WHEN AUTOMORPHISMS OF $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ ARE TRIVIAL OFF A SMALL SET

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Abstract. It is shown that if $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then every automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ is trivial outside of a set of cardinality $2^{\aleph_0}$.

1. Introduction

The study of automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ was initiated by W. Rudin in [4, 5] who showed that the Continuum Hypothesis can be used to construct non-trivial autohomeomorphisms of $\beta\mathbb{N}/\mathbb{N}$, in other words, homeomorphisms from $\beta\mathbb{N}/\mathbb{N}$ to $\beta\mathbb{N}/\mathbb{N}$ that are not induced by any function from $\mathbb{N}$ to $\mathbb{N}$. Using Stone duality, this means that there are automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ that are not induced by a function from $\mathbb{N}$ to $\mathbb{N}$. A major advance was provided by S. Shelah in [6] who showed that it is consistent with set theory that every automorphism of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ is induced by a function from ω to ω. B. Velickovic later showed in [7] that the conjunction of OCA and MA implies that every automorphism of $\mathcal{P}(\omega_1)/[\omega_1]^{<\aleph_0}$ is induced by a function from ω_1 to ω_1. Moreover, he showed that PFA implies that if κ is uncountable then every automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ is induced by a function from κ to κ.

However, finding extensions of Rudin's result on the existence on non-trivial automorphisms of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ has proven to be much harder. This article will provide some reasons for this. In particular, a positive answer to the following question from [7] will be provided for cardinals below the first inaccessible: Can it be shown from MA and OCA alone that for every uncountable κ, every automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ is induced by a function from κ to κ?

The main result to be established in this article is that if $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then for every automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ there is a set $X \subseteq \kappa$ such that $|\kappa \setminus X| \leq 2^{\aleph_0}$ and the restriction of the automorphism to $\mathcal{P}(X)/[X]^{<\aleph_0}$ is induced by a function from κ to κ.

The question of cardinals not greater than the continuum has been dealt with by P. Larson and P. McKenney [3] who have shown the following.

Theorem 1.1 (Larson & McKenney). *If $\kappa \leq 2^{\aleph_0}$ and Ψ is an automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ such that $\Psi \restriction \mathcal{P}(X)/[X]^{<\aleph_0}$ is trivial for each $X \in [\kappa]^{\aleph_1}$ then Ψ is trivial.*

Without too much extra work, these two results then provide the partial answer to Velickovic's question.

Corollary 1.1. *If MA and OCA both hold and if κ is less than the first inaccessible then every automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$ is trivial.*

Proof. Let κ be less than the first inaccessible cardinal and let Φ be an automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\aleph_0}$. By Theorem 3.1 there is $X \subseteq \kappa$ such that $|X| = 2^{\aleph_0}$ and there is a function that induces $\Phi \restriction \mathcal{P}(\kappa \setminus X)/[\kappa \setminus X]^{<\aleph_0}$. It therefore suffices to show that $\Phi \restriction \mathcal{P}(X)/[X]^{<\aleph_0}$ is trivial. In order for this to be true, it suffices by Theorem 1.1 to show that if $Y \subseteq X$ and $|Y| = \aleph_1$ then $\Phi \restriction \mathcal{P}(Y)/[Y]^{<\aleph_0}$ is trivial. But this follows from the theorem of Velickovic from [7] which states that the conjunction of OCA and MA implies that every automorphism of $\mathcal{P}(\omega_1)/[\omega_1]^{<\aleph_0}$ is trivial. Hence Φ is trivial. ☐

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2. Terminology and Notation

Notation 2.1. For $X \subseteq \kappa$ let $[X]$ be the equivalence class of X modulo the ideal $[X]^{<\aleph_0}$. For any infinite set X the notation $\mathcal{P}(X)/\mathcal{F}in$ will be used in place of $\mathcal{P}(X)/[X]^{<\aleph_0}$ it being understood that $\mathcal{F}in$ refers to $[X]^{<\aleph_0}$ as appropriate. For any function f and any set x let $f\langle x\rangle$ denote the image of x under f.

Definition 2.1. A homomorphism $\Phi : \mathcal{P}(\kappa)/\mathcal{F}in \to \mathcal{P}(\kappa)/\mathcal{F}in$ will be said to be λ-trivial if there is a set $S \in [\kappa]^\lambda$ and a one-to-one function $F : \kappa \setminus S \to \kappa$ such that $F\langle X\rangle \in \Phi([X])$ for every $X \subseteq \kappa \setminus S$. If $\Phi : \mathcal{P}(\kappa)/\mathcal{F}in \to \mathcal{P}(\kappa)/\mathcal{F}in$ is a homomorphism let $\hat{\Phi}$ be some lifting of Φ. In other words, $\hat{\Phi}(X)$ is a function from $\mathcal{P}(\kappa)$ to $\mathcal{P}(\kappa)$ such that $\hat{\Phi}(X) \in \Phi([X]))$ for each $X \subseteq \kappa$ and $\hat{\Phi}$ is constant on each $[X]$.

Theorem 2.1 (Balcar and Frankiewicz [1])**.** *For any cardinal κ and any automorphism Ψ of $\mathcal{P}(\kappa)/\mathcal{F}in$ if $\Psi([X]) = [Y]$ and $|X| \geq \aleph_1$ and $|Y| \geq \aleph_1$ then $|X| = |Y|$ and, moreover, if $|X| = \aleph_0$ then $|Y| \leq \aleph_1$.*

In particular, there is no isomorphism from $\mathcal{P}(\kappa)/\mathcal{F}in$ to $\mathcal{P}(\lambda)/\mathcal{F}in$ unless $\kappa = \lambda$ or $\{\kappa, \lambda\} = \{\aleph_0, \aleph_1\}$. In this context, it is interesting to note that it is shown in [2] that if there is an isomorphism from $\mathcal{P}(\omega_1)/\mathcal{F}in$ to $\mathcal{P}(\omega)/\mathcal{F}in$ then there is a non-trivial automorphism of $\mathcal{P}(\omega)/\mathcal{F}in$ itself.

Definition 2.2. If $\mathfrak{B}$ is a subalgebra of $\mathcal{P}(X)$ and λ is an infinite cardinal, define $\mathcal{I}_\lambda(\mathfrak{B}) = [X]^{<\lambda} \cap \mathfrak{B}$ and define $I_\lambda(\mathfrak{B}) = \bigcup \mathcal{I}_\lambda(\mathfrak{B})$. The subalgebra of $\mathcal{P}(X \setminus I_\lambda(\mathfrak{B}))$ consisting of $\{B \setminus I_\lambda(\mathfrak{B}) \mid B \in \mathfrak{B}\}$ will be denoted by $\mathfrak{B}_\lambda$.

Note that $\mathfrak{B}_\lambda$ may be trivial if $\lambda \geq |X|$ or if $[X]^{<\lambda} \subseteq \mathfrak{B}$.

Definition 2.3. If Φ is an automorphism of $\mathcal{P}(X)/\mathcal{F}in$ and $\mathfrak{B}$ is a subalgebra of $\mathcal{P}(X)$ then define $\mathfrak{B}^\Phi$ to be the algebra generated by $\hat{\Phi}\langle\mathfrak{B}\rangle$, noting that $\hat{\Phi}\langle\mathfrak{B}\rangle$ may not itself be a subalgebra. The notation $\mathfrak{B}^\Phi_\lambda$ will be used to denote the more cumbersome $(\mathfrak{B}^\Phi)_\lambda$. If λ is an uncountable cardinal, define $\Phi_\lambda : \mathfrak{B}_\lambda \to \mathfrak{B}^\Phi_\lambda$ by $\Phi_\lambda(B \setminus I_\lambda(\mathfrak{B})) = \hat{\Phi}(B) \setminus I_\lambda(\mathfrak{B}^\Phi)$ for $B \in \mathfrak{B}$.

Lemma 2.1. *If $\Phi : \mathcal{P}(X)/\mathcal{F}in \to \mathcal{P}(X)/\mathcal{F}in$ is a one-to-one homomorphism, $\mathfrak{B}$ is a subalgebra of $\mathcal{P}(X)$, $\lambda > \aleph_1$ and $|\mathfrak{B}| < \mathbf{cof}(\lambda)$ then Φ_λ is a well defined isomorphism from $\mathfrak{B}_\lambda$ to $\mathfrak{B}^\Phi_\lambda$.*

Proof. To see that the mapping is well defined, begin by noting that $|I_\lambda(\mathfrak{B})| < \lambda$. Now suppose that $B \setminus I_\lambda(\mathfrak{B}) = B' \setminus I_\lambda(\mathfrak{B})$. It follows that $|B \triangle B'| < \lambda$ and hence, by Theorem 2.1, that $|\hat{\Phi}(B \triangle B')| < \lambda$. However $\hat{\Phi}(B) \triangle \hat{\Phi}(B') \equiv^* \hat{\Phi}(B \triangle B')$ and so $\hat{\Phi}(B) \triangle \hat{\Phi}(B') \subseteq I_\lambda(\mathfrak{B}^\Phi)$. In other words, $\Phi_\lambda(B) = \Phi_\lambda(B')$.

To see that Φ_λ is a homomorphism suppose that $\Phi_\lambda(B) \not\subseteq \Phi_\lambda(B')$. Then $|\hat{\Phi}(B) \setminus \hat{\Phi}(B')| \geq \lambda$. Since $\hat{\Phi}(B) \setminus \hat{\Phi}(B') \equiv^* \hat{\Phi}(B \setminus B')$ it follows from Theorem 2.1 that $|B \setminus B'| \geq \lambda$ and so $B \setminus I_\lambda(\mathfrak{B}) \not\subseteq B' \setminus I_\lambda(\mathfrak{B})$.

Since Φ is a homomorphism, in order to see that Φ_λ is one-to-one it suffices to show that if $\Phi_\lambda(B) = \emptyset$ then $B \subseteq I_\lambda(\mathfrak{B})$. This is immediate from Theorem 2.1.

To see that Φ_λ is onto let $B \in \mathfrak{B}^\Phi_\lambda$. By definition there is some $B' \in \mathfrak{B}_\lambda$ such that $\hat{\Phi}(B') \equiv^* B$ and hence $\hat{\Phi}(B') \triangle B \subseteq I_\lambda(\mathfrak{B}^\Phi)$. In other words, $\Phi_\lambda(B') \setminus I_\lambda(\mathfrak{B}^\Phi) = B \setminus I_\lambda(\mathfrak{B}^\Phi)$ as required. □

3. The lemmas needed for the main result

The main result of this article is the following.

Theorem 3.1. *If $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then every automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ is $2^{\aleph_0}$-trivial.*

The proof of Theorem 3.1 proceeds by induction on κ. This section contains the statements of the lemmas needed to prove it. The first is immediate.

Lemma 3.1. *If every automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ is λ-trivial and every automorphism of $\mathcal{P}(\lambda)/\mathcal{F}in$ is μ-trivial then every automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ is μ-trivial.*

The next lemma deals with the successors of cardinals with uncountable cofinality.

Lemma 3.2. *If $\kappa^{\aleph_0} = \kappa$ and $\nu \leq 2^\kappa$ then every automorphism of $\mathcal{P}(\nu)/\mathcal{F}in$ is κ-trivial.*

The singular cardinals of uncountable cofinality are easy to handle, as the next lemma demonstrates.

Lemma 3.3. *If κ is singular of uncountable cofinality and if for every cardinal $\lambda < \kappa$ every automorphism of $\mathcal{P}(\lambda)/\mathcal{F}in$ is $2^{\aleph_0}$-trivial then so is every automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$.*

Proof. Let Φ be an automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$. Let $\{\kappa_\xi\}_{\xi\in\mathbf{cof}(\kappa)}$ be an increasing, cofinal sequence of cardinals in κ. Using the hypothesis, for each $\xi \in \mathbf{cof}(\kappa)$ choose X_ξ and $F_\xi : \kappa_\xi \setminus X_\xi \to \kappa$ such that

- $|X_\xi| \leq 2^{\aleph_0}$
- $F_\xi \langle A\rangle \in \Phi([A])$ for each $A \subseteq \kappa_\xi \setminus X_\xi$.

Then let

$$X = \left(\bigcup_{\xi\in\mathbf{cof}(\kappa)} X_\xi\right) \cup \left(\bigcup_{\{\xi,\eta\}\in[\mathbf{cof}(\kappa)]^2} \{\zeta \in \kappa \mid F_\xi(\zeta) \neq F_\eta(\zeta)\}\right)$$

and note that $|X| \leq \mathbf{cof}(\kappa)\cdot 2^{\aleph_0}$. Let $F = \bigcup_{\xi\in\mathbf{cof}(\kappa)} F_\xi \restriction (\kappa \setminus X)$ and note that F is a function.

To see that $F\langle A\rangle \in \Phi([A])$ for each $A \subseteq \kappa\setminus X$ observe that otherwise there is some infinite $A \subseteq \kappa\setminus X$ such that $F\langle A\rangle\cap\Phi([A])$ is finite. Moreover, A can then be chosen to be countable and, hence, $A \subseteq \kappa_\xi\setminus X_\xi$ for some $\xi \in \mathbf{cof}(\kappa)$ contradicting that $F\langle A\rangle = F_\xi\langle A\rangle$.

If $\mathbf{cof}(\kappa) \leq 2^{\aleph_0}$ there is nothing else to do. Otherwise, it has been shown that Φ is $\mathbf{cof}(\kappa)$-trivial. Since $\mathbf{cof}(\kappa) < \kappa$, the result now follows from Lemma 3.1. $\square$

The next two lemmas deal with the harder case of singular cardinals of countable cofinality.

Lemma 3.4. *If κ has cofinality ω and for every $\lambda < \kappa$ every automorphism of $\mathcal{P}(\lambda)/\mathcal{F}in$ is $2^{\aleph_0}$-trivial then every automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ is $2^{\aleph_0}$-trivial.*

Lemma 3.5. *Suppose that*

- *κ has countable cofinality*
- *$2^\mu < \kappa$ for each $\mu < \kappa$*
- *if $\mu \leq \kappa$ then every automorphism of $\mathcal{P}(\mu)/\mathcal{F}in$ is $2^{\aleph_0}$-trivial.*

Then every automorphism of $\mathcal{P}(\rho)/\mathcal{F}in$ is κ-trivial for every cardinal ρ such that $\kappa^+ \leq \rho \leq \kappa^{\aleph_0}$.

Proof of Theorem 3.1. If the result fails let λ be the least cardinal such that there is an automorphism Φ of $\mathcal{P}(\lambda)/\mathcal{F}in$ that is not $2^{\aleph_0}$-trivial. By Lemma 3.3 and 3.4 it must be that λ is a regular cardinal. Since λ is less than the first inaccessible, there must be some least $\kappa < \lambda$ such that $2^\kappa \geq \lambda$. From Lemma 3.2 applied to κ and λ, it follows that $\kappa^{\aleph_0} > \kappa$. If $\kappa^{\aleph_0} < \lambda$ then Lemma 3.2 applied to $\kappa^{\aleph_0}$ and λ yields that Φ is $\kappa^{\aleph_0}$-trivial. A contradiction then follows from the induction hypothesis and Lemma 3.1. It can therefore be assumed that $\kappa^{\aleph_0} \geq \lambda$.

From Lemma 3.5 it then follows that either κ must have uncountable cofinality or there must be $\nu < \kappa$ such that $2^\nu \geq \kappa$. (The third alternative cannot fail because of the minimality of λ.) If the second alternative holds then the minimality of the choice of κ implies that $2^\nu < \lambda$. Since $2^\nu = (2^\nu)^{\aleph_0}$ and $2^{2^\nu} \geq 2^\kappa \geq \lambda$ it is once again possible to apply Lemma 3.2 to 2^ν and λ to get the contradiction that Φ is 2^ν-trivial and $2^\nu < \lambda$. So it suffices to consider the case that κ has uncountable cofinality and $2^\nu < \kappa$ for each $\nu < \kappa$. This implies that $\nu^{\aleph_0} < \kappa$ for each $\nu < \kappa$ and, hence, that $\kappa^{\aleph_0} = \kappa$. This possibility has already been ruled out. $\square$

4. Automorphisms of $\mathcal{P}(2^\kappa)/\mathcal{F}in$ when $\kappa^{\aleph_0} = \kappa$

Proof of Lemma 3.2. Assume that $\kappa^{\aleph_0} = \kappa$ and $\nu \leq 2^\kappa$ and and note that there is nothing to prove if $\nu \leq \kappa$. Let Ψ be an automorphism of $\mathcal{P}(\nu)/\mathcal{F}in$ and $\hat{\Psi}$ a lifting of Ψ. Begin by noting that if $(2^X)_\sigma$

denotes the space 2^X with the topology generated by G_δ sets in 2^X with the usual product topology, then the density of $(2^\nu)_\sigma$ is less than or equal to the density of $(2^{2^\kappa})_\sigma$ and this, in turn, is less than or equal to $\kappa^{\aleph_0} = \kappa$. Hence there is a family $\mathcal{D} \subseteq \mathcal{P}(\nu)$ of cardinality κ such that for any two disjoint countable sets $A \subseteq \nu$ and $B \subseteq \nu$ there is $D \in \mathcal{D}$ such that $A \subseteq D$ and $B \cap D = \emptyset$.

Let $\mathfrak{M}$ be an elementary submodel of $(H((2^\kappa)^+), \mathcal{D}, \hat{\Psi}, \in)$ such that $[\mathfrak{M}]^{\aleph_0} \subseteq \mathfrak{M}$ and $|\mathfrak{M}| = \kappa$. Let $\mathfrak{A} = \mathcal{P}(\nu) \cap \mathfrak{M}$ and note that $\mathfrak{A}$ is a Boolean σ-sub-algebra of $P(\nu)$ containing $\mathcal{D}$. Moreover, $\mathfrak{A}$ contains the Boolean σ-subalgebra generated by $\hat{\Psi}\langle\mathfrak{A}\rangle$. By Lemma 2.1 it follows that $\Psi_{\kappa^+} : \mathfrak{A}_{\kappa^+} \to \mathfrak{A}^{\Psi}_{\kappa^+}$ is an isomomorphism, but, as will be shown, it is in fact a σ-isomorphism. The following lemma will yield this result.

Lemma 4.1. *If $\mathcal{C} \subseteq \mathfrak{A}_{\kappa^+}$ is countable then $\Psi_{\kappa^+}(\bigcup \mathcal{C}) = \bigcup \Psi_{\kappa^+}\langle\mathcal{C}\rangle$.*

Proof. Since Ψ_{κ^+} is a homomorphism it follows that $\Psi_{\kappa^+}(\bigcup \mathcal{C}) \supseteq \bigcup \Psi_{\kappa^+}\langle\mathcal{C}\rangle$, so suppose that

$$\Psi_{\kappa^+}\left(\bigcup \mathcal{C}\right) \setminus \bigcup \Psi_{\kappa^+}\langle\mathcal{C}\rangle \neq \emptyset. \tag{4.1}$$

It follows that

$$Z = \hat{\Psi}\left(\bigcup \mathcal{C}\right) \setminus \bigcup \hat{\Psi}\langle\mathcal{C}\rangle \neq \emptyset \tag{4.2}$$

and, moreover, $Z \in \mathfrak{M}$ because $\hat{\Psi}$ and $\mathcal{C}$ both belong to $\mathfrak{M}$. From (4.1) it follows that $|Z| \geq \kappa^+$. By elementarity it follows that there is some $Z^* \subseteq \nu$ such that $\hat{\Psi}(Z^*) \equiv^* Z$. By Theorem 2.1 it follows that $|Z^*| \geq \kappa^+$ and hence there is $C \in \mathcal{C}$ such that $|C \cap Z^*| \geq \kappa^+$. Hence $\hat{\Psi}(C \cap Z^*) \subseteq^* Z \cap \hat{\Psi}(C)$. By Theorem 2.1 $|\hat{\Psi}(C \cap Z^*)| \geq \kappa^+$ contradicting that $Z \cap \hat{\Psi}(C) = \emptyset$. □

Corollary 4.1. *Ψ_{κ^+} is a σ-isomorphism.*

Now, for $\xi \in \nu \setminus I_{\kappa^+}(\mathfrak{A})$ let $U(\xi)$ be the ultrafilter on $\mathfrak{A}_{\kappa^+}$ defined by $U(\xi) = \{A \in \mathfrak{A}_{\kappa^+} \mid \xi \in A\}$. Let $U(\xi, \Psi)$ be the image of $U(\xi)$ under Ψ_{κ^+} and note that $U(\xi, \Psi)$ is also an ultrafilter on $\mathfrak{A}^{\Psi}_{\kappa^+}$.

Lemma 4.2. *For all but finitely many $\xi \in \nu \setminus I_{\kappa^+}(\mathfrak{A})$ the cardinality of $\bigcap U(\xi, \Psi)$ is at most 1.*

Proof. Let $B = \{\xi \in \nu \setminus I_{\kappa^+}(\mathfrak{A}) \mid |\bigcap U(\xi, \Psi)| > 1\}$ and suppose that B is infinite. Choose a countable $\bar{B} \subseteq B$ and then choose $\{\beta_0, \beta_1\} \in [\bigcap U(\beta, \Psi)]^2$ for each $\beta \in \bar{B}$. Let $B_i = \left\{\beta_i \mid \beta \in \bar{B}\right\}$. Observe that since $\mathcal{D} \subseteq \mathfrak{A}$ it follows that if $\xi \neq \eta$ then $(\bigcap U(\xi, \Psi)) \cap (\bigcap U(\eta, \Psi)) = \emptyset$ and, hence, $\{\beta_0, \beta_1\} \cap \{\bar{\beta}_0, \bar{\beta}_1\} = \emptyset$ if β and $\bar{\beta}$ are distinct elements of $\bar{B}$. Therefore $B_0 \cap B_1 = \emptyset$.

Then let C_0 be such that $\hat{\Psi}(C_0) \equiv^* B_0$. Now, even though B_0 is countable, Theorem 2.1 does not rule out the possibility that $|C_0| = \aleph_1$. In this case choose a countable $C_0^* \subseteq C_0$ and let $W = \left\{\beta \in \bar{B} \;\middle|\; \beta_0 \in \hat{\Psi}(C_0^*)\right\}$. Then let C_1 be such that $\hat{\Psi}(C_1) \equiv^* \{\beta_1 \mid \beta \in W\}$. Once again it might be that C_1 is uncountable. If this is the case, choose a countable $B_1^* \subseteq C_1$ and let $\bar{W} = \left\{\beta \in W \;\middle|\; \beta_1 \in \hat{\Psi}(B_1^*)\right\}$. Let $\bar{B}_i = \left\{\beta_i \mid \beta \in \bar{W}\right\}$ and then let B_0^* be such that $\hat{\Psi}(B_0^*) \equiv^* \bar{B}_0$. It follows that $\hat{\Psi}(B_i^*) \equiv^* \bar{B}_i$ for each i and that each B_i^* is countable.

Since $\bar{B}_0 \cap \bar{B}_1 = \emptyset$ a contradiction will be obtained if it can be shown that $B_i^* \supseteq^* \bar{W}$ for each i. So suppose that $Z \subseteq \bar{W} \setminus B_i^*$ is countably infinite for some $i \in 2$. Using the fact that $\mathfrak{A} \supseteq \mathcal{D}$ and that Z and B_i^* are both countable it follows that there is some $D \in \mathcal{D}$ such that $Z \subseteq D$ and $D \cap B_i^* = \emptyset$. It then follows that

$$|\bar{B}_i \cap \Psi_{\kappa^+}(D)| < \aleph_0$$

contradicting, by Lemma 4.1, that

$$|\bar{B}_i \cap \Psi_{\kappa^+}(D)| \geq |\bigcup_{\beta \in Z} \bar{B}_i \cap \left(\bigcap U(\beta, \Psi)\right)| \geq |Z| = \aleph_0 \tag{4.3}$$

for each $i \in 2$. □

Lemma 4.3. *For all but κ many $\xi \in \nu \setminus I_{\kappa^+}(\mathfrak{A})$ the cardinality of $\bigcap U(\xi, \Psi)$ is at least 1.*

Proof. Let $B = \{\xi \in \nu \setminus I_{\kappa^+}(\mathfrak{A}) \mid \bigcap U(\xi, \Psi) = \emptyset\}$ and suppose that $|B| > \kappa$. By Theorem 2.1 it follows that $|\hat{\Psi}(B)| > \kappa$ also and hence it is possible to find a countable $\bar{B} \subseteq B \setminus I_{\kappa^+}(\mathfrak{A})$ such that $\hat{\Psi}(\bar{B}) \cap I_{\kappa^+}(\mathfrak{A}^\Psi) \equiv^* \emptyset$. As in the proof of Lemma 4.2, there is no loss of generality in assuming that $\hat{\Psi}(\bar{B})$ is countable. Then observe that $\bar{B} \subseteq \bigcup_{\beta \in \bar{B}} V(\beta)$ for all $V \in \prod_{\beta \in \bar{B}} U(\beta)$. Hence, using Corollary 4.1, it follows that for any $V \in \prod_{\beta \in \bar{B}} U(\beta)$

$$\hat{\Psi}(\bar{B}) \subseteq^* \hat{\Psi}\left(\bigcup_{\beta \in \bar{B}} V(\beta)\right) \setminus I_{\kappa^+}(\mathfrak{A}^\Psi) = \Psi_{\kappa^+}\left(\bigcup_{\beta \in \bar{B}} V(\beta)\right) = \bigcup_{\beta \in \bar{B}} \Psi_{\kappa^+}(V(\beta))$$

and, since $\hat{\Psi}(\bar{B})$ is countable, it follows that there is some $\beta_1 \in \bar{B}$ and $\beta_2 \in \hat{\Psi}(\bar{B})$ such that $\beta_2 \in \Psi_{\kappa^+}(V(\beta_1))$ for cofinally many $V \in \prod_{\beta \in \bar{B}} U(\beta)$ where $\prod_{\beta \in \bar{B}} U(\beta)$ is given the natural partial order of coordinatewise inclusion. But then, since each ultrafilter is countably closed by Corollary 4.1, it follows that β_2 belongs to a cofinal subset of $\Psi_{\kappa^+}(U(\beta_1)) = U(\beta_1, \Psi)$ and hence $\beta_2 \in \bigcap U(\beta_1, \Psi)$. This contradicts that $\beta_1 \in \bar{B} \subseteq B$. □

It now follows that if $H(\xi)$ is defined to be the unique, if it exists, element of ν such that $\bigcap U(\xi, \Psi) = \{H(\xi)\}$ then H is defined for all but κ elements of $\nu \setminus I_{\kappa^+}(\mathfrak{A})$. The arguments of Lemmas 4.2 and 4.3 show that H^{-1} is defined for all but κ elements of $\nu \setminus I_{\kappa^+}(\mathfrak{A}^\Psi)$. It follows that H is a bijection whose domain is E and range is E_Ψ such that $|\nu \setminus E| \leq \kappa$ and $|\nu \setminus E_\Psi| \leq \kappa$. It may further be assumed that $E_\Psi \cap I_{\kappa^+}(\mathfrak{A}^\Psi) = \emptyset$.

All that remains to be shown is that $[H\langle A\rangle] = \Psi([A])$ for each $A \subseteq E$. Since Ψ is an isomorphism and H induces an isomorphism, it suffices to show that if $A \subseteq E$ and $B \subseteq E$ and $H\langle A\rangle \equiv^* \hat{\Psi}(B)$ then $A \equiv^* B$. So suppose that A and B provide a counterexample to this. It is then possible to find a countable $C \subseteq H\langle A\rangle \cap \hat{\Psi}(B)$ such that there are countable $A^* \subseteq A$ and $B^* \subseteq B$ such that $A^* \cap B^* = \emptyset$ and $H\langle A^*\rangle \equiv^* \hat{\Psi}(B^*) \equiv^* C$. That C, A^* and B^* can be assumed countable uses an argument similar to that of Lemma 4.2.

Now use the density property of $\mathcal{D}$ to find $D \in \mathcal{D} \subseteq \mathfrak{A}$ such that $A^* \subseteq D$ and $D \cap B^* = \emptyset$. Note that it is easy to see that $H\langle F \cap E\rangle = \Psi_{\kappa^+}(F) \cap E_\Psi$ for any $F \in \mathfrak{A}_{\kappa^+}$ and, in particular, $H\langle D \cap E\rangle = \Psi_{\kappa^+}(D) \cap E_\Psi$. Using this and the fact that $I_{\kappa^+}(\mathfrak{A}^\Psi) \cap E_\Psi = \emptyset$, it follows that

$$C \subseteq^* H\langle A^*\rangle \cap \hat{\Psi}(B^*) \cap E_\Psi \subseteq^* H\langle D \cap E\rangle \cap \hat{\Psi}(\nu \setminus D) \cap E_\Psi = \Psi_{\kappa^+}(D) \cap \Psi_{\kappa^+}(\nu \setminus D) \cap E_\Psi$$

contradicting that Ψ_{κ^+} is an isomorphism. □

5. Automorphisms of $\mathcal{P}(\kappa)/\mathcal{F}in$ when $\mathbf{cof}(\kappa) = \omega$

Proof of Lemma 3.4. Let $\{\kappa_n\}_{n \in \omega}$ be an increasing sequence of cardinals cofinal in κ. Suppose that Φ is an automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ and $\hat{\Phi}$ is a lifting of it. Let $\hat{\Phi}^{-1}$ be a lifting of Φ^{-1} and note that $\hat{\Phi}(\hat{\Phi}^{-1}(X)) \equiv^* X$ for all X.

Using the hypothesis, let F_n and S_n be such that

(1) $S_n \subseteq \kappa_n$ and $|S_n| \leq 2^{\aleph_0}$
(2) $F_n : \kappa_n \setminus S_n \to \kappa$
(3) $F_n\langle X\rangle \in \Phi([X])$ for every $X \subseteq \kappa_n \setminus S_n$.

Let $S = \bigcup_n S_n$.

Claim 1. For $n \in m \in \omega$ there is a finite set $A_{n,m}$ such that $F_n \restriction (\kappa_n \setminus (S \cup A_{n,m})) \subseteq F_m$.

Proof. To see this suppose not. There is then an infinite set $A \subseteq \kappa_n \setminus S$ such that $F_n\langle A\rangle \cap F_m\langle A\rangle = \emptyset$. This contradicts that $F_n\langle A\rangle \in \Phi([A])$ and $F_m\langle A\rangle \in \Phi([A])$. □

Now let $\bar{S} = S \cup \bigcup_{n,m} A_{n,m}$ and let $F = \bigcup_n F_n \restriction (\kappa \setminus \bar{S})$. It suffices to show that $F\langle X\rangle \in \Phi([X])$ for every $X \subseteq \kappa \setminus \bar{S}$.

To this end, suppose not and choose recursively $X_\xi \subseteq \kappa \setminus \bar{S}$, for $\xi \in \omega_1$, such that

(4) $|X_\xi| = \aleph_0$
(5) $X_\xi \cap X_\eta = \emptyset$ if $\xi \neq \eta$
(6) $X_\xi \cap F^{-1}(\hat{\Phi}(\bigcup_{\eta\in\xi} X_\eta)) = \emptyset$
(7) $X_\xi \cap \bigcup_{\eta\in\xi} \hat{\Phi}^{-1}(F\langle X_\eta\rangle) = \emptyset$
(8) $F\langle X_\xi\rangle \cap \hat{\Phi}(X_\xi)$ is finite.

To see that this is possible note that if

$$S^\xi = \bar{S} \cup F^{-1}\left(\hat{\Phi}\left(\bigcup_{\eta\in\xi} X_\eta\right)\right) \cup \bigcup_{\eta\in\xi} \hat{\Phi}^{-1}(F\langle X_\eta\rangle) \cup \bigcup_{\eta\in\xi} X_\eta$$

then by Theorem 2.1 $|S^\xi \setminus \bar{S}| \leq \aleph_1$ for every $\xi \in \omega_1$. Hence, if it is not possible to find X_ξ, then F and S^ξ witness that Φ is $2^{\aleph_0}$-trivial.

Now let $X = \bigcup_{\xi\in\omega_1} X_\xi$.

Claim 2. $\hat{\Phi}(X) \cap F\langle X_\xi\rangle$ is finite for every $\xi \in \omega_1$.

Proof. Let $\xi \in \omega_1$ be given. Then

$$\hat{\Phi}(X) \equiv^* \hat{\Phi}\left(\bigcup_{\eta\in\xi} X_\eta\right) \cup \hat{\Phi}(X_\xi) \cup \hat{\Phi}\left(\bigcup_{\eta>\xi} X_\eta\right)$$

By Condition (8) it follows that $F\langle X_\xi\rangle \cap \hat{\Phi}(X_\xi)$ is finite and by Condition (6) it follows that

$$F\langle X_\xi\rangle \cap \hat{\Phi}\left(\bigcup_{\eta\in\xi} X_\eta\right) = \emptyset.$$

By Condition (7) it follows that $\hat{\Phi}^{-1}(F\langle X_\xi\rangle) \cap \left(\bigcup_{\alpha>\xi} X_\alpha\right) = \emptyset$ and hence that

$$\hat{\Phi}(\hat{\Phi}^{-1}(F\langle X_\xi\rangle)) \cap \hat{\Phi}\left(\bigcup_{\alpha>\xi} X_\alpha\right) \equiv^* \emptyset$$

and so $F\langle X_\xi\rangle \cap \hat{\Phi}\left(\bigcup_{\alpha>\xi} X_\alpha\right)$ is finite as well. □

Now let $g : \omega_1 \to \omega$ be such that $\hat{\Phi}(X) \cap F\langle X_\xi\rangle \subseteq F\langle \kappa_{g(\xi)}\rangle$ for each $\xi \in \omega_1$. Then define $g^+ : \omega_1 \to \omega$ such that $F\langle X_\xi \cap \kappa_{g^+(\xi)}\rangle \not\subseteq F\langle \kappa_{g(\xi)}\rangle$ for each $\xi \in \omega_1$. To see that this is possible, note that the failure to find a suitable $g^+(\xi)$ would imply that $F\langle X_\xi\rangle \subseteq F\langle \kappa_{g(\xi)}\rangle$ and, hence, that $X_\xi \subseteq \kappa_{g(\xi)}$. Since $F_{g(\xi)}$ induces $\Phi \restriction \mathcal{P}(\kappa_{g(\xi)})/[\kappa_{g(\xi)}]^{<\aleph_0}$ it would follow that $F\langle X_\xi\rangle \equiv^* \hat{\Phi}(X_\xi)$ contradicting Condition (8). Now choose $m \in \omega$ and $k \in \omega$ such that there is an infinite $Z^* \subseteq \omega_1$ such that $g(\xi) = m$ and $g^+(\xi) = k$ for $\xi \in Z^*$

Now let $Z = \bigcup_{\xi\in Z^*} X_\xi \cap \kappa_k$. Since the X_ξ are pairwise disjoint and $\hat{\Phi}(Z) \subseteq^* \hat{\Phi}(X)$, it follows that

- $\hat{\Phi}(Z) \cap F\langle X_\xi\rangle \subseteq^* F\langle \kappa_m\rangle$ for all $\xi \in Z^*$
- $\hat{\Phi}(Z) \cap F\langle X_\xi\rangle \subseteq F\langle \kappa_m\rangle$ for all but finitely many $\xi \in Z^*$

and, hence, that $\hat{\Phi}(Z) \cap \left(\bigcup_{\xi\in Z^*} F\langle X_\xi\rangle\right) \subseteq^* F\langle \kappa_m\rangle$. Furthermore, $Z \subseteq \kappa_k \setminus \bar{S}$ and so $\hat{\Phi}(Z) \equiv^* F\langle Z\rangle$. Moreover $F\langle Z\rangle \setminus F\langle \kappa_m\rangle$ is infinite because of the definition of g^+ and the fact that the X_ξ are pairwise disjoint. This contradicts the fact that $F\langle Z\rangle \setminus F\langle \kappa_m\rangle \subseteq^* \hat{\Phi}(Z)\cap F\langle Z\rangle \subseteq \hat{\Phi}(Z)\cap \left(\bigcup_{\xi\in Z^*} F\langle X_\xi\rangle\right) \subseteq^* F\langle \kappa_m\rangle$. □

6. Automorphisms of $\mathcal{P}(\kappa^{\aleph_0})/\mathcal{F}in$ when κ has countable cofinality

Proof of Lemma 3.5. Suppose that $\kappa^+ \leq \rho \leq \kappa^{\aleph_0}$ and that ρ is the least such cardinal such that there is an automorphism of $\mathcal{P}(\rho)/\mathcal{F}in$ which is not κ-trivial. Note that by Lemmas 3.3 and 3.4 it must be that ρ is a regular cardinal and by hypothesis $\kappa \neq 2^{\aleph_0}$ and so $\rho > (2^{\aleph_0})^+$. Let $\{\kappa_n\}_{n\in\omega}$ be an increasing sequence of cardinals converging to κ. Let $R \subseteq \prod_{n\in\omega}\kappa_n$ be such that $|R| = \rho$ and suppose that there is an automorphism Φ of $\mathcal{P}(R)/\mathcal{F}in$ that is not κ-trivial.

For $k \in \omega$ let $T_k = \prod_{n\in k}\kappa_n$, so that T_k consists of sequences t of length k such that $t(j) \in \kappa_j$ for each $j \in k = \mathbf{domain}(t)$. For $W \subseteq T_k$ let $C(W) = \{f \in R \mid f\restriction k \in W\}$. Let $\mathfrak{B}$ be the Boolean subalgebra of $\mathcal{P}(R)$ generated by sets of the form $C(W)$ where $W \subseteq T_k$ for some $k \in \omega$ and note that $|\mathfrak{B}| = \kappa$. Let $\hat{\Phi}$ be a lifting of Φ and $\Phi_{\kappa^+} : \mathfrak{B}_{\kappa^+} \to \mathfrak{B}^{\Phi}_{\kappa^+}$. This is a well defined isomorphism by Lemma 2.1. Since no confusion can arise because of it, the notation $C(W)$ will continue to be used to denote $C(W)\setminus I_{\kappa^+}(\mathfrak{B})$ in the algebra $\mathfrak{B}_{\kappa^+}$. Note however, that it may be that $C(W) = \emptyset$ for some non-empty W; this depends on R of course. As well, for singletons $\{t\}$ the notation $C(t)$ will be used in place of the more cumbersome $C(\{t\})$.

Next observe that

$$(\forall m\in\omega)(\forall W\subseteq T_m)\ \ \Phi_{\kappa^+}(C(W)) = \bigcup_{t\in W}\Phi_{\kappa^+}(C(t)) \tag{6.1}$$

or, in other words,

$$|\hat{\Phi}(C(W))\setminus \bigcup_{t\in W}\hat{\Phi}(C(t))| \leq \kappa.$$

To see this, suppose not and let $m\in\omega$, $W\subseteq T_m$ and $A\subseteq R$ be such that

$$\hat{\Phi}(A) \equiv^* \hat{\Phi}(C(W))\setminus \bigcup_{t\in W}\hat{\Phi}(C(t))$$

and note that $|A| \geq \kappa_m^+$ by Theorem 2.1. Hence, since $A\subseteq^* C(W)$, there is $t\in W$ such that $|A\cap C(t)| \geq \kappa_m^+$. Since Φ is a homomorphism, this contradicts that $\hat{\Phi}(A)\cap\hat{\Phi}(C(t)) \equiv^* \emptyset$. Furthermore,

$$\text{if } s\subseteq t \text{ then } \Phi_{\kappa^+}(C(s)) \supseteq \Phi_{\kappa^+}(C(t)). \tag{6.2}$$

For each $f\in R\setminus I_{\kappa^+}(\mathfrak{B})$ let $U(f)$ be the ultrafilter $\{B\in\mathfrak{B}_{\kappa^+} \mid f\in B\}$ and let $V(f)$ be the ultrafilter $\left\{B\in\mathfrak{B}^{\Phi}_{\kappa^+} \mid f\in B\right\}$. Let $U_\Phi(f)$ be the ultrafilter $\left\{\Phi_{\kappa^+}(B)\in\mathfrak{B}^{\Phi}_{\kappa^+} \mid B\in U(f)\right\}$.

Claim 3. Letting $S = \{f\in R\setminus I_{\kappa^+}(\mathfrak{B}) \mid (\exists h\in R\setminus I_{\kappa^+}(\mathfrak{B}))\ V(h) = U_\Phi(f)\}$ it follows that

$$|R\setminus(S\cup I_{\kappa^+}(\mathfrak{B}))| \leq 2^{\aleph_0}.$$

Proof. If the claim fails then let $A\subseteq R\setminus(S\cup I_{\kappa^+}(\mathfrak{B}))$ be such that $|A| > 2^{\aleph_0}$. There must then be some $k\in\omega$ such that $|\{f\restriction k \mid f\in A\}| > 2^{\aleph_0}$. Hence there is $A_1\subseteq A$ of cardinality greater than $2^{\aleph_0}$ such that the mapping $f\mapsto f\restriction k$ is one-to-one on A_1.

Since $A_1\cap S=\emptyset$ it follows that $\bigcap_{n\in\omega}\Phi_{\kappa^+}(C(f\restriction n)) = \emptyset$ for each $f\in A_1$. Since $\Phi_{\kappa^+}(C(f\restriction k))\cap\hat{\Phi}(A_1)$ is finite for each $f\in A_1$ it follows that for each $f\in A_1$ there is $m_f\geq k$ such that

$$\Phi_{\kappa^+}(C(f\restriction m_f))\cap\Phi_{\kappa^+}(C(f\restriction k))\cap\hat{\Phi}(A_1) = \emptyset$$

and hence

$$\Phi_{\kappa^+}(C(f\restriction m_f))\cap\hat{\Phi}(A_1) = \emptyset \tag{6.3}$$

because $m_f\geq k$ and (6.2) holds.

Now let $A_2 \subseteq A_1$ be infinite and m be such that $m_f = m$ for all $f \in A_2$ and let $Z = \{f \restriction m \mid f \in A_2\}$. Then $A_2 \subseteq C(Z)$ and $A_2 \cap I_{\kappa^+}(\mathfrak{B}) = \emptyset$ and so $C(Z) \neq \emptyset$. Moreover, from Equation 6.1 it follows that

$$\Phi_{\kappa^+}(C(Z)) = \bigcup_{f \in A_2} \Phi_{\kappa^+}(C(f \restriction m))$$

and hence, by (6.3) that $\Phi_{\kappa^+}(C(Z)) \cap \hat{\Phi}(A_1) = \emptyset$. However, this contradicts that $\hat{\Phi}(A_2) \subseteq^* \hat{\Phi}(A_1)$ and $\hat{\Phi}(A_2) \subseteq^* \Phi_{\kappa^+}(C(Z))$. □

Claim 4. Letting $\tilde{S} = \{f \in R \setminus I_{\kappa^+}(\mathfrak{B}) \mid (\forall h \in R \setminus I_{\kappa^+}(\mathfrak{B}))$ if $V(h) = V(f)$ then $h = f\}$ it follows that $|R \setminus (\tilde{S} \cup I_{\kappa^+}(\mathfrak{B}))| \leq 2^{\aleph_0}$.

Proof. Suppose that $\{(f_\xi, g_\xi)\}_{\xi \in \mathfrak{c}^+}$ are disjoint pairs such that $V(f_\xi) = V(g_\xi)$ and $\{f_\xi, g_\xi\} \cap I_{\kappa^+}(\mathfrak{B}) = \emptyset$ for each $\xi \in \mathfrak{c}^+$. Let A be such that $\hat{\Phi}(A) = \{f_\xi\}_{\xi \in \mathfrak{c}^+}$. Using arguments as in Claim 3, find $k \in \omega$ and $A_1 \subseteq A$ and $Z \subseteq \mathfrak{c}^+$ such that

- $|A_1| = |Z| = \mathfrak{c}^+$
- the mapping $f \mapsto f \restriction k$ is one-to-one on A_1
- $\hat{\Phi}(A_1) \equiv^* \{f_\xi\}_{\xi \in Z}$.

Then find B, $Z^* \subseteq Z$ and $k^* \geq k$ such that

- $|B| = |Z^*| = \mathfrak{c}^+$
- the mapping $f \mapsto f \restriction k^*$ is one-to-one on B
- $\hat{\Phi}(B) \equiv^* \{g_\xi\}_{\xi \in Z^*}$.

Since A_1 and B are almost disjoint, it is possible to find $m \geq k^*$ and an infinite $\bar{A} \subseteq A_1$ such that if $f \in \bar{A}$ and $g \in B$ then $f \restriction m \neq g \restriction m$. Let $W = \{f \restriction m \mid \xi \in \bar{A}\}$.

Hence $\hat{\Phi}(\bar{A}) \subseteq^* \Phi_{\kappa^+}(C(W))$ and $\hat{\Phi}(B) \cap \Phi_{\kappa^+}(C(W))$ is finite. It follows that for all but finitely many ξ such that $f_\xi \in \hat{\Phi}(\bar{A})$ it must be the case that $\Phi_{\kappa^+}(C(W)) \in V(f_\xi)$ but $\Phi_{\kappa^+}(C(W)) \notin V(g_\xi)$ contradicting that $V(f_\xi) = V(g_\xi)$.

□

A similar argument shows the following.

Claim 5. Letting $S^* = \{f \in R \setminus I_{\kappa^+}(\mathfrak{B}) \mid (\forall h \in R \setminus I_{\kappa^+}(\mathfrak{B}))$ if $U_\Phi(h) = U_\Phi(f)$ then $f = h\}$ it follows that $|R \setminus (S \cup I_{\kappa^+})(\mathfrak{B})^*| \leq 2^{\aleph_0}$.

It follows from Claims 3, 4 and 5 that letting $E = S \cap \tilde{S} \cap S^* \setminus I_{\kappa^+}(\mathfrak{B})$ there is a well defined, one-to-one function $F : E \to \prod_{n \in \omega} \kappa_n$ such that $V(F(f)) = U_\Phi(f)$ for every $f \in E$. Observe that

$$(\forall A \in \mathfrak{B}_{\kappa^+})\ F\left\langle A \cap \tilde{E} \right\rangle = \Phi_{\kappa^+}(A) \cap F\left\langle \tilde{E} \right\rangle \equiv^* \hat{\Phi}(A) \cap F\left\langle \tilde{E} \right\rangle \tag{6.4}$$

holds for any $\tilde{E} \subseteq E$ by the definition of F.

Claim 6. Suppose that $W \subseteq E$ and $\Psi : W \to W$ are such that $\Psi\langle A \rangle \equiv^* \hat{\Phi}(A)$ for each $A \subseteq W$. Then $|\{w \in W \mid \Psi(w) \neq F(w)\}| \leq 2^{\aleph_0}$.

Proof. If the claim fails then it is possible to find $k \in \omega$ and $Z \subseteq \{w \in W \cap E \mid \Psi(w) \neq F(w)\}$ such that $|Z| > 2^{\aleph_0}$ and the mapping $z \mapsto z \restriction k$ is one-to-one. For each $z \in Z$ there is some $m_z \geq k$ and $t^0_z : m_z \to \kappa$ and $t^1_z : m_z \to \kappa$ such that $t^0_z \neq t^1_z$ and the following hold:

$$\Psi(z) \in \Phi_{\kappa^+}(C(t^0_z)) \tag{6.5}$$

$$F(z) \in \Phi_{\kappa^+}(C(t^1_z)). \tag{6.6}$$

Observe that the definition of F implies that $t^1_z = z \restriction m_z$. Let $\bar{Z} \in [Z]^{\aleph_1}$ and m be such that $m_z = m$ for all $z \in \bar{Z}$.

Since the t^1_z are all distinct, there is an infinite $\bar{\bar{\bar{Z}}} \subseteq \bar{Z}$ such that if $W_i = \left\{t^i_z \;\middle|\; z \in \bar{\bar{\bar{Z}}}\right\}$ then $W_0 \cap W_1 = \emptyset$ and, hence, $C(W_0) \cap C(W_1) = \emptyset$. The construction of Φ_{κ^+} guarantees that

$$\Phi_{\kappa^+}(C(W_0)) \cap \Phi_{\kappa^+}(C(W_1)) = \emptyset. \tag{6.7}$$

By Equality 6.4 and the hypothesis on Ψ it follows that

$$F\langle C(W_i) \cap W \cap E\rangle \equiv^* \hat{\Phi}(C(W_i)) \cap F\langle W \cap E\rangle \equiv^* \Psi\langle C(W_i) \cap W \cap E\rangle \tag{6.8}$$

Since $\bar{\bar{\bar{Z}}} \subseteq C(W_1)$ it follows that

$$\Psi\left\langle \bar{\bar{\bar{Z}}}\right\rangle \subseteq^* \Psi\langle C(W_1) \cap W \cap E)\rangle \equiv^* \hat{\Phi}(C(W_1) \cap W \cap E) \subseteq^* \hat{\Phi}(C(W_1)).$$

On the other hand,

$$\Psi\left\langle \bar{\bar{\bar{Z}}}\right\rangle \subseteq \bigcup_{z \in \bar{\bar{\bar{Z}}}} \Phi_{\kappa^+}(C(t^0_z)) = \Phi_{\kappa^+}(C(W_0)) \subseteq \hat{\Phi}(C(W_0))$$

by (6.1) and (6.5) and the definition of Φ_{κ^+}. This, of course, contradicts that

$$\hat{\Phi}(C(W_0)) \cap \hat{\Phi}(C(W_1)) \equiv^* \emptyset.$$

□

Now let $\{r_\xi\}_{\xi\in\rho}$ be an enumeration of $E \subseteq R$ and define $E_\alpha = \{r_\xi\}_{\xi\in\alpha}$. Two cases need to be considered, the first one being that $\rho = \kappa^+$. Observe that if $\mathfrak{M}$ is an elementary submodel of $(H(\rho^+), \{r_\xi\}_{\xi\in\rho}, \hat{\Phi}, \in)$ and $\mathfrak{M} \cap \rho^+$ is an ordinal of uncountable cofinality then $\hat{\Phi}(\{r_\xi \mid \xi \in \mathfrak{M}\}) \equiv^* \{r_\xi \mid \xi \in \mathfrak{M}\}$. Recalling that ρ is a regular cardinal greater than $\mathfrak{c}^+$, it follows that

$$Y = \left\{\xi \in \rho \;\middle|\; \hat{\Phi}(E_\xi) \equiv^* E_\xi \text{ and } \mathbf{cof}(\xi) > \mathfrak{c}\right\}$$

is a stationary set. The third hypothesis of the lemma then yields that for each $\xi \in Y$ the restriction of Φ to $\mathcal{P}(E_\xi)/\mathcal{F}in$ is an automorphism of $\mathcal{P}(E_\xi)/\mathcal{F}in$ that must be $2^{\aleph_0}$-trivial.

The other case is that $\rho > \kappa^+$. In this case

$$Y = \left\{\xi \in \rho \;\middle|\; \hat{\Phi}(E_\xi) \equiv^* E_\xi \text{ and } \mathbf{cof}(\xi) = \kappa^+\right\}$$

is a stationary set. In this case the minimality of ρ yields that for each $\xi \in Y$ the restriction of Φ to $\mathcal{P}(E_\xi)/\mathcal{F}in$ is an automorphism of $\mathcal{P}(E_\xi)/\mathcal{F}in$ that must be κ-trivial.

In either case there is some $\alpha(\xi) \in \xi$ and a function Ψ such that $\hat{\Phi}(A) \equiv^* \Psi\langle A\rangle$ for each $A \subseteq E_\xi \setminus \alpha(\xi)$. By Claim 6 there is then $\beta(\xi) \in \xi \setminus \alpha(\xi)$ such that $\Psi(\eta) = F(\eta)$ provided that $\beta(\xi) \in \eta \in \xi$. The stationarity of Y then yields a $\beta \in \rho$ such that $F\langle A\rangle \equiv^* \hat{\Phi}(A)$ for every bounded $A \subseteq \rho \setminus \beta$. This is enough to conclude that Φ is $|\beta|$-trivial and Lemma 3.1 and the minimality of ρ then imply that Φ is κ-trivial.

□

7. Remarks and questions

Observe that it follows from Theorems 1.1 and 3.1 that if κ is less than the first inaccessible cardinal and Φ is a non-trivial automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ then there is $X \in [\kappa]^{\aleph_1}$ such that $\Phi \restriction \mathcal{P}(X)/\mathcal{F}in$ is also non-trivial. Hence it is of interest to understand the non-trivial automorphisms of $\mathcal{P}(\omega_1)/\mathcal{F}in$. Of course, there are non-trivial automorphisms of $\mathcal{P}(\omega_1)/\mathcal{F}in$ that are obtained by taking a non-trivial automorphisms of $\mathcal{P}(\omega)/\mathcal{F}in$ and extending it to all of $\mathcal{P}(\omega_1)/\mathcal{F}in$ by the identity on sets disjoint from from ω. Hence the following definition is natural in this context.

Definition 7.1. An automorphism Φ of $\mathcal{P}(\omega_1)/\mathcal{F}in$ will be called non-trivially non-trivial if it is non-trivial (as defined in §1) but $\Phi \restriction Z$ is trivial for every countable $Z \subseteq \omega_1$.

Question 7.1. Is it consistent with set theory that there is an automorphism Φ of $\mathcal{P}(\omega_1)/\mathcal{F}in$ that is non-trivially non-trivial?

Recall from §1 that Velickovic showed in [7] that the conjunction of OCA and MA implies that every automorphism of $\mathcal{P}(\omega_1)/\mathcal{F}in$ is trivial. However it might be that there is a stronger result answering the following question.

Question 7.2. Does it follow from the fact that every automorphism of $\mathcal{P}(\omega)/\mathcal{F}in$ is trivial, that every automorphism of $\mathcal{P}(\omega_1)/\mathcal{F}in$ is trivial?

Note that in any model with an automorphism of Φ of $\mathcal{P}(\omega_1)/\mathcal{F}in$ providing a positive answer to Question 7.1 there must be a family of injection $f_\xi : \xi \to \omega_1$ for each $\xi \in \omega_1$ such that if $\xi \in \eta$ then $f_\xi \equiv^* f_\eta \restriction \xi$ and $\Phi \restriction \mathcal{P}(\xi)/\mathcal{F}in$ is induced by f_ξ but there is no $f : \omega_1 \to \omega_1$ threading the f_ξ — in other words, there is no f such that $f_\xi \equiv^* f \restriction \xi$ for all ξ.

So given a potential family of functions $\{f_\xi\}_{\xi\in\omega_1}$, what needs to be done is to define the values of $\Phi(X)$ for uncountable $X \subseteq \omega_1$ without adding a function f threading the f_ξ. The natural partial order for adding the unique element of $\mathcal{P}(\omega_1)/\mathcal{F}in$ that must be equal to $\Phi(X)$ will preserve ω_1 for certain, carefully constructed families of functions $\{f_\xi\}_{\xi\in\omega_1}$. However, the countable support iteration poses several problems. Larson and McKenney and, independently, Rinot and Steprans have shown that certain instances of this partial order can be iterated, with countable support, to yield a model where there is a non-trivial automorphism of the Boolean sub algebra of $\mathcal{P}(\omega_1)/\mathcal{F}in$ generated by the countable sets and a maximal independent family of subsets of ω_1 which is trivial when restricted to any countable set.

On the other hand, there is also the question of inaccessible κ.

Question 7.3. Is it consistent, relative to the consistency of an inaccessible cardinal, that there is a non-trivially non-trivial automorphism of $\mathcal{P}(\kappa)/\mathcal{F}in$ where κ is inaccessible?

Observe that if κ provides positive answer to Question 7.3 then κ cannot be too large a cardinal. For example, if κ is the least cardinal such that there is an a non-trivially non-trivial automorphism Φ of $\mathcal{P}(\kappa)/\mathcal{F}in$ then it cannot carry a normal κ-additive ultrafilter. To see this note that for each $\xi \in \kappa$ there must be some $f_\xi : \xi \to \kappa$ such that $\Phi \restriction \mathcal{P}(\xi)/\mathcal{F}in$ is induced by f_ξ. It must, of course, be the case that $f_\xi \equiv^* f_\eta \restriction \xi$ for $\xi \in \eta$. In other words, for each $\xi \in \eta \in \kappa$ there is $a_{\xi,\eta} \in [\xi]^{<\aleph_0}$ such that $f_\eta \restriction \xi \setminus a_{\xi,\eta} = f_\xi \restriction \xi \setminus a_{\xi,\eta}$. The normal κ-additive ultrafilter yields a set $X \subseteq \kappa$ of cardinality κ an a such that $a_{\xi,\eta} = a$ for all $\{\xi, \eta\} \in [X]^2$. Hence $\bigcup_{\xi\in X} f_\xi \restriction \xi \setminus a$ induces Φ.

Theorem 3.1 suggests an alternate version of non-trivial non-triviality.

Definition 7.2. An automorphism Φ of $\mathcal{P}(\omega_1)/\mathcal{F}in$ will be called very non-trivially non-trivial if $\Phi \restriction Z$ is non-trivial for every co-countable $Z \subseteq \omega_1$.

Results about the existence or non-existence of very non-trivially non-trivial would also be of interest.

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