ON THE COFINALITY OF THE SPLITTING NUMBER

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Abstract. The splitting number $\mathfrak{s}$ can be singular. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets introduced by Blass and the second author [*Ultrafilters with small generating sets*, Israel J. Math., **65**, (1989)]

1. Introduction

The cardinal invariants of the continuum discussed in this article are very well known (see [5, van Douwen, p111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^{\aleph_0}$ if S is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $\mathfrak{s}$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (equivalently every $S \in [\omega]^{\aleph_0}$ is *split* by some member of $\mathcal{Y}$). It is mentioned in [2] that it is currently unknown if $\mathfrak{s}$ can be a singular cardinal.

Proposition 1.1. *The cofinality of the splitting number is not countable.*

Proof. Assume that θ is the supremum of $\{\kappa_n : n \in \omega\}$ and that there is no splitting family of cardinality less than θ. Let $\mathcal{Y} = \{Y_\alpha : \alpha < \theta\}$ be a family of subsets of ω. Let $S_0 = \omega$ and by induction on n, choose an infinite subset S_{n+1} of S_n so that S_{n+1} is not split by the family $\{Y_\alpha : \alpha < \kappa_n\}$. If S is any pseudointersection of $\{S_n : n \in \omega\}$, then S is not split by any member of $\mathcal{Y}$. □

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One can easily generalize the previous result and proof to show that the cofinality of the splitting number is at least $\mathfrak{t}$. In this paper we prove the following.

Theorem 1.2. *If κ is any uncountable regular cardinal, then there is a $\lambda > \kappa$ with $\mathrm{cf}(\lambda) = \kappa$ and a ccc forcing $\mathbb{P}$ satisfying that $\mathfrak{s} = \lambda$ in the forcing extension.*

To prove the theorem, we construct $\mathbb{P}$ using matrix iterations.

2. A special splitting family

Definition 2.1. *Let us say that a family $\{x_i : i \in I\} \subset [\omega]^\omega$ is θ-Luzin (for an uncountable cardinal θ) if for each $J \in [I]^\theta$, $\bigcap\{x_i : i \in J\}$ is finite and $\bigcup\{x_i : i \in J\}$ is cofinite.*

Clearly a family is θ-Luzin if every θ-sized subfamily is θ-Luzin. We leave to the reader the easy verification that for a regular uncountable cardinal θ, each θ-Luzin family is a splitting family. A poset being θ-Luzin preserving will have the obvious meaning. For example, any poset of cardinality less than a regular cardinal θ is θ-Luzin preserving.

Lemma 2.2. *If θ is a regular uncountable cardinal then any ccc finite support iteration of θ-Luzin preserving posets is again θ-Luzin preserving.*

Proof. We prove this by induction on the length of the iteration. Fix any θ-Luzin family $\{x_i : i \in I\}$ and let $\langle\langle \mathbb{P}_\alpha : \alpha \leq \gamma\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle\rangle$ be a finite support iteration of ccc posets satisfying that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is ccc and θ-Luzin preserving, for all $\alpha < \gamma$.

If γ is a successor ordinal $\beta + 1$, then for any $\mathbb{P}_\beta$-generic filter G_β, the family $\{x_i : i \in I\}$ is a θ-Luzin family in $V[G_\beta]$. By the hypothesis on $\dot{\mathbb{Q}}_\beta$, this family remains θ-Luzin after further forcing by $\dot{\mathbb{Q}}_\beta$.

Now we assume that α is a limit. Let $\dot{J}_0$ be any $\mathbb{P}_\gamma$-name of a subset of I and assume that $p \in \mathbb{P}_\gamma$ forces that $|\dot{J}_0| = \theta$. We must produce a $q < p$ that forces that $\dot{J}_0$ is as in the definition of θ-Luzin. There is a set $J_1 \subset I$ of cardinality θ satisfying that, for each $i \in J_1$, there is a $p_i < p$ with $p_i \Vdash i \in \dot{J}_0$. The case when the cofinality of α not equal to θ is almost immediate. There is a $\beta < \alpha$ such that $J_2 = \{i \in J_1 : p_i \in \mathbb{P}_\beta\}$ has cardinality θ. There is a $\mathbb{P}_\beta$-generic filter G_β such that $J_3 = \{i \in J_2 : p_i \in G_\beta\}$ has cardinality θ. By the induction hypothesis, the family $\{x_i : i \in I\}$ is θ-Luzin in $V[G_\beta]$ and so we have that $\bigcap\{x_i : i \in J_3\}$ is finite and $\bigcup\{x_i : i \in J_3\}$ is co-finite. Choose any $q < p$ in G_β and a name $\dot{J}_3$ for J_3 so that q forces this property for $\dot{J}_3$. Since q forces that $\dot{J}_3 \subset \dot{J}_0$, we have that q forces the same property for $\dot{J}_0$.

Finally we assume that α has cofinality θ. Naturally we may assume that the collection $\{\mathrm{dom}(p_i) : i \in J_1\}$ forms a Δ-system with root contained in some $\beta < \alpha$. Again, we may choose a $\mathbb{P}_\beta$-generic filter G_β satisfying that $J_2 = \{i \in J_1 : p_i \restriction \beta \in G_\beta\}$ has cardinality θ. In $V[G_\beta]$, let $\{J_{2,\xi} : \xi \in \omega_1\}$ be a partition of J_2 into pieces of size θ. For each $\xi \in \omega_1$, apply the induction hypothesis in the model $V[G_\beta]$, and so we have that $\bigcap\{x_i : i \in J_{2,\xi}\}$ is finite and $\bigcup\{x_i : i \in J_{2,\xi}\}$ is co-finite. For each $\xi \in \omega_1$ let m_ξ be an integer large enough so that $\bigcap\{x_i : i \in J_{2,\xi}\} \subset m_\xi$ and $\bigcup\{x_i : i \in J_{2,\xi}\} \supset \omega \setminus m_\xi$. Let m be any integer such that $m_\xi = m$ for uncountably many ξ. Choose any condition $\bar{p} \in \mathbb{P}_\alpha$ so that $\bar{p} \restriction \beta \in G_\beta$. We prove that for each $n > m$ there is a $\bar{p}_n < \bar{p}$ so that $\bar{p}_n \Vdash n \notin \bigcap\{x_i : i \in \dot{I}\}$ and $\bar{p}_n \Vdash n \in \bigcup\{x_i : i \in \dot{I}\}$. Choose any $\xi \in \omega_1$ so that $m_\xi = m$ and $\mathrm{dom}(p_i) \cap \mathrm{dom}(\bar{p}) \subset \beta$ for all $i \in J_{2,\xi}$. Now choose any $i_0 \in J_{2,\xi}$ so that $n \notin x_{i_0}$. Next choose a distinct ξ' with $m_{\xi'} = m$ so that $\mathrm{dom}(p_i) \cap (\mathrm{dom}(\bar{p}) \cup \mathrm{dom}(p_{i_0})) \subset \beta$ for all $i \in J_{2,\xi'}$. Now choose $i_1 \in J_{2,\xi'}$ so that $n \in x_{i_1}$. We now have that $\bar{p} \cup p_{i_0} \cup p_{i_1}$ is a condition that forces $\{i_0, i_1\} \subset \dot{I}$. □

Next we introduce a σ-centered poset that will render a given family non-splitting.

Definition 2.3. *For a filter $\mathfrak{D}$ on ω, we define the Laver style poset $\mathbb{L}(\mathfrak{D})$ to be the set of trees $T \subset \omega^{<\omega}$ with the property that T has a minimal branching node* stem(T) *and for all* stem$(T) \subseteq t \in T$*, the branching set $\{k : t^\frown k \in T\}$ is an element of $\mathfrak{D}$. If $\mathfrak{D}$ is a filter base for a filter $\mathfrak{D}^*$, then $\mathbb{L}(\mathfrak{D})$ will also denote $\mathbb{L}(\mathfrak{D}^*)$.*

The name $\dot{L} = \{(k, T) : (\exists t)\ t^\frown k \subset \mathrm{stem}(T)\}$ will be referred to as the canonical name for the real added by $\mathbb{L}(\mathfrak{D})$.

If $\mathfrak{D}$ is a principal (fixed) ultrafilter on ω, then $\mathbb{L}(\mathfrak{D})$ has a minimum element and so is forcing isomorphic to the trivial poset. If $\mathfrak{D}$ is principal but not an ultrafilter, then $\mathbb{L}(\mathfrak{D})$ is isomorphic to Cohen forcing. If $\mathfrak{D}$ is a free filter, then $\mathbb{L}(\mathfrak{D})$ adds a dominating real and has similarities to Hechler forcing. As usual, for a filter (or filter base) $\mathfrak{D}$ of subsets of ω, we use $\mathfrak{D}^+$ to denote the set of all subsets of ω that meet every member of $\mathfrak{D}$.

Definition 2.4. *If E is a dense subset of $\mathbb{L}(\mathfrak{D})$, then a function ρ_E from $\omega^{<\omega}$ into ω_1 is a rank function for E if $\rho_E(t) = 0$ if and only if $t =$* stem(T) *for some $T \in E$, and for all $t \in \omega^{<\omega}$ and $0 < \alpha \in \omega_1$, $\rho_E(t) \leq \alpha$ providing the set $\{k \in \omega : \rho_E(t^\frown k) < \alpha\}$ is in $\mathfrak{D}^+$.*

When $\mathfrak{D}$ is a free filter, then $\mathbb{L}(\mathfrak{D})$ has cardinality $\mathfrak{c}$, but nevertheless, if $\mathfrak{D}$ has a base of cardinality less than a regular cardinal θ, $\mathbb{L}(\mathfrak{D})$ is θ-Luzin preserving.

Lemma 2.5. *If $\mathfrak{D}$ is a free filter on ω and if $\mathfrak{D}$ has a base of cardinality less than a regular uncountable cardinal θ, then $\mathbb{L}(\mathfrak{D})$ is θ-Luzin preserving.*

Proof. Let $\{x_i : i \in \theta\}$ be a θ-Luzin family with θ as in the Lemma. Let $\dot{J}$ be a $\mathbb{L}(\mathfrak{D})$-name of a subset of θ. We prove that if $\bigcap\{x_i : i \in \dot{J}\}$ is not finite, then $\dot{J}$ is bounded in θ. By symmetry, it will also prove that if $\bigcup\{x_i : i \in \dot{J}\}$ is not cofinite, then $\dot{J}$ is bounded in θ. Let $\dot{y}$ be the $\mathbb{L}(\mathfrak{D})$-name of the intersection, and let T_0 be any member of $\mathbb{L}(\mathfrak{D})$ that forces that $\dot{y}$ is infinite. Let M be any $< \theta$-sized elementary submodel of $H((2^{\mathfrak{c}})^+)$ such that $T_0, \mathfrak{D}$, $\dot{J}$, and $\{x_i : i \in \theta\}$ are all members of M and such that $M \cap \mathfrak{D}$ contains a base for $\mathfrak{D}$. Let $i_M = \sup(M \cap \theta)$. If $x \in M \cap [\omega]^\omega$, then $I_x = \{i \in \theta : x \subset x_i\}$ is an element of M and has cardinality less than θ. Therefore, if $i \in \theta \setminus i_M$, then x_i does not contain any infinite subset of ω that is an element of M. We prove that x_i is forced by T_0 to also not contain $\dot{y}$. This will prove that $\dot{J}$ is bounded by i_M. Let $T_1 < T_0$ be any condition in $\mathbb{L}(\mathfrak{D})$ and let $t_1 = \mathrm{stem}(T_1)$. We show that T_1 does not force that $x_i \supset \dot{y}$. We define the relation $\Vdash_w$ on $T_0 \times \omega$ to be the set

$$\{(t,n) \in T_0 \times \omega : \text{ there is no } T \leq T_0, \mathrm{stem}(T) = t, \text{s.t. } T \Vdash n \notin \dot{y}\} \ .$$

For convenience we may write, for $T \leq T_0$, $T \Vdash_w n \in \dot{y}$ providing $(\mathrm{stem}(T), n)$ is in $\Vdash_w$, and this is equivalent to the relation that T has no stem preserving extension forcing that n is not in $\dot{y}$. Let $T_2 \in M$ be any extension of T_0 with stem t_1. Let L denote the set of $\ell \in \omega$ such that $T_2 \Vdash_w \ell \in \dot{y}$. If L is infinite, then, since $L \in M$, there is an $\ell \in L \setminus x_i$. This implies that T_1 does not force $x_i \supset \dot{y}$, since $T_2 \Vdash_w j \in \dot{y}$ implies that T_1 fails to force that $\ell \notin \dot{y}$.

Therefore we may assume that L is finite and let ℓ be the maximum of L. Define the set $E \subset \mathbb{L}(\mathfrak{D})$ according to $T \in E$ providing that either $t_1 \notin T$ or there is a $j > \ell$ such that $T \Vdash_w j \in \dot{y}$. Again this set E is in M and is easily seen to be a dense subset of $\mathbb{L}(\mathfrak{D})$. By the choice of ℓ, we note that $\rho_E(t_1) > 0$. If $\rho_E(t_1) > 1$, then the set $\{k \in \omega : 0 < \rho_E(t_1^\frown k) < \rho_E(t_1)\}$ is in $\mathfrak{D}^+$ and so there is a k_1 in this set such that $t_1^\frown k_1 \in T_1 \cap T_2$. By a finite induction, we can choose an extension $t_2 \supseteq t_1$ so that $t_2 \in T_1 \cap T_2$ and $\rho_E(t_2) = 1$. Now, there is a set $D \in \mathfrak{D} \cap M$ contained in $\{k : t_2^\frown k \in T_1 \cap T_2\}$ since M contains a base for $\mathfrak{D}$. Also, $D_E = \{k \in D : \rho_E(t_2^\frown k) = 0\}$ is in $\mathfrak{D}^+$. For each $k \in D_E$, choose the minimal j_k so that $T_2^\frown k \Vdash j_k \in \dot{y}$. The set

$\{j_k : k \in D_E\}$ is an element of M. This set is not finite because if it were then there would be a single j such that $\{k \in D_E : j_k = j\} \in \mathcal{D}^+$, which would contradict that $\rho_E(t_2) > 0$. This means that there is a $k \in D_E^+$ with $j_k \notin x_i$, and again we have shown that T_1 fails to force that x_i contains $\dot{y}$. □

3. Matrix Iterations

The terminology "matrix iterations" is used in [3], see also forthcoming preprint (F1222) from the second author. The paper [3] nicely expands on the method of matrix iterated forcing first introduced in [1].

Let us recall that a poset $(P, <_P)$ is a complete suborder of a poset $(Q, <_Q)$ providing $P \subset Q$, $<_P \subset <_Q$, and each maximal antichain of $(P, <_P)$ is also a maximal antichain of $(Q, <_Q)$. Note that it follows that incomparable members of $(P, <_P)$ are still incomparable in $(Q, <_Q)$, i.e. $p_1 \perp_P p_2$ implies $p_1 \perp_Q p_2$. We use the notation $(P, <_P) <\!\circ\, (Q, <_Q)$ to abbreviate the complete suborder relation, and similarly use $P <\!\circ\, Q$ if $<_P$ and $<_Q$ are clear from the context. An element p of P is a reduction of $q \in Q$ if $r \not\perp_Q q$ for each $r <_P p$. If $P \subset Q$, $<_P \subset <_Q$, $\perp_P \subset \perp_Q$, and each element of Q has a reduction in P, then $P <\!\circ\, Q$. The reason is that if $A \subset P$ is a maximal antichain and $p \in P$ is a reduction of $q \in Q$, then there is an $a \in A$ and an r less than both p and a in P, such that $r \not\perp_Q q$.

Definition 3.1. *We will say that an object $\underline{\mathbf{P}}$ is a matrix iteration if there is an infinite cardinal κ and an ordinal γ (thence a (κ, γ)-matrix iteration) such that $\underline{\mathbf{P}} = \langle \langle \mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha} : i \le \kappa, \alpha \le \gamma \rangle, \langle \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha} : i \le \kappa, \alpha < \gamma \rangle \rangle$ where, for each $(i, \alpha) \in \kappa + 1 \times \gamma$ and each $j < i$,*

1. *$\mathbb{P}^{\underline{\mathbf{P}}}_{j,\alpha}$ is a complete suborder of the poset $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}$ (i.e. $\mathbb{P}^{\underline{\mathbf{P}}}_{j,\alpha} <\!\circ\, \mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}$),*
2. *$\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha}$ is a $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}$-name of a ccc poset, $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha+1}$ is equal to $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha} * \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha}$,*
3. *for limit $\delta \le \gamma$, $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\delta}$ is equal to the union of the family $\{\mathbb{P}^{\underline{\mathbf{P}}}_{i,\beta} : \beta < \delta\}$*
4. *$\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa,\alpha}$ is the union of the chain $\{\mathbb{P}^{\underline{\mathbf{P}}}_{j,\alpha} : j < \kappa\}$.*

When the context makes it clear, we omit the superscript $\underline{\mathbf{P}}$ when discussing a matrix iteration. Throughout the paper, κ will be a fixed uncountable regular cardinal

Definition 3.2. *A sequence $\vec{\lambda}$ is κ-tall if $\vec{\lambda} = \langle \mu_\xi, \lambda_\xi : \xi < \kappa \rangle$ is a sequence of pairs of regular cardinals satisfying that $\mu_0 = \omega < \kappa < \lambda_0$ and, for $0 < \eta < \kappa$, $\mu_\eta < \lambda_\eta$ where $\mu_\eta = (2^{\sup\{\lambda_\xi : \xi < \eta\}})^+$.*

Also for the remainder of the paper, we fix a κ-tall sequence $\vec{\lambda}$ and λ will denote the supremum of the set $\{\lambda_\xi : \xi \in \kappa\}$. For simpler notation, whenever we discuss a matrix iteration $\underline{\mathbf{P}}$ we shall henceforth assume that it is a (κ,γ)-matrix iteration for some ordinal γ. We may refer to a forcing extension by $\underline{\mathbf{P}}$ as an abbreviation for the forcing extension by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa,\gamma}$.

For any poset P, any P-name $\dot{D}$, and P-generic filter G, $\dot{D}[G]$ will denote the valuation of $\dot{D}$ by G. For any ground model x, $\check{x}$ denotes the canonical name so that $\check{x}[G] = x$. When x is an ordinal (or an integer) we will suppress the accent in $\check{x}$. A P-name $\dot{D}$ of a subset of ω will be said to be *nice* or *canonical* if for each integer $j \in \omega$, there is an antichain A_j such that $\dot{D} = \bigcup\{\{j\} \times A_j : j \in \omega\}$. We will say that $\dot{\mathfrak{D}}$ is a nice P-name of a family of subsets of ω just to mean that $\dot{\mathfrak{D}}$ is a collection of nice P-names of subsets of ω. We will use $(\dot{\mathfrak{D}})_P$ if we need to emphasize that we mean the P-name. Similarly if we say that $\dot{\mathfrak{D}}$ is a nice P-name of a filter (base) we mean that $\dot{\mathfrak{D}}$ is a nice P-name such that, for each P-generic filter, the collection $\{\dot{D}[G] : \dot{D} \in \dot{\mathfrak{D}}\}$ is a filter (base) of infinite subsets of ω.

Following these conventions, the following notation will be helpful.

Definition 3.3. *For a (κ,γ)-matrix $\underline{\mathbf{P}}$ and $i < \kappa$, we let $\mathbb{B}^{\underline{\mathbf{P}}}_{i,\gamma}$ denote the set of all nice $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$-names of subsets of ω. We note that this then is the nice $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$-name for the power set of ω. As usual, when possible we suppress the $\underline{\mathbf{P}}$ superscript.*

For a nice $\underline{\mathbf{P}}$-name $\dot{\mathfrak{D}}$ of a filter (or filter base) of subsets of ω, we let $(\dot{\mathfrak{D}})^+$ denote the set of all nice $\underline{\mathbf{P}}$-names that are forced to meet every member of $\dot{\mathfrak{D}}$. It follows that $(\dot{\mathfrak{D}})^+$ is the nice $\underline{\mathbf{P}}$-name for the usual defined notion $(\dot{\mathfrak{D}})^+$ in the forcing extension by $\underline{\mathbf{P}}$. We let $\langle\dot{\mathfrak{D}}\rangle$ denote the nice $\underline{\mathbf{P}}$-name of the filter generated by $\dot{\mathfrak{D}}$. We use the same notational conventions if, for some poset $\mathbb{P}$, $\dot{\mathfrak{D}}$ is a nice $\mathbb{P}$-name of a filter (or filter base) of subsets of ω.

The main idea for controlling the splitting number in the extension by $\underline{\mathbf{P}}$ will involve having many of the subposets being θ-Luzin preserving for $\theta \in \{\lambda_\xi : \xi \in \kappa\}$. Motivated by the fact that posets of the form $\mathbb{L}(\mathfrak{D})$ (our proposed iterands) are θ-Luzin preserving when $\mathfrak{D}$ is sufficiently small we adopt the name $\vec{\lambda}$-thin for this next notion.

Definition 3.4. *For a κ-tall sequence $\vec{\lambda}$, we will say that a (κ,γ)-matrix-iteration $\underline{\mathbf{P}}$ is $\vec{\lambda}$-thin providing that for each $\xi < \kappa$ and $\alpha \leq \gamma$, $\mathbb{P}^{\underline{\mathbf{P}}}_{\xi,\alpha}$ is λ_ξ-Luzin preserving.*

Now we combine the notion of $\vec{\lambda}$-thin matrix-iteration with Lemma 2.2. We adopt Kunen's notation that for a set I, $\mathrm{Fn}(I,2)$ denotes the usual poset for adding Cohen reals (finite partial functions from I into 2 ordered by superset).

Lemma 3.5. *Suppose that* $\underline{\mathbf{P}}$ *is a* $\vec{\lambda}$*-thin* (κ,γ)*-matrix iteration for some* κ*-tall sequence* $\vec{\lambda}$*. Further suppose that* $\dot{\mathbb{Q}}_{i,0}$ *is the* $\mathbb{P}_{i,0}$*-name of the poset* $\mathrm{Fn}(\lambda_\xi,2)$ *for each* $\xi\in\kappa$*, and therefore* $\mathbb{P}_{\kappa,1}$ *is isomorphic to* $\mathrm{Fn}(\lambda,2)$*. Let* $\dot{g}$ *denote the generic function from* λ *onto* 2 *added by* $\mathbb{P}_{\kappa,1}$ *and, for* $i<\lambda$*, let* $\dot{x}_i$ *be the canonical name of the set* $\{n\in\omega : \dot{g}(i+n)=1\}$*. Then the family* $\{\dot{x}_i : i<\lambda\}$ *is forced by* $\underline{\mathbf{P}}$ *to be a splitting family.*

Proof. Let $G_{\kappa,\gamma}$ be a $\mathbb{P}_{\kappa,\gamma}$-generic filter. For each $\xi\in\kappa$ and $\alpha\le\gamma$, let $G_{\xi,\alpha}=G_{\kappa,\gamma}\cap\mathbb{P}_{\xi,\alpha}$. Let $\dot{y}$ be any nice $\mathbb{P}_{\kappa,\gamma}$-name for a subset of ω. Since $\dot{y}$ is a countable name, we may choose a $\xi<\kappa$ so that $\dot{y}$ is a $\mathbb{P}_{\xi,\gamma}$-name. It is easily shown, and very well-known, that the family $\{\dot{x}_i : i<\lambda_\xi\}$ is forced by $\mathbb{P}_{\xi,1}$ (i.e. $\mathrm{Fn}(\lambda_\xi,2)$) to be a λ_ξ-Luzin family. By the hypothesis that $\underline{\mathbf{P}}$ is $\vec{\lambda}$-thin, we have, by Lemma 2.2, that $\{\dot{x}_i : i<\lambda_\xi\}$ is still λ_ξ-Luzin in $V[G\cap\mathbb{P}_{\xi,\gamma}]$. Since $\dot{y}$ is a $\mathbb{P}_{\xi,\gamma}$-name, there is an $i<\lambda_\xi$ such that $\dot{y}[G_{\xi,\gamma}]\cap\dot{x}_i[G_{\xi,\gamma}]$ and $\dot{y}[G_{\xi,\gamma}]\setminus\dot{x}_i[G_{\xi,\gamma}]$ are infinite. □

4. The construction of $\underline{\mathbf{P}}$

When constructing a matrix-iteration by recursion, we will need notation and language for extension. We will use, for an ordinal γ, $\underline{\mathbf{P}}^\gamma$ to indicate that $\underline{\mathbf{P}}^\gamma$ is a (κ,γ)-matrix iteration.

Definition 4.1.
(1) *A matrix iteration* $\underline{\mathbf{P}}^\gamma$ *is an extension of* $\underline{\mathbf{P}}^\delta$ *providing* $\delta\le\gamma$*, and, for each* $\alpha\le\delta$ *and* $i\le\kappa$*,* $\mathbb{P}^{\underline{\mathbf{P}}^\delta}_{i,\alpha}=\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\alpha}$*. We can use* $\underline{\mathbf{P}}^\gamma\restriction\delta$ *to denote the unique* (κ,δ)*-matrix iteration extended by* $\underline{\mathbf{P}}^\gamma$*.*
(2) *If, for each* $i<\kappa$*,* $\dot{\mathbb{Q}}_{i,\gamma}$ *is a* $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$*-name of a ccc poset satisfying that, for each* $i<j<\kappa$*,* $\mathbb{P}_{i,\gamma}*\dot{\mathbb{Q}}_{i,\gamma}$ *is a complete subposet of* $\mathbb{P}_{j,\gamma}*\dot{\mathbb{Q}}_{j,\gamma}$*, then we let* $\underline{\mathbf{P}}*\langle\dot{\mathbb{Q}}_{i,\gamma} : i<\kappa\rangle$ *denote the* $(\kappa,\gamma+1)$*-matrix* $\langle\langle\mathbb{P}_{i,\alpha} : i\le\kappa,\alpha\le\gamma+1\rangle,\langle\dot{\mathbb{Q}}_{i,\alpha} : i\le\kappa,\alpha<\gamma+1\rangle\rangle$*, where* $\dot{\mathbb{Q}}_{\kappa,\gamma}$ *is the* $\underline{\mathbf{P}}$*-name of the union of* $\{\dot{\mathbb{Q}}_{i,\gamma} : i<\kappa\}$ *and, for* $i\le\kappa$*,* $\mathbb{P}_{i,\gamma}=\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$*,* $\mathbb{P}_{i,\gamma+1}=\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}*\dot{\mathbb{Q}}_{i,\gamma}$*, and for* $\alpha<\gamma$*,* $(\mathbb{P}_{i,\alpha},\dot{\mathbb{Q}}_{i,\alpha})=(\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha},\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha})$*.*

The following, from [3, Lemma 3.10], shows that extension at limit steps is canonical.

Lemma 4.2. *If γ is a limit and if $\{\underline{\mathbf{P}}^\delta : \delta < \gamma\}$ is a sequence of matrix iterations satisfying that for $\beta < \delta < \gamma$, $\underline{\mathbf{P}}^\delta \restriction \beta = \underline{\mathbf{P}}^\beta$, then there is a unique matrix iteration $\underline{\mathbf{P}}^\gamma$ such that $\underline{\mathbf{P}}^\gamma \restriction \delta = \underline{\mathbf{P}}^\delta$ for all $\delta < \gamma$.*

Proof. For each $\delta < \gamma$ and $i < \kappa$, we define $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$ to be $\mathbb{P}^{\underline{\mathbf{P}}^\delta}_{i,\delta}$ and $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$ to be $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^{\delta+1}}_{i,\delta}$. It follows that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$ is a $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$-name. Since γ is a limit, the definition of $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\gamma}$ is required to be $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\delta} : \delta < \gamma\}$ for $i < \kappa$. Similarly, the definition of $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\kappa,\gamma}$ is required to be $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\gamma} : i < \kappa\}$. Let us note that $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\kappa,\gamma}$ is also required to be the union of the chain $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\kappa,\delta} : \delta < \gamma\}$, and this holds by assumption on the sequence $\{\underline{\mathbf{P}}^\delta : \delta < \gamma\}$.

To prove that $\underline{\mathbf{P}}^\gamma$ is a (κ, γ)-matrix it remains to prove that for $j < i \leq \kappa$, and each $q \in \mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\gamma}$, there is a reduction p in $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{j,\gamma}$. Since γ is a limit, there is an $\alpha < \gamma$ such that $q \in \mathbb{P}^{\underline{\mathbf{P}}^\alpha}_{i,\alpha}$ and, by assumption, there is a reduction, p, of q in $\mathbb{P}^{\underline{\mathbf{P}}^\alpha}_{j,\alpha}$. By induction on β ($\alpha \leq \beta \leq \gamma$) we note that $q \in \mathbb{P}^{\underline{\mathbf{P}}^\beta}_{i,\beta}$ and that p is a reduction of q in $\mathbb{P}^{\underline{\mathbf{P}}^\beta}_{j,\beta}$. For limit β it is trivial, and for successor β it follows from condition (1) in the definition of matrix iteration. $\square$

We also will need the next result taken from [3, Lemma 13], which they describe as well known, for stepping diagonally in the array of posets.

Lemma 4.3. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders such that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion and let $\dot{\mathbb{B}}$ be a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subset \dot{\mathbb{B}}$, and every $\mathbb{P}$-name of a maximal antichain of $\dot{\mathbb{A}}$ is also forced by $\mathbb{Q}$ to be a maximal antichain of $\dot{\mathbb{B}}$. Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$*

Let us also note if $\dot{\mathbb{B}}$ is equal to $\dot{\mathbb{A}}$ in Lemma 4.3, then the hypothesis and the conclusion of the Lemma are immediate. On the other hand, if $\dot{\mathbb{A}}$ is the $\mathbb{P}$-name of $\mathbb{L}(\dot{\mathfrak{D}})$ for some $\mathbb{P}$-name of a filter $\dot{\mathfrak{D}}$, then the $\mathbb{Q}$-name of $\mathbb{L}(\dot{\mathfrak{D}})$ is not necessarily equal to $\dot{\mathbb{A}}$.

Lemma 4.4 ([6, 1.9])**.** *Suppose that $\mathbb{P}, \mathbb{Q}$ are posets with $\mathbb{P} \lessdot \mathbb{Q}$. Suppose also that $\dot{\mathfrak{D}}_0$ is a $\mathbb{P}$-name of a filter on ω and $\dot{\mathfrak{D}}_1$ is a $\mathbb{Q}$-name of a filter on ω. If $\Vdash_{\mathbb{Q}} \dot{\mathfrak{D}}_0 \subseteq \dot{\mathfrak{D}}_1$ then $\mathbb{P} * \mathbb{L}(\dot{\mathfrak{D}}_0)$ is a complete subposet of $\mathbb{Q} * \mathbb{L}(\dot{\mathfrak{D}}_1)$ if either of the two equivalent conditions hold:*

(1) $\Vdash_{\mathbb{Q}} ((\dot{\mathfrak{D}}_0)^+)_{\mathbb{P}} \subseteq \dot{\mathfrak{D}}_1^+$,
(2) $\Vdash_{\mathbb{Q}} \dot{\mathfrak{D}}_1 \cap V^{\mathbb{P}} \subseteq \langle \dot{\mathfrak{D}}_0 \rangle$ (where $V^{\mathbb{P}}$ is the class of $\mathbb{P}$-names).

Proof. Let $\dot{E}$ be any $\mathbb{P}$-name of a maximal antichain of $\mathbb{L}(\dot{\mathfrak{D}}_0)$. By Lemma 4.3, it suffices to show that $\mathbb{Q}$ forces that every member of

$\mathbb{L}(\dot{\mathfrak{D}}_1)$ is compatible with some member of $\dot{E}$. Let G be any $\mathbb{Q}$-generic filter and let E denote the valuation of $\dot{E}$ by $G \cap \mathbb{P}$. Working in the model $V[G \cap \mathbb{P}]$, we have the function ρ_E as in Lemma 2.4. Choose $\delta \in \omega_1$ satisfying that $\rho_E(t) < \delta$ for all $t \in \omega^{<\omega}$. Now, working in $V[G]$, we consider any $T \in \mathbb{L}(\dot{\mathfrak{D}}_1)$ and we find an element of E that is compatible with T. In fact, by induction on $\alpha < \delta$, one easily proves that for each $T \in \mathbb{L}(\dot{\mathfrak{D}}_1)$ with $\rho_E(\mathrm{stem}(T)) \leq \alpha$, T is compatible with some member of E. □

Definition 4.5. *For a (κ, γ)-matrix-iteration $\underline{\mathbf{P}}$, and ordinal $i_\gamma < \kappa$, we say that an increasing sequence $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases, if for each $i < j < \kappa$*

(1) $\dot{\mathfrak{D}}_i$ is a subset of $\mathbb{B}_{i,\gamma}$ (hence a nice $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$-name)
(2) $\Vdash_{\mathbb{P}_{i,\gamma}} \dot{\mathfrak{D}}_i$ is a filter with a base of cardinality at most μ_{i_γ},
(3) $\Vdash_{\mathbb{P}_{j,\gamma}} \langle \dot{\mathfrak{D}}_j \rangle \cap \mathbb{B}_{i,\gamma} \subseteq \langle \dot{\mathfrak{D}}_i \rangle$.

Notice that a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases can be (essentially) eventually constant. Thus we will say that a sequence $\langle \dot{\mathfrak{D}}_i : i \leq j \rangle$ (for some $j < \kappa$) is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases if the sequence $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases where $\dot{\mathfrak{D}}_i$ is the $\mathbb{P}_{i,\gamma}$-name for $\mathbb{B}_{i,\gamma} \cap \langle \dot{\mathfrak{D}}_j \rangle$ for $j < i \leq \kappa$. When $\underline{\mathbf{P}}$ is clear from the context, we will use $\vec{\lambda}(i_\gamma)$-thin as an abbreviation for $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin.

Corollary 4.6. *For a (κ, γ)-matrix-iteration $\underline{\mathbf{P}}$, ordinal $i_\gamma < \kappa$, and a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases $\langle \dot{\mathfrak{D}}_\xi : i < \kappa \rangle$, $\underline{\mathbf{P}} * \langle \dot{\mathbb{Q}}_{i,\gamma} : i \leq \kappa \rangle$ is a $\gamma + 1$-extension of $\underline{\mathbf{P}}$, where, for each $i \leq i_\gamma$, $\dot{\mathbb{Q}}_{i,\gamma}$ is the trivial poset, and for $i_\gamma \leq i < \kappa$, $\dot{\mathbb{Q}}_{i,\gamma}$ is $\mathbb{L}(\dot{\mathfrak{D}}_i)$.*

Definition 4.7. *Whenever $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases, let $\underline{\mathbf{P}} * \mathbb{L}(\langle \dot{\mathfrak{D}}_i : i_\gamma \leq i < \kappa \rangle)$ denote the $\gamma + 1$-extension described in Corollary 4.6.*

This next corollary is immediate.

Corollary 4.8. *If $\underline{\mathbf{P}}$ is a $\vec{\lambda}$-thin (κ, γ)-matrix and if $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases, then $\underline{\mathbf{P}} * \mathbb{L}(\langle \dot{\mathfrak{D}}_i : i_\gamma \leq i < \kappa \rangle)$ is a $\vec{\lambda}$-thin $(\kappa, \gamma + 1)$-matrix.*

We now describe a first approximation of the scheme, $\mathcal{K}(\vec{\lambda})$, of posets that we will be using to produce the model.

Definition 4.9. *For an ordinal $\gamma > 0$ and a (κ, γ)-matrix iteration $\underline{\mathbf{P}}$, we will say that $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ providing for each $0 < \alpha < \gamma$,*

(1) *for each* $i \leq \kappa$, $\mathbb{P}^{\underline{\mathbf{P}}}_{i,1}$ *is* $\mathrm{Fn}(\lambda_i, 2)$, *and*
(2) *there is an* $i_\alpha = i^{\underline{\mathbf{P}}}_\alpha < \kappa$ *and a* $(\underline{\mathbf{P}} \restriction \alpha, \vec{\lambda}(i_\alpha))$*-thin sequence* $\langle \dot{\mathfrak{D}}^\alpha_i : i < \kappa \rangle$ *of filter bases, such that* $\underline{\mathbf{P}} \restriction \alpha + 1$ *is equal to* $\underline{\mathbf{P}} \restriction \alpha * \mathbb{L}(\langle \dot{\mathfrak{D}}^\alpha_i : i_\alpha \leq i < \kappa \rangle)$.

For each $0 < \alpha < \gamma$*, we let* $\dot{\mathfrak{D}}^\alpha_\kappa$ *denote the* $\underline{\mathbf{P}} \restriction \alpha$*-name of the union* $\bigcup \{ \dot{\mathfrak{D}}^\alpha_i : i_\alpha \leq i < \kappa \}$*, and we let* $\dot{L}_\alpha$ *denote the canonical* $\underline{\mathbf{P}} \restriction \alpha{+}1$*-name of the subset of* ω *added by* $\mathbb{L}(\mathfrak{D}^\alpha_\kappa)$.

Let us note that each $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ is $\vec{\lambda}$-thin. Furthermore, by Lemma 3.5, this means that each $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ forces that $\mathfrak{s} \leq \lambda$. We begin a new section for the task of proving that there is a $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ that forces that $s \geq \lambda$.

It will be important to be able to construct $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequences of filter bases, and it seems we will need some help.

Definition 4.10. *For an ordinal* $\gamma > 0$ *and a* (κ, γ)*-matrix iteration* $\underline{\mathbf{P}}$ *we will say that* $\underline{\mathbf{P}} \in \mathcal{H}(\vec{\lambda})$ *if* $\underline{\mathbf{P}}$ *is in* $\mathcal{K}(\vec{\lambda})$ *and for each* $0 < \alpha < \gamma$*, if* $i_\alpha = i^{\underline{\mathbf{P}}}_\alpha > 0$ *then* $\omega_1 \leq \mathrm{cf}(\alpha) \leq \mu_{i_\alpha}$ *and there is a* $\beta_\alpha < \alpha$ *such that*
(1) *for* $\beta_\alpha \leq \xi < \alpha$, $i_\xi \in \{0, i_\alpha\}$,
(2) *if* $\beta_\alpha \leq \eta < \alpha$, $i_\eta > 0$ *and* $\xi = \eta + \omega_1 \leq \alpha$*, then* $\dot{L}_\eta \in \dot{\mathcal{D}}^\xi_{i_\xi}$*, and* $\mathbb{P}_{i_\xi,\xi} \Vdash \dot{\mathcal{D}}^\alpha_{i_\xi}$ *has a descending mod finite base of cardinality* ω_1,
(3) *if* $\beta_\alpha < \xi \leq \alpha$, $i_\xi > 0$*, and* $\eta + \omega_1 < \xi$ *for* $\eta < \xi$*, then* $\{\dot{L}_\eta : \beta_\alpha \leq \eta < \alpha, \mathrm{cf}(\eta) \geq \omega_1\}$ *is a base for* $\dot{\mathcal{D}}^\xi_{i_\xi}$.

5. Producing $\vec{\lambda}$-thin filter sequences

In this section we prove this main lemma.

Lemma 5.1. *Suppose that* $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ *and that* $\mathcal{Y}$ *is a set of fewer than* λ *nice* $\underline{\mathbf{P}}^\gamma$*-names of subsets of* ω*, then there is a* $\delta < \gamma + \lambda$ *and an extension* $\underline{\mathbf{P}}^\delta$ *of* $\underline{\mathbf{P}}^\gamma$ *in* $\mathcal{H}(\vec{\lambda})$ *that forces that the family* $\mathcal{Y}$ *is not a splitting family.*

The main theorem follows easily.

Proof of Theorem 1.2. Let θ be any regular cardinal so that $\theta^{<\lambda} = \theta$ (for example, $\theta = (2^\lambda)^+$). Construct $\underline{\mathbf{P}}^\theta \in \mathcal{H}(\vec{\lambda})$ so that for all $\mathcal{Y} \subset \mathbb{B}_{\kappa,\theta}$ with $|\mathcal{Y}| < \lambda$, there is a $\gamma < \delta < \theta$ so that $\mathcal{Y} \subset \mathbb{B}_{\kappa,\gamma}$ and, by applying Lemma 5.1, such that $\underline{\mathbf{P}}^\theta \restriction \delta$ forces that $\mathcal{Y}$ is not a splitting family. $\square$

We begin by reducing our job to simply finding a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence.

Definition 5.2. *For a (κ,γ)-matrix-iteration $\underline{\mathbf{P}}^\gamma$, we say that a subset $\mathcal{E}$ of $\mathbb{B}_{\kappa,\gamma}$ is $(\underline{\mathbf{P}}^\gamma,\vec{\lambda}(i_\gamma))$-thin filter subbase if, $i_\gamma<\kappa$, $|\mathcal{E}|\le\mu_{i_\gamma}$, and the sequence $\langle\langle\mathcal{E}\cap\mathbb{B}_{i,\gamma}\rangle : i<\kappa\rangle$ is a $(\underline{\mathbf{P}}^\gamma,\vec{\lambda}(i_\gamma))$-thin sequence of filter bases.*

Lemma 5.3. *For any $\underline{\mathbf{P}}^\gamma\in\mathcal{H}(\vec{\lambda})$, and any $(\underline{\mathbf{P}}^\gamma,\vec{\lambda}(i_\gamma))$-thin filter base $\mathcal{E}$, there is an $\alpha\le\gamma+\mu_{i_\gamma}+1$ and extensions $\underline{\mathbf{P}}^\alpha,\underline{\mathbf{P}}^{\alpha+1}$ of $\underline{\mathbf{P}}^\gamma$ in $\mathcal{H}(\vec{\lambda})$, such that, $\underline{\mathbf{P}}^{\alpha+1}=\underline{\mathbf{P}}^\alpha*\mathbb{L}(\langle\dot{\mathfrak{D}}^\alpha_i : i_\alpha\le i<\kappa\rangle)$ and $\underline{\mathbf{P}}^\alpha$ forces that $\mathcal{E}\cap\mathbb{B}_{i,\gamma}$ is a subset of $\dot{\mathfrak{D}}^\alpha_i$ for all $i<\kappa$.*

Proof. The case $i_\gamma=0$ is trivial, so we assume $i_\gamma>0$. There is no loss of generality to assume that $\mathcal{E}\cap\mathbb{B}_{i_\gamma,\gamma}$ has character μ_{i_γ}. Let $\{\dot{E}_\xi:\xi<\mu_{i_\gamma}\}\subset\mathcal{E}\cap\mathbb{B}_{i_\gamma,\gamma}$ enumerate a filter base for $\langle\mathcal{E}\rangle\cap\mathbb{B}_{i_\gamma,\gamma}$. We can assume that this enumeration satisfies that $\dot{E}_\xi\setminus\dot{E}_{\xi+1}$ is forced to be infinite for all $\xi<\mu_{i_\gamma}$. Let $\mathcal{A}$ be any countably generated free filter on ω that is not principal mod finite. By induction on $\xi<\mu_{i_\gamma}$ we define $\underline{\mathbf{P}}^{\gamma+\xi}$ by simply defining $i_{\gamma+\xi}$ and the sequence $\langle\dot{\mathfrak{D}}^{\gamma+\xi}_i : i_{\gamma+\xi}\le i\le\kappa\rangle$. We will also recursively define, for each $\xi<\mu_{i_\gamma}$, a $\underline{\mathbf{P}}^{\gamma+\xi}$-name $\dot{D}_\xi$ such that $\underline{\mathbf{P}}^{\gamma+\xi}$ forces that $\dot{D}_\xi\subset\dot{E}_\xi$. An important induction hypothesis is that $\{\dot{D}_\eta:\eta<\xi\}\cup\{\dot{E}_\zeta:\zeta<\mu_{i_\gamma}\}\cup\mathcal{E}$ is forced to have the finite intersection property.

For each $\xi<\gamma+\omega_1$, let $i_\xi=0$ and $\dot{\mathfrak{D}}^\xi_i$ be the $\underline{\mathbf{P}}^\xi$-name $\langle\mathcal{A}\rangle\cap\mathbb{B}_{i,\xi}$ for all $i\le\kappa$. The definition of $\dot{D}_0$ is simply $\dot{E}_0$. By recursion, for each $\eta<\omega_1$ and $\xi=\eta+1$, we define $\dot{D}_\xi$ to be the intersection of $\dot{D}_\eta$ and $\dot{E}_\xi$. For limit $\xi<\omega_1$, we note that $\mathbb{P}_{i_\gamma,\xi}$ forces that $\mathbb{L}(\langle\mathcal{A}\rangle)$ is isomorphic to $\mathbb{L}(\langle\{\dot{D}_\eta\cap\dot{E}_\xi:\eta<\xi\}\rangle)$. Therefore, we can let $\dot{D}_\xi$ be a $\underline{\mathbf{P}}^{\xi+1}$-name for the generic real added by $\mathbb{L}(\langle\{\dot{D}_\eta\cap\dot{E}_\xi:\eta<\xi\}\rangle)$. A routine density argument shows that this definition satisfies the induction hypothesis.

The definition of $i_{\gamma+\omega_1}$ is i_γ and the definition of $\dot{\mathfrak{D}}^{\gamma+\omega_1}_{i_\gamma}$ is the filter generated by $\{\dot{D}_\xi:\xi<\omega_1\}$. The definition of $\dot{D}_{\omega_1}$ is $\dot{L}_{\gamma+\omega_1}$.

Let S denote the set of $\eta<\mu_{i_\gamma}$ with uncountable cofinality. We now add additional induction hypotheses:

(1) if $\zeta=\sup(S\cap\xi)<\xi$ and $\xi=\nu+1$, then $\dot{D}_\xi=\dot{D}_\nu\cap\dot{E}_\xi$, and $i_\xi=0$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_i=\langle\mathcal{A}\rangle$ for all $i\le\kappa$
(2) if $\zeta=\sup(S\cap\xi)<\xi$ and ξ is a limit of countable cofinality, then $i_\xi=0$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_i=\langle\mathcal{A}\rangle$ for all $i\le\kappa$, and $\dot{D}_\xi$ is forced by $\underline{\mathbf{P}}^{\gamma+\xi+1}$ to be the generic real added by $\mathbb{L}(\{\dot{D}_\eta\cap\dot{E}_\xi:\zeta\le\eta<\xi\})$,
(3) if $\zeta=\sup(S\cap\xi)$ and $\xi=\zeta+\omega_1$, then $i_\xi=i_\gamma$, $\dot{\mathfrak{D}}^{\gamma+\xi}_{i_\xi}$ is the filter generated by $\{\dot{E}_\xi\cap\dot{D}_\eta:\zeta\le\eta<\xi\}$ and $\dot{D}_\xi$ is $\dot{L}_{\gamma+\xi}$,

(4) if $S \cap \xi$ is cofinal in ξ and $\mathrm{cf}(\xi) > \omega$, then $i_\xi = i_\gamma$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_{i_\xi}$ is the filter generated by $\{\dot{D}_{\gamma+\eta} : \eta \in S \cap \xi\}$ and $\dot{D}_\xi = \dot{L}_{\gamma+\xi}$,

(5) if $S \cap \xi$ is cofinal in ξ and $\mathrm{cf}(\xi) = \omega$, then $i_\xi = 0$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_i = \langle \mathcal{A} \rangle$ for all $i \leq \kappa$, and $\dot{D}_\xi$ is forced by $\underline{\mathbf{P}}^{\gamma+\xi+1}$ to be the generic real added by $\mathbb{L}(\{\dot{D}_{\eta_n} \cap \dot{E}_\xi : n \in \omega\})$, where $\{\eta_n : n \in \omega\}$ is some increasing cofinal subset of $S \cap (\gamma, \xi)$.

It should be clear that the induction continues to stage μ_{i_γ} and that $\underline{\mathbf{P}}^{\gamma+\xi} \in \mathcal{H}(\vec{\lambda}(i_\gamma))$ for all $\xi \leq \mu_{i_\gamma}$, with $\beta_{\gamma_\xi} = \gamma$ being the witness to Definition 4.10 for all ξ with $\mathrm{cf}(\xi) > \omega$.

The final definition of the sequence $\langle \dot{\mathfrak{D}}^\delta_i : i_\delta = i_\gamma \leq i \leq \kappa \rangle$, where $\delta = \gamma + \mu_{i_\gamma}$ is that $\dot{\mathfrak{D}}^\delta_{i_\gamma}$ is the filter generated by $\{\dot{L}_{\gamma+\xi} : \mathrm{cf}(\xi) > \omega\}$, and for $i_\gamma < i \leq \kappa$, $\dot{\mathfrak{D}}^\delta_i$ is the filter generated by $\dot{\mathfrak{D}}^\delta_{i_\gamma} \cup (\mathcal{E} \cap \mathbb{B}_{i,\gamma}$. □

Lemma 5.4. *Suppose that $\mathcal{E}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter base. Also assume that $i < \kappa$ and $\alpha \leq \gamma$ and $\mathcal{E}_1 \subset \mathbb{B}_{i,\alpha}$ is a $(\underline{\mathbf{P}}^\alpha, \vec{\lambda}(i_\gamma))$-thin filter base satisfying that $\langle \mathcal{E} \rangle \cap \mathbb{B}_{i,\alpha} \subset \langle \mathcal{E}_1 \rangle$, then $\mathcal{E} \cup \mathcal{E}_1$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase.*

Proof. Let $\mathcal{E}_2$ be equal to $\mathcal{E} \cup \mathcal{E}_1$. The fact that each member of the sequence $\langle \dot{\mathcal{D}}_j = \langle \mathcal{E}_2 \cap \mathbb{B}_{j,\gamma} \rangle : j < \kappa \rangle$ is a name of a filter base with character at most μ_{i_γ} is immediate. Now we verify that if $j_1 < j_2 < \kappa$, then $\Vdash_{\mathbb{P}_{j_2,\gamma}} \dot{\mathcal{D}}_{j_2} \cap \mathbb{B}_{j_1,\gamma} \subset \dot{\mathcal{D}}_{j_1}$. Let $\dot{b} \in \mathbb{B}_{j_2,\gamma}$ and suppose there are $p \in \mathbb{P}_{j_2,\gamma}$, $\dot{E}_0 \in \mathcal{E} \cap \mathbb{B}_{j_2,\gamma}$, and $\dot{E}_1 \in \mathcal{E}_1$ such that $p \Vdash b \cap \dot{E}_0 \cap \dot{E}_1$. It suffices to produce an $\dot{E} \in \langle \mathcal{E}_2 \rangle \cap \mathbb{B}_{j_1,\gamma}$ satisfying that $p \Vdash \dot{b} \cap \dot{E} = \emptyset$. First, using that $\mathcal{E}$ is $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin, choose $\dot{E}_2 \in \langle \mathcal{E} \rangle \cap \mathbb{B}_{j_1,\gamma}$ such that $p \Vdash (\dot{b} \setminus \dot{E}_0) \cap \dot{E}_2 = \emptyset$. Equivalently, we have that $p \Vdash (\dot{b} \cap \dot{E}_2) \subset \dot{E}_0$, and therefore $p \Vdash (\dot{b} \cap \dot{E}_2) \cap \dot{E}_1 = \emptyset$. Since $\dot{E}_1$ is a $\mathbb{P}_{j_2,\alpha}$-name, there is a $\mathbb{P}_{j_1,\alpha}$-name (which we can denote as) $(\dot{b} \cap \dot{E}_2) \restriction \alpha$ satisfying that $p \Vdash \dot{E}_2 \cap (\dot{b} \cap \dot{E}_2) \restriction \alpha$ is empty and that $p \Vdash (\dot{b} \cap \dot{E}_2) \subset (\dot{b} \cap \dot{E}_2) \restriction \alpha$. Now using that $\mathcal{E}_1$ is $(\underline{\mathbf{P}}^\alpha, \vec{\lambda}(i_\gamma))$-thin, choose $\dot{E}_3 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{j_1,\alpha}$ so that $p \Vdash \dot{E}_3 \cap (\dot{b} \cap \dot{E}_2) \restriction \alpha$ is empty. Naturaly we have that $p \Vdash \dot{E}_3 \cap (\dot{b} \cap \dot{E}_2)$ is also empty. This completes the proof since $\dot{E}_2 \cap \dot{E}_3$ is in $\langle \mathcal{E}_2 \rangle \cap \mathbb{B}_{j_1,\gamma}$. □

Let $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ and let $\dot{y} \in \mathbb{B}_{\kappa,\gamma}$. For a family $\mathcal{E} \subset \mathbb{B}_{\kappa,\gamma}$ and condition $p \in \underline{\mathbf{P}}^\gamma$ say that p forces that $\mathcal{E}$ measures $\dot{y}$ if $p \Vdash_{\underline{\mathbf{P}}^\gamma} \{\dot{y}, \omega \setminus \dot{y}\} \cap \langle \mathcal{E} \rangle \neq \emptyset$. Naturally we will just say that $\mathcal{E}$ measures $\dot{y}$ if 1 forces that $\mathcal{E}$ measures $\dot{y}$.

Given Lemma 5.3, it will now suffice to prove.

Lemma 5.5. *If $\mathcal{Y} \subset \mathbb{B}_{\kappa,\gamma}$ for some $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ and $|\mathcal{Y}| \le \mu_{i_\gamma}$ for some $i_\gamma < \kappa$, then there is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter $\mathcal{E} \subset \mathbb{B}_{\kappa,\gamma}$ that measures every element of $\mathcal{Y}$.*

In fact, to prove Lemma 5.5, it is evidently sufficient to prove:

Lemma 5.6. *If $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$, $\dot{y} \in \mathbb{B}_{\kappa,\gamma}$, and if $\mathcal{E}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter, then there is a family $\mathcal{E}_1 \supset \mathcal{E}$ measuring $\dot{y}$ that is also a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter.*

Proof. Throughout the proof we suppress mention of $\underline{\mathbf{P}}^\gamma$ and refer instead to component member posets $\mathbb{P}_{i,\alpha}, \dot{\mathbb{Q}}_{i,\alpha}$ of $\underline{\mathbf{P}}^\gamma$. Let $i_{\dot{y}}$ be minimal such that $\dot{y}$ is in $\mathbb{B}_{i_{\dot{y}},\gamma}$. Proceeding by induction, we can assume that the lemma holds for all $\dot{x} \in \mathbb{B}_{j,\gamma}$ and all $j < i_{\dot{y}}$.

We can replace $\dot{y}$ by any $\dot{x} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ that has the property that $1 \Vdash \dot{x} \in \{\dot{y}, \omega \setminus \dot{y}\}$ since if we measure $\dot{x}$ then we also measure $\dot{y}$. With this reduction then we can assume that no condition forces that $\omega \setminus \dot{y}$ is in the filter generated by $\mathcal{E}$.

Fact 1. If $i_{\dot{y}} \le i_\gamma$, then there is a $\dot{E} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ such that $\mathcal{E} \cup \{\dot{E}\}$ is contained a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter that measures $\dot{y}$.

Proof of Fact 1. It is immediate that $\langle \{\dot{y}\} \cup (\mathbb{B}_{i_{\dot{y}},\gamma} \cap \mathcal{E})\rangle$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter. Therefore, by Lemma 5.4, $\mathcal{E} \cup \{\dot{y}\}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase. □

We may thus assume that $0 < i_{\dot{y}}$ and that the Lemma has been proven for all members of $\mathbb{B}_{i,\gamma}$ for all $i < i_{\dot{y}}$. Similarly, let $\alpha_{\dot{y}}$ be minimal so that $\dot{y} \in \mathbb{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$, and assume that the Lemma has been proven for all members of $\mathbb{B}_{i_{\dot{y}},\beta}$ for all $\beta < \alpha_{\dot{y}}$. We skip proving the easy case when $\alpha_{\dot{y}} = 1$ and henceforth assume that $1 < \alpha_{\dot{y}}$. Notice also that $\alpha_{\dot{y}}$ has countable cofinality since $\mathbb{P}_{i_{\dot{y}},\gamma}$ is ccc.

Now choose an elementary submodel M of $H((2^{\lambda\cdot\gamma})^+)$ containing $\vec{\lambda}, \underline{\mathbf{P}}^\gamma, \mathcal{E}, \dot{y}$ and so that M has cardinality equal to μ_{i_γ} and, by our cardinal assumptions, $M^{\lambda_j} \subset M$ for all $j < i_\gamma$. Naturally this implies that $M^\omega \subset M$.

By the inductive assumption we may assume that there is an $\mathcal{E}_1 \supset \mathcal{E}$ that is $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin and measures every element of $M \cap \mathbb{B}_{j,\gamma}$ for $j < i_{\dot{y}}$ as well as every element of $M \cap \mathbb{B}_{i_{\dot{y}},\beta}$ for all $\beta \in M \cap \alpha_{\dot{y}}$. Moreover, it is easily checked that we can assume that $\mathcal{E}_1$ is a subset of M. Furthermore, we may assume that $\mathcal{E}_1$ contains a maximal family of subsets of $M \cap \mathbb{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$ that forms a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase.

Fact 2. There is a maximal antichain $A \subset \mathbb{P}_{i_{\dot{y}},\gamma}$ and a subset $A_1 \subset A$ such that

(1) each $p \in A_1$ forces that $\mathcal{E}_1$ measures $\dot{y}$,
(2) for each $p \in A \setminus A_1$, p forces that there is an $i_p < i_{\dot{y}}$ such that $\mathbb{B}_{i_p,\gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{y}\}\rangle$ is not generated by the elements in M,
(3) for each $p \in A \setminus A_1$, p forces that there is a $j_p < i_{\dot{y}}$ such that $i_p \leq j_p$ and $\mathbb{B}_{j_p,\gamma} \cap \langle \mathcal{E}_1 \cup \{\omega \setminus \dot{y}\}\rangle$ is not generated by the elements in M.

Proof of Fact 2. Suppose that $p \in \mathbb{P}_{i_{\dot{y}},\gamma}$ forces that the conclusion (2) fails. We have already arranged that $p \Vdash_{\mathbb{P}_{i_{\dot{y}},\gamma}} \dot{y} \in \langle \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\gamma}\rangle^+$. Define $\dot{E} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ so that p forces $\dot{E} = \dot{y}$ and each $q \in \mathbb{P}_{i_{\dot{y}},\gamma} \cap p^\perp$ forces that $\dot{E} = \omega$. It is easily checked that $\mathbb{B}_{i_{\dot{y}},\gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{E}\}\rangle$ is then $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin and that p forces that it measures $\dot{y}$. This condition ensures that p is compatible with an element of A_1.

If (2) holds but (3) fails, then by a symmetric argument as in the previous paragraph we can again define $\dot{E}$ so that $\mathbb{B}_{i_{\dot{y}},\gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{E}\}\rangle$ is then $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin and that p forces that it measures $\omega \setminus \dot{y}$. □

If by increasing M we can enlarge A_1 we simply do so. Since $\underline{\mathbf{P}}^\gamma$ is ccc we may assume that this is no longer possible, and therefore we may also assume that A is a subset of M. Now we choose any $p \in A \setminus A_1$. It suffices to produce an $\dot{E}_p \in \mathbb{B}_{i_{\dot{y}},\gamma}$ that can be added to $\mathcal{E}_1$ that measures $\dot{y}$ and satisfies that $q \Vdash \dot{E}_p = \omega$ for all $q \in p^\perp$. This is because we then have that $\mathcal{E}_1 \cup \{\dot{E}_p : p \in A \setminus A_1\}$ is contained in a $\vec{\lambda}(i_\gamma)$-thin filter that measures $\dot{y}$.

Fact 3. There is an α such that $\alpha_{\dot{y}} = \alpha + 1$.

Proof of Fact 3. Otherwise, let $j = i_p$ and for each $r < p$ in $\mathbb{P}_{i_{\dot{y}},\alpha_{\dot{y}}}$, choose $\beta \in M \cap \alpha_{\dot{y}}$ such that $r \in \mathbb{P}_{i_{\dot{y}},\beta}$, and define a name $\dot{y}[r]$ in $M \cap \mathbb{B}_{j,\gamma}$ according to $(\ell, q) \in \dot{y}[r]$ providing there is a pair $(\ell, p_\ell) \in \dot{y}$ such that $q <_j p_\ell$ and $q \restriction \beta$ is in the set $M \cap \mathbb{P}_{j,\beta} \setminus (r \wedge p_\ell \restriction \beta)^\perp$. This set, namely $\dot{y}[r]$, is in M because $\mathbb{P}_{j,\beta}$ is ccc and $M^\omega \subset M$.

We prove that r forces that $\dot{y}[r]$ contains $\dot{y}$. Suppose that $r_1 < r$ and there is a pair $(\ell, p_\ell) \in \dot{y}$ with $r_1 < p_\ell$. Choose an $r_2 \in \mathbb{P}_{j,\gamma}$ so that $r_2 <_j r_1$. It suffices to show $r_2 \Vdash \ell \in \dot{y}[r]$. Let $q <_j p_\ell$ with $q \in M$. Then $r_2 \not\perp p_\ell$ implies $r_2 \not\perp q$. Since r_2 was any $<_j$-projection of r_1 we can assume that $r_2 < q$. Since $r_2 \restriction \beta$ is in $(\mathbb{P}_{j,\beta} \cap (r \wedge p_\ell \restriction \beta)^\perp)^\perp$, it follows that $q \restriction \beta \notin (r \wedge p_\ell \restriction \beta)^\perp$. This implies that $(\ell, q) \in \dot{y}[r]$ and completes the proof that $r_2 \Vdash \ell \in \dot{y}[r]$.

Now assume that $\beta < \alpha_{\dot{y}}$ and $r \Vdash \dot{b} \cap \dot{E} \cap \dot{y}$ is empty for some $r < p$ in $\mathbb{P}_{i_{\dot{y}},\beta}$, $\dot{b} \in \mathbb{B}_{j,\gamma}$, and $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\gamma}$. Let $\dot{x} = (\dot{E} \cap \dot{y})[r]$ (defined as above for $\dot{y}[r]$). We complete the proof of Fact 3 by proving that $r \Vdash \dot{b} \cap \dot{x}$ is empty. Since each are in $\mathbb{B}_{j,\gamma}$, we may choose any $r_1 <_j r$, and assume that $r_1 \Vdash \ell \in \dot{b} \cap \dot{x}$. In addition we can suppose that there is a pair $(\ell, q) \in \dot{x}$ such that $r_1 < q$. The fact that $(\ell, q) \in \dot{x}$ means there is a p_ℓ with (ℓ, p_ℓ) in the name $\dot{E} \cap \dot{y}$ such that $q <_j p_\ell$. Since $r_1 \in \mathbb{P}_{j,\gamma}$ and $r_1 < q$, it follows that $r_1 \not\perp p_\ell$. Now it follows that r_1 has an extension forcing that $\ell \in \dot{b} \cap (\dot{E} \cap \dot{y})$ which is a contradiction. □

Fact 4. $i_{\dot{y}} = i_\alpha$ and so also $i_p < i_\alpha$.

Proof of Fact 4. Since $\mathbb{P}_{i,\alpha+1} = \mathbb{P}_{i,\alpha}$ for $i < i_\alpha$, we have that $i_\alpha \le i_{\dot{y}}$. Now assume that $i_\alpha < i_{\dot{y}}$ and we proceed much as we did in Fact 3 to prove that i_p does not exist. Assume that $r < p$ (in $\mathbb{P}_{i_{\dot{y}},\alpha+1}$ and $r \Vdash \dot{b} \cap (\dot{E} \cap \dot{y})$ is empty for some $\dot{E} \in M \cap \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_{\dot{y}},\gamma}$ and $\dot{b} \in \mathbb{B}_{i_p,\gamma}$. It follows from Lemma 5.4 that we can simply assume that $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\alpha+1}$, and similarly that $\dot{b} \in \mathbb{B}_{i_p,\alpha+1}$.

Let $\dot{T}_\alpha$ be the $\mathbb{P}_{i_{\dot{y}},\alpha}$-name such that $r \restriction \alpha \Vdash r(\alpha) = \dot{T}_\alpha \in \mathbb{L}(\mathfrak{D}^\alpha_{i_{\dot{y}}})$. We may assume that there is a $t_\alpha \in \omega^{<\omega}$ such that $r \restriction \alpha \Vdash t_\alpha = \text{stem}(\dot{T}_\alpha)$.

Choose any $M \cap \mathbb{P}_{i_\alpha,\alpha}$-generic filter $\bar{G}$ such that $r \restriction \alpha \in \bar{G}^+$. Since $\mathbb{P}_{i_\alpha,\alpha}$ is ccc and $M^\omega \subset M$, it follows that $M[\bar{G}]$ is closed under ω-sequences in the model $V[\bar{G}]$.

In this model, define an $\mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$-name $\dot{x}$. A pair $(\ell, T_\ell) \in \dot{x}$ if $t_\alpha \le \text{stem}(T_\ell) \in T_\ell \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$ and for each $\text{stem}(T_\ell) \le t \in T_\ell$, there is a pair $(\ell, q_{\ell,t}) \in M$ in the name $(\dot{y} \cap \dot{E})$ such that $q_{\ell,t} \restriction \alpha \in \bar{G}^+$, $q_{\ell,t} \restriction \alpha \Vdash t = \text{stem}(q_{\ell,t}(\alpha))$, and $(q_{\ell,t} \restriction \alpha \wedge r \restriction \alpha)$ does not force (over the poset $\bar{G}^+$) that $t \notin \dot{T}_\alpha$. We will show that r forces over the poset $\bar{G}^+$ that $\dot{x}$ contains $\dot{E} \cap \dot{y}$ and that $\dot{x} \cap \dot{b}$ is empty. This proves that p forces that $\langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_p,\alpha+1}$ generates $\langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle \cap \mathbb{B}_{i_p,\alpha+1}$ since $\dot{x}$ must be forced to be in $\langle \mathcal{E}_1 \rangle$. It then follows from Lemma 5.4 that $\mathcal{E}_1 \cap \mathbb{B}_{i_p,\gamma}$ generates $\langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle \cap \mathbb{B}_{i_p,\gamma}$, contradicting the assumption on i_p.

To prove that r forces that $\dot{x}$ contains $\dot{y} \cap \dot{E}$, we consider any $r_\ell < r$ that forces over $\bar{G}^+$ that $\ell \in \dot{y} \cap \dot{E}$. We may choose $(\ell, p_\ell) \in M$ in the name $(\dot{E} \cap \dot{y})$ such that (wlog) $r_\ell < p_\ell$. We may assume that $r_\ell \restriction \alpha$ forces a value t on $\text{stem}(r_\ell(\alpha))$ and that this equals $\text{stem}(p_\ell(\alpha))$. Now show there is a $T_\ell \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$. In fact, assume $t \in T_\ell$ with $q_{\ell,t}$ as the witness. Let $L^- = \{k : t^\frown k \notin T_\ell\}$; it suffices to show that $L^- \notin (\mathfrak{D}^\alpha_{i_\alpha})^+$.

By assumption that $q_{t,\ell}$ is the witness, there is an $r_t < (q_{\ell,t} \restriction \alpha \wedge r \restriction \alpha)$ such that $r_t \Vdash t \in \dot{T}_\alpha$ and $r_t \Vdash t = \text{stem}(q_{\ell,t}(\alpha))$. By strengthening r_t we can assume that r_t forces a value $\dot{D} \in \dot{\mathcal{D}}^\alpha_{i_{\dot{y}}}$ on $\{k : t^\frown k \in \dot{T}_\alpha \cap q_{\ell,t}(\alpha)\}$. But now, it follows that r_t forces that $\dot{D}$ is disjoint from L^- since if $r_{t,k} \Vdash k \in \dot{D}$ for some $r_{t,k} < r_t$, $r_{t,k}$ is the witness to $(\ell, q_{\ell,t^\frown k})$ is in $(\dot{y} \cap \dot{E})$ etc., where $q_{\ell,t^\frown k} \restriction \alpha = q_{\ell,t} \restriction \alpha$ and $q_{\ell,t^\frown k}(\alpha) = (q_{\ell,t}(\alpha))_{t^\frown k}$.

Since some condition forces that L^- is not in $(\dot{\mathfrak{D}}^\alpha_{i_{\dot y}})^+$ it follows that L^- is not in $(\dot{\mathfrak{D}}^\alpha_{i_\alpha})^+$

Finally we must show that r forces over $\bar G^+$ that $\dot b$ is disjoint from $\dot x$. Since each are $\mathbb{P}_{i_p,\alpha+1}$-names, it suffices to assume that $\bar r \in \bar G^+$ is some $\mathbb{P}_{i_p,\alpha+1}$-reduct of r that forces some ℓ is in $\dot b \cap \dot x$, and to then show that r fails to force that $\ell \notin \dot b \cap (\dot E \cap \dot y)$. Choose $(\ell, q_{\ell,t}) \in (\dot y \cap \dot E)$ witnessing that $\bar r \Vdash \ell \in \dot x$. That is, we may assume that $\bar r \restriction \alpha \Vdash t = \mathrm{stem}(\bar r(\alpha))$, that $q_{\ell,t} \restriction \alpha \in \bar G^+$, and $(q_{\ell,t} \wedge r \restriction \alpha)$ does not force over $\bar G^+$ that $t \notin \dot T_\alpha$. Of course this means that the condition $\bar r \wedge r \wedge [[t \in \dot T_\alpha]] \wedge q_{\ell,t}$ is not 0. This condition forces that ℓ is in $\dot b \cap (\dot E \cap \dot y)$ as required. □

Fact 5. The character of $\mathfrak{D}^\alpha_{i_\alpha}$ is greater than μ_{i_γ}.

Proof of Fact 5. We know that $\mathfrak{D}^\alpha_{i_\alpha}$ is forced to have an ω-closed base (in fact, descending mod finite with uncountable cofinality). Even more, $\mathbb{P}_{i_\alpha,\alpha}$ forces that for all $T \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$, there is a $D \in \mathfrak{D}^\alpha_{i_\alpha}$ such that the condition $([D]^{<\omega})_{\mathrm{stem}(T)}$ is below T. Let χ_α be the cofinality of α and fix a list $\{\dot D_\beta : \beta < \chi_\alpha\} \in M$ (closed under mod finite changes) of $\mathbb{P}_{i_\alpha,\alpha}$-names of elements of $\dot{\mathfrak{D}}^\alpha_{i_\alpha}$ that is forced to be a base.

Now, suppose that $\dot b \in \mathbb{B}_{i_p,\alpha+1} = \mathbb{B}_{i_p,\alpha}$ and there is an $\dot E \in \mathcal{E}_1$ and an $r < p$ forcing that $\dot b \cap (\dot E \cap \dot y)$ is empty. We prove there is an $\dot x \in \mathcal{E}_1$ and an $r_2 < r \restriction \alpha$ in $\mathbb{P}_{i_\alpha,\alpha}$ such that $r_2 \Vdash \dot b \cap \dot x$ is empty. We may assume that r_2 forces a value t on $\mathrm{stem}(r(\alpha))$ and that, for some $\beta < \chi_\alpha$, $r_2 \Vdash (\dot D_\beta^{<\omega})_t < r(\alpha)$. Let

$$\dot x = \{(\ell, q_\ell \restriction \alpha) : (\ell, q_\ell) \in (\dot E \cap \dot y) \text{ and } \ q_\ell \restriction \alpha \Vdash q_\ell(\alpha) \leq (\dot D_\beta^{<\omega})_t\}\ .$$

It is immediate that $\dot x \in M$ and that $(r_2 \wedge r) \Vdash_{\mathbb{P}_{i_\alpha,\alpha+1}} \dot x \supseteq (\dot E \cap \dot y)$. Since $\dot E \cap \dot y$ is forced to be in $\mathcal{E}_1^+$, it follows that $\dot x$ is forced by r_2 to be in $\langle \mathcal{E}_1 \rangle$. Now we verify that $r_2 \Vdash \dot b \cap \dot x$ is empty. Assume that $r_3 < r_2$ in $\mathbb{P}_{i_\alpha,\alpha}$ and that $r_3 \Vdash \ell \in \dot b \cap \dot x$. We may assume there is $(\ell, q_\ell \restriction \alpha) \in \dot x$ such that $r_3 < q_\ell \restriction \alpha$. But now $r_2 \Vdash q_\ell(\alpha) \leq r(\alpha)$ and so $r_2 \wedge r \Vdash \ell \in \dot b \cap (\dot E \cap \dot y)$ – a contradiction.

The conclusion now follows from Lemma 5.4. □

Definition 5.7. *For each $t \in \omega^{<\omega}$, define that $\mathbb{P}_{i_\alpha,\alpha}$-name $\dot E_t$ according to the rule that $r \Vdash \ell \in \dot E_t$ providing $r \in \mathbb{P}_{i_\alpha,\alpha}$ forces that there is a $\dot T$ with $r \Vdash \dot T \in \mathbb{L}(\dot{\mathfrak{D}}^\alpha_{i_\alpha})$, $r \Vdash t = \mathrm{stem}(\dot T)$, and $r \cup \{(\alpha, \dot T)\} \Vdash \ell \notin \dot y$.*

Fact 6. There is a $\dot T \in \mathbb{L}(\dot{\mathfrak{D}}^\alpha_{i_\alpha}) \cap M$ such that $p \restriction \alpha$ forces the statement: $\dot E_t \in \mathcal{E}_1$ for all t such that $\mathrm{stem}(\dot T) \leq t \in \dot T$.

Proof of Fact 6. By elementarity, there is a maximal antichain of $\mathbb{P}_{i_\alpha,\alpha}$ each element of which decides if there is a $\dot T$ with $\dot E_t \in \mathcal{E}_1$ for all $t \in \dot T$ above stem$(\dot T)$. Since $p \in A \setminus A_1$ it follows that there is an $i_p < i_\alpha$ as in condition (2) of Fact 2. Let $t_0 \in \omega^{<\omega}$ so that $p \restriction \alpha \Vdash t_0 = \text{stem}(p(\alpha))$. By the maximum principle, there is a $\dot b \in \mathbb{B}_{i_p,\gamma}$ and a $\dot E_0 \in \mathcal{E}_1$ satisfying that $p \Vdash \dot b \cap \dot E_0 \cap \dot y$ is empty, while $p \Vdash \dot b \cap \dot E$ is infinite for all $\dot E \in \langle \mathcal{E}_1 \rangle$. This means that p forces that $\dot b \cap \dot E_0$ is an element of $\langle \mathcal{E}_1 \rangle^+$ that is contained in $\omega \setminus \dot y$. As in the proof of Lemma 5.4, there is an $\dot E_2 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_p,\gamma}$ such that p forces that $\dot b \cap \dot E_2$ is contained in $\dot E_0$. We also have that $(\dot b \cap \dot E_2) \restriction \alpha$ is forced to be contained in $\omega \setminus \dot y$. It now follows that $p \restriction \alpha$ forces that for all $t_0 \le t \in p(\alpha)$, $p \restriction \alpha$ forces that $\dot E_t$ contains $(\dot b \cap \dot E_2) \restriction \alpha$ and so is in $\langle \mathcal{E}_1 \rangle^+$. Since $\dot E_t$ is also measured by $\mathcal{E}_1$, we have that $p \restriction \alpha$ forces that such $\dot E_t$ are in $\mathcal{E}_1$. This completes the proof. $\square$

Now we show how to extend $\mathcal{E}_1 \cap \mathbb{B}_{i_\alpha,\gamma}$ so as to measure $\dot y$. Let $\beta = \sup(M \cap \alpha)$. By Fact 5, $\beta < \alpha$ and by the definition of $\mathcal{H}(\vec\lambda)$, $M \cap \dot{\mathfrak{D}}^\alpha_{i_\alpha}$ is a subset of $\langle \dot{\mathfrak{D}}^\beta_{i_\beta} \rangle$, $\dot L_\beta \in \dot{\mathfrak{D}}^\alpha_{i_\alpha}$, and $i_\beta = i_\alpha$. We also have that the family $\{\dot L_\xi : \text{cf}(\xi) \ge \omega_1 \text{ and } \beta_\alpha \le \xi \in M \cap \beta\}$ is a base for $\dot{\mathfrak{D}}^\beta_{i_\beta}$. For convenience let $q <_M p$ denote the relation that q is an $M \cap \mathbb{P}_{i_\alpha,\alpha+1}$-reduct of p. Let $\bar p$ be any condition in $\mathbb{P}_{i_\beta,\beta+1}$ satisfying that $\bar p \restriction \beta = p \restriction \alpha$ and $\bar p \restriction \beta \Vdash \text{stem}(\bar p(\beta)) = t_\alpha$ (recall that $p \restriction \alpha \Vdash t_\alpha = \text{stem}(p(\alpha))$.

Let us note that for each $q \in M \cap \mathbb{P}_{\alpha,i_\alpha+1}$, $q \restriction \alpha = q \restriction \beta$ and $q \restriction \beta \Vdash q(\alpha)$ is also a $\mathbb{P}_{\beta,i_\beta}$-name of an element of $\mathbb{L}(\dot{\mathfrak{D}}^\beta_{i_\beta})$. Let $\dot x$ be the following $\mathbb{P}_{i_\beta,\beta+1}$-name

$$\dot x = \{(\ell, q \restriction \beta \cup \{(\beta, q(\beta))\}) : (\ell, q) \in \dot y \cap M \text{ and } q <_M p\} .$$

We will complete the proof by showing that there is an extension of p that forces that $\mathcal{E}_1 \cup \{\omega \setminus (\dot x[\dot L_\beta])\}$ measures $\dot y$ and that 1 forces that $\langle \mathcal{E}_1 \cup \{\omega \setminus (\dot x[\dot L_\beta])\}\rangle \cap \mathbb{B}_{i_{\dot y},\beta+1}$ is $\vec\lambda(i_\gamma)$-thin. Here $\dot x[\dot L_\beta]$ abbreviates the $\mathbb{P}_{i_\beta,\beta+1}$-name

$$\{(\ell, r) : (\exists q)\ (\ell, q) \in \dot x,\ q \restriction \beta = r \restriction \beta,\ \text{and } r \Vdash \text{stem}(q(\beta)) \in \dot L_\beta^{<\omega}\}.$$

The way to think of $\dot x[\dot L_\beta]$ is that if $\bar p$ is in some $\mathbb{P}_{i_\alpha,\alpha}$-generic filter G, then $\dot y[G]$ is now an $\mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$-name, $L_\beta^{<\omega} = (\dot L_\beta[G])^{<\omega}$ is in $\mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$, and $(\dot x[\dot L_\beta])[G]$ is equal to $\{\ell : L_\beta^{<\omega} \not\Vdash \ell \notin \dot y\}$. We will use the properties of $\dot x$ to help show that $\mathcal{E}_1 \cup \{\omega \setminus (\dot x[\dot L_\beta])\}$ is $\vec\lambda(i_\gamma)$-thin. This semantic description of $\dot x[\dot L_\beta]$ makes clear that $\bar p \cup \{(\alpha, (\dot L_\beta)^{<\omega})\} \in \mathbb{P}_{i_\alpha,\alpha+1}$ forces that $\dot x[\dot L_\beta]$ contains $\dot y$. This implies that $\mathcal{E}_1 \cup \{\omega \setminus (\dot x[\dot L_\beta])\}$ measures $\dot y$.

Claim: It is forced by $\bar p$ that $\omega \setminus \dot x$ is not measured by $\mathcal{E}_1$.

Each element of $\mathcal{E}_1$ is in M and simple elementarity will show that for any condition in q in M that forces $\dot{E} \cap (\omega \setminus \dot{y})$ is infinite, the corresponding $\bar{q} = q \restriction \alpha \cup \{(\beta, q(\alpha))\}$ will also force that $\dot{E} \cap (\omega \setminus \dot{x})$ is infinite.

It follows from Fact 5, with $\omega \setminus \dot{x}$ playing the role of $\dot{y}$, that $\mathcal{E}_1 \cup \{\omega \setminus \dot{x}\}$ is $\vec{\lambda}(i_\gamma)$-thin. Recall that $q \Vdash \dot{x} = \emptyset$ for all $q \perp \bar{p}$. Now to prove that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ is also $\vec{\lambda}(i_\gamma)$-thin, we prove that

$$\langle \mathcal{E}_1 \cup \{\omega \setminus \dot{x}\}\rangle \cap \mathbb{B}_{i,\alpha} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}\rangle \cap \mathbb{B}_{i,\alpha}$$

for all $i < i_\alpha$. In fact, first we prove

$$\langle \mathcal{E}_1 \cup \{\omega \setminus \dot{x}\}\rangle \cap \mathbb{B}_{i,\beta} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}\rangle \cap \mathbb{B}_{i,\beta}$$

for all $i < i_\alpha$.

We begin with this main Claim.

Claim 1. If $\dot{b} \in \mathbb{B}_{i,\beta}$ $(i < i_\beta)$ and there is an $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_\alpha,\beta}$ and a $\bar{p} \geq q \in \mathbb{P}_{i_\beta,\beta+1}$ such that $q \Vdash \dot{b} \cap (\dot{E} \setminus \dot{x}) = \emptyset$ then $q \restriction \beta \Vdash (\exists \dot{E} \in \mathcal{E}_1)\ \dot{b} \cap \dot{E} = \emptyset$.

Proof of Claim: We may assume that $q \restriction \beta$ forces a value t on stem$(q(\beta))$. Recall that $q \restriction \beta$ forces the statement: there is a $\dot{D} \in M \cap \dot{\mathfrak{D}}^\alpha_{i_\alpha}$ such that $(\dot{D}^{<\omega})_t \leq q(\beta)$. The definition of $\dot{x}$ ensures that $q \restriction \beta \cup \{(\alpha, (\dot{D}^{<\omega})_t)\} \Vdash$ $\dot{b} \cap (\dot{E} \setminus \dot{y})$ is empty. There is a $\mathbb{P}_{i_\alpha,\alpha}$-name $\dot{E}_1 \in M$ such that $q \restriction \alpha \Vdash$ $\dot{E}_1 = \{\ell : (\dot{D}^{<\omega})_t \not\Vdash \ell \notin (\dot{E} \setminus \dot{y})\}$. By assumption $q \restriction \alpha \Vdash \dot{E}_1 \in \langle \mathcal{E}_1 \rangle$. Since $\dot{b}$ is also a $\mathbb{P}_{i,\alpha}$-name, we have that $q \restriction \alpha \Vdash \dot{b} \cap \dot{E}_1 = \emptyset$. $\square$

Now assume that $\dot{b} \in \mathbb{B}_{i_\beta,\beta}$ and $q \Vdash \dot{b} \cap (\dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$ is empty for some $q < \bar{p}$ in $\mathbb{P}_{i_\beta,\beta+1}$. By Lemma 5.4 it suffices to assume that $\dot{E} \in \mathbb{B}_{i_\beta,\beta}$. To prove that q forces that $\dot{b} \notin \langle \mathcal{E}_1 \rangle^+$, it suffices to prove that there is some $\dot{E}_1 \in \mathcal{E}_1$ such that $q \Vdash \dot{b} \cap (\dot{E}_1 \cap (\omega \setminus \dot{x}))$ is finite. We proceed by contradiction.

We may again assume that $q \restriction \beta$ forces that $q(\beta)$ is $(\dot{D}^{<\omega})_t$ for some $t \supset t_\alpha$ and some $\dot{D} \in \dot{\mathfrak{D}}^\alpha_{i_\alpha} \cap M$. Let H be the range of t. Let, for the moment, G be a $\mathbb{P}_{i_\alpha,\alpha}$-generic filter with $q \in G$. Now in $M[G]$ we have the value L_β of $\dot{L}_\beta$ and $H \subset L_\beta$. We can also let E denote the value of $\dot{E}[G]$. Recall that for each $s \in H^{<\omega}$, E_s denotes the set of $\ell \in E$ such that there is some $T \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$ with $s = \text{stem}(T)$ and $T \Vdash \ell \notin \dot{y}$. We have shown in Fact 6 that there is a $T \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha}) \cap M$ such that $E_s \in \mathcal{E}_1$ for all $s \in T$ above stem(T). This means that there is an $\ell \in b \cap E$ such that $\ell \in E_s$ for each of the finitely many suitable s. For each s, choose $T_s \subset T$ witnessing $\ell \in E_s$. As before, and since there are only finitely many s involved, we can assume that $\dot{T}_s = (\dot{D}^{<\omega})_s$ for some $H \subset \dot{D} \in \dot{\mathfrak{D}}^{\alpha}_{i_\alpha} \cap M$ and we then define an extension q of q so that $q'(\beta) = (\dot{D}^{<\omega})_{t_\alpha}$ ensures that $(\dot{L}^{<\omega}_{\beta})_s < T_s$ for each s. Note that such a condition q' we have that $q' \cup \{(\alpha, (\dot{L}_\beta)^{<}\omega)\}$ forces that $\ell \notin \dot{y}$. But then it should be clear that q' that forces $\ell \notin \dot{x}[\dot{L}_\beta]$. This contradicts that q forces $\ell \notin \dot{b} \cap (\dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$.

□

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