On a question about families of entire functions*

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Abstract

We show that the existence of a continuum sized family $\mathcal{F}$ of entire functions such that for each complex number z, the set $\{f(z) : f \in \mathcal{F}\}$ has size less than continuum is undecidable in ZFC plus the negation of CH.

1 Introduction

In [2], Erdös asked the following (for some history on this, see [3])

Question 1.1. *Is there a continuum sized family $\mathcal{F}$ of analytic functions from $\mathbb{C}$ to $\mathbb{C}$ such that for each $z \in \mathbb{C}$, $\{f(z) : f \in \mathcal{F}\}$ has size less than continuum?*

In the same paper, answering a question of Wetzel, he showed that CH is equivalent to the following: There is an uncountable family $\mathcal{F}$ of analytic functions from $\mathbb{C}$ to $\mathbb{C}$ such that for each $z \in \mathbb{C}$, $\{f(z) : f \in \mathcal{F}\}$ is countable. We show here that the answer to Question 1.1 is undecidable in ZFC plus the negation of CH.

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2 No such family in the Cohen real model

The following theorem implies that there is no such family in the Cohen real model which is obtained by adding $\aleph_2$ Cohen reals to L.

Theorem 2.1. *Suppose $V \models \mathfrak{c} = \lambda \geq cf(\lambda) > \kappa = \omega_1$. Let $\mathbb{P}$ add κ Cohen reals. Then in $V^{\mathbb{P}}$, whenever $\mathcal{F}$ is a continuum sized family of pairwise distinct entire functions, there exists $z \in \mathbb{C}$ such that $|\{f(z) : f \in \mathcal{F}\}| = \mathfrak{c}$.*

Proof of Theorem 2.1: Let $r \in {}^{\kappa}2$ be the Cohen generic sequence added by $\mathbb{P}$. Clearly, $V[r] \models \mathfrak{c} = \lambda$. Suppose $\langle f_\alpha : \alpha < \lambda\rangle$ is a sequence of pairwise distinct entire functions in $V[r]$. Note that each f_α is coded in $V[r \upharpoonright \xi_\alpha]$ for some $\xi_\alpha < \kappa$. As $cf(\lambda) > \kappa$, we can choose $X \in [\lambda]^\lambda$, $\xi_\star < \kappa$ such that for each $\alpha \in X$, f_α is coded in $V[r \upharpoonright \xi_\star]$. Let $z_\star \in \mathbb{C}$ be Cohen over $V[r \upharpoonright \xi_\star]$ so that it avoids every meager subset of the complex plane coded in $V[r \upharpoonright \xi_\star]$. Since two distinct entire functions only agree on a countable set, it follows that $\langle f_\alpha(z_\star) : \alpha \in X\rangle$ are pairwise distinct. $\square$

3 Consistency with failure of CH

We now show that a positive answer to 1.1 is also consistent with the failure of CH.

Theorem 3.1. *It is consistent with ZFC plus the negation of CH that there is a family $\mathcal{F}$ of entire functions such that $|\mathcal{F}| = \mathfrak{c}$ and for every $z \in \mathbb{C}$, $|\{f(z) : z \in \mathbb{C}\}| < \mathfrak{c}$.*

Before we begin the proof of Theorem 3.1, let us recall Erdös' construction in [2] under CH. Let $\{z_i : i < \omega_1\} = \mathbb{C}$. Inductively construct $\langle f_i : i < \omega_1\rangle$ such that each $f_i : \mathbb{C} \to \mathbb{C}$ is entire and for every $j < i < \omega_1$, $f_i \neq f_j$ and $f_i(z_j)$ is a rational complex number. This is possible because for every countable $X \subseteq \mathbb{C}$, there is a non constant entire function sending X into the set of rational complex numbers.

We adopt a slightly different strategy that exploits the singularity of continuum as follows. Starting with a model where $\mathfrak{c} = \omega_{\omega_1}$, we perform a finite support iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \omega_1\rangle$ such that, at each stage $i < \omega_1$, via a ccc forcing $\mathbb{Q}_i$ of size ω_{i+1}, we add a family $\mathcal{F}_i$ of entire functions such that $|\mathcal{F}_i| = \omega_{i+1}$ and for every $j \leq i$, letting W_j be the set of first ω_{j+1} members of $V^{\mathbb{P}_i} \cap \mathbb{C}$ in some fixed enumeration, we have that $(\forall z \in W_j)(|\{f(z) : f \in \mathcal{F}_i\}| \leq \omega_{j+1})$. So $\mathcal{F} = \bigcup\{\mathcal{F}_i : i < \omega_1\}$ will be the required family in $V^{\mathbb{P}}$. The possible set of values for $\{f(z) : f \in \mathcal{F}_i\}$ is not fixed beforehand but added generically together with $\mathcal{F}$ - This is the major point of difference with Erdös' construction. The main problem then is to ensure that $\mathbb{Q}_i$ is ccc. We do this by requiring that the finite approximations to members of $\{f(z) : z \in W_i\}$ can be chosen quite independently of those for $\{g(z) : z \in W_i\}$, for $f \neq g \in \mathcal{F}_i$. This is materialized by using strongly almost disjoint families in $[\omega_{i+1}]^{\omega_{i+1}}$. The next lemma says that such families can consistently exist.

Lemma 3.2. *The following is consistent.*

(a) $\mathfrak{c} = \omega_{\omega_1}$

(b) *There is a family* $\{A_\alpha : \alpha < \omega_{\omega_1}\}$ *such that each* $A_\alpha \in [\omega_{\omega_1}]^{\omega_{\omega_1}}$

(c) *For every* $\alpha < \beta < \omega_{\omega_1}$, $A_\alpha \cap A_\beta$ *is finite*

(d) *For every* $i < \omega_1$ *and* $\alpha < \omega_{\omega_1}$, $|A_\alpha \cap \omega_{i+1}| = \omega_{i+1}$

Proof of Lemma 3.2: We use Baumgartner's thinning out forcing - See Theorem 6.1 in [1]. Let $V \models$ GCH. Put $\lambda = \omega_{\omega_1}$ and $\lambda_i = \omega_{i+1}$. For each $1 \leq i < \omega_1$, define $\mathbb{P}_i$ as follows. Let $K_i = \{\nu \in [\omega_2, \lambda_i] : \nu = \mathrm{cf}(\nu)\}$. $p \in \mathbb{P}_i$ iff

(i) $p = \langle p_\nu : \nu \in K_i \rangle$

(ii) Each p_ν is a function with $\mathrm{dom}(p_\nu) \in [\lambda]^{<\nu}$

(iii) For each $\alpha \in \mathrm{dom}(p_\nu)$, $p_\nu(\alpha) \in [\lambda_i]^{<\nu}$

(iv) If $\nu < \nu'$, then $\mathrm{dom}(p_\nu) \subseteq \mathrm{dom}(p_{\nu'})$ and for each $\alpha \in \mathrm{dom}(p_\nu)$, $p_\nu(\alpha) \subseteq p_{\nu'}(\alpha)$

For $p, q \in \mathbb{P}_i$, write $p \leq_i q$ iff

(a) For each $\nu \in K_i$, $\mathrm{dom}(p_\nu) \subseteq \mathrm{dom}(q_\nu)$

(b) For each $\alpha, \beta \in \mathrm{dom}(p_\nu)$, $p_\nu(\alpha) \subseteq q_\nu(\alpha)$ and if $\alpha \neq \beta$, then $p_\nu(\alpha) \cap p_\nu(\beta) = q_\nu(\alpha) \cap q_\nu(\beta)$

Let $\mathbb{P} = \prod\{\mathbb{P}_i : i < \kappa\}$ be the full support product of $\{\mathbb{P}_i : i < \kappa\}$. So $p \in \mathbb{P}$ iff $p = \langle p(i) : i < \kappa \rangle$ and for every $i < \kappa$, $p(i) \in \mathbb{P}_i$. For $p, q \in \mathbb{P}$, $p \leq q$ iff for every $i < \kappa$, $p(i) \leq_i q(i)$.

Claim 3.3. $\mathbb{P}$ *preserves all regular cardinals below* λ.

Proof of Claim 3.3: The proof is almost identical to that of Lemma 6.6 in [1] but we provide a sketch. Let G be $\mathbb{P}$-generic over V. Let $\tau < \lambda$ be a regular cardinal in V and suppose $V[G] \models \tau > \mathrm{cf}(\tau) = \mu$. Note that $\mathbb{P}$ is ω_2-closed so $\mu \geq \omega_2$. Fix $1 \leq i_\star < \omega_1$ such that $\mu = \lambda_{i_\star}$.

Let $\mathbb{Q} = \{\langle p(i) \restriction [\lambda_{i_\star+1}, \infty) : i < \omega_1\rangle : p \in \mathbb{P}\}$ and $H = \{\langle p(i) \restriction [\lambda_{i_\star+1}, \infty) : i < \omega_1\rangle : p \in G\}$. Then $\mathbb{Q}$ is $\lambda_{i_\star+1}$-closed and H is $\mathbb{Q}$-generic over V. In $V[H]$, for $i_\star < i < \omega_1$ and $\alpha < \lambda$, let $E_{i,\alpha} = \bigcup\{p(i)(\lambda_{i_\star+1})(\alpha) : p \in H\}$ and for $i \leq i_\star$, $\alpha < \lambda$, let $E_{i,\alpha} = \lambda_i$. Let $\mathbb{Q}' = \{\langle p(i) \restriction [0, \lambda_{i_\star}] : i < \omega_1\rangle : p \in \mathbb{P} \wedge (\forall \alpha \in \mathrm{dom}(p(i)(\lambda_{i_\star})))p_i(\lambda_{i_\star})(\alpha) \subseteq E_{i,\alpha}\}$ and $K = \{\langle p(i) \restriction [0, \lambda_{i_\star}] : i < \omega_1\rangle : p \in G\}$. Then it is easily verified that K is $\mathbb{Q}'$-generic over $V[H]$ and $V[G] = V[H][K]$. As $\mathbb{Q}$ is $\lambda_{i_\star+1}$-closed, $\mathrm{cf}(\tau) \geq \lambda_{i_\star+1}$ in $V[H]$. Since $\lambda_{i_\star} \geq \omega_2$, a Δ-system argument shows that $V[H] \models \mathbb{Q}'$ satisfies $\lambda_{i_\star+1}$-c.c. (see Lemma 6.3 in [1]) hence $V[G] = V[H][K] \models \mathrm{cf}(\tau) \geq \lambda_{i_\star+1} > \mu$: A contradiction. $\square$

Let G be $\mathbb{P}$-generic over V and $V_1 = V[G]$. In V_1, for $\alpha < \lambda$, let $F_\alpha = \bigcup\{F_{i,\alpha} \cap [\omega_i, \omega_{i+1}) : i < \omega_1\}$ where $F_{i,\alpha} = \bigcup\{q_{\omega_2}(\alpha) : (\exists p \in G)(q = p(i))\}$. Then each F_α is unbounded in ω_{i+1} for $1 \leq i < \omega_1$ and their pairwise intersections have sizes $\leq \omega_1$. In V_1, define $\mathbb{P}_1$ by $p \in \mathbb{P}_1$ iff p is a function, $\mathrm{dom}(p) \in [\lambda]^{<\aleph_1}$ and for each $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in [F_\alpha \cup \omega_1]^{<\aleph_1}$. For $p, q \in \mathbb{P}_1$, $p \leq q$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $\alpha, \beta \in \mathrm{dom}(p)$, $p(\alpha) \subseteq q(\alpha)$ and if $\alpha \neq \beta$, then $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$. As CH holds in V_1, a Δ-system argument shows that $\mathbb{P}_1$ satisfies $\aleph_2$-cc. Since it is also countably closed, all cofinalities from V_1 are preserved. Let G_1 be $\mathbb{P}_1$-generic over V_1 and $V_2 = V_1[G_1]$. For $\alpha < \lambda$, put $F'_\alpha = \bigcup\{p(\alpha) : p \in G_1\}$. Then each F'_α is unbounded in ω_{i+1} for $i < \omega_1$ and their pairwise intersections are countable. In V_2, define $\mathbb{P}_2$ by $p \in \mathbb{P}_2$ iff p is a function, $\mathrm{dom}(p) \in [\lambda]^{<\aleph_0}$ and for each $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in [F'_\alpha]^{<\aleph_0}$. For $p, q \in \mathbb{P}_1$, $p \leq q$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $\alpha, \beta \in \mathrm{dom}(p)$, $p(\alpha) \subseteq q(\alpha)$ and if $\alpha \neq \beta$, then $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$. A Δ-system argument shows that $\mathbb{P}_2$ satisfies ccc so all cofinalities are preserved. Let G_2 be $\mathbb{P}_2$-generic over V_2 and $V_3 = V_2[G_2]$. For $\alpha < \lambda$, put $A_\alpha = \bigcup\{p(\alpha) : p \in G_2\}$. Then each A_α is unbounded below ω_{i+1} for each $i < \omega_1$ and their pairwise intersections are finite. As $\{A_\alpha \cap \omega_1 : \alpha < \lambda\}$ is a mod finite almost disjoint family, $V_3 \models \mathfrak{c} \geq \lambda$. The other inequality follows from a name counting argument using $V_2 \models \lambda^{\aleph_0} = \lambda$. $\square$

Proof of Theorem 3.1: Let V be a model satisfying the clauses of Claim 3.2. We'll construct a finite support iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \omega_1\rangle$ of ccc forcings with limit $\mathbb{P}$ satisfying the following.

(1) $|\mathbb{P}| = \omega_{\omega_1}$

(2) $\Vdash_{\mathbb{P}_i} \langle \mathring{z}_{i,\alpha} : \alpha < \omega_{\omega_1} \rangle$ lists $\mathbb{C}$ and $\mathring{Z}_i = \{\mathring{z}_{j,\alpha} : j \leq i, \alpha < \omega_{i+1}\}$

(3) $\langle \mathring{y}_\alpha : \alpha < \omega_{i+1} \rangle \in V^{\mathbb{P}_i}$ is such that $\Vdash_{\mathbb{P}_i} \langle \mathring{y}_\alpha : \alpha < \omega_{i+1} \rangle$ is a one-one listing of $\mathring{Z}_i$ - So $\{\mathring{y}_\alpha : \alpha < \omega_{\omega_1}\} = \mathbb{C} \cap V^{\mathbb{P}}$

(4) In $V^{\mathbb{P}_i}$, $\mathbb{Q}_i$ is a ccc forcing of size λ_i that adds a family $\mathcal{F}_i$ of entire functions of size ω_{i+1} such that for every $j \leq i$, $\Vdash_{\mathbb{Q}_i} |\{f(y_\alpha) : \alpha < \omega_{j+1}\}| \leq \omega_{j+1}$

Put $\mathcal{F} = \bigcup_{i<\omega_1} \mathcal{F}_i$. If $\mathring{z} \in V^{\mathbb{P}} \cap \mathbb{C}$, then for some $i_\star < \omega_1$ and $\alpha < \omega_{i_\star+1}$, $\mathring{z} = y_\alpha$. Hence $|\{f(\mathring{z}) : f \in \mathcal{F}\}| \leq |\bigcup_{i<i_\star} \{f(\mathring{z}) : f \in \mathcal{F}_i\}| + |\bigcup_{i>i_\star} \{f(y_\alpha) : f \in \mathcal{F}_i\}| \leq \omega_{i_\star+1} + \omega_1 \cdot \omega_{i_\star+1} = \omega_{i_\star+1} < \mathfrak{c}$. The following lemma shows that $\mathbb{Q}_i$'s can be constructed.

Lemma 3.4. *Suppose κ is regular uncountable. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be such that for every $\alpha < \beta < \kappa$ and uncountable cardinal $\mu \leq \kappa$, $A_\alpha \cap \mu \in [\mu]^\mu$ and $A_\alpha \cap A_\beta$ is finite (so $\kappa \leq \mathfrak{c}$). Let $\langle y_\alpha : \alpha < \kappa \rangle$ be a sequence of distinct complex numbers. Then there exists a ccc forcing $\mathbb{Q}$ of size κ such that the following hold in $V^{\mathbb{Q}}$.*

(a) There is a family $\mathcal{F}$ of entire functions of size κ

(b) For every uncountable cardinal $\mu \leq \kappa$, $|\{f(y_\alpha) : \alpha < \mu, f \in \mathcal{F}\}| = \mu$

Proof of Lemma 3.4: For $\xi < \kappa$, let $h_\xi : \kappa \to A_\xi$ be such that $h_\xi(\alpha)$ is the αth member of A_ξ. Note that for every regular uncountable $\mu < \kappa$, $h_\xi[\mu] = A_\xi \cap \mu$. Define $\mathbb{Q}$ as follows: $p \in \mathbb{Q}$ iff

$p = (n_p, m_p, u_p, v_p, w_p, \langle m^p_{\xi,\alpha} : \xi \in u_p, \alpha \in v_p \rangle, \langle f^p_\xi : \xi \in u_p \rangle, \langle B^p_{\gamma,m} : \gamma \in w_p, m < m_p \rangle)$ where

(1) $1 \leq n_p < \omega$, $1 \leq m_p < \omega$

(2) $u_p, v_p, w_p \in [\kappa]^{<\aleph_0}$ and for every $\alpha \in v_p$, $|y_\alpha| < n_p$

(3) $w_p \supseteq \{h_\xi(\alpha) : \xi \in u_p, \alpha \in v_p\}$

(4) $m^p_{\xi,\alpha} < m_p$ for every $\xi \in u_p, \alpha \in v_p$

(5) For each $\xi \in u_p$, $f^p_\xi = f^p_\xi(x, x'_\alpha, x''_\alpha)_{\alpha \in v_p}$ is a rational function in the $2|v_p| + 1$ variables $\{x\} \cup \{x'_\alpha, x''_\alpha : \alpha \in v_p\}$ over the rational complex field which can be expressed as a polynomial in x whose coefficients are rational functions of $\{x'_\alpha, x''_\alpha : \alpha \in v_p\}$ that satisfies: For every $\beta \in v_p$, $f^p_\xi(x'_\beta, x'_\alpha, x''_\alpha)_{\alpha \in v_p} = x''_\beta$

(6) For every $\gamma \in w_p$ and $m < m_p$, $B^p_{\gamma,m}$ is a closed disk in complex plane with rational complex center and rational radius that satisfies: If $z_{\alpha,\xi,m} \in B^p_{h_\xi(\alpha),m}$, for $\xi \in u_p$, $\alpha \in v_p$ and $m < m_p$, then $f_\xi(x, y_\alpha, z_{\alpha,\xi,m^p_{\xi,\alpha}})_{\alpha \in v_p}$ is well defined (no vanishing denominators).

Informally, p promises that for $\xi \in u_p$, the ξth entire function $\mathring{f}_\xi$ added by $\mathbb{Q}$ is approximated by $f^p_\xi(x, y_\alpha, z_\alpha)_{\alpha \in v_p}$ uniformly on the disk $\{x \in \mathbb{C} : |x| \leq n_p\}$ with an error $\leq 2^{-n_p}$ where z_α is an arbitrary point in $B^p_{h_\xi(\alpha),m^p_{\xi,\alpha}}$. It also promises that $\mathring{f}_\xi$ will map y_α (for $\alpha \in v_p$) into $B^p_{h_\xi(\alpha),m^p_{\xi,\alpha}}$. The parameter m in $B^p_{\gamma,m}$ allows us a countable amount of freedom to choose $\mathring{f}_\xi(y_\alpha)$ (this is useful to increase v_p, see Claim 3.5(c) below).

For $p, q \in \mathbb{Q}$, define $p \leq q$ iff

(6) $n_p \leq n_q, m_p \leq m_q$

(7) $u_p \subseteq u_q, v_p \subseteq v_q$ and $w_p \subseteq w_q$

(8) If $\xi \in u_p, \alpha \in v_p$, then $m^q_{\xi,\alpha} = m^p_{\xi,\alpha}$

(9) $B^q_{\gamma,m} \subseteq B^p_{\gamma,m}$ for every $\gamma \in w_p$ and $m < m_p$

(10) Whenever $|z| < n_p$, $\xi \in u_p$, $z_{\xi,\alpha} \in B^q_{h_\xi(\alpha),m^q_{\xi,\alpha}}$ for $\alpha \in v_q$, we have

$$|f^p_\xi(z, y_\alpha, z_{\xi,\alpha})_{\alpha \in v_p} - f^q_\xi(z, y_\alpha, z_{\xi,\alpha})_{\alpha \in v_q}| \leq 1/2^{n_p} - 1/2^{n_q}$$

Claim 3.5. *The following are dense in* $\mathbb{Q}$

(a) $\{p \in \mathbb{Q} : \xi \in u_p\}$ *for* $\xi < \kappa$

(b) $\{p \in \mathbb{Q} : (\gamma \in w_p) \wedge (n_p, m_p \geq N)\}$ *for* $N < \omega$ *and* $\gamma < \kappa$

(c) $\{p \in \mathbb{Q} : \beta \in v_p\}$ *for* $\beta < \kappa$

(d) $\{p \in \mathbb{Q} : (\forall m < m_p)(\forall \gamma \in w_p)(\mathsf{diam}(B^p_{\gamma,m}) < 2^{-N})\}$ *for* $N < \omega$

(e) $\{p \in \mathbb{Q} : (\forall \gamma_1, \gamma_2 \in w_p)(\forall m_1, m_2 < m_p)((\gamma_1, n_1) \neq (\gamma_2, n_2) \implies B^p_{\gamma_1,m_1} \cap B^p_{\gamma_2,m_2} = \phi)\}$

Proof of Claim 3.5: Clauses (b), (d) and (e) should be clear. Let us check (a) and (c).

(a) Suppose $q \in \mathbb{Q}$ with $\xi_\star \in \kappa \setminus u_q$. If $v_q = \phi$, then we can add $\xi_\star$ to u_q and set $f^q_{\xi_\star}(x) = 0$. So assume $v_q = \{\alpha_i : 1 \leq i \leq k\}$. Define $g_i = g_i(x, x'_{\alpha_j}, x''_{\alpha_j})_{1 \leq j \leq i}$ for $1 \leq i \leq k$ recursively as follows

$$g_1 = x + x''_{\alpha_1} - x'_{\alpha_1}$$

$$g_{i+1} = g_i + \left(\prod_{1 \leq j \leq i} (x - x'_{\alpha_j})\right)\left(\frac{x''_{\alpha_{i+1}} - g_i(x'_{\alpha_{i+1}}, x'_{\alpha_j}, x''_{\alpha_j})_{1 \leq j \leq i}}{\prod_{1 \leq j \leq i}(x'_{\alpha_{i+1}} - x'_{\alpha_j})}\right)$$

Define $p \geq q$ as follows. Put $n_p = n_q$, $m_p = m_q$, $u_p = u_q \cup \{\xi_\star\}$, $v_p = v_q$, $w_p = w_q \cup \{h_{\xi_\star}(\alpha) : \alpha \in v_q\}$, $f^p_{\xi_\star} = g_k$ and $f^p_\xi = f^q_\xi$ for $\xi \in u_q$. Set $m^p_{\xi,\alpha} = m^q_{\xi,\alpha}$ and $B^p_{\gamma,m} = B^q_{\gamma,m}$ if already defined otherwise choose them arbitrarily.

(c) Suppose $q \in \mathbb{Q}$ and $\beta \in \kappa \setminus v_q$. By increasing n_q, we can assume $|y_\beta| < n_q$. For each $\xi \in u_q$, define f^p_ξ by

$$f^p_\xi = f^q_\xi + \left(\prod_{\alpha \in v_q} (x - x'_\alpha)\right)\left(\frac{x''_\beta - f^q_\xi(x'_\beta, x'_\alpha, x''_\alpha)_{\alpha \in v_q}}{\prod_{\alpha \in v_q}(x'_\beta - x'_\alpha)}\right)$$

where, we take a product over the empty index set to be 1.

Let $\varepsilon = \min\{|y_\beta - y_\alpha| : \alpha \in v_q\}$ if $v_q \neq \phi$ and $\varepsilon = 1$ otherwise. Put $u_p = u_q$, $v_p = v_q \cup \{\beta\}$, $w_p = w_q \cup \{h_\xi(\beta) : \xi \in u_q\}$, $n_p = n_q + 1$, $m_p = m_q + |u_q|$. For each $\xi \in u_q$, choose $m^p_{\xi,\beta} \geq m_q$ such that $\xi_1 \neq \xi_2$ implies $m^p_{\xi_1,\beta} \neq m^p_{\xi_2,\beta}$. We need to choose $B^p_{\gamma,m}$'s such that whenever $|z| < n_q$, $\xi \in u_q$ and $z_{\xi,\alpha} \in B^p_{h_\xi(\alpha), m^p_{\xi,\alpha}}$ for $\alpha \in v_p$,

$$\left|\left(\prod_{\alpha \in v_q} (z - y_\alpha)\right)\left(\frac{z_{\xi,\beta} - f^q_\xi(y_\beta, y_\alpha, z_{\xi,\alpha})_{\alpha \in v_q}}{\prod_{\alpha \in v_q}(y_\beta - y_\alpha)}\right)\right| \leq \frac{1}{2^{n_q}} - \frac{1}{2^{n_q+1}}$$

For this it is enough to have

$$\left|z_{\xi,\beta} - f^q_\xi(y_\beta, y_\alpha, z_{\xi,\alpha})_{\alpha \in v_q}\right| \leq \frac{\varepsilon^k}{(2n_q)^k (2^{n_q+1})}$$

where $k = |v_q|$. But this is easily arranged by first shrinking $B^q_{h_\xi(\alpha),m^q_{\xi,\alpha}}$'s for $\alpha \in v_q$ and then choosing $B^p_{h_\xi(\beta),m^p_{\xi,\beta}}$ accordingly.

$\square$

Let G be $\mathbb{Q}$-generic over V. For $\gamma < \kappa$ and $m < \omega$, let $a_{\gamma,m}$ be the unique member of $\bigcap\{B^p_{\gamma,m} : p \in G\}$.

For $\xi < \lambda$, define $f_\xi : \mathbb{C} \to \mathbb{C}$ as follows. Choose $\{p_k : k < \omega\} \subseteq G$ such that $\xi \in u_{p_k}$ and $n_{p_k} \geq k$ for every $k < \omega$, and put $f_\xi(z) = \lim_k f^{p_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{p_k}_{\xi,\alpha}})_{\alpha\in v_{p_k}}$. Since we have uniform convergence on compact sets, f_ξ is analytic. Note that the definition of f_ξ is independent of the choice of $\{p_k : k < \omega\} \subseteq G$. For suppose $\{q_k : k < \omega\} \subseteq G$ is such that $\xi \in u_{q_k}$ and $n_{q_k} \geq k$ for every $k < \omega$. Let $r_k \in G$ be a common extension of p_k, q_k. Then, for every $z \in \mathbb{C}$ with $|z| < k$, we have $|f^{p_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{p_k}_{\xi,\alpha}})_{\alpha\in v_{p_k}} - f^{q_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{q_k}_{\xi,\alpha}})_{\alpha\in v_{q_k}}| \leq 2^{-k+1}$ since it is at most the sum of $|f^{p_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{p_k}_{\xi,\alpha}})_{\alpha\in v_{p_k}} - f^{r_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{r_k}_{\xi,\alpha}})_{\alpha\in v_{r_k}}|$ and $|f^{q_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{q_k}_{\xi,\alpha}})_{\alpha\in v_{q_k}} - f^{r_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha),m^{r_k}_{\xi,\alpha}})_{\alpha\in v_{r_k}}|$ and hence the two limits must be the same.

Put $\mathcal{F} = \{f_\xi : \xi < \kappa\}$. For $\xi, \alpha < \kappa$, let $m_{\xi,\alpha}$ be such that for some $p \in G$, $\xi \in u_p$, $\alpha \in v_p$ and $m^p_{\xi,\alpha} = m_{\xi,\alpha}$. Note that, for every $\xi, \alpha < \kappa$, by considering a sequence $\{p_k : k < \omega\} \subseteq G$ with $\alpha \in v_{p_k}$, we can infer that $f_\xi(y_\alpha) = a_{h_\xi(\alpha),m_{\xi,\alpha}}$. Next suppose $\xi_1 < \xi_2 < \kappa$. Choose $\alpha < \kappa$ such that $h_{\xi_1}(\alpha) \neq h_{\xi_2}(\alpha)$. Then $f_{\xi_1}(y_\alpha) = a_{h_{\xi_1}(\alpha),m_{\xi_1,\alpha}} \neq a_{h_{\xi_2}(\alpha),m_{\xi_2,\alpha}} = f_{\xi_2}(y_\alpha)$. So f_ξ's are pairwise distinct. Finally, for every uncountable $\mu \leq \kappa$, we have $|\{f_\xi(y_\alpha) : \alpha < \mu, \xi < \kappa\}| \leq |\{a_{h_\xi(\alpha),m_{\xi,\alpha}} : \xi < \kappa, \alpha < \mu\}| \leq |\{a_{\gamma,m} : \gamma < \mu, m < \omega\}| = \mu$.

So it suffices to show that $\mathbb{Q}$ is ccc. Suppose $A \subseteq \mathbb{Q}$ is uncountable. Choose $S \subseteq A$ uncountable such that the following hold.

(1) $n_p = n_\star$, $m_p = m_\star$, $|u_p| = n^1_\star$ and $|v_p| = n^2_\star$ do not depend on $p \in S$

(2) $\langle u_p : p \in S\rangle$ is a Δ-system with root $u_\star$ and $\langle v_p : p \in S\rangle$ is a Δ-system with root $v_\star$

(3) If $\xi_1 \neq \xi_2$ are from $u_\star$ and $h_{\xi_1}(\alpha_1) = h_{\xi_2}(\alpha_2)$, then $\{\alpha_1, \alpha_2\} \cap (v_p \setminus v_\star) = \phi$ for every $p \in S$ - This uses the fact that $A_{\xi_1} \cap A_{\xi_2}$ is finite (countable suffices)

(4) By possibly extending $p \in S$, we can assume $1 \leq |v_\star| < n^2_\star$ (so v_p and $v_p \setminus v_\star \neq \phi$)

(5) $u_p = \{\xi_{p,j} : j < n^1_\star\}$, $v_p = \{\alpha_{p,k} : k < n^2_\star\}$ list members in increasing order and $r^1_\star \subseteq n^1_\star$, $r^2_\star \subseteq n^2_\star$ are such that $u_\star = \{\xi_{p,k} : j \in r^1_\star\}$ and $v_\star = \{\alpha_{p,k} : k \in r^2_\star\}$

(6) For every $j < n^1_\star$, $k < n^2_\star$ and $m < m_\star$, we have $f^p_{\xi_{p,j}} = f_j(x, x'_{\alpha_{p,k}}, x''_{\alpha_{p,k}})_{k<n^2_\star}$, $m^p_{\xi_{p,j},\alpha_{p,k}} = m_{j,k}$ and $B^p_{h_{\xi_{p,j}}(\alpha_{p,k}),m} = B_{j,k,m}$ where f_j, $m_{j,k}$, $B_{j,k,m}$ do not depend on $p \in S$,

(7) $0 < \varepsilon_1 < 2^{-(n_\star+1)}$, ε_1 is smaller than the radius of every $B_{j,k,m}$, and for every $p \in S$, for every $k_1 < k_2 < n^2_\star$, $|y_{\alpha_{p,k_1}} - y_{\alpha_{p,k_2}}| > \varepsilon_1$

(8) Each point of $X = \{\langle y_{\alpha_{p,k}} : k < n^2_\star\rangle : p \in S\}$ is a condensation point of $X \subseteq \mathbb{C}^{n^2_\star}$

Suppose $p, p' \in S$ and we would like to find a common extension q. This boils down to constructing f^q_ξ for $\xi \in u_p \cup u_{p'}$. For $\xi \in (u_p \cup u_{p'}) \setminus u_\star$, this is similar to the proof of Claim 3.5(c). To construct f^q_ξ for $\xi \in u_\star$, we'll make use of the following lemma.

Lemma 3.6. *Suppose*

(*i*) $1 \leq n_\star < \omega$, $0 < \varepsilon_1 < 0.5$

(*ii*) $f = f(z, x_k, y_k)_{k<k_\star}$ *is a rational function in the variables* $\{z\} \cup \{x_k, y_k : k < k_\star\}$ *over the rational complex field which can be expressed as a polynomial in* z *whose coefficients are rational functions of* x_k, y_k *for* $k < k_\star$ *over the rational complex field, satisfying* $f(x_l, x_k, y_k)_{k<k_\star} = y_l$ *for every* $l < k_\star$

(*iii*) $a_k, b_k \in \mathbb{C}$ *for* $k < k_\star$, *for each* $k < k_\star$, $|a_k| < n_\star$ *and for every* $k_1 < k_2 < k_\star$, $|a_{k_1} - a_{k_2}| > \varepsilon_1$

(*iv*) *If* $|a'_k - a_k| < \varepsilon_1$ *and* $|b'_k - b_k| < \varepsilon_1$ *for* $k < k_\star$, *then* $f(z, a'_k, b'_k)_{k<k_\star}$ *is well defined (no vanishing denominators)*

(*v*) $v_\star \subseteq k_\star$, $v_\star \notin \{\phi, k_\star\}$

Then there exist $0 < \varepsilon_2 < \varepsilon_1/8$ *and* $g = g(z, x_l, y_l, x^1_k, x^2_k, y^1_k, y^2_k)_{l\in v_\star, k\in k_\star\setminus v_\star}$ *such that whenever* $|a^2_k - a_k| < \varepsilon_2$ *for* $k \in k_\star \setminus v_\star$, *letting* $b^2_k = f(a^2_k, a_j, b_j)_{j<k_\star}$ *we have* $|b^2_k - b_k| < \varepsilon_1 - 2\varepsilon_2$ *for* $k \in k_\star \setminus v_\star$ *and the following hold.*

(*a*) g *is a polynomial in* z *whose coefficients are rational functions of the other variables over the rational complex field satisfying* $z = x_l$ *implies* $g = y_l$ *for* $l \in v_\star$ *and* $z = x^j_k$ *implies* $g = y^j_k$ *for* $j = 1, 2$ *and* $k \in k_\star \setminus v_\star$

(b) *Letting* $a^1_k = a_k$, $b^1_k = b_k$ *for* $k \in k_\star \setminus v_\star$ *we have the following. For every* c_l, c^j_k *satisfying* $|c_l - b_l| < \varepsilon_2$, $|c^j_k - b^j_k| < \varepsilon_2$ *for* $l \in v_\star$, $j = 1, 2$, $k \in k_\star \setminus v_\star$, *for every* $|z| < n_\star$ *and* $j = 1, 2$, *we have*

$$\left| f(z, a_l, a^j_k, c_l, c^j_k)_{l \in v_\star, k \in k_\star \setminus v_\star} - g(z, a_l, c_l, a^1_k, a^2_k, c^1_k, c^2_k)_{l \in v_\star, k \in k_\star \setminus v_\star} \right| < \varepsilon_1$$

Proof of Lemma 3.6: Put

$$g = f(z, x_l, x^1_k, y_l, y^1_k)_{l \in v_\star, k \in k_\star \setminus v_\star} + \sum_{j \in k_\star \setminus v_\star} G_j$$

where

$$G_j = \frac{F_j(z)[y^2_j - f(x^2_j, x_l, x^1_k, y_l, y^1_k)_{l \in v_\star, k \in k_\star \setminus v_\star}]}{F_j(x^2_j)}$$

where

$$F_j(z) = \prod_{k \in k_\star \setminus v_\star}^{k \neq j} (z - x^2_k) \prod_{l \in v_\star} (z - x_l) \prod_{k \in k_\star \setminus v_\star} (z - x^1_k)$$

Clause (a) is easily verified. We need to find $0 < \varepsilon_2 < \varepsilon_1/8$ such that clause (b) holds. Note that for all sufficiently small $\varepsilon_2 < \varepsilon_1/8$, if $|a^2_j - a_j| < \varepsilon_2$, then $|b^2_j - b_j| = |f(a^2_j, a_k, b_k)_{k < k_\star} - f(a_j, a_k, b_k)_{k < k_\star}| < 3\varepsilon_1/4 < \varepsilon_1 - 2\varepsilon_2$. Fix c_l, c^j_k as in clause (b) and consider

$$\left| f(z, a_l, a^j_k, c_l, c^j_k)_{l \in v_\star, k \in k_\star \setminus v_\star} - g(z, a_l, c_l, a^1_k, a^2_k, c^1_k, c^2_k)_{l \in v_\star, k \in k_\star \setminus v_\star} \right|$$

This is at most the sum of

$$\left| f(z, a_l, a^1_k, c_l, c^1_k)_{l \in v_\star, k \in k_\star \setminus v_\star} - f(z, a_l, a^2_k, c_l, c^2_k)_{l \in v_\star, k \in k_\star \setminus v_\star} \right|$$

and

$$\sum_{j \in k_\star \setminus v_\star} \left| G_j(z, a_l, c_l, a^1_k, a^2_k, c^1_k, c^2_k)_{l \in v_\star, k \in k_\star \setminus v_\star} \right|$$

The former term is easily bounded by $\varepsilon_1/2$ by choosing sufficiently small ε_2. For the latter, notice that

$$\left| \frac{F_j(z)}{F_j(a^2_j)} \right| < \left(\frac{4n_\star}{\varepsilon_1} \right)^{2k_\star}$$

So it suffices to ensure that

$$\left|c_j^2 - f(a_j^2, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star, k\in k_\star\setminus v_\star}\right| < \frac{\varepsilon_1^{2k_\star+1}}{k_\star(4n_\star)^{2k_\star}}$$

The expression on the left side is at most

$$|c_j^2 - b_j^2| + \left|b_j^2 - f(a_j^2, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star, k\in k_\star\setminus v_\star}\right|$$

Recalling our choice of b_j^2, this is bounded by

$$\varepsilon_2 + \left|f(a_j^2, a_l, a_k^1, b_l, b_k^1)_{l\in v_\star, k\in k_\star\setminus v_\star} - f(a_j^2, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star, k\in k_\star\setminus v_\star}\right|$$

It is clear that this can be made arbitrarily small by choosing sufficiently small ε_2. $\square$

Fix $p \in S$. For each $j \in r_\star^1$, using Lemma 3.6, get $\varepsilon_2 = \varepsilon_{2,j}$ and $g = g_j$ w.r.t. $f = f_j$, $a_k = y_{\alpha_{p,k}}$, $b_k =$ the center of $B_{j,k,m_{j,k}}$ and $v_\star = r_\star^2$. Let $\varepsilon_3 = \min\{\varepsilon_{2,j} : j \in r_\star^1\}$. Choose $p' \neq p$ from S such that for each $k < n_\star^2$, $|y_{\alpha_{p,k}} - y_{\alpha_{p',k}}| < \varepsilon_3$. We'll construct a common extension q of p, p'.

Put $n_q = n_\star + 1$, $m_q = m_\star + n_\star^1 n_\star^2$, $u_q = u_p \cup u_{p'}$, $v_q = v_p \cup v_{p'}$ and $w_q = w_p \cup w_{p'} \cup \{h_\xi(\alpha) : \xi \in u_q, \alpha \in v_q\}$. Choose $m_{\xi,\alpha}^q$'s such that $\{m_{\xi,\alpha}^q : (\xi \in u_p \setminus u_\star \wedge \alpha \in v_p \setminus v_\star)$ or $(\xi \in u_{p'} \setminus u_\star \wedge \alpha \in v_{p'} \setminus v_\star)\}$ are pairwise distinct integers in $[m_\star, m_p)$. Next choose f_ξ^q, $B_{\gamma,m}^q$ for $\xi \in u_q$, $\gamma \in w_q$ and $m < m_q$ as follows.

(1) If $\xi \in u_p \setminus u_\star$, let f_ξ^q be as in the proof of Claim 3.5(c) applying the process $|v_{p'} - v_\star|$ times. Define $B_{h_\xi(\alpha), m_{\xi,\alpha}^q}^q$ for $\alpha \in v_p$ by shrinking $B_{h_\xi(\alpha), m_{\xi,\alpha}^p}^p$ and choose $B_{h_\xi(\alpha), m_{\xi,\alpha}^q}^q$ for $\alpha \in v_{p'} \setminus v_\star$ accordingly.

(2) If $\xi \in u_{p'} \setminus u_\star$, we define f_ξ^q and $B_{h_\xi(\alpha), m_{\xi,\alpha}^q}^q$ analogously.

(3) If $\xi \in u_\star$, choose $j \in r_\star^1$ such that $\xi_{p,j} = \xi$ and put $f_\xi^q = g_j$. Obtain b_k^2 for $k \in n_\star^2 \setminus r_\star^2$ as in Lemma 3.6 w.r.t. $a_k^2 = y_{\alpha_{p',k}}$. For $k \in n_\star^2$, choose $B_{h_\xi(\alpha_{p,k}), m_{j,k}}^q$ to be a rational disk contained in $B_{j,k,m_{j,k}}$ with center b_k and radius less than ε_3. For $k \in n_\star^2 \setminus r_\star^2$, choose $B_{h_\xi(\alpha_{p',k}), m_{j,k}}^q$ to be a rational disk with center b_k^2 and radius less than ε_3 (so it is contained in $B_{j,k,m_{j,k}}$). Notice that if $\xi_1 \neq \xi_2$ are from $u_\star$ and $\{\alpha_1, \alpha_2\} \cap (v_q \setminus v_\star) \neq \phi$, then $h_{\xi_1}(\alpha_1) \neq h_{\xi_2}(\alpha_2)$ so there is no conflict in doing this. $\square$

4 Regular continuum

We conclude with the following.

Question 4.1. *Is a positive answer to Question 1.1 consistent with $2^{\aleph_0} = \aleph_2$?*

One way to get this would be to construct a model where $2^{\aleph_0} = \aleph_2$ and for some $A \in [\mathbb{C}]^{\aleph_1}$, for every $X \in [\mathbb{C}]^{\aleph_1}$, there is a non constant entire function sending X into A. We do not know if this is possible.

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