An Extension of M. C. R. Butler's theorem on endomorphism rings

Manfred Dugas, Daniel Herden, and Saharon Shelah

Dedicated to the memory of Rüdiger Göbel

Abstract. We will prove the following

Theorem. Let D be the ring of algebraic integers of a finite Galois field extension F of $\mathbb{Q}$ and E a D-algebra such that E is a locally free D-module of countable rank and all elements of E are algebraic over F. Then there exists a left D-submodule $M \supseteq E$ of $FE = E \otimes_D F$ such that the left multiplications by elements of E are the only D-linear endomorphisms of M.

1. Introduction

The main purpose of this paper is to honor the memory of Rüdiger Göbel, a dear friend and colleague, who passed away much too early. We would like to make a few personal remarks about our time with Rüdiger:

1.1. Manfred Dugas' Statement: It was a cold month of January in 1976, when I started work as Rüdiger's assistant in the mathematics department at Essen University, Essen, Germany. Rüdiger had just come back from an extended visit to the University of Texas at Austin. There he had worked on general relativity, a branch of theoretical physics. He was also interested in infinite cartesian products of groups. We immediately started working together and wrote a paper [**4**], in German, on cartesian products of copies of $\mathbb{Z}$, the ring of integers. The paper [**5**] soon followed. After that, we wrote all our joint papers in English. If I counted correctly, we managed to write a total of twenty-eight papers together on a variety of topics. Rüdiger had come to Essen just one year before I did, and he had already attracted several students, who attended our very active research seminar. We worked through A. L. S. Corner's seminal paper [**2**] and also [**36**]. This lead to our arguably most important and consequential paper [**6**], which started a whole cottage industry of realizing rings as endomorphism rings of abelian groups and modules. We also worked through Eklof's splendid account [**9**] of Shelah's ingenious solution to the famous Whitehead problem. After that, applying set theory to

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algebra became our main staple. The people in our seminar went to Oxford for two weeks in the summer of 1979 to meet and work with Tony Corner. This was an unforgettable event for everybody, and Tony became our friend. Rüdiger's big talent for organization made this trip possible!

In 1984, Rangaswamy invited me to visit the University of Colorado at Colorado Springs for a year. So my two small daughters, my wife, and I packed up and moved to Colorado. We came to love our life in Colorado and we were very fortunate that the university offered me a tenure-track position, which I gratefully accepted. After three years in Colorado, Baylor University, Waco, Texas (not far from Austin!), made me an offer that I could not refuse. We have lived in Texas ever since. Rüdiger and I stayed in touch and we visited each other frequently. Whenever we met, we wrote at least one paper!

Here is an overview of our joint work on "realizations results" in chronological order of publication:

1982: • *Every cotorsion-free ring is an endomorphism ring*, Proc. London Math. Soc. **45** (1982), 319–336.

Here we use the weak diamond principle to construct large κ-free abelian groups of cardinality κ with prescribed, cotorsion-free endomorphism ring.

• *On endomorphism rings of primary abelian groups*, Math. Ann. **261** (1982), 359–385.

Let R be a ring whose additive group is the p-adic completion of a free p-adic module. Then there exist large separable abelian p-groups such that $\mathrm{End}(G) = R \oplus E_s(G)$, where $E_s(G)$ is the ideal of all small endomorphisms.

• *Every cotorsion free algebra is an endomorphism algebra*, Math. Z. **181** (1982), 451–470.

Let R be a Dedekind domain and A some cotorsion-free R-algebra. We used a combinatorial principle due to S. Shelah to construct, in standard ZFC set theory, large R-modules with endomorphism ring A.

1983: • *Endomorphism algebras of torsion modules II*, Abelian Group Theory (Honolulu, HI, 1983), 400–411.

Here we generalized the results of the previous Math. Ann. paper to R-modules over suitable rings R.

1984: • *Torsion-free abelian groups with prescribed finitely topologized endomorphism rings*, Proc. Amer. Math. Soc. **90** (1984), 519–527.

Let R be a complete Hausdorff topological ring R such that the topology is induced by a family of right ideals I such that R/I is cotorsion-free. Then there exist large cotorsion-free abelian groups G such that $\mathrm{End}(G) \cong R$ as topological rings, where $\mathrm{End}(G)$ carries the finite topology.

• (With B. Goldsmith) *Representation of algebras over a complete discrete valuation ring*, Quart. J. Math. Oxford Ser. **35** (1984), 131–146.

Let R be a complete discrete valuation domain with maximal ideal pR, and A some torsion-free R-algebra, Hausdorff in the p-adic topology. Then there exist large, torsion-free R modules G such that $\mathrm{End}(G) =$

$A \oplus \mathrm{Ines}(G)$, where $\mathrm{Ines}(G) = \{\varphi \in \mathrm{End}(G) : \widehat{\varphi}(\widehat{G}) \subseteq G\}$ where $\widehat{G}$ is the p-adic completion of G and $\widehat{\varphi}$ is the unique extension of φ to $\widehat{\varphi} : \widehat{G} \to \widehat{G}$. (Note that Ines is the name of Rüdiger's daughter.)

• *Almost Σ-cyclic abelian p-groups in L*, Abelian Groups and Modules (Udine, 1984), CISM Courses and Lectures **287**, Springer (1984), 87–105.

Here we revisit the topic in the Math. Ann. paper and use $V = L$ to construct κ-cyclic p-groups.

1985: • *Endomorphism rings of separable torsion-free abelian groups*, Houston J. Math. **11** (1985), 471–483.

For every ring A with free additive structure $A^+ = \bigoplus_{\alpha\in\kappa} \mathbb{Z}e_\alpha$ and $\overline{A} = \widehat{A} \cap \prod_{\alpha\in\kappa} \mathbb{Z}e_\alpha$ an A-submodule of its $\mathbb{Z}$-adic completion $\widehat{A}$, we construct arbitrarily large homogeneous, separable torsion-free abelian groups G of type $\mathbb{Z}$ with $\mathrm{End}(G) = A \oplus \mathrm{Ines}(G)$.

1987: • *All infinite groups are Galois groups over any field*, Trans. Amer. Math. Soc. **304** (1987), 355–384.

Let G be a monoid with identity and right cancellation and F a field. Then there exist field extensions K of F such that the endomorphism monoid $\mathrm{End}(K)$ modulo Frobenius homomorphisms is isomorphic to G.

• *Field extensions in L – A solution of C.U. Jensen's \$25 problem*, Abelian Group Theory (Oberwolfach, 1985), Gordon and Breach (1987), 509–529.

We used $V = L$ to embed any field F with $\mathrm{char}(F) \neq 2$ into a rigid field K, i.e. $\mathrm{Aut}(K) = \{\mathrm{id}_K\}$.

1990: • *Torsion-free nilpotent groups and E-modules*, Arch. Math. **54** (1990), 340–351.

Large nilpotent groups G of class 2 are constructed such that $\mathrm{Aut}(G) = \mathrm{Aut}(R) \ltimes \mathrm{Stab}(G)$, where R is some torsion-free ring which is p-reduced for infinitely many prime numbers p.

• *Separable abelian p-groups having certain prescribed chains*, Israel J. Math. **72** (1990), 289–298.

Similar to the 1983 paper, but the p-groups have certain filtrations related to p^σ-projective p-groups.

1991: • *Outer automorphisms of groups*, Illinois J. Math. **35** (1991), 27–46.

Essentially, given a group H, large groups G are constructed with $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes H$. (One of my favorite papers.)

1992: • *Automorphisms of torsion-free nilpotent groups of class two*, Trans. Amer. Math. Soc. **332** (1992), 633–646.

Given some group H, we used bilinear forms f to construct abelian groups G with $\mathrm{End}(G) = S[H] \oplus f_G$ where $S[H]$ is a group ring and f_G is some ideal derived from f. In a second step, some multiplication $\cdot$ is defined on G turning G into a nilpotent group of class 2, whose automorphisms can be described.

1993: • *On locally finite p-groups and a problem of Philip Hall's*, J. Algebra **159** (1993), 115–138.

Here we construct groups containing a given subgroup with prescribed outer automorphism group.

1994: • *Automorphism groups of fields*, Manuscripta Math. **85** (1994), 227–242.

Similar to the 1987 papers. Let F be a field and G a group. Then there exists a field extension K of cardinality $\leq \max\{\aleph_0, |F|, |G|\}$ with $G = \mathrm{Aut}(K)$. Also, some results for ordered fields are obtained.

1996: • *Applications of abelian groups and model theory to algebraic structures*, Infinite groups 1994 (Ravello), de Gruyter (1996), 41–62.

This paper surveys applications to fields and noncommutative groups of infinite combinatorial tools developed by Saharon Shelah.

1997: • *Automorphism groups of fields II*, Comm. Algebra **25** (1997), 3777–3785.

This is a continuation of the 1994 paper.

• *Endomorphism rings of B_2-groups of infinite rank*, Israel J. Math. **101** (1997), 141–156.

B_2-groups are some infinite rank analog to the classical, finite rank torsion-free Butler groups. If R is a ring whose additive group is a B_2-group, then R is the endomorphism ring of arbitrarily large B_2-groups.

2001: • *Automorphism groups of geometric lattices*, Algebra Universalis **45**, 425–433.

For a change, groups are realized as automorphism groups of lattices.

2007: • *An extension of Zassenhaus' theorem on endomorphism rings*, Fund. Math. **194** (2007), 239–251.

Sadly, this was to be our last joint paper. Zassenhaus' result [**36**] is extended to rings E with free countable rank additive group such that all $a \in E$ are algebraic over $\mathbb{Q}$.

Somehow, we got back to where it all started.

Soon after [**36**] was written, Butler generalized Zassenhaus' result replacing "free abelian of finite rank" by "locally free abelian of finite rank." Reid and Vinsonhaler [**25**] extended this result by replacing the ring of integers by certain Dedekind domains. We will prove:

Theorem 1. *Let D be the ring of algebraic integers of a finite Galois field extension F of $\mathbb{Q}$ and E a D-algebra such that E is a locally free D-module of countable rank and all elements of E are algebraic over F. Then there exists a left D-submodule $M \supseteq E$ of $FE = E \otimes_D F$ such that the left multiplications by elements of E are the only D-linear endomorphisms of M.*

1.2. Saharon Shelah's Statement: The Oberwolfach conference January 12–17, 1981 may be considered a precursor for my longstanding research collaboration with Rüdiger. Organized by him and Elbert Walker as a memorial tribute to Reinhold Baer, this seminal event brought together leading abelian group theorists from all over the world to present the state of the art of their discipline. State of the art, back then, was the freshly discovered link between set-theory, model theory and infinite abelian groups, and the Oberwolfach conference devoted a full plenary session to combinatorial principles in set theory and their applications to algebra.

The starting point for this development was my 1974 publication [**30**], the first in a long line of contributions to the theory of infinite abelian groups, rings and modules. This paper covered two fundamental results which by now have become folklore, the construction of large rigid families and the independence of the Whitehead problem of ZFC. In particular, the discovered set-theoretic aspects of Whitehead groups sparked immediate interest and further research: by 1975, all Whitehead groups were shown to be free under V=L, including the celebrated Singular Compactness Theorem [**31**]. In 1976, an expository account by Paul Eklof [**9**] opened the techniques of my construction to the larger public. A proof of the independence of the Whitehead problem of ZFC+CH followed in [**32, 33**]. All of these developments reverberated in the Oberwolfach plenary talks. Frank Tall discussed Martin's axiom MA and its applications. Keith Devlin introduced the Weak Diamond [**3**], a combinatorial principle that had just been applied by Manfred and Rüdiger [**6**] to realize cotorsion-free rings as endomorphism rings of abelian groups. I myself was represented in Oberwolfach by Gershon Sageev. He and Paul Eklof presented results on the structure of $\mathrm{Ext}(A, \mathbb{Z})$ for weakly compact cardinals [**26**] and CH [**10, 27**].

The years 1980/81 also mark my first closer contact with Rüdiger. My work [**34**] on endo-rigid, strongly $\aleph_1$-free abelian groups of size $\aleph_1$ in ZFC brought me together with him and Manfred [**6**]. In 1982, Rüdiger and I submitted our first joint paper [**16**] on the existence of semi-rigid classes of cotorsion-free abelian groups. This collaboration intensified in 1983/84, when Rüdiger visited Jerusalem to work on endomorphism ring realizations [**17, 18**] based on my latest combinatorial principle, the Black Box [**35**]. Many more visits should follow. For almost 16 years, starting in 1995, our highly successful partnership received ongoing funding by the German Israeli Foundation (GIF). It brought forth 34 publications along with an intensive exchange between the University of Duisburg-Essen and the Hebrew University of Jerusalem. This included work with a number of Rüdiger's postdocs and young pupils, among them Manfred, Daniel and Lutz Strüngmann. I value Rüdiger as a colleague and close friend. His warm-hearted nature, his relentless passion for mathematics, and his exceptional instinct for problems and applications are well remembered.

It follows a list of our joint work on realizing endomorphism rings. The selection will be quite strictly limited to the topic of endomorphisms, omitting related topics such as dual modules, reflexive modules, splitters or localizations.

1985: • *Modules over arbitrary domains* (GbSh 219), Math. Z. **188** (1985), 325–337.

Given an R-algebra A with torsion-free and reduced R-module structure and some cardinal λ with $\lambda^\kappa = \lambda \geq |A|$, we use a Black Box construction to find a rigid family of R-modules G_i $(i < 2^\lambda)$ with $\mathrm{End}(G_i) = A \oplus \mathrm{Ines}(G_i)$. For R-algebras A with cotorsion-free R-module structure we can achieve $\mathrm{End}(G_i) = A$.

1986: • *Modules over arbitrary domains II* (GbSh 224), Fund. Math. **126** (1986), 217–243.

We generalize the result of the previous paper to R-algebras A with $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced R-module structure with respect to some multiplicatively closed countable set $\mathbb{S} \subseteq R$. The construction makes use of the $\mathbb{S}$-completion of a free A-module.

1995: • *On the existence of rigid $\aleph_1$-free abelian groups of cardinality $\aleph_1$* (GbSh 519), Abelian Groups and Modules (Padova, 1994), Math. Appl. **343**, Kluwer (1995), 227–237.

For any ring R with R^+ free and $|R| < \lambda \leq 2^{\aleph_0}$, there exists in ZFC an $\aleph_1$-free abelian group G of cardinality λ with $\mathrm{End}(G) = R$.

1996: • *G.C.H. implies existence of many rigid almost free abelian groups* (GbSh 579), Abelian Groups and Modules (Colorado Springs, CO, 1995), Lecture Notes in Pure and Appl. Math. **182**, Dekker (1996), 253–271.

Given a countable, commutative ring R with $\mathbb{S}$-topology, some regular cardinal μ with $\lambda = 2^\mu = \mu^+$ and and a strongly μ-free A-module H of cardinality μ with $\mathrm{End}(H) = A$, A_R free and $|A| < \mu$, then we can find a strongly λ-free A-module G of cardinality λ with $\mathrm{End}(G) = A$.

1998: • *Indecomposable almost free modules – the local case* (GbSh 591), Canad. J. Math. **50** (1998), 719–738.

We generalize the results of our 1995 paper, this time realizing endomorphism rings of modules over countable PIDs which are not fields.

• *Endomorphism rings of modules whose cardinality is cofinal to omega* (GbSh 568), Abelian Groups, Module Theory, and Topology (Padova, 1997), Lecture Notes in Pure and Appl. Math. **201**, Dekker (1998), 235–248.

Once more we generalize the result of our 1995 paper: Given a commutative ring R with $\mathbb{S}$-topology, A an R-algebra with $\aleph_0$-cotorsion-free R-module structure, and cardinals μ, λ with $|A| \leq \mu = \mu^{\aleph_0} < \lambda \leq 2^\mu$, there exists an $\aleph_0$-cotorsion-free R-module G of cardinality λ with $\mathrm{End}(G) = A \oplus \mathrm{Ines}(G)$. Note, that this construction includes the critical case $\mathrm{cf}(\lambda) = \omega$. A similar result holds for the endomorphism ring realization of separable torsion-free R-modules.

2000: • *Cotorsion theories and splitters* (GbSh 647), Trans. Amer. Math. Soc. **352** (2000), 5357–5379.

Given a ring A with free additive group A^+ of cardinality $|A| < \kappa$ for some regular cardinal $\kappa = \kappa^{\aleph_0}$, and an n-free-by-1 abelian group G' for some $n > 0$, there exists a torsion-free abelian group G of rank κ such that $\mathrm{Ext}(G', G) = 0$ and $\mathrm{End}(G) = A$.

2003: • *Philip Hall's problem on non-abelian splitters* (GbSh 738), Math. Proc. Cambridge Philos. Soc. **134** (2003), 23–31.

In answering a question by Philip Hall, we construct arbitrarily large non-abelian groups G with $G \times G \cong G$, $\mathrm{Ext}(G, G) = 1$, and $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes S_\omega$, where S_ω is the full symmetric group acting on a countable set.

• *Characterizing automorphism groups of ordered abelian groups* (GbSh 780), Bull. London Math. Soc. **35** (2003), 289–292.

In this short note, we characterize automorphism groups $\mathrm{Aut}(G)$ of ordered abelian groups $(G, +, <)$ as right ordered groups $(H, \cdot, <)$. Furthermore, for every right ordered group $(H, \cdot, <)$ there exists some $\aleph_1$-free ordered abelian groups $(G, +, <)$ with $\mathrm{Aut}(G) \cong H$.

• (with L. Strüngmann) *Almost-free E-rings of cardinality* $\aleph_1$ (GShS 785), Canad. J. Math. **55** (2003), 750–765.

We revisit our 1995 paper, constructing in ZFC for all regular cardinals $\aleph_1 \leq \lambda \leq 2^{\aleph_0}$ a commutative ring R of cardinality λ with $\aleph_1$-free additive structure R^+ and $\mathrm{End}(R^+) = R$. The result is generalized to

F-algebras over a countable PID F with infinitely many pairwise coprime elements.

2004: • *Uniquely transitive torsion-free abelian groups* (GbSh 650), Rings, Modules, Algebras, and Abelian Groups, Lecture Notes in Pure and Appl. Math. **236**, Dekker (2004), 271–290.

In answering a question by Emmanuel Dror Farjoun, we construct for any successor cardinal $\lambda = \mu^+$ with $\mu = \mu^{\aleph_0}$ an $\aleph_1$-free abelian group G of cardinality λ such that for any ordered pair of pure elements in G there is a unique automorphism mapping the first element onto the second one.

• (with L. Strüngmann) *Generalized E-rings* (GShS 681), Rings, Modules, Algebras, and Abelian Groups, Lecture Notes in Pure and Appl. Math. **236**, Dekker (2004), 291–306.

In pursuit of a question by László Fuchs, we set out on a search for non-commutative rings R with $\text{End}(R^+) \cong R$.

2005: • *On Crawley modules* (GbSh 833), Comm. Algebra **33** (2005), 4211–4218.

An R-module G is a Crawley module if for any pair M, N of pure and dense submodules of corank 1 there is an automorphism mapping the first submodule onto the second one. We demonstrate for $R = J_p$ the undecidability of the question whether all torsion-free, reduced Crawley R-modules of rank $\aleph_1$ are free. This includes the realization of counterexamples under the assumption of Martin's Axiom.

2006: • *Generalized E-algebras via λ-calculus I* (GbSh 867), Fund. Math. **192** (2006), 155–181.

We continue our quest for generalized E-rings which started in 2004.

2007: • *Absolutely indecomposable modules* (GbSh 880), Proc. Amer. Math. Soc. **135** (2007), 1641–1649.

Let R be a domain with infinitely many comaximal primes. For $|R| \leq \lambda < \kappa(\omega)$, the first ω-Erdős cardinal, we construct an absolute fully rigid family M_U $(U \subseteq \lambda)$ of torsion-free R-modules M_U of size λ. In particular, $\text{End}(M_U) = R$ $(U \subseteq \lambda)$ holds in any generic extension of the given universe of set theory.

Our remaining publications for the years 2009 to 2014 are joint work with Daniel.

1.3. Daniel Herden's Statement: My memories of Rüdiger mirror those of Manfred in some quite uncanny detail. They provide vivid proof of how Rüdiger's organization and networking skills turned his research group "Algebra/Logic" at the University of Duisburg-Essen into a bustling center for front-line research on rings, modules, abelian groups, and their interaction with logic and set theory. For over three decades Essen has served as meeting point of a close-knit community of leading German and European scientists, even attracting regular visits from across the pond. And Rüdiger has been the heart and soul of this community!

Born at a time when Rüdiger and Manfred published their first groundbreaking joint papers, I enrolled at the University of Duisburg-Essen as a student in fall 2000. Having completed my first year math courses during civil service enabled me to jump into second year courses straight away, with Rüdiger teaching the advanced abstract algebra class. His compelling lecture style and his interest in us young pupils immediately attracted most of the students of my year, introducing half a dozen enthusiastic undergraduates into his research group Algebra/Logic, then centered around Simone Wallutis, Lutz Strüngmann and Georg Hennecke. This group should provide home for my academic development for more than a decade.

Rüdiger introduced us to current research right away, each of us being assigned a seminar project which, over time, would develop into their diploma or PhD topic. I was set the topic of endomorphisms and automorphisms acting transitively on the pure elements of an $\aleph_1$-free abelian group. A seminar paper on A- and E-transitive groups should introduce me to Manfred and Saharon [**7**]. From here we went on to [**19**] and the more demanding task of constructing sharply transitive groups, which was to become the central topic of both my diploma and PhD thesis. The final result was a shortened new construction which provided a surprising link between sharply transitive groups and E-rings and let to our first paper [**11**].

Rüdiger's magnum opus of my early years and the goal of all our joint efforts was the compilation of Göbel, Trlifaj [**21**], an extensive compendium of his contributions to algebra and set theory. After finishing my PhD in 2005 (László Fuchs was a guest of my defense!), I was all set for my years of wandering and learning: I spend the years 2007/08 with Saharon at the Hebrew University of Jerusalem, Israel. Constructing generalized $E(R)$-algebras [**13**] has been our main project of these years, an extremely demanding task which should become the first in a series of three highly successful joint publications with Rüdiger and Saharon. In 2009/10, I moved on to the Institute of Mathematical Logic and Fundamental Research at the University of Münster, Germany pursuing the construction of absolute E-rings [**14**].

In 2010, I returned to Essen. By this time I had risen to the position of Rüdiger's assistant, taking significant part in supervising his PhD students and progressing the further research of the group Algebra/Logic. Prior to this, I had already been assisting with [**20**], a paper on the existence of $\aleph_n$-free groups in ZFC with trivial duals. The realization of $\aleph_n$-free structures in ZFC should now become the main focus of our research: In [**15**], we realized $\aleph_n$-free groups with prescribed endomorphism rings while Héctor Gabriel Salazar Pedroza's PhD thesis [**28**] gave a similar construction for $\aleph_n$-free separable groups. The combinatorial background of the Easy Black Box and a blueprint for general $\aleph_n$-free constructions was provided in my Habilitationsschrift [**23**] and applied to the constriction of $\aleph_n$-free E-rings in ZFC.

In 2012, the second edition of [**22**] was published. Due to the immense amount of new results amassed during the past six years, the content had grown by 50% in material and filled now two volumes instead of one. In May 2013, Rüdiger was struck by sudden and severe illness, forcing him to leave office immediately. It was a terrible shock for all of us! Henceforth, I took over running Rüdiger's research group by proxy. In summer 2013, I did complete my habilitation. I should spend the winter of 2013/14 with Jan Trlifaj at the Charles University in Prague, Czechia and the summer of 2014 with Saharon back in Jerusalem. By that time, and thanks to Manfred's communication, I had already accepted an offer at Baylor University (Waco, TX). Montakarn Petapirak's PhD defense took place on July 4, 2014. It was the perfect opportunity for a last gathering of colleagues and friends around Rüdiger, with Jan Trlifaj and Ulrich Albrecht coming in from abroad. The event was altogether merry, and Rüdiger was immensely happy to see his academic inheritance secured and prospering: In August 2014, I was to start my position in Waco, and Katrin Leistner, the remaining PhD student of the research group, was to finish by early 2015. Rüdiger, you will be never forgotten and always be missed!

Here is an overview of our joint work on E-rings and endomorphism rings:

2007: • *$E(R)$-algebras that are sharply transitive modules*, J. Algebra **311** (2007), 319–336.

Let R be a PID with $\widehat{R}$ ($= \widehat{R}_p$ for some prime $p \in R$) of transcendence degree ≥ 2 over R, and let $\aleph_0 \leq |R| < \kappa$ be a successor cardinal. Under the presumption of the weak diamond principle $\Phi(\kappa)$, we construct a PID A of cardinality κ, which is an $E(R)$-algebra with $\aleph_1$-free R-module structure. In addition, for any ordered pair of pure elements in A there is a unique automorphism mapping the first element onto the second one.

• *Constructing sharply transitive R-modules of rank $\leq 2^{\aleph_0}$*, J. Group Theory **10** (2007), 467–475.

This is a continuation of our J. Algebra paper. This time we give an elementary proof of the existence of sharply transitive R-modules M of rank $\leq 2^{\aleph_0}$ based on a similar construction for E-rings by Faticoni.

2008: • *The existence of large $E(R)$-algebras that are sharply transitive modules*, Commun. Algebra **36** (2008), 120–131.

This is a continuation of our J. Algebra paper. We eliminate the weak diamond condition $\Phi(\kappa)$ by means of a Black Box construction.

2009: • (with S. Shelah) *Skeletons, bodies and generalized $E(R)$-algebras* (GbHeSh 943), J. Eur. Math. Soc. **11** (2009), 845–901.

In answering a fifty year old problem by László Fuchs, we provide non-commutative R-algebras A with $\mathrm{End}_R(A) \cong A$ for any p-cotorsion-free commutative ring R with 1. The proof combines a standard Black Box construction with model-theoretic arguments, introducing skeletons and bodies as genuine new model-theoretic objects which allow a combinatorial-geometric description with the help of decorated trees and a carefully chosen family of permitted small cancelations.

2011: • (with S. Shelah) *Absolute E-rings* (GbHeSh 948), Adv. Math. **226** (2011), 235–253.

Given any cardinal $\aleph_0 \leq \lambda < \kappa(\omega)$ (the first ω-Erdős cardinal), we construct an absolute E-ring R of cardinality λ. In particular, $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$ for some family X of λ commuting free variables, and $\mathrm{End}(R^+) = R$ holds in any generic extension of the given universe of set theory.

2014: • (with S. Shelah) *Prescribing endomorphism algebras of $\aleph_n$-free modules* (GbHeSh 970), J. Eur. Math. Soc. **16** (2014), 1775–1816.

Given a p-cotorsion-free domain R and an R-algebra A with free R-module structure, we construct in ZFC for any natural number k arbitrarily large $\aleph_k$-free A-modules G with $\mathrm{End}_R(G) = A$. This construction combines Shelah's newly developed Easy Black Box with a Strong Black Box argument.

2015: • (with H. G. Salazar Pedroza) *$\aleph_k$-free separable groups with prescribed endomorphism ring*, Fund. Math. **231** (2015), 39–55.

This is a thematic continuation of our 2014 paper. For every ring A with free additive structure $A^+ = \bigoplus_{\alpha\in\kappa} \mathbb{Z}e_\alpha$ and $\overline{A} = \widehat{A} \cap \prod_{\alpha\in\kappa} \mathbb{Z}e_\alpha$ an A-submodule of its $\mathbb{Z}$-adic completion $\widehat{A}$, we construct in ZFC arbitrarily large $\aleph_k$-free A-modules G such that G is separable as an abelian group with $\mathrm{End}(G) = A \oplus \mathrm{Fin}(G)$.

2. The Results

NOTATION 1. *Let D denote a countable Dedekind domain of characteristic zero and with infinitely many prime ideals. Let F be its field of fractions. It follows that for any prime ideal P of D, the localization D_P of D at P is a PID with unique maximal ideal pD_P for some $p \in P$. Let $\widehat{D}_P$ denote the P-adic closure of D_P. Let $f(x) \in F[x]$. Then $f(x) \in D_P[x]$ with the leading coefficient a unit in D_P for all but finitely many prime ideals P of D. Define $N_P(f)$ to be the number of roots of $f(x)$ in $\widehat{D}_P$. We call D an* ***admissible domain*** *if for all $f(x) \in F[x]$ the set of prime ideals P of D with $N_P(f) \geq 1$ is infinite. If E is some D-module, then we call E* ***torsion-free*** *if $se = 0$ for $s \in D$ and $e \in E$ implies $s = 0$ or $e = 0$. Moreover, E is called* ***locally free****, if the localization $E_P = E \otimes_D D_P$ is a free D_P-module for all prime ideals P of D.*

Our main result will be the following

THEOREM 2. *Let D be an admissible domain and E a countable, torsion-free and locally free D-algebra such that each $a \in E$ is algebraic over F. Then there exists a locally free left E-submodule M of $FE = E \otimes_D F$ such that $E \subseteq M$ and $\mathrm{End}_D(M) = E\cdot$, the ring of left multiplications by elements of E.*

2.1. The Proof of Theorem 1.

Before we turn to the proof of Theorem 2, note that Theorem 1 will be an immediate consequence provided

PROPOSITION 1. *Let D be the ring of algebraic integers of some finite Galois field extension of $\mathbb{Q}$. Then D is an admissible Dedekind domain.*

We need to show for all $f(x) \in F[x]$ the existence of infinitely many prime ideals P of D with $N_P(f) \geq 1$. We will line up some results from algebraic number theory to obtain this proposition. Note that for any $f(x) \in F[x]$ there exists some $d \in D$ with $df(x) \in F[x]$. Thus, we may restrict to polynomials $f(x) \in D[x]$. Furthermore, any polynomial is a product of irreducible ones and we may restrict to irreducible $f(x) \in D[x]$.

We recall the following, well known version of Hensel's Lemma [**24**, Proposition 2, p. 43]:

LEMMA 1. *Let $1 \in S$ be a commutative ring and $\mathfrak{m}$ an ideal of S such that S is complete in the $\mathfrak{m}$-adic topology. Let $f(x) \in S[x]$ and $a \in A$ such that $f(a) \in f'(a)^2\mathfrak{m}$. Then there exists some $b \in S$ such that $f(b) = 0$ and $b - a \in f'(a)^2\mathfrak{m}$.*

Applying this to our situation:

REMARK 1. *Let P be a prime ideal of D and let $f(x) \in D[x]$ be irreducible of degree n over F. Then $f(x)$ has only simple roots and thus has non-zero discriminant $\Delta(f)$. Let P be a prime ideal of D such that $\Delta(f) \notin P$. Then $f(x) \bmod P$ has no multiple roots. Assume that $a \in \widehat{D}_P$ is such that $f(a) \in p\widehat{D}_P$. Then $f'(a) \notin p\widehat{D}_P$ and we may apply Lemma 1 to obtain $b \in \widehat{D}_P$ with $f(b) = 0$ and $b - a \in p\widehat{D}_P$.*

Thus, for irreducible $f(x) \in D[x]$, $f(a) \in p\widehat{D}_P$ implies $N_P(f) \geq 1$.

By the above it is sufficient to show that for any irreducible $f(x) \in D[x]$, there are infinitely many prime ideals P of D such that $f(x) \bmod P$ has a root in D/P. Hensel's Lemma will then provide a root of $f(x)$ in $\widehat{D}_P$.

First we recall some well-known definitions that are in [**24**] and many other sources.

Let k be an algebraic number field and K a Galois extension of k with Galois group G. Let $\mathcal{O}_k$ ($\mathcal{O}_K$) denote the ring of algebraic integers in k (K). Let $\mathfrak{p}$ be a prime (ideal) of $\mathcal{O}_k$ and $\mathfrak{P}$ a prime of $\mathcal{O}_K$ lying over $\mathfrak{p}$. Then $\mathcal{O}_K/\mathfrak{P}$ is a finite extension of the finite field $\mathcal{O}_k/\mathfrak{p}$ and thus a finite field of order $n_{\mathfrak{P}}$ with cyclic Galois group $G = \langle\overline{\sigma}\rangle$ over $\mathcal{O}_k/\mathfrak{p}$ where $\overline{\sigma}(x) = x^{n_{\mathfrak{P}}} \bmod \mathfrak{P}$. Let $G_{\mathfrak{P}} = \{g \in G : g\mathfrak{P} = \mathfrak{P}\}$ denote the *decomposition group* of $\mathfrak{P}$ and $T_{\mathfrak{P}} = \{g \in G : \overline{g} = \mathrm{id}_{\mathcal{O}_K/\mathfrak{P}}\}$ the *inertia group* of $\mathfrak{P}$. Then there exists some coset $\sigma T_{\mathfrak{P}} \in G_{\mathfrak{P}}/T_{\mathfrak{P}}$ which induces $\overline{\sigma}$. Any element of that coset is called a *Frobenius automorphism* which we denote by $\sigma(\mathfrak{P}, K/k)$. Now we need a celebrated theorem due to Chebotarev [**24**, Theorem 10, page 169]:

THEOREM 3. *[Chebotarev]. Let K be a Galois extension of k with Galois group G. Let $\emptyset \neq C \subseteq G$ be some set invariant under conjugations with $|C| = c$ and $[K : k] = n$. Let*

$$M = \{\textit{primes } \mathfrak{p} \textit{ of } k \mid \mathfrak{p} \textit{ is unramified in } K \textit{ and there is some prime } \mathfrak{P} \textit{ of } K \textit{ lying over } \mathfrak{p} \textit{ such that } \sigma(\mathfrak{P}, K/k) \in C\}.$$

Then the set M has a density and this density is $\frac{c}{n}$. Moreover, $0 < \frac{c}{n} < 1$ for all $C \subsetneq G$.

The definition of density in this context can be found in [**24**, page 167]. All we need to know is that only infinite sets have a positive density.

Now we need a result from [**29**]. We maintain our current notations.

THEOREM 4. *Let $f(x) \in \mathcal{O}_k[x]$ have degree $n \geq 2$ and be irreducible over k. Let $N_{\mathfrak{p}}(f)$ be the number of roots of $f(x)$ $(\bmod\, \mathcal{O}_k/\mathfrak{p})$ in $\mathcal{O}_k/\mathfrak{p}$. Let*

$$P_0(f) = \{\mathfrak{p} \textit{ prime in } \mathcal{O}_k \mid N_{\mathfrak{p}}(f) = 0\}.$$

Then $P_0(f)$ has density $\frac{c}{n}$. Moreover, $0 < \frac{c}{n} < 1$.

This shows that the set of all primes $\mathfrak{p}$ **not** in $P_0(f)$ has positive density and thus is infinite, completing the proof of Proposition 1.

Here is an outline of Serre's argument [**29**, page 432]: First, disregard all (finitely many) primes $\mathfrak{p}$ of $\mathcal{O}_k$ that are ramified or contain non-zero coefficients of $f(x)$. Let K be the splitting field of $f(x)$ over k with Galois group G and $\sigma = \sigma(\mathfrak{P}, K/k)$. Moreover, let X be the set of the n distinct roots of $f(x)$ in K. It turns out that $N_{\mathfrak{p}}(f)$ is the number of fixed points of $\sigma \restriction X$. Now put

$$G_0 = \{g \in G \mid g \restriction X \text{ has no fixed point}\}$$

and note that G_0 is invariant under conjugation, with $G_0 \subsetneq G$ since $\mathrm{id}_K \notin G_0$. Now apply Theorem 3 with $C = G_0$.

2.2. The Proof of Theorem 2. We start with an easy observation.

Proposition 2. *Let $1 \in S$ be a commutative ring, A some S-algebra, and $\tau \in A$. Let $f(x) = \sum_{i=0}^{m} f_i x^i \in S[x]$, the polynomial ring over S. Then*

$$f(x) = f(\tau) + (x - \tau)(f_m \tau^{m-1} + g(\tau, x))$$

where $g(\tau, x) \in \mathrm{span}_{\mathbb{Z}[x, f_0, \dots, f_m]}\{\tau^j : 0 \le j \le m-2\}$.

Proof. We evaluate

$$\begin{aligned} f(x) &= f((x-\tau)+\tau) = \sum_{i=0}^{m} f_i[(x-\tau)+\tau]^i = \sum_{i=0}^{m} f_i \left[\sum_{j=0}^{i} \binom{i}{j}(x-\tau)^j \tau^{i-j}\right] = \\ &= \sum_{i=0}^{m} f_i \left[\tau^i + \sum_{j=1}^{i} \binom{i}{j}(x-\tau)^j \tau^{i-j}\right] = \\ &= f(\tau) + \sum_{i=0}^{m} f_i (x-\tau) \sum_{j=1}^{i} \binom{i}{j}(x-\tau)^{j-1}\tau^{i-j} = \\ &= f(\tau) + (x-\tau)\left[\sum_{i=0}^{m} f_i \sum_{j=1}^{i} \binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}\right]. \end{aligned}$$

The highest power of τ that might occur in $\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}$ is τ^{i-1}. Note that

$$\begin{aligned} \sum_{j=1}^{m} \binom{m}{j}(x-\tau)^{j-1}\tau^{m-j} &= \sum_{j=1}^{m}\binom{m}{j}\left[\sum_{k=0}^{j-1}\binom{j-1}{k}x^k(-\tau)^{j-1-k}\right]\tau^{m-j} \\ &= \sum_{j=1}^{m}\binom{m}{j}\left[\sum_{k=0}^{j-1}\binom{j-1}{k}x^k\tau^{m-1-k}(-1)^{j-1-k}\right]. \end{aligned}$$

Thus τ^{m-1} only occurs for $k = 0$ and with coefficient $\sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}$. Recall that $\sum_{j=0}^{m}\binom{m}{j}(-1)^j = 0$ and thus $1 = \binom{m}{0} = -\sum_{j=1}^{m}\binom{m}{j}(-1)^j = \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}$.

This shows that $f(x) = f(\tau) + (x-\tau)\left[f_m\tau^{m-1} + g(\tau, x)\right]$ where $g(\tau, x) \in \mathrm{span}_{\mathbb{Z}[x, f_0, \dots, f_m]}\{\tau^j : 0 \le j \le m-2\}$. □

Corollary 1. *Same notation as in the proposition. Let S be an integral domain with Q its field of fractions and $c \in S$ such that $f(c) \neq 0 = f(\tau)$. Then*

$$(c-\tau)^{-1} = \frac{1}{f(c)}(f_m\tau^{m-1} + g(\tau, x)) \in QA.$$

We also want to list:

Proposition 3. *Let F be a field and V some vector space over F. If $\tau \in \mathrm{End}_F(V)$ is algebraic over F, then τ has only finitely many eigenvalues.*

Proof. There exists some monic polynomial $f(x) \in F[x]$ such that $f(\tau) = 0$. Let $0 \neq v \in V$ be an eigenvector of τ with eigenvalue λ. Then $I = \{g(x) \in F[x] : g(\tau \restriction_{vF}) = 0\} = (x-\lambda)F[x]$ is an ideal of $F[x]$ and $f(x) \in I$. This shows that λ is a root of $f(x)$, of which there are only finitely many. □

LEMMA 2. *Let $\tau \in \mathrm{End}_D(E^+)$ such that τ is algebraic over F. Let $0 \neq e \in E$ and Π a finite number of prime ideals of D. Then there exists a prime ideal $P \notin \Pi$ of D and $c \in D$ such that $c - \tau$ is an automorphism of FE^+ and $e \notin E_P(c-\tau)$. Moreover, $E_P(c-\tau)^{-1} \subseteq p^{-k}E_P$ for some natural number k where $PD_P = pD_P$.*

PROOF. Let $g(x) = \sum_{i=0}^{n} g_i x^i \in F[x]$ be the minimal polynomial of τ over F with $g_n = 1$. Let $V = eF[\tau]$, a finite dimensional τ-invariant F-subspace of FE. Put $\theta = \tau \restriction_V$, the restriction of τ to V, and $f(x) = \sum_{i=0}^{m} f_i x^i \in F[x]$ the monic minimal polynomial of θ. Then $f(x)$ is a divisor of $g(x)$ and the set of all prime ideals Q of D for which $h(x) \notin D_Q[x]$ for any monic divisor $h(x)$ of $g(x)$ is finite. We may enlarge Π to contain the finitely many exceptions. By Proposition 2, we have, for any $s \in D$, that $g(s) = (s-\tau)(\tau^{n-1} + \sum_{i=0}^{n-2} s_i \tau^i)$ where $s_i \in D_Q$ for all prime ideals $Q \notin \Pi$. We infer that $s - \tau$ is an automorphism of FE^+ whenever $g(s) \neq 0$. In this case, we have that $E_Q(s-\tau)^{-1} \subseteq \frac{1}{g(s)}E_Q$. A similar statement holds for $s - \theta$.

Since D is admissible, there is an infinite set of prime ideals Q of D such that $f(x)$ has a root γ in the Q-adic completion of the discrete valuation domain D_Q. We choose such a prime ideal $P \notin \Pi$. Let $P = D \cap pD_P$ for some $p \in P$.

Let $V = eF[\tau] = e \cdot \mathrm{span}_F\{1, \tau, \tau^2, ..., \tau^{m-1}\}$ be the τ-invariant subspace of FE generated by e. Note that $\{e, e\theta, e\theta^2, ..., e\theta^{m-1}\}$ is a basis of V over F. Let $V_P = V \cap E_P$, which is a free D_P-module of rank m. Let $W_P = e \cdot \mathrm{span}_{D_P}\{1, \theta, \theta^2, ..., \theta^{m-1}\}$, a free D_P-module of rank m. Since D_P is a PID, the Stacked Basis Theorem for finite rank free modules holds and we infer that $p^h V_P \subseteq W_P$ for some natural number h.

Let $\gamma_0 \in D$ such that $\gamma \equiv \gamma_0 \bmod p^{h+1}D_P$. Then $f(\gamma_0 + p^{h+j}) \equiv 0 \bmod p^{h+1}D_P$ for all natural numbers $j \geq 1$. We infer the existence of some $c \in D$ such that

(1) $g(c) \neq 0$
(2) $f(c) \equiv 0 \bmod p^{h+1}D_P$.

Note that this implies $f(c) \neq 0$ and $g(c) \equiv 0 \bmod p^{h+1}D_P$ as well.

It follows from the above that $c - \tau \in \mathrm{End}_F(FE^+)$ is bijective with $E_P(c-\tau)^{-1} \subseteq \frac{1}{g(c)}E_P$. Moreover, $c - \theta \in \mathrm{End}_F(V)$ is bijective as well.

Assume that $e(c-\theta)^{-1} \in E_P$.

Since $e(c-\theta)^{-1} \in eF[\theta] = V$ as well, we infer that $e(c-\theta)^{-1} \in V_P$ and thus $p^h e(c-\theta)^{-1} = \frac{p^h}{f(c)}\left[e\theta^{m-1} + e\psi\right] \in W_P$ for some $\psi \in \mathrm{span}_{D_P}\{1, \theta, \theta^2, ..., \theta^{m-2}\}$. This is a contradiction since $\frac{1}{p}e\theta^{m-1} \notin W_P$. □

COROLLARY 2. *Let Π be a finite set of prime ideals of D and $0 \neq \psi \in \mathrm{End}_F(FE^+)$ such that $1\psi = 0$. Let $t \in E$ such that $0 \neq t\psi$. Then there is a prime ideal $P \notin \Pi$ of D and a free D_P-submodule M_P of FE^+ such that*

(1) *$E_P \subseteq M_P$,*
(2) *$M_P\psi \nsubseteq M_P$ and*
(3) *For each $x \in FE$ we have $xM_P \subseteq M_P$ if and only if $x \in E_P$.*

Note that (2) holds for any $\varphi \in \mathrm{End}_F(FE^+)$ in place of ψ such that $1\varphi = 0$ and $t\psi = t\varphi$.

PROOF. Let $0 \neq e = t\psi$. We may assume that $e \in E$. Define $\tau \in \mathrm{End}_F(FE^+)$ by $\tau(x) = xt$ for all $x \in FE$. Then $\tau E \subseteq E$ since E is a ring. Since t is algebraic over F, so is τ and we can apply Lemma 2 and find a prime ideal $P \notin \Pi$ and $c \in D$

such that $\sigma = c - \tau \in \operatorname{End}_F(FE^+)$ is bijective, $e \notin E_P\sigma$ and $E_P\sigma \subseteq E_P$. Moreover, $E_P\sigma^{-1} \subseteq p^{-k}E_P$ for some natural number k. We infer $p^kE_P \subseteq E_P\sigma \subseteq E_P$.

Let $M_P = p^{-k}E_P\sigma$. Since σ is injective, M_P is a free D_P-module.

Then $E \subseteq E_P \subseteq p^{-k}E_P\sigma = M_P$ and (1) holds. Moreover, $E_P \cdot M_P \subseteq M_P$ since the multiplication in FE is associative.

Since $1\psi = 0$, we have $-1\sigma\psi = -(c1 - \tau)\psi = t\psi = e$ and $p^{-k}e \in M_P\psi$ but $p^{-k}e \notin p^{-k}E_P\sigma = M_P$. This shows that $M_P\psi \nsubseteq M_P$ and we have (2).

Let $x \in FE$. Then $x(p^{-k}E_P\sigma) = p^{-k}(xE_P)\sigma$ is contained in $p^{-k}E_P\sigma$ if and only if $xE_P \subseteq E_P$ by the injectivity of σ. Since $1 \in E_P$, this holds if and only if $x \in E_P$, and (3) follows. □

Let $\operatorname{End}^0(FE^+) = \{\varphi \in \operatorname{End}_F(FE^+) : 1\varphi = 0\}$ be the set of all linear transformations of FE^+ that map the identity element of E to zero. Then $\operatorname{End}_F(FE^+) = \operatorname{End}^0(FE^+) \oplus ((FE^+)\cdot)$. There exists a countable subset $1 \notin B$ of E such that $FE = \operatorname{span}_F(B \cup \{1\})$. Note that if $0 \neq \varphi \in \operatorname{End}^0(FE^+)$, then there exists some $b \in B$ such that $b\varphi \neq 0$. Moreover, $b\varphi$ is an element of the *countable* (E is countable, cf. Notation 1 and Theorem 2!) set FE. This shows that there exists a countable list $\{\varphi_n : n \in \mathbb{N}\}$ of elements of $\operatorname{End}^0(FE^+)$ such that for all $\tau \in \operatorname{End}^0(FE^+)$ there exists some $n \in \mathbb{N}$ and $b \in B$ such that $\tau(b) = \varphi_n(b) \neq 0$. We apply Corollary 2 repeatedly to find a sequence of distinct prime ideals P_n of D and free D_{P_n}-modules M_{P_n} with properties

(1_n) $E_{P_n} \subseteq M_{P_n}$,
(2_n) $M_{P_n}\varphi_n \nsubseteq M_{P_n}$ and
(3_n) If $x \in FE$, then $xM_{P_n} \subseteq M_{P_n}$ if and only if $x \in E_{P_n}$.

If Q is a prime ideal not in the list $\{P_n : n \in \mathbb{N}\}$, we put $M_Q = E_Q$. Then we have

(1) $E_P \subseteq M_P$ for all prime ideals P of D and also
(3) For each $x \in FE$, we have $xM_P \subseteq M_P$ if and only if $x \in E_P$.

Now let $M = \bigcap_P M_P$, where the intersection runs over all prime ideals P of D.

Then $E \subseteq M$ by (1), and M is locally free since all M_P are free D_P-modules. Recall that $\operatorname{End}_D(M) = \bigcap_P \operatorname{End}_{D_P}(M_P)$. By (3) we get that

$$((FE)\cdot) \cap \operatorname{End}_D(M) = (E\cdot).$$

Let $0 \neq \psi \in \operatorname{End}^0(FE^+)$. Then there exists some $n \in \mathbb{N}$ such that, for some $b \in B$, we have $b\psi = b\varphi_n \neq 0$. By ($2_n$), we have that $M_{P_n}\psi \nsubseteq M_{P_n}$ which shows that $\operatorname{End}^0(FE^+) \cap \operatorname{End}_D(M) = \{0\}$. Let $\varphi \in \operatorname{End}_D(M)$. Then $\varphi = \psi + (x\cdot)$ for some $x \in FE$ and $\psi \in \operatorname{End}^0(FE^+)$. Pick $0 \neq s \in D$ with $sx \in E$. Then $s\varphi = s\psi + s(x\cdot) = s\psi + (sx\cdot)$, where $sx \in E$, and we infer

$$s\varphi - (sx\cdot) = s\psi \in \operatorname{End}^0(FE^+) \cap \operatorname{End}_D(M) = \{0\}.$$

Thus $\psi = 0$ and $\varphi = x\cdot$ for some $x \in E$ by condition (3). We conclude that $\operatorname{End}_D(M) = E\cdot$, as promised.

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