Mad families and non-meager filters

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Abstract

We prove the consistency of $ZF + DC +$ "there are no mad families" $+$ "there exists a non-meager filter on ω" relative to ZFC, answering a question of Neeman and Norwood. We also introduce a weaker version of madness, and we strengthen the result from [HwSh:1090] by showing that no such families exist in our model.[1]

Introduction

This paper is a continuation of [HwSh:1090], which is part of the ongoing effort to investigate the possible non-existence and definability of mad families. In [HwSh:1090] we proved that $ZF + DC +$ "there are no mad families" is equiconsistent with ZFC (previous results by Mathias and Toernquist established the consistency of that statement relative to large cardinals, see [Ma1] and [To]). In this paper we extend our results from [HwSh:1090] to address the following question by Neeman and Norwood:

Question ([NN]): If there are no mad families, does it follow that every filter is meager?

By a result of Mathias ([Ma2]), if every set of reals has the Ramsey property, then every filter is meager.

We shall construct a model of $ZF + DC$ where there are no mad families, but there is a non-meager filter on ω. Our proof relies heavily on [HwSh:1090], the main change is that now we're dealing with a class K_2 consisting of pairs $(\mathbb{P}, \mathcal{A})$ such that $\mathbb{P}$ is ccc and forces $MA_{\aleph_1}$, and in addition, $\mathbb{P}$ forces that $\mathcal{A}$ is independent (we shall require more, see definition 2). In order to imitate the proof from [HwSh:1090], we need to prove analogous amalgamation results for an appropriate subclass of K_2. As in [HwSh:1090], our final model is obtained by forcing with $\mathbb{P}$ where $(\mathbb{P}, \mathcal{A})$ is a "very large" object in a subclass of K_2, and the non-meager filter will be constructed from $\mathcal{A}$, which should contain many Cohen reals.

Finally, we consider the notion of nearly mad families (see definition 14), which was also introduced in [NN]. We introduce the notion of a somewhat mad family, which includes both mad and nearly mad families, and we prove that no somewhat mad families exist in our model.

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A non-meager filter without mad families

Hypothesis 1: We fix μ and λ such that $\aleph_2 \leq \mu$, $\lambda = \lambda^{<\mu}$, $\mu = cf(\mu)$ and $\alpha < \mu \rightarrow |\alpha|^{\aleph_1} < \mu$.

Definition 2: A. Let K_2 be the class of $\mathbf{k}$ such that:

a. Each $\mathbf{k}$ has the form $(\mathbb{P}, \mathcal{A}) = (\mathbb{P}_{\mathbf{k}}, \mathcal{A}_{\mathbf{k}})$.

b. $\mathbb{P}$ is a ccc forcing such that $\Vdash_{\mathbb{P}} MA_{\aleph_1}$.

c. $\mathcal{A}$ is a set of canonical $\mathbb{P}$-names of subsets of ω.

d. $\Vdash_{\mathbb{P}}$ "$\mathcal{A}$ is independent, i.e. every finite non-trivial Boolean combination of elements of $\mathcal{A}$ is infinite".

B. For $\mathbf{k} \in K_2$ and a $\mathbb{P}_{\mathbf{k}}$-name $\underset{\sim}{b}$, let $\Vdash_{\mathbb{P}_{\mathbf{k}}}$ "$\underset{\sim}{b} \in pos(\mathbf{k})$" mean $\Vdash_{\mathbb{P}_{\mathbf{k}}}$ "$\underset{\sim}{b} \in [\omega]^{\omega}$and there is no non-trivial Boolean combination of sets from $\mathcal{A}_{\mathbf{k}}$ that is almost disjoint to $\underset{\sim}{b}$".

C. Let $\leq_1$ be the following partial order on K_2: $\mathbf{k}_1 \leq_1 \mathbf{k}_2$ if and only if:

a. $\mathbb{P}_{\mathbf{k}_1} \lessdot \mathbb{P}_{\mathbf{k}_2}$.

b. $\mathcal{A}_{\mathbf{k}_1} \subseteq \mathcal{A}_{\mathbf{k}_2}$.

D. Let $\leq_2$ be the following partial order on K_2:

$\mathbf{k}_1 \leq_2 \mathbf{k}_2$ if and only if:

a. As in B(a).

b. As in B(b).

c. If $\underset{\sim}{b}$ is a $\mathbb{P}_{\mathbf{k}_1}$-name then $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$" implies $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_2})$".

E. Let K_2^+ be the class of $\mathbf{k} \in K_2$ such that $\Vdash_{\mathbb{P}_{\mathbf{k}}}$ "$\mathcal{A}_{\mathbf{k}}$ is a maximal independent set everywhere", where $\mathcal{A}$ is a maximal independent set everywhere if for every $a_0, ..., a_{n-1} \in \mathcal{A}$ without repetition, $b := \underset{l<n}{\cap} a_l^{\text{if l is even}} \in [\omega]^{\omega}$ and $\{a \cap b : a \in \mathcal{A} \setminus \{a_0, ..., a_{n-1}\}\}$ is a maximal independent set in $[b]^{\omega}$.

F. When we write "$\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in \mathcal{A}$", we mean that $(\underset{\sim}{a}_i : i < n)$ is without repetition, moreover, $i < j < n \rightarrow \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_i \neq \underset{\sim}{a}_j$.

Observation 3: a. $\leq_1$ and $\leq_2$ are partial orders, and if $\mathbf{k}_1, \mathbf{k}_2 \in K_2^+$ then $\mathbf{k}_1 \leq_1 \mathbf{k}_2 \rightarrow \mathbf{k}_1 \leq_2 \mathbf{k}_2$.

b. If $\mathbf{k}_1 \leq_2 \mathbf{k}_2$ and $\underset{\sim}{b}$ is a $\mathbb{P}_{\mathbf{k}_1}$-name, then $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$" iff $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_2})$".

Proof: We shall prove the second claim of 3(a), everything else should be clear. Suppose that $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$", but for some $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in \mathcal{A}_{\mathbf{k}_2}$ and $p \in \mathbb{P}_{\mathbf{k}_2}$, $p \Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{b} \cap (\underset{l<n}{\cap} \underset{\sim}{a}_l^{\text{l is even}})$ is finite". Let $G \subseteq \mathbb{P}_{\mathbf{k}_1}$ be generic over V such that $p \in G$ and we shall work over $V[G]$. WLOG there is $n_1 < n$ such that $\underset{\sim}{a}_l \in \mathcal{A}_{\mathbf{k}_1}$ iff

$l < n_1$, and denote $\underset{\sim}{a}_* = \bigcap_{l<n_1} \underset{\sim}{a}_l^{\text{l is even}}$. As $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$", it follows that $\underset{\sim}{b} \cap \underset{\sim}{a}_*$ is infinite. It's enough to show that for some Boolean combination $\underset{\sim}{a}_{**}$ from $\mathcal{A}_{\mathbf{k}_1}$, $\underset{\sim}{a}_{**} \subseteq \underset{\sim}{a}_*$ and $\underset{\sim}{a}_{**} \subseteq^* \underset{\sim}{b}$, as then $\underset{\sim}{a}_{**} \cap (\bigcap_{n_1 \leq l<n} \underset{\sim}{a}_l^{\text{l is even}}) \subseteq^* \underset{\sim}{b} \cap (\bigcap_{l<n} \underset{\sim}{a}_l^{\text{l is even}})$, and therefore it's finite, contradicting the definition of $\mathcal{A}_{\mathbf{k}_2}$. As $\mathbf{k}_1 \in K_2^+$, it follows that $\{\underset{\sim}{a}_* \cap \underset{\sim}{c} : \underset{\sim}{c} \in \mathcal{A}_{\mathbf{k}_1} \setminus \{\underset{\sim}{a}_l : l < n_1\}\}$ is a maximal independent set in $[\underset{\sim}{a}_*]^\omega$, hence there are $\underset{\sim}{c}_0, ..., \underset{\sim}{c}_{m-1} \in \mathcal{A}_{\mathbf{k}_1} \setminus \{\underset{\sim}{a}_l : l < n_1\}$ such that $(\bigcap_{l<n_1} \underset{\sim}{a}_l^{\text{if l is even}}) \cap (\bigcap_{k<m} \underset{\sim}{c}_m^{\text{if m is even}}) \subseteq^* \underset{\sim}{b}$, so $\underset{\sim}{a}_{**} = (\bigcap_{l<n_1} \underset{\sim}{a}_l^{\text{if l is even}}) \cap (\bigcap_{k<m} \underset{\sim}{c}_m^{\text{if m is even}})$ is as required. $\square$

Observation 4: $\mathbf{k}_1 \leq_2 \mathbf{k}_2$ and $\mathbf{k}_1 \in K_2^+$ when the following hold for some κ:

a. $\mathbf{k}_2 \in K_2^+$.

b. $\mathbf{k}_2 \in H(\kappa)$.

c. M is a model such that $\mathbf{k}_2 \in M \prec_{\mathcal{L}_{\aleph_2,\aleph_2}} (H(\kappa), \in)$.

d. $\mathbf{k}_1 = \mathbf{k}_2^M$.

Proof: By observation 3, recalling that "$\mathbb{P} \models ccc$" and "$\mathbb{P} \models MA_{\aleph_1}$" are $\mathcal{L}_{\aleph_2,\aleph_2}$-expressible .

Claim 5: For every $\mathbf{k} \in K_2$ there is $\mathbf{k}' \in K_2^+$ such that $\mathbf{k} \leq_1 \mathbf{k}'$. Moreover, if $|\mathbb{P}_{\mathbf{k}}| < \mu$ then we can find such $\mathbf{k}'$ that satisfies $|\mathbb{P}_{\mathbf{k}'}| < \mu$.

Proof: If $|\mathbb{P}_{\mathbf{k}}| < \mu$, let $\lambda_* = \mu$, otherwise, let λ_* be a regular cardinal greater than $(2 + |\mathbb{P}_{\mathbf{k}}|)^{\aleph_1}$, such that $\alpha < \lambda_* \rightarrow |\alpha|^{\aleph_1} < \lambda_*$.

we try to choose a sequence $(\mathbf{k}_\alpha : \alpha < \lambda_*)$ by induction on $\alpha < \lambda_*$ such that:

1. $\mathbf{k}_0 = \mathbf{k}$.

2. $(\mathbf{k}_\beta : \beta \leq \alpha)$ is an increasing continuous sequence of members of K_2 (with respect to $\leq_1$).

3. $|\mathbb{P}_{\mathbf{k}_\alpha}| < \mu$.

4. For every $\alpha < \lambda_*$, if $\mathbf{k}_\alpha \notin K_2^+$, we choose a Boolean combination $\underset{\sim}{a}_\alpha$ from $\mathcal{A}_{\mathbf{k}_\alpha}$ and $\underset{\sim}{b}_\alpha \subseteq \underset{\sim}{a}_\alpha$ witnessing the failure of the condition from Definition 2(E). We then define $\mathbb{P}_{\mathbf{k}_{\alpha+1}}$ as an extension (with respect to $\lessdot$) of $\mathbb{P}_{\mathbf{k}_\alpha} \star Cohen$ to a ccc forcing that forces $MA_{\aleph_1}$, we let $\underset{\sim}{\eta}_\alpha$ be the relevant Cohen generic real and we let $\mathcal{A}_{\mathbf{k}_{\alpha+1}} = \mathcal{A}_{\mathbf{k}\alpha} \cup \{\underset{\sim}{b}_\alpha \cup \underset{\sim}{\eta}_\alpha^{-1}(\{1\})\}$.

5. If $\alpha < \lambda_*$ is a limit ordinal, we define $\mathbf{k}_\alpha$ as in the proof of claim 7 below.

Why can we carry the induction at stage $\alpha + 1$ for α as in (4)? We shall prove that for each α, $\Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}}$ "$\mathcal{A}_{\mathbf{k}_{\alpha+1}}$ is independent". Let $\underset{\sim}{a}_\alpha^* = \underset{\sim}{b}_\alpha \cup \underset{\sim}{\eta}_\alpha^{-1}(\{1\})$, note that $\Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}}$ "$\underset{\sim}{a}_\alpha^* \notin \mathcal{A}_{\mathbf{k}_\alpha}$", as otherwise there are $p \in \mathbb{P}_{\mathbf{k}_{\alpha+1}}$, $n < \omega$ and $\underset{\sim}{a}' \in \mathcal{A}_{\mathbf{k}_\alpha}$ such that $p \Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}}$ "$\underset{\sim}{a}' \setminus n = \underset{\sim}{a}_\alpha^* \setminus n$", and therefore $p \Vdash_{\mathbb{P}_{\mathbf{k}_{\alpha+1}}}$ "$\underset{\sim}{\eta}_\alpha^{-1}(\{1\}) \restriction (\omega \setminus$

$\underset{\sim}{b}_\alpha \setminus n) = \underset{\sim}{a}^*_\alpha \setminus \underset{\sim}{b}_\alpha \setminus n = \underset{\sim}{a}' \setminus \underset{\sim}{b}_\alpha \setminus n \in V^{\mathbb{P}_{\mathbf{k}_\alpha}}$", a contradiction (as $\underset{\sim}{\eta}_\alpha$ is Cohen and $\omega \setminus \underset{\sim}{b}_\alpha \setminus n$ is infinite). Now if $\underset{\sim}{a}_\alpha = \bigcap_{l<n} \underset{\sim}{a}_{\alpha,l}^{\text{if l is even}}$ and $\underset{\sim}{c} = \underset{\sim}{a}_\alpha \cap (\bigcap_{l<m} \underset{\sim}{d}_l^{\text{if l is even}})$ where $\underset{\sim}{d}_0, ..., \underset{\sim}{d}_{m-1} \in \mathcal{A}_{\mathbf{k}_\alpha} \setminus \{\underset{\sim}{a}_{\alpha,0}, ..., \underset{\sim}{a}_{\alpha,n-1}\}$, then $\underset{\sim}{c} \setminus \underset{\sim}{a}^*_\alpha$ and $\underset{\sim}{c} \cap \underset{\sim}{a}^*_\alpha$ are infinite (as $\underset{\sim}{\eta}_\alpha$ is Cohen and $\underset{\sim}{c} \setminus \underset{\sim}{b}$ is infinite), so $\mathcal{A}_{\mathbf{k}_{\alpha+1}}$ is forced to be independent.

If for some $\alpha < \lambda_*$, $\mathbf{k}_\alpha \in K_2^+$, when we're done. Otherwise, by Fodor's lemma, there are $\alpha < \beta < \lambda_*$ such that $(\underset{\sim}{a}_\alpha, \underset{\sim}{b}_\alpha) = (\underset{\sim}{a}_\beta, \underset{\sim}{b}_\beta)$, a contradiction. $\square$

Definition 6: We say that $(\mathbf{k}_\alpha : \alpha < \beta)$ is increasing continuous if $\alpha_1 < \alpha_1 \rightarrow \mathbf{k}_{\alpha_1} \leq_2 \mathbf{k}_{\alpha_2}$, and for every limit $\delta < \beta$, $\bigcup_{i<\delta} \mathbb{P}_{\mathbf{k}_i} \lessdot \mathbb{P}_{\mathbf{k}_\delta}$.

Claim 7: Every increasing continuous sequence in $(K_2^+, \leq_2)$ has an upper bound. Moreover, if the length of the sequence has cofinality $> \aleph_1$, then the union is an upper bound in K_2^+.

Proof: Given an increasing continuous sequence $(\mathbf{k}_\alpha : \alpha < \beta)$, we choose $\mathbb{P}_{\mathbf{k}_\beta}$ as in [HwSh:1090] and we let $\mathcal{A}_{\mathbf{k}_\beta} = \bigcup_{\alpha<\beta} \mathcal{A}_{\mathbf{k}_\alpha}$. This is enough for $\leq_1$, so by claim 5 we're done. $\square$

Claim 8: A. If $\mathbf{k}_1 \in K_2$ then there are $\mathbf{k}_2$ and $\underset{\sim}{a}$ such that:

a. $\mathbf{k}_1 \leq_1 \mathbf{k}_2$.

b. $\mathcal{A}_{\mathbf{k}_1} \cup \{\underset{\sim}{a}\} \subseteq \mathcal{A}_{\mathbf{k}_2}$.

c. $\underset{\sim}{a}$ is Cohen over $V^{\mathbb{P}_{\mathbf{k}_1}}$.

d. $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}|)^{\aleph_1}$.

B. Moreover, we may require that $\mathbf{k}_2 \in K_2^+$.

Proof: A. Let $\mathbb{P} = \mathbb{P}_{\mathbf{k}_1} \star \mathbb{C}$ where $\mathbb{C}$ is Cohen forcing, now let $\mathbb{P}_{\mathbf{k}_2}$ be a ccc forcing such that $\mathbb{P} \lessdot \mathbb{P}_{\mathbf{k}_2}$, $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$MA_{\aleph_1}$" and $|\mathbb{P}_{\mathbf{k}_2}| \leq (2+|\mathbb{P}|)^{\aleph_1}$. Finally, let $\mathcal{A}_{\mathbf{k}_2} = \mathcal{A}_{\mathbf{k}_1} \cup \{\underset{\sim}{a}\}$ where $\underset{\sim}{a}$ is a name for a Cohen real added by $\mathbb{P}_{\mathbf{k}_2}$, it's easy to see that $(\mathbb{P}_{\mathbf{k}_2}, \mathcal{A}_{\mathbf{k}_2})$ are as required.

B. By claim 5. $\square$

Definition 9: We define the amalgamation property in the context of K_2^+ as follows: K_2^+ has the amalgamation property if A implies B where:

A. a. $\mathbf{k}_l \in K_2^+$ $(l = 0, 1, 2)$.

b. $\mathbf{k}_0 \leq_2 \mathbf{k}_l$ $(l = 1, 2)$.

c. $\mathbb{P}_{\mathbf{k}_1} \cap \mathbb{P}_{\mathbf{k}_2} = \mathbb{P}_{\mathbf{k}_0}$.

B. There exists $\mathbf{k}_3 = (\mathbb{P}_{\mathbf{k}_3}, \mathcal{A}_{\mathbf{k}_3}) \in K_2^+$ such that $\mathbf{k}_l \leq_2 \mathbf{k}_3$ $(l = 1, 2)$.

Claim 10: a. $(K_2^+, \leq_2)$ has the amalgamation property.

b. Suppose that $\mathbf{k}_0, \mathbf{k}_1$ and $\mathbf{k}_2 \in K_2^+$, $g : \mathbb{P}_{\mathbf{k}_0} \to \mathbb{P}_{\mathbf{k}_1}$ is an embedding such that $(g(\mathbb{P}_{\mathbf{k}_0}), g(\mathcal{A}_{\mathbf{k}_0})) \leq_2 \mathbf{k}_1$ and $\mathbf{k}_0 \leq_2 \mathbf{k}_2$, then there exist $\mathbf{k}, \mathbf{k}'_2 \in K_2^+$ and f such that:

1. $\mathbf{k}_1 \leq_2 \mathbf{k}$ and $|\mathbb{P}_\mathbf{k}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}| + |\mathbb{P}_{\mathbf{k}_2}|)^{\aleph_1}$.
2. $(g(\mathbb{P}_{\mathbf{k}_0}), g(\mathcal{A}_{\mathbf{k}_0})) \leq_2 \mathbf{k}'_2 \leq_2 \mathbf{k}$.
3. $f : \mathbb{P}_{\mathbf{k}_2} \to \mathbb{P}_{\mathbf{k}'_2}$ is an isomorphism mapping $\mathcal{A}_{\mathbf{k}_2}$ to $\mathcal{A}_{\mathbf{k}'_2}$
4. $g \subseteq f$.

Proof: a. We shall first prove that $\mathcal{A}_{\mathbf{k}_1} \cap \mathcal{A}_{\mathbf{k}_2} = \mathcal{A}_{\mathbf{k}_0}$. Note that $\mathcal{A}_{\mathbf{k}_0} \subseteq \mathcal{A}_{\mathbf{k}_1} \cap \mathcal{A}_{\mathbf{k}_2}$ is true by the definition of $\leq_2$, so suppose that $\underset{\sim}{a} \in \mathcal{A}_{\mathbf{k}_1} \setminus \mathcal{A}_{\mathbf{k}_0}$, we need to show that $\underset{\sim}{a} \notin \mathcal{A}_{\mathbf{k}_2}$. As $\mathbf{k}_0 \leq_2 \mathbf{k}_1$, $\underset{\sim}{a}$ is not a $\mathbb{P}_{\mathbf{k}_0}$-name. Therefore, it's not a $\mathbb{P}_{\mathbf{k}_2}$-name, hence $\underset{\sim}{a} \notin \mathcal{A}_{\mathbf{k}_2}$.

Now construct $\mathbb{P}$ as in [HwSh:1090], i.e. we take the amalgamation $\mathbb{P}' = \mathbb{P}_{\mathbf{k}_1} \underset{\mathbb{P}_{\mathbf{k}_0}}{\times} \mathbb{P}_{\mathbf{k}_2}$ and then we take $\mathbb{P} \in K$ such that $\mathbb{P}' \lessdot \mathbb{P}$ and $|\mathbb{P}| \leq (2 + |\mathbb{P}'|)^{\aleph_1}$. Now let $\mathcal{A} := \mathcal{A}_{\mathbf{k}_1} \cup \mathcal{A}_{\mathbf{k}_2}$. We need to show that $\mathcal{A}$ is as required, i.e. we need to prove that $(\mathbb{P}, \mathcal{A})$ satisfy requirements (A)(d) and (D)(c) in Definition 2 (in the end, we will use claim 5 for the requirement in Definition (2)(D)(c)). By symmetry, it's enough to show that if $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$" and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in \mathcal{A}$, then $\Vdash_\mathbb{P}$ "$\underset{\sim}{b} \cap (\underset{l<n}{\cap} \underset{\sim}{a}_l^{\text{if (l is even)}}) \in [\omega]^\omega$". Let $n_2 := n$, wlog there are $n_0 < n_1 < n_2$ such that $\underset{\sim}{a}_l \in \mathcal{A}_{\mathbf{k}_0} \iff l < n_0$, $\underset{\sim}{a}_l \in \mathcal{A}_{\mathbf{k}_1} \iff l \in [0, n_1)$ and $\underset{\sim}{a}_l \in \mathcal{A}_{\mathbf{k}_2} \iff l \in [0, n_0) \cup [n_1, n_2)$. It's enough to show that the last statement is forced by $\mathbb{P}'$, so let $k < \omega$ and $p = (p_1, p_2) \in \mathbb{P}'$, we shall find $q \in \mathbb{P}'$ and $m < \omega$ such that $p \leq q$, $k \leq m$ and $q \Vdash_{\mathbb{P}'}$ "$m \in \underset{\sim}{b} \cap (\underset{l<n}{\cap} \underset{\sim}{a}_l^{\text{if (l is even)}})$". Let $p_0 \in \mathbb{P}_{\mathbf{k}_0}$ witness "$(p_1, p_2) \in \mathbb{P}'$", i.e. $p_0 \Vdash_{\mathbb{P}_{\mathbf{k}_0}}$ " $\underset{l=1,2}{\wedge} p_l \in \mathbb{P}_{\mathbf{k}_l}/\mathbb{P}_{\mathbf{k}_0}$". Let $\underset{\sim}{b}^* = \{m : p_1 \nVdash_{\mathbb{P}_{\mathbf{k}_1}/\mathbb{P}_{\mathbf{k}_0}}$ "$m \notin \underset{\sim}{b} \cap (\cap \{\underset{\sim}{a}_l^{\text{if l is even}} : l \in [n_0, n_1)\})$"$\}$ (so $\underset{\sim}{b}^*$ is a $\mathbb{P}_{\mathbf{k}_0}$-name) and let $p_0 \in G_0 \subseteq \mathbb{P}_{\mathbf{k}_0}$ be generic over V, then $\underset{\sim}{b}^* = \underset{\sim}{b}^*[G_0] \in V[G_0]$ and as $p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{b} \in pos(\mathcal{A}_{\mathbf{k}_1})$", it follows that $p_0 \Vdash_{\mathbb{P}_{\mathbf{k}_0}}$ "$\underset{\sim}{b}^* \in [\omega]^\omega$", moreover, $p_0 \Vdash_{\mathbb{P}_{\mathbf{k}_0}}$ "$\underset{\sim}{b}^* \in pos(\mathcal{A}_{\mathbf{k}_0})$". Let b^{**} be the $\mathbb{P}_{\mathbf{k}_0}$-name defined as b^* if p_0 is in the generic set, and as ω otherwise. As $\mathbf{k}_0 \leq_2 \mathbf{k}_2$, it follows that $p_2 \Vdash_{\mathbb{P}_{\mathbf{k}_2}/G_0}$ "$\underset{\sim}{b}^{**} \cap (\underset{l \in [0,n_0) \cup [n_1,n_2)}{\cap} \underset{\sim}{a}_l^{\text{if (l is even)}}) \in [\omega]^\omega$".

Therefore, in $V[G_0]$ there are (p'_2, m) such that:

a. $p_2 \leq p'_2 \in \mathbb{P}_{\mathbf{k}_2}/G_0$.

b. $m > k$.

c. $p'_2 \Vdash_{\mathbb{P}_{\mathbf{k}_2}/G_0}$ "$m \in \underset{\sim}{b}^{**} \cap (\underset{l \in [0,n_0) \cup [n_1.n_2)}{\cap} \underset{\sim}{a}_l^{\text{if (l is even)}})$".

Note that as $\underset{\sim}{b}^{**} \in V[G_0]$, $V[G_0] \models$ "$m \in \underset{\sim}{b}^{**}[G_0] = \underset{\sim}{b}^*[G_0]$". Therefore, by the definitions of $\underset{\sim}{b}$ and $\underset{\sim}{b}^*$, there is $p'_1 \in \mathbb{P}_{\mathbf{k}_1}/G_0$ above p_1 such that $p'_1 \Vdash_{\mathbb{P}_{\mathbf{k}_1}/G_0}$ "$m \in \underset{\sim}{b} \cap (\cap \{\underset{\sim}{a}_l^{\text{if l is even}} : l \in [n_0, n_1)\})$". Therefore, there is $p_0 \leq p'_0 \in G_0$ forcing (in $\mathbb{P}_{\mathbf{k}_0}$) all of the aforementioned statements about (p'_1, p'_2) in $V[G_0]$. Now it's easy to check that $q = (p'_1, p'_2)$ is as required. Finally, extend $(\mathbb{P}, \mathcal{A})$ (with respect to $\leq_1$) to a member of K_2^+. By observation 3, we're done.

b. Follows from (a) by changing names. $\square$

Claim 11: There exists $\mathbf{k} = (\mathbb{P}_{\mathbf{k}}, \mathcal{A}_{\mathbf{k}}) = (\mathbb{P}, \mathcal{A}) \in K_2^+$ such that $|\mathbb{P}_{\mathbf{k}}| = \lambda$ and:

1. For every $X \subseteq \mathbb{P}$ of cardinality $< \mu$, there exists $\mathbf{k}' = (\mathbb{Q}, \mathcal{A}') \in K_2^+$ such that $X \subseteq \mathbb{Q}$, $\mathbf{k}' \leq_2 \mathbf{k}$ and $|\mathbb{Q}| < \mu$.

2. If $\mathbf{k}_1, \mathbf{k}_2 \in K_2^+$, $|\mathbb{P}_{\mathbf{k}_1}|, |\mathbb{P}_{\mathbf{k}_2}| < \mu$, $\mathbf{k}_1 \leq_2 \mathbf{k}_2$ and $f_1 : \mathbb{P}_{\mathbf{k}_1} \to \mathbb{P}$ is a complete embedding such that $(f_1(\mathbb{P}_{\mathbf{k}_1}), f_1(\mathcal{A}_{\mathbf{k}_1})) \leq_2 (\mathbb{P}, \mathcal{A})$, then there is a complete embedding f_2 such that $f_1 \subseteq f_2$ and $(f_2(\mathbb{P}_{\mathbf{k}_2}), f_2(\mathcal{A}_{\mathbf{k}_2})) \leq_2 (\mathbb{P}, \mathcal{A})$.

Proof: The first property is satisfied by every $\mathbf{k} \in K_2^+$ by observation 4. The proof of (2) is as in [HwSh:1090]. $\square$

Claim 12: A implies B where:

A. a. $\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2 \in K_2^+$, $\mathbf{k}_0 \leq_2 \mathbf{k}_l$ $(l = 1, 2)$ and $\mathbb{P}_{\mathbf{k}_0} = \mathbb{P}_{\mathbf{k}_1} \cap \mathbb{P}_{\mathbf{k}_2}$.

b. $\underset{\sim}{D}$ is a $\mathbb{P}_{\mathbf{k}_0}$-name of a nonprincipal ultrafilter on ω.

c. For $l = 1, 2$, $\underset{\sim l}{a}$ and $\underset{\sim}{b_l}$ are canonical $\mathbb{P}_{\mathbf{k}_l}$-names of a member of $[\omega]^\omega$.

d. For $l = 1, 2$, $\Vdash_{\mathbb{P}_{\mathbf{k}_l}}$ "$\underset{\sim l}{a} \cap \underset{\sim l}{b}$ is infinite and $\underset{\sim l}{a}$ contains no members of $\underset{\sim}{D}$ from $V^{\mathbb{P}_{\mathbf{k}_0}}$.

e. $\mathbb{P}_{\mathbf{k}_1} \cap \mathbb{P}_{\mathbf{k}_2} = \mathbb{P}_{\mathbf{k}_0}$.

f. For $l = 1, 2$, $\Vdash_{\mathbb{P}_{\mathbf{k}_l}}$ "$\underset{\sim}{b_l}$ is a pseudo intersection of $\underset{\sim}{D}$".

B. There is $\mathbf{k} \in K_2^+$ such that:

a. $|\mathbb{P}_{\mathbf{k}}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}| + |\mathbb{P}_{\mathbf{k}_2}|)^{\aleph_1}$.

b. $\mathbf{k}_l \leq_2 \mathbf{k}$ $(l = 1, 2)$.

c. $\Vdash_{\mathbb{P}_{\mathbf{k}}}$ "$\underset{\sim}{a_2} \setminus \underset{\sim}{a_1}$ and $\underset{\sim}{a_1} \setminus \underset{\sim}{a_2}$ are infinite".

Proof: Let $\mathbf{k} \in K_2^+$ be the object constructed by the proof of claim 10. We need to prove that $\mathbf{k}$ satisfies clause (B)(c). For $l = 0, 1, 2$, let $\mathbb{P}_l = \mathbb{P}_{\mathbf{k}_l}$ and let $\mathbb{P}'$ be as in the proof of claim 10, so it suffices to prove that $\Vdash_{\mathbb{P}'}$ "$\underset{\sim}{a_2} \setminus \underset{\sim}{a_1}$ and $\underset{\sim}{a_1} \setminus \underset{\sim}{a_2}$ are infinite". Let $p = (p_1, p_2) \in \mathbb{P}'$, $k < \omega$ and let $p_0 \in \mathbb{P}_0$ be a witness of "$p = (p_1, p_2) \in \mathbb{P}'$". Now let $G_0 \subseteq \mathbb{P}'$ be generic over V such that $p_0 \in G_0$ and let $D = \underset{\sim}{D}[G_0]$. By the assumptions, for $l = 1, 2$, $p_l \Vdash_{\mathbb{P}_1/G_0}$ "$\underset{\sim}{a_l} \cap \underset{\sim}{b_l}$ is an infinite pseudo intersection of D". In $V[G_0]$, let $b_l^* = \{m : p_l \nVdash_{\mathbb{P}_l/G_0}$ "$m \notin \underset{\sim}{a_l} \cap \underset{\sim}{b_l}$"$\}$, then $p_l \Vdash_{\mathbb{P}_l/G_0}$ "$\underset{\sim}{a_l} \cap \underset{\sim}{b_l} \subseteq \underset{\sim}{b_l^*}$, hence b_l^* infinite". As $p_l \Vdash_{\mathbb{P}_l/G_0}$ "$\underset{\sim}{a_l} \cap \underset{\sim}{b_l}$ is a pseudo intersection of D", necessarily $V[G_0] \models b_l^* \in D$.

Let $a_l^* = \{m : p_l \Vdash_{\mathbb{P}_l/G_0} m \in \underset{\sim}{a_l}\}$, so $a_l^* \in V[G_0]$ and $p_l \Vdash_{\mathbb{P}_l/G_0}$ "$a_l^* \subseteq \underset{\sim}{a_l}$". Now recall that $\Vdash_{\mathbb{P}_{\mathbf{k}_l}}$ "$\underset{\sim l}{a}$ contains no member of $\underset{\sim}{D}$ from $V^{\mathbb{P}_{\mathbf{k}_0}}$", therefore $p_l \Vdash_{\mathbb{P}_l/G_0}$ "$a_l^* \notin \underset{\sim}{D}$".

Hence in $V[G_0]$ (recalling D is an ultrafilter), $b := (b_1^* \cap b_2^*) \setminus (a_1^* \cup a_2^*) \in D$. Let $m \in b$ be such that $k < m$. By the definition of b_l^*, there is $p_l' \in \mathbb{P}_l/G_0$ above p_l such that $p_l' \Vdash_{\mathbb{P}_l/G_0}$ "$m \in \underset{\sim}{a}_l \cap \underset{\sim}{b}_l$". By the definition of a_l^*, there is $p_l'' \in \mathbb{P}_l/G_0$ above p_l such that $p_l'' \Vdash_{\mathbb{P}_l/G_0}$ "$m \notin \underset{\sim}{a}_l$". Let $p_0' \in G_0$ be a condition above p_0 forcing the above statements, so p_0' is witnessing the fact that $(p_1', p_2''), (p_1'', p_2') \in \mathbb{P}'$ are above $p = (p_1, p_2)$. Now $m > k$, $(p_1', p_2'') \Vdash$ "$m \in \underset{\sim}{a}_1 \setminus \underset{\sim}{a}_2$" and $(p_1'', p_2') \Vdash$ "$m \in \underset{\sim}{a}_2 \setminus \underset{\sim}{a}_1$", which completes the proof. $\square$

Definition 13: Let $\mathbb{P} = \mathbb{P}_{\mathbf{k}}$ be the forcing from claim 11, let $G \subseteq \mathbb{P}$ be generic over V and in $V[G]$, let $V_1 = HOD(\mathbb{R}^{<\mu} \cup \{\mathcal{A}_{\mathbf{k}}\})$.

Definition 14 ([NN]): A family $\mathcal{F} \subseteq [\omega]^\omega$ is nearly mad if $|A \cap B| < \aleph_0$ or $|A \Delta B| < \aleph_0$ for every $A \neq B \in \mathcal{F}$, and $\mathcal{F}$ is maximal with respect to this property.

Theorem 15: $V_1 \models ZF + DC_{<\mu}$+"there are no mad families"+"there are no nearly mad families"+ "there exists a non-meager filter on ω".

Proof: 1. In order to see that there exists a non-meagre filter in V_1, let $\underset{\sim}{D}$ be the filter generated by $\mathcal{A}_{\mathbf{k}}$ and the cofinite sets. By claim 8 and the choice of $\mathbf{k}$, $\underset{\sim}{D}$ contains many Cohen reals and therefore is non-meager.

2. The proof of the non-existence of mad families is exactly as in [HwSh:1090], where $(K_2^+, \leq_2)$ here replaces $(K, \lessdot)$ there, and claim 10 is used for the amalgamation arguments. Alternatively, see the proof of (3) below.

3. The non-existence of a nearly mad family in V_1 will follow from the proofs below. $\square$

Somewhat mad families

Definition 16: A family $\mathcal{F} \subseteq [\omega]^\omega$ is somewhat mad if:

a. For every $a_1, a_2 \in \mathcal{F}$, $|a_1 \cap a_2| < \aleph_0$ or $a_1 \subseteq^* a_2$ or $a_2 \subseteq^* a_1$.

b. If $b \in [\omega]^\omega$ then for some $a \in \mathcal{F}$, $|a \cap b| = \aleph_0$.

Observation 17: Nearly mad families are somewhat mad. $\square$

Definition 18: Let $Pr(\mathbf{k}_1, \mathbf{k}_2, \underset{\sim}{D}, \underset{\sim}{b}_2)$ mean:

a. $\mathbf{k}_1 \leq_2 \mathbf{k}_2$, $\underset{\sim}{b}_2$ is a $\mathbb{P}_{\mathbf{k}_2}$-name and $\underset{\sim}{D}$ is a $\mathbb{P}_{\mathbf{k}_1}$-name such that $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{b}_2 \in [\omega]^\omega$" and $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{D}$ is a nonprincipal ultrafilter on ω".

b. If $G_1 \subseteq \mathbb{P}_{\mathbf{k}_1}$ is generic over V, $p_1 \in \mathbb{P}_{\mathbf{k}_2}/G_1$, $b_0^* = \{n : p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_1} G_1}$ "$n \in \underset{\sim}{b}_2$"$\}$ and $b_1^* = \{n : p_1 \nVdash_{\mathbb{P}_{\mathbf{k}_1}/G_1}$ "$n \notin \underset{\sim}{b}_2$"$\}$, then $V[G_1] \models b_1^* \setminus b_0^* \in D$.

Claim 19: (A) implies (B) where:

A. a. $\mathbf{k}_1 \in K_2^+$.

b. $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{D}$ is a nonprincipal ultrafilter on ω".

c. $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{S}_1$ is somewhat mad".

d. $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{S}_1 \cap \underset{\sim}{D} = \emptyset$".

B. There is $\mathbf{k}_2$ such that:

a. $\mathbf{k}_2 \in K_2^+$.

b. $\mathbf{k}_1 \leq_2 \mathbf{k}_2$.

c. $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}|)^{\aleph_1}$.

d. (α) implies (β) where:

α. $(\mathbf{k}_3, \underset{\sim}{S}_2)$ satisfy the following properties:

1. $\mathbf{k}_2 \leq_2 \mathbf{k}_3 \in K_2^+$.

2. $\Vdash_{\mathbb{P}_{\mathbf{k}_3}}$ "$\underset{\sim}{S}_2$ is somewhat mad and $\underset{\sim}{S}_1 \subseteq \underset{\sim}{S}_2$".

3. $\Vdash_{\mathbb{P}_{\mathbf{k}_3}}$ "no member of $\underset{\sim}{S}_2 \setminus \underset{\sim}{S}_1$ contains a member of $\underset{\sim}{D}$".

β. For some $\mathbb{P}_{\mathbf{k}_3}$-name $\underset{\sim}{a}$, $\Vdash_{\mathbb{P}_{\mathbf{k}_3}}$ "$\underset{\sim}{a} \in \underset{\sim}{S}_2$" and $Pr(\mathbf{k}_1, \mathbf{k}_3, \underset{\sim}{D}, \underset{\sim}{a})$.

Proof: Using Mathias forcing restricted to $\underset{\sim}{D}$, it's easy to see that there is $\mathbf{k}_2$ and a $\mathbb{P}_{\mathbf{k}_2}$ name $\underset{\sim}{b}$ such that $\mathbf{k}_1 \leq_2 \mathbf{k}_2 \in K_2^+$, $|\mathbb{P}_{\mathbf{k}_2}| \leq (2 + |\mathbb{P}_{\mathbf{k}_1}|)^{\aleph_1}$ and $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{b}$ is a pseudo intersection of $\underset{\sim}{D}$". Therefore, $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{b} \in [\omega]^\omega$ is almost disjoint to every $\underset{\sim}{a} \in \underset{\sim}{S}_1$" (by the fact that $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "$\underset{\sim}{S}_1 \cap \underset{\sim}{D} = \emptyset$").

We shall now prove that $\mathbf{k}_2$ satisfies (B)(d). Suppose that $(\mathbf{k}_3, \underset{\sim}{S}_2)$ are as there. By the somewhat madness of $\underset{\sim}{S}_2$, there is $\underset{\sim}{a}$ such that $\Vdash_{\mathbb{P}_{\mathbf{k}_3}}$ "$\underset{\sim}{a} \in \underset{\sim}{S}_2$ and $|\underset{\sim}{a} \cap \underset{\sim}{b}| = \aleph_0$". Therefore, $\Vdash_{\mathbb{P}_{\mathbf{k}_3}}$ "$\underset{\sim}{a} \cap \underset{\sim}{b} \in [\omega]^\omega$ is a pseudo intersection of $\underset{\sim}{D}$". Now let $G_1 \subseteq \mathbb{P}_{\mathbf{k}_1}$ be generic over V. If $p_1 \in \mathbb{P}_{\mathbf{k}_3}/G_1$ then $b^* = \{n : p_1 \nVdash_{\mathbb{P}_{\mathbf{k}_3}/G_1}$ "$n \notin \underset{\sim}{a} \cap \underset{\sim}{b}$"$\} \in \underset{\sim}{D}[G_1]$ by the fact that $\underset{\sim}{a} \cap \underset{\sim}{b}$ is a pseudo intersection of $\underset{\sim}{D}$. Now let $a^* = \{n : p_1 \Vdash_{\mathbb{P}_{\mathbf{k}_3/G_1}}$ "$n \in \underset{\sim}{a}$"$\}$, then $p_1 \Vdash_{\mathbb{P}_{\mathbf{K}_3}/G_1}$ "$a^* \subseteq \underset{\sim}{a}$ is infinite". If $a^* \in \underset{\sim}{D}[G_1]$, then p_1 forces that $\underset{\sim}{a}$ (which belongs to $\underset{\sim}{S}_2 \setminus \underset{\sim}{S}_1$) contains a member of $\underset{\sim}{D}[G_1]$, contradicting $(\alpha)(3)$. Therefore, $a^* \notin \underset{\sim}{D}[G_1]$, and $\underset{\sim}{a}$ is as required in the definition of $Pr(\mathbf{k}_1, \mathbf{k}_3, \underset{\sim}{D}, \underset{\sim}{a})$. $\square$

Claim 20: There is no somewhat mad family in V_1.

Proof: Suppose towards contradiction that $\underset{\sim}{S}$ is a $\mathbb{P}$-name of a somewhat mad family. As in [HwSh:1090], let $\underset{\sim}{D}$ be a $\mathbb{P}$-name of a Ramsey ultrafilter on ω such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S} \cap \underset{\sim}{D} = \emptyset$". By claim 11(a), there is $\mathbf{k}_1 \leq_2 \mathbf{k}$ such that $\mathbf{k}_1 \in K_2^+$, $|\mathbb{P}_{\mathbf{k}_1}| < \mu$ and $\underset{\sim}{S}$ is definable using a $\mathbb{P}_{\mathbf{k}_1}-$name. Let $K_{\mathbb{P}}^+$ be the set of $\mathbf{k}' \in K_2^+$ such that $\mathbf{k}' \leq_2 \mathbf{k}$, $|\mathbb{P}_{\mathbf{k}'}| < \mu$, $\underset{\sim}{S} \restriction \mathbb{P}_{\mathbf{k}'}$ is a canonical $\mathbb{P}_{\mathbf{k}'}$-name of a somewhat mad family in $V^{\mathbb{P}_{\mathbf{k}'}}$ and $\underset{\sim}{D} \restriction \mathbb{P}_{\mathbf{k}'}$ is a $\mathbb{P}_{\mathbf{k}'}$-name of a Ramsey ultrafilter on ω. As in [HwSh:1090],

$K^+_{\mathbb{P}}$ is $\leq_2$-dense in K^+_2, so there exists $\mathbf{k}_2 \in K^+_{\mathbb{P}}$ such that $\mathbf{k}_1 \leq \mathbf{k}_2$. Let $\mathbf{k}_3 \in K^+_2$ be as in claim 19 for $(\mathbf{k}_2, \underset{\sim}{S} \restriction \mathbb{P}_{\mathbf{k}_2})$, wlog $\mathbf{k}_3 \leq_2 \mathbf{k}$ (see claim 11). Choose $\mathbf{k}_4 \in K^+_{\mathbb{P}}$ such that $\mathbf{k}_3 \leq_2 \mathbf{k}_4$ and let $\underset{\sim}{S}_4 := \underset{\sim}{S} \restriction \mathbb{P}_{\mathbf{k}_4}$. Let $\underset{\sim}{a}$ be a $\mathbb{P}_{\mathbf{k}_4}$-name such that $Pr(\mathbf{k}_3, \mathbf{k}_4, \underset{\sim}{D}, \underset{\sim}{a})$ holds, as guaranteed by claim 19.

As in [HwSh:1090], there are $\mathbf{k}_5, \mathbf{k}_6, \in K^+_{\mathbb{P}}$ and an isomorphism f from $\mathbf{k}_4$ to $\mathbf{k}_5$ over $\mathbf{k}_2$ such that $\mathbb{P}_{\mathbf{k}_5}$ adds a generic for $\mathbb{M}_{\underset{\sim}{D} \restriction \mathbb{P}_{\mathbf{k}_4}}$ (Mathias forcing restricted to the ultrafilter $\underset{\sim}{D} \restriction \mathbb{P}_{\mathbf{k}_4}$) and $(\mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6)$ here are as $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ in claim 10, and wlog $\mathbf{k}_6 \leq_2 \mathbf{k}$. By the choice of f, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}, f(\underset{\sim}{a}) \in \underset{\sim}{S}$".

By claim 12, with $(\mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6, \underset{\sim}{a}, f(\underset{\sim}{a}))$ standing for $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}, \underset{\sim}{a}_1, \underset{\sim}{a}_2)$ there, it's forced by $\mathbb{P}_{\mathbf{k}_5}$, and hence by $\mathbb{P}$, that $\underset{\sim}{a} \setminus f(\underset{\sim}{a})$ and $f(\underset{\sim}{a}) \setminus \underset{\sim}{a}$ are infinite. As in [HwSh:1090], $\Vdash_{\mathbb{P}}$ "$|\underset{\sim}{a} \cap f(\underset{\sim}{a})| = \aleph_0$". As $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}, f(\underset{\sim}{a}) \in \underset{\sim}{S}$", we get a contradiction. $\square$

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