that $q \Vdash_{\mathbb{S}_\gamma} \dot{x} \in K + z$. We will construct sequences

$$\langle x_k = (n_{x_k}, t_{x_k}, s_{x_k}) : k \in \omega \rangle$$

and $\langle \xi_k : k \in \omega \rangle$ such that

(1) $x_0 = \mathbf{0}$,
(2) $\forall \xi < \gamma \; \exists^\infty k \; \xi_k = \xi$,
(3) $x_{k+1} \geq x_k$,
(4) $\mathsf{rk}_{\vec{p}}(x_k) = \infty$,
(5) x_{k+1} responds to every $(\vec{p}, x_k)$-challenge (v, ξ_k).

Suppose that x_k is already constructed. To get x_{k+1} apply Lemma 35 with $\xi = \xi_k$.

Let $\bar{T} = \bigcup_k t_{x_k}$ and $z = \bigcup_k s_{x_k}$. It follows that $[\bar{T}] + z \subseteq [\widetilde{T}] = K$, i.e., $[\bar{T}] \subseteq K + z$.

Claim 40. *There exists $q \in \mathbb{S}_\gamma$ such that $\bar{q} = F^{-1}([\bar{T}])$.*

The claim finishes the proof, as $q \Vdash_{\mathbb{S}_\gamma} \dot{x} \in [\bar{T}] \subseteq K + z$.

Proof of claim. Let $Q = F^{-1}([\bar{T}])$, we want to show that there is $q \in \mathbb{S}_\gamma$ such that $\bar{q} \subseteq Q$. It suffices to show that for every $\beta < \gamma$ and every $x \in (2^\omega)^\beta$, $((Q)_x)_\beta$ is a perfect set provided that $((Q)_x)_\beta \neq \emptyset$. In other words, whenever x simulates the first β Sacks reals, $((Q)_x)_\beta$ is supposed to be a Sacks condition determined by x. Note that $((Q)_x)_\beta$ is a closed set, so it is a set of branches of some tree. Choose a $v \in 2^{<\omega}$ such that $[v] \cap ((Q)_x)_\beta \neq \emptyset$. It remains to check that v has two incompatible extensions t_0, t_1 such $[t_0] \cap ((Q)_x)_\beta \neq \emptyset$ and $[t_1] \cap ((Q)_x)_\beta \neq \emptyset$. Let $x^\star \in (2^\omega)^\gamma$ be such that $x^\star \restriction \beta = x$ and $v \subseteq x^\star(\beta)$ and let $y^\star = F(x^\star)$. By Lemma 27 for each $n \in \omega$ there is A_n such that $F^{-1}([y^\star \restriction n]) = \{x : \forall \alpha \in A_n \; \mathsf{proj}_\alpha(y^\star \restriction n) \subseteq x(\alpha)\}$. Let n and k be chosen so large that

(1) $\beta = \xi_k$,
(2) $v \subseteq \mathsf{proj}_\beta(y^\star{\restriction}n)$,
(3) $y^\star{\restriction}n$ is a maximal node in t_{x_k}.

In other words, at this step we will produce nodes t_0, t_1 such that

(1) $y^\star{\restriction}n \subseteq t_0, t_1$,
(2) $\mathsf{proj}_\beta(t_0)$, $\mathsf{proj}_\beta(t_1)$ are incompatible,
(3) $\forall \zeta < \beta$ $\mathsf{proj}_\zeta(t_0) = \mathsf{proj}_\zeta(t_1)$.

It follows that t_0 and t_1 are two incompatible extensions of v in $((Q)_x)_\beta$. □

7. Coherent club-guessing principles

The argument in Section 8 uses the coherent club-guessing principle given by Remark 47 below, which is obtained easily from the one in Theorem 45. First we prove Theorem 41, a stronger version of the restriction of Theorem 45 to the case of successors of regular cardinals. The material in this section is entirely due to the third author, but the proof of Theorem 41 was provided to us by Assaf Rinot. The sets C_α in Theorem 41 are not closed, but they are cofinal. In Theorem 45 the condition of cofinality is dropped as well.

Theorem 41. *Let λ be a regular uncountable cardinal, let $\theta < \lambda$ be a limit ordinal, and let S be a stationary subset of λ^+ consisting of ordinals of cofinality $\mathsf{cof}(\theta)$. Then there exists a sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ such that each C_α is a subset of the corresponding α, and, for each club $E \subseteq \lambda^+$, there exists an $\alpha \in S$ such that*

(1) $\sup(C_\alpha) = \alpha$,
(2) $\mathsf{ot}(C_\alpha) = \theta$,
(3) $C_\beta = C_\alpha \cap \beta$ *for all* $\beta \in C_\alpha$,
(4) $C_\alpha \subseteq S \cap E$.

Proof. For each ordinal $\alpha < \lambda^+$, fix an injection $d_\alpha : \alpha \to \lambda$, and for each $\beta < \lambda$, let a^β_α denote $d^{-1}_\alpha[\beta]$. Then for each $\alpha < \lambda^+$, $\langle a^\beta_\alpha \mid \beta < \lambda\rangle$ is a continuous, $\subseteq$-increasing chain in $[\alpha]^{<\lambda}$ with union α. For each $\alpha < \lambda^+$, let F_α be the set of $\gamma < \lambda$ such that, for all $\beta \in a^\gamma_\alpha$, $a^\gamma_\beta = a^\gamma_\alpha \cap \beta$. Then each F_α is club subset of λ.

Since $\theta < \lambda$, club many ordinals below λ^+ of cofinality $\mathsf{cof}(\theta)$ contain a cofinal set of ordertype θ.

Given a set $E \subseteq \lambda^+$, and $\beta < \lambda$, let $E(\beta)$ be the set of all $\alpha \in S$ for which the following hold:

(1) $\beta \in F_\alpha$;
(2) $\mathsf{ot}(E \cap S \cap \alpha) = \alpha$;
(3) $\sup(E \cap S \cap a^\beta_\alpha) = \alpha$;
(4) $\mathsf{ot}(E \cap S \cap a^\beta_\alpha) = \mathsf{ot}(a^\beta_\alpha)$ contains a cofinal subset of ordertype θ.

Note that if $E \subseteq E'$ are subsets of λ and $\beta < \lambda$, then $E(\beta) \subseteq E'(\beta)$.

Claim 42. *There exists a $\beta^* < \lambda$ for which $E(\beta^*)$ is nonempty whenever E is a club in λ^+.*

Proof. Suppose otherwise. Then for each $\beta < \lambda$ we may pick a club $E_\beta \subseteq \lambda^+$ for which $E_\beta(\beta) = \emptyset$. Let $E = \bigcap_{\beta<\lambda} E_\beta \setminus \lambda$. Since E is club in λ^+, we may fix an $\alpha \in E \cap S$ such that $\mathsf{ot}(E \cap S \cap \alpha) = \alpha$.

As $\mathsf{cof}(\alpha) < \mathsf{cof}(\lambda)$, the set $D = \{\beta < \lambda : \sup(E \cap S \cap a^\beta_\alpha) = \alpha\}$ is co-bounded in λ. Furthermore, continuity entails that the set

$$D' = \{\beta \in D : \mathsf{ot}(E \cap S \cap a^\beta_\alpha) = \mathsf{ot}(a^\beta_\alpha)\}$$

is club in λ. Pick $\beta \in D' \cap F_\alpha$ such that $\mathsf{ot}(a^\beta_\alpha)$ contains a cofinal subset of ordertype θ. Then since $E \subseteq E_\beta$, we get that $\mathsf{ot}(E_\beta \cap S \cap a^\beta_\alpha) = \mathsf{ot}(a^\beta_\alpha)$. So $\alpha \in E_\beta(\beta)$, contradicting the choice of E_β. This completes the proof of Claim 42. □

Let $\beta^* < \lambda$ be as given by Claim 42.

Claim 43. *There exists a club $E^* \subseteq \lambda^+$ such that for every club $D \subseteq \lambda^+$, the set $\{\alpha \in E^*(\beta^*) : a^{\beta^*}_\alpha \cap E^* \subseteq D\}$ is nonempty.*

Proof. Suppose otherwise. Then there exists a $\subseteq$-decreasing sequence $\langle G_\beta : \beta < \lambda\rangle$ of club subsets of λ^+ such that

(1) $G_0 = \lambda^+$,
(2) for every $\beta < \lambda$, the set $\{\alpha \in G_\beta(\beta^*) : a^{\beta^*}_\alpha \cap G_\beta \subseteq G_{\beta+1}\}$ is empty,
(3) for every limit ordinal $\gamma < \lambda$, $G_\gamma = \bigcap_{\beta<\gamma} G_\beta$.

Let $G = \bigcap_{\beta<\lambda} G_\beta$, and pick $\alpha \in G(\beta^*)$. Then $\alpha \in G_\beta(\beta^*)$ for all $\beta < \lambda$, hence

$$\langle a^{\beta^*}_\alpha \cap G_\beta : \beta < \lambda\rangle$$

must be a strictly decreasing sequence of subsets of $a^{\beta^*}_\alpha$, contradicting the fact that $|a^{\beta^*}_\alpha| < \lambda$. This completes the proof of Claim 43. □

Let $E^* \subseteq \lambda^+$ be as given by Claim 43.

Claim 44. *There exists an ordinal $\tau^* < \lambda$ which contains a cofinal subset of ordertype θ such that for every club $D \subseteq \lambda^+$, the set*

$$\{\alpha \in E^*(\beta^*) : a^{\beta^*}_\alpha \cap E^* \subseteq D,\ \mathsf{ot}(a^{\beta^*}_\alpha) = \tau^*\}$$

is nonempty.

Proof. Again, suppose otherwise. Then for every ordinal $\tau < \lambda$ which contains a cofinal subset of ordertype θ, there exists a club $D_\tau \subseteq \lambda^+$ for which

$$\{\alpha \in E^*(\beta^*) : a^{\beta^*}_\alpha \cap E^* \subseteq D_\tau,\ \mathsf{ot}(a^{\beta^*}_\alpha) = \tau\}$$

is empty. Let D be the intersection of these sets D_τ. By the choice of E^* we may pick an $\alpha \in E^*(\beta^*)$ such that $a^{\beta^*}_\alpha \cap E^* \subseteq D$. Let $\tau = \mathsf{ot}(a^{\beta^*}_\alpha)$. Since $\alpha \in E^*(\beta^*)$, τ contains a cofinal subset of ordertype θ, contradicting the fact $a^{\beta^*}_\alpha \cap E^* \subseteq D_\tau$. This completes the proof of Claim 44. $\square$

Now we finish the proof of the theorem. Let τ^* be as given by Claim 44. As τ^* contains a cofinal subset of ordertype θ, we may fix a cofinal subset $u \subseteq \tau^*$ of ordertype θ. For each $\alpha < \lambda^+$, let

$$C_\alpha = \{\beta \in E^* \cap S \cap a^{\beta^*}_\alpha : \mathsf{ot}(a^{\beta^*}_\beta \cap E^* \cap S) \in u\}.$$

Let us see that $\langle C_\alpha : \alpha < \lambda^+\rangle$ works. Suppose that we are given a club $E \subseteq \lambda^+$. Applying the choice of τ^*, pick $\alpha \in E^*(\beta^*)$ such that $a^{\beta^*}_\alpha \cap E^* \subseteq E$ and $\mathsf{ot}(a^{\beta^*}_\alpha) = \tau^*$. Then:

(1) $\alpha \in S$;
(2) $\sup(E^* \cap S \cap a^{\beta^*}_\alpha) = \alpha$;
(3) $C_\alpha \subseteq E^* \cap S \cap a^{\beta^*}_\alpha \subseteq S \cap E$;
(4) $\beta^* \in F_\alpha$, so for all $\gamma \in a^{\beta^*}_\alpha$, we have $a^{\beta^*}_\gamma = a^{\beta^*}_\alpha \cap \gamma$, and $C_\gamma = C_\alpha \cap \gamma$;
(5) $\mathsf{ot}(E^* \cap S \cap a^{\beta^*}_\alpha) = \mathsf{ot}(a^{\beta^*}_\alpha) = \tau^*$;
(6) $\mathsf{ot}(C_\alpha) = \mathsf{ot}(u) = \theta$.

This completes the proof of Theorem 41. $\square$

Given a set C of ordinals, and an ordinal $\beta < \sup(C)$, we let $\mathsf{next}_C(\beta)$ denote

$$\min(C \setminus (\beta + 1)).$$

Theorem 45. *Suppose that λ is an uncountable cardinal, and let γ be a countable ordinal. There exists a sequence $\bar{C} = \{C_\alpha : \alpha < \lambda^+\}$ such that*

(1) $\forall\alpha < \lambda^+\ C_\alpha \subseteq \alpha$,
(2) if $\beta \in C_\alpha$ then $C_\beta = C_\alpha \cap \beta$,
(3) $S = \{\alpha < \lambda^+ : \mathsf{ot}(C_\alpha) = \gamma\}$ is stationary,
(4) if $E \subseteq \lambda^+$ is a club then the set

$$\mathsf{gd}(E) = \{\alpha \in S \cap E : \forall\beta \in C_\alpha \setminus \{\sup(C_\alpha)\}\ [\beta, \mathsf{next}_{C_\alpha}(\beta)) \cap E \neq \emptyset\}$$

is stationary.

Before beginning the proof of Theorem 45 we recall the definition of the set $I[\lambda]$.

Definition 46. *For a regular uncountable cardinal λ, $I[\lambda]$ is the set of $A \subseteq \lambda$ such that $\{\delta \in A : \mathsf{cof}(\delta) = \delta\}$ is nonstationary and, for some $\langle P_\alpha : \alpha < \lambda\rangle$, we have that*

- *for each $\alpha < \lambda$, $P_\alpha \subseteq \mathcal{P}(\alpha)$ and $|P_\alpha| < \lambda$;*
- *for each limit ordinal $\alpha \in A$ with $\mathsf{cof}(\alpha) < \alpha$, there exists a cofinal $x \subseteq \alpha$ with $\mathsf{ot}(x) < \alpha$ such that, for all $\zeta < \alpha$, $x \cap \zeta \in \{P_\gamma : \gamma < \alpha\}$.*

It is straightforward to verify that $I[\lambda]$ is an ideal.

Proof of Theorem 45. In the case where λ is regular, this follows from Theorem 41, by replacing each C_α given there (with S as the set of all ordinals below λ^+ of countable cofinality) with $C_\alpha \cap \beta$ for β minimal violating condition (2) of the statement of this theorem (and leaving C_α as is if there is no such β).

We now prove the theorem assuming only $\lambda \geq \omega_2$, following the argument on pages 93-94 of [11]. By Conclusion 1.7 of [9] (and the fact that $I[\lambda^+]$ is closed under subsets), there is a stationary set $S_0 \subseteq \lambda^+ \setminus (\omega_1 + 1)$ in $I[\lambda^+]$ consisting of ordinals of cofinality $\aleph_1$. By Claim 1.3 of [9], then, there exist a club $E_0 \subseteq \lambda^+$ and a sequence $\langle C^0_\alpha : \alpha < \lambda^+\rangle$ such that

(1) each C^0_α is a closed subset of the corresponding α,
(2) each nonaccumulation point of each C^0_α is a successor ordinal,
(3) whenever $\beta \in C^0_\alpha$ is a nonaccumulation point of C^0_α, $C^0_\beta = C^0_\alpha \cap \beta$,
(4) for every $\alpha \in S_0 \cap E_0$, $\mathsf{ot}(C^0_\alpha) = \omega_1$ and $\alpha = \sup(C^0_\alpha)$.

We may assume that $S_0 \subseteq E_0$. Let $\langle C^1_\alpha : \alpha < \lambda^+\rangle$ be the sequence formed by removing from each C^0_α all of its accumulation points. Then $\langle C^1_\alpha : \alpha < \lambda^+\rangle$ retains properties (1) - (4), except that the sets C^1_α need not be closed.

Given sets C, F, let $gl(C, F)$ denote the set $\{\sup(\beta \cap F) : \beta \in C \setminus (\min(F) + 1)\}$. Following [10] (Sh365, Claim 2.3 (2), for id^b), we will show that there is a club $E_1 \subseteq \lambda^+$ such that for each club $E \subseteq E_1$, the set of $\alpha \in S_0$ for which $gl(C^1_\alpha, E_1) \subseteq E$ is stationary. To see this, suppose otherwise, and choose F_γ $(\gamma \leq \omega)$ satifying the following conditions:

- $F_0 = \lambda^+$;
- for each $\gamma < \omega_2$, $F_{\gamma+1}$ is $F^0_\gamma \cap G_\gamma$, for some pair of club subsets F^0_γ, G_γ of λ^+ such that $F^0_\gamma \subseteq F_\gamma$ and $\{\alpha \in S_0 : gl(C^1_\alpha, F_\gamma) \subseteq F^0_\gamma\} \cap G_\gamma = \emptyset$;
- for each limit ordinal $\delta \leq \omega_2$, $F_\delta = \bigcap_{\gamma<\delta} F_\gamma$.

Now fix an $\alpha \in S_0$ which is a limit point of F_{ω_2}, with $\mathsf{ot}(\alpha \cap F_{\omega_2}) = \alpha$. For each $\beta \in C^1_\alpha$ above $\min(F_{\omega_2})$, the sequence $\langle \sup(\beta \cap F_\gamma) : \gamma < \omega_2\rangle$ is nonincreasing, and therefore eventually constant. We may fix then for each such β ordinal $\gamma_\beta < \omega_2$ and $\zeta_\beta \leq \beta$ such that $\sup(\beta \cap F_\gamma) = \zeta_\beta$ for all $\gamma \in [\gamma_\beta, \omega_2)$. Since $|C^1_\alpha| = \aleph_1$, we may fix a $\gamma_* \in \omega_2$ which is greater than each γ_β. It follows that each element of $gl(C^1_\alpha, F_{\gamma_*})$ has the form $\sup(\beta \cap F_{\gamma_*+1})$ for some β, and is therefore a member of F_{γ_*+1}. Since α is in F_{γ_*+1}, this contradicts the choice of F_{γ_*+1}. This shows that an E_1 as desired exists.

For each $\alpha \in \lambda^+$, let

$$C^2_\alpha = \{\beta \in C^1_\alpha : \beta = \min(C^1_\alpha \setminus \sup(\beta \cap E_1)) > \min(E_1)\}.$$

We claim that $\langle C^2_\alpha : \alpha < \lambda^+\rangle$ satisfies item (4) of the conclusion of the theorem (using S_0, which will be a subset of the desired S). To see this, fix $E \subseteq \lambda^+$ club. It suffices to consider the case where E consists of limit points of E_1. Fix $\alpha \in S_0 \cap E$ for which $gl(C^1_\alpha, E_1) \subseteq E$, and fix $\beta \in C^2_\alpha$. Let $\beta' = \mathsf{next}_{C^2_\alpha}(\beta)$. Then $\beta < \sup(\beta' \cap E_1)$, since

$\beta' = \min(C^1_\alpha \setminus \sup(\beta' \cap E_1))$, and $\sup(\beta' \cap E_1) \in E \cap \beta'$, since $gl(C^1_\alpha, E_1) \subseteq E$ and β' is a successor ordinal. Then $\sup(\beta' \cap E_1)$ is as desired.

Finally, for each $\alpha < \lambda^+$, let $C_\alpha = \{\beta \in C^2_\alpha : \mathsf{ot}(C^2_\alpha \cap \beta) < \gamma\}$. Then the sequence $\langle C_\alpha : \alpha < \lambda^+\rangle$ is as desired. $\square$

Condition (4) implies that for stationary many α is S there is an element of E between any two consecutive elements of C_α. By removing the least element of C_α we can also assume that $\min(C_\alpha) \cap E \neq \emptyset$ whenever $\alpha \in \mathsf{gd}(E)$ and E is a club. Observe that coherence condition (2) implies that for any $\alpha, \beta \in S$, if $\delta = \sup(C_\alpha \cap C_\beta)$ then $C_\alpha \cap \delta = C_\beta \cap \delta$.

In the following remark, we thin the sets given in Theorem 45 so that we catch club strictly in between successive members of our C_α.

Remark 47. *Suppose that $\bar{C}$ and γ are as in Theorem 45, and that γ is a limit ordinal. Define C'_α, for $\alpha < \lambda^+$ by letting each C'_α be the set of $\beta \in C_\alpha$ for which the ordertype of $C_\alpha \cap \beta$ has the form $\rho\omega + k$, for ρ an ordinal and $k \in \omega$ even. Then $\{C'_\alpha : \alpha < \lambda^+\}$ also satisfies the conclusion of the theorem, with part (4) strengthened so that*

$$\mathsf{gd}'(E) = \{\alpha \in S \cap E : \forall \beta \in C_\alpha \setminus \{\sup(C_\alpha)\}\ (\beta, \mathsf{next}_{C_\alpha}(\beta)) \cap E \neq \emptyset\}$$

is stationary. The corresponding strengthened version of Theorem 45 for nonlimit γ can be obtained similarly, starting from a sequence $\bar{C}$ corresponding to some limit ordinal $\gamma' \geq \gamma$.

8. The ZFC result

In this section we prove the first part of Theorem 38, and thereby the second conclusion of Theorem 5 in the corresponding case. Specifically, we show the following.

Theorem 48. *Let K be a nowhere dense compact subset of 2^ω. If for some $\gamma < \omega_1$ there exists a $\vec{p} \in \mathbf{Q}_\gamma$ such that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) < \omega_1$ then there exist a set $Y \subseteq 2^\omega$ of size $2^{\aleph_0}$ and an ideal $\mathcal{J}$ on Y such that*

(1) K is $\mathcal{J}$-small, and
(2) $\mathsf{cov}(\mathcal{J}) = 2^{\aleph_0}$.

If CH holds, we can let $Y = 2^\omega$ and $\mathcal{J}$ be the ideal of meager sets. We will assume then that it fails.

Given an infinite cardinal λ and a countable ordinal γ, let us say that a *catching (λ^+, γ)-sequence* is a sequence $\bar{C} = \langle C_\alpha : \alpha < \lambda^+\rangle$ such that

$$\gamma = \sup\{\mathsf{ot}(C_\alpha) : \alpha < \lambda\}$$

and, letting S be $\{\alpha < \lambda^+ : \mathsf{ot}(C_\alpha) = \gamma\}$,

- $C_\alpha \subseteq \alpha$ for each $\alpha < \lambda^+$;
- $\mathsf{gd}'(E)$ is a stationary subset of λ^+, for each club $E \subseteq \lambda^+$.

We say that $\bar{C}$ is a *coherent* catching (λ^+,γ)-sequence if in addition, for each $\alpha<\lambda^+$ and all $\beta\in C_\alpha$, $C_\beta=C_\alpha\cap\beta$. By Remark 47, for each uncountable cardinal λ and each countable ordinal γ, there is a coherent catching sequence $\bar{C}$ for λ^+ such that $\gamma=\sup\{\mathsf{ot}(C_\alpha):\alpha<\lambda\}$.

Given a (λ^+,γ)-catching sequence $\bar{C}$, we define $\mathcal{I}_{\bar{C}}$ to be the ideal on the corresponding set S generated by the collection of all sets of the form $S\setminus\mathsf{gd}'(E)$, for E a club subset of λ^+.

Lemma 49. *If $\bar{C}$ is a (λ^+,γ)-catching sequence, for some infinite cardinal λ and some $\gamma<\omega_1$, then additivity of $\mathcal{I}_{\bar{C}}$ is λ^+. In particular, $\mathsf{cov}(\mathcal{I}_{\bar{C}})=\lambda^+$.*

Proof. Given $\{I_\alpha:\alpha<\lambda\}\subseteq\mathcal{I}_{\bar{C}}$, let E_α $(\alpha<\lambda^+)$ be club subsets of λ^+ such that $I_\alpha\subseteq S\setminus\mathsf{gd}'(E_\alpha)$ for all $\alpha<\lambda$. Then $\bigcup_{\alpha<\lambda}I_\alpha\subseteq S\setminus\mathsf{gd}'(\bigcap_{\alpha<\lambda}E_\alpha)\in\mathcal{I}_{\bar{C}}$. □

We now turn to the proof of Theorem 48. Fix a nowhere dense compact $K\subseteq 2^\omega$ with associated tree $\widetilde{T}$. Given a countable ordinal γ and $\vec{p}\in\mathbf{Q}_\gamma$, we will produce a set $Y_*\subseteq 2^\omega$ and an ideal $\mathcal{J}$ on Y_* with $\mathsf{cov}(\mathcal{J})=2^{\aleph_0}$. We will show that if K is not $\mathcal{J}$-small, then $\mathsf{rk}_{\vec{p}}(\mathbf{0})=\infty$. This will give the theorem.

Fix $\gamma\in\omega_1$ and $\vec{p}\in\mathbf{Q}_\gamma$. As in the remarks after Definition 25, we let $F'=F\circ h$, for some homeomorphism $h\colon(2^\omega)^\gamma\to\bar{p}$. Let $\hat{\gamma}$ be the least limit ordinal greater than or equal to γ. Say that an ordinal is *odd* (*even*) if it is of the form $\omega\rho+k$, for some ordinal ρ and some odd (even) $k\in\omega$. Given a set of ordinal C, let $O(C)$ denote the set of $\beta\in C$ such that $\mathsf{ot}(C\cap\beta)$ is odd, and let $O_\gamma(C)$ be the set of $\beta\in O(C)$ such that $\mathsf{ot}(\beta\cap O(C))<\gamma$.

Let $\langle x_\xi:\xi<2^{\aleph_0}\rangle$ be an enumeration of 2^ω. Let Λ be a cofinal set of cardinals in the interval $[\aleph_1,2^{\aleph_0})$ (recall that we are assuming that CH fails). For each $\lambda\in\Lambda$, fix a a coherent catching $(\lambda^+,\hat{\gamma})$-sequence $\bar{C}_\lambda=\langle C^\lambda_\alpha:\alpha<\lambda^+\rangle$. For each $\lambda\in\Lambda$, let S^λ be the set S corresponding to $\bar{C}_\lambda$, and, for each $\alpha\in S^\lambda$, let $y^\lambda_\alpha=F'(O_\gamma(C^\lambda_\alpha))$. Then by the definition of proj_δ, $\bigcup\{\mathsf{proj}_\delta(y^\lambda_\alpha\restriction n):n\in\omega\}=h(x_\xi)$ whenever ξ is the δ-th element of $O(C^\lambda_\alpha)$. For each $\lambda\in\Lambda$, let $Y_\lambda=\{y^\lambda_\alpha:\alpha\in S^\lambda\}$. Now let $Y_*=\bigcup_{\lambda\in\Lambda}Y_\lambda$, and let $\mathcal{J}$ be the set of $Z\subseteq Y_*$ such that $\{\alpha\in S^\lambda:y^\lambda_\alpha\in Z\}\in\mathcal{I}_{\bar{C}_\lambda}$ for all $\lambda\in\Lambda$. By Lemma 49, $\mathsf{cov}(\mathcal{J})=2^{\aleph_0}$ as desired.

We suppose now that K is not $\mathcal{J}$-small. This means that there exist $\lambda\in\Lambda$ and $z\in 2^\omega$ such that the set $S_z=\{\alpha\in S_\lambda:y^\lambda_\alpha\in K+z\}$ is not in $\mathcal{I}_{\bar{C}_\lambda}$. Fix such λ and z. For the rest of this section, we drop λ from our subscripts and superscripts when this seems unlikely to cause confusion (so $\bar{C}_\lambda$ becomes $\bar{C}$, S^λ becomes S, etc.).

Lemma 50. *The set $\mathsf{gd}'(E)\cap S_z$ is stationary for every club $E\subseteq\lambda^+$.*

Proof. If $\mathsf{gd}'(E)\cap S_{z^\star}\cap E'=\emptyset$ for some club E' then $\mathsf{gd}'(E\cap E')\cap S_{z^\star}=\emptyset$. In particular $S_z\in\mathcal{I}_{\bar{C}}$. □

Lemma 51. *There exists a perfect tree $Q \subseteq 2^{<\omega}$ such that for every node $t \in Q$*

$$\{\alpha \in S_z : t \subseteq y_\alpha\} \notin \mathcal{I}_{\bar{C}}.$$

Proof. Let $Z_0 = \{y_\alpha : \alpha \in S_z\} = \{y_\alpha : y_\alpha \in z + K\}$. By the Cantor-Bendixon theorem there exist a perfect tree $Q_0 \subseteq 2^{<\omega}$ and a countable set C_0 such that $\mathsf{cl}(Z_0) = [Q_0] \cup C_0$. For $t \in Q_0$ let $R_t = \{\alpha \in S_z : t \subseteq y_\alpha\}$ and let

$$Q_1 = \{t \in Q_0 : S_t \notin \mathcal{J}_{\bar{C}}\}.$$

Since $\mathcal{I}_{\bar{C}}$ is an ideal, Q_1 is a nontempty tree without terminal nodes.

Let $Z_1 = [Q_1]$. If Z_1 is uncountable then by applying the Cantor-Bendixon theorem again we get a perfect tree Q such that $Z_1 = [Q] \cup C_1$. The tree Q has the required property.

Suppose toward a contradiction that Z_1 is countable. Let

$$F_0 = \lambda^+ \setminus \{\alpha : y_\alpha \in Z_1 \cup C_0\},$$

and for $t \in Q_0 \setminus Q_1$ let E_t be a club subset of λ^+ such that $E_t \cap R_t = \emptyset$. Let

$$E = F_0 \cap \bigcap_{t \in Q_0 \setminus Q_1} E_t.$$

Then $S_z \cap E = \emptyset$, giving a contradiction. □

The following lemma introduces a useful sequence of clubs and elementary submodels. Observe that (3)(f) is the only condition imposing dependence between different sequences $\bar{N}_\xi$.

Lemma 52. *There exists a sequence $\langle E_\xi, \bar{N}_\xi : \xi < \omega_1\rangle$ such that for each $\xi < \omega_1$,*

(1) E_ξ is a club subset of λ^+,
(2) $E_\xi \subseteq \bigcap_{\zeta<\xi} E_\zeta$,
(3) $\bar{N}_\xi$ is a sequence $\langle N_{\xi,\alpha} : \alpha \in E_\xi\rangle$ such that for each $\alpha \in E_\xi$,
  - *(a) $N_{\xi,\alpha} \prec \mathbf{H}(\lambda^{++})$,*
  - *(b) $\lambda + 1 \subseteq N_{\xi,\alpha}$,*
  - *(c) $z, \bar{C}, \widetilde{T}, Y \in N_{\xi,\alpha}$,*
  - *(d) $|N_{\xi,\alpha}| = \lambda$,*
  - *(e) for all $\beta \in \alpha \cap E_\xi$, $N_{\xi,\beta} \subseteq N_{\xi,\alpha}$, and if α is a limit point of E_ξ then $N_{\xi,\alpha} = \bigcup_{\beta\in\alpha\cap E_\xi} N_{\xi,\beta}$,*
  - *(f) for all $\beta < \alpha$, $\langle N_{\xi,\delta} : \delta \in \beta \cap E_\xi\rangle \in N_{\xi,\alpha}$,*
  - *(g) $\{E_\zeta : \zeta < \xi\} \in N_{\xi,\alpha}$,*
  - *(h) $N_{\xi,\alpha} \cap \lambda^+ = \alpha$.*

Proof. Suppose that $\langle \bar{N}_\zeta, E_\zeta\rangle$ for $\zeta < \xi$ are already chosen. Let $\langle N_\alpha : \alpha < \lambda^+\rangle$ be a continuous sequence of models satisfying condition (3)(a)-(f). Let C be

$$\{\alpha : N_\alpha \cap \lambda^+ = \alpha\}.$$

Since $\langle N_\alpha : \alpha < \lambda^+ \rangle$ is continuous, C is a club. Put $E_\xi = C \cap \bigcap_{\zeta<\xi} E_\zeta$ and let

$$\bar{N}_\xi = \langle N_\alpha : \alpha \in E_\xi \rangle.$$

Observe that E_ξ and $\bar{N}_\xi$ are as required. □

To show that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$, we identify the following class of suitable pairs.

Definition 53. *Suppose that $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$. A pair $(x, \bar{\alpha})$ is* suitable *if*

(1) $x = (n_x, t_x, s_x) \in \mathsf{obj}(T)$,

(2) $s_x = z \restriction n_x$,

(3) $\bar{\alpha} = \langle \alpha_v : v \in t_x \cap 2^{n_x} \rangle$ *is such that, for each* $v \in t_x \cap 2^{n_x}$,

 (a) $\alpha_v \in \mathsf{gd}'(E_\xi) \cap S_z$,

 (b) $v \subseteq y_{\alpha_v}$.

Recall that $\alpha_v \in S_z$ means that $y_{\alpha_v} \in z + K$. Condition (3) of the definition of suitability implies then that $\mathsf{rk}_{\vec{p}}(x) > 0$ for every suitable pair $(x, \bar{\alpha})$. Observe that $(\mathbf{0}, \langle\rangle)$ is suitable. The following lemma shows that $\mathsf{rk}_{\vec{p}}(x) = \infty$ for every suitable pair $(x, \bar{\alpha})$, which completes the proof.

Lemma 54. *Suppose that $(x, \bar{\alpha})$ is a suitable pair, and that (v, δ) is a $(\vec{p}, x)$-challenge. Then there exists a suitable pair $(y, \bar{\beta})$ such that y is a response to (v, δ).*

Proof. Fix $(x, \bar{\alpha}_x)$ and (v, δ). Let $\gamma_* \in C_{\alpha_v}$ be such that $\mathsf{ot}(C_{\alpha_v} \cap \gamma_*)$ is even, and that $\mathsf{ot}(O(C_{\alpha_v}) \cap \gamma_*) = \delta$. Let γ_{**} be the least element of C_{α_v} above γ_*. Let Z be the collection of all pairs (γ', α') such that

(1) $\alpha' \in \mathsf{gd}'(E_\zeta) \cap S_z$,

(2) $\gamma_* \in C_{\alpha'}$,

(3) γ' is the least element of $C_{\alpha'}$ above γ_*,

(4) $v \subseteq y_{\alpha'}$.

Then

(1) for all $(\gamma', \alpha') \in Z$, $C_{\alpha'} \cap \gamma_* = C_{\alpha_v} \cap \gamma_* = C_{\gamma_*}$,

(2) $(\gamma_{**}, \alpha_v) \in Z$.

Since $\alpha_v \in \mathsf{gd}'(E_\xi)$, we may fix $\rho \in (\gamma_*, \gamma_{**}) \cap E_\xi$. Then all parameters from the definition of Z are in $N_{\xi,\rho}$.

Claim 55. *The set $H = \{\gamma' : \exists \alpha' \ (\gamma', \alpha') \in Z\}$ is unbounded in λ^+.*

Proof. If H were bounded it would be the same set in $N_{\zeta,\rho}$ as in $\mathbf{H}(\lambda^{++})$. However, $\gamma_{**} \in H$, and $\gamma_{**} \notin N_{\zeta,\rho}$. □

Fix $\gamma' \in H$ such that $\gamma' \neq \gamma_{**}$, and let α' be such that $(\gamma', \alpha') \in Y$. Since

$$\gamma_* \in C_{\alpha'} \cap C_{\alpha_v}$$

it follows that $C_{\alpha'} \cap \gamma_* = C_{\alpha_v} \cap \gamma_*$. Recall that each y_α was defined to be $F'(O_\gamma(C_\alpha))$. Consequently,

$$\bigcup\{\mathsf{proj}_\eta(y_{\alpha'} \restriction n) : n \in \omega\} = \bigcup\{\mathsf{proj}_\eta(y_{\alpha_v} \restriction n) : n \in \omega\}$$

for all $\eta < \delta$. On the other hand since the δ-th elements of $O_\gamma(C_\alpha)$ and $O_\gamma(C_{\alpha_v})$ are different,

$$\bigcup\{\mathsf{proj}_\delta(y_{\alpha'} \restriction n) : n \in \omega\} \neq \bigcup\{\mathsf{proj}_\delta(y_{\alpha_v} \restriction n) : n \in \omega\}.$$

Define $y \geq x$ as follows. First find $n_y \in \omega$ such that

$$\mathsf{proj}_\delta(y_{\alpha'})\restriction n_y \neq \mathsf{proj}_\delta(y_{\alpha_v})\restriction n_y.$$

Next let $s_y = z\restriction n_y$. Let $t_y = \{y_{\alpha'}\restriction n_y\} \cup \{y_{\alpha_w}\restriction n_y : w \in t_x \cap 2^{n_x}\}$. Finally, let

$$\bar{\beta} = \{\beta_w : w \in t_y \cap 2^{n_y}\}$$

be defined as follows:

$$\beta_w = \begin{cases} \alpha' & \text{if } w = y_{\alpha'}\restriction n_y \\ \alpha_v & \text{if } w = y_{\alpha_v}\restriction n_y \\ \alpha_s & \text{if } w = y_{\alpha_s}\restriction n_y \text{ for } s \in t_x \cap 2^{n_x} \setminus \{v\} \end{cases}$$

By the choice of n_y, the node v gets two distinct extensions, $y'_\alpha\restriction n_y$ and $y_{\alpha_v}\restriction n_y$, and one is assigned α' and the other α_v. All other nodes follow appropriate reals and have the same ordinals assigned to them. □

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