

A VERSION OF κ -MILLER FORCING

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ABSTRACT. Let κ be an uncountable cardinal such that $2^{<\kappa} = \kappa$ or just $\text{cf}(\kappa) > \omega$, $2^{2^{<\kappa}} = 2^\kappa$, and $([\kappa]^\kappa, \supseteq)$ collapses 2^κ to ω . We show under these assumptions the κ -Miller forcing with club many splitting nodes collapses 2^κ to ω and adds a κ -Cohen real.

1. INTRODUCTION

Many of the tree forcings on the classical Baire space have various analogues for higher cardinals. Here we are concerned with Miller forcing [?]. For a κ -version of Miller forcing, in addition to superperfectness one usually requires (see, e.g., [?, Section 5.2]) limits of length $< \kappa$ of splitting nodes as well and that splitting mean splitting into a club. In this paper we investigate a version of κ -Miller forcing where this latter requirement is waived. We show: If $\text{cf}(\kappa) > \omega$, $\text{cf}(\kappa) = \kappa$ or $\text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$, $2^{2^{<\kappa}} = 2^\kappa$, and there is a κ -mad family of size 2^κ , then this variant of Miller forcing is related to the forcing $([\kappa]^\kappa, \supseteq)$ and collapses 2^κ to ω . In particular, if $\omega < \kappa^{<\kappa} = \kappa$, then our four premises are fulfilled.

Throughout the paper we let κ be an uncountable cardinal. We write \trianglelefteq for end extension of functions whose domains are ordinals. If $\text{dom}(t), i$ are ordinals, we write $t \hat{\ } i$ for the concatenation of t with the singleton function $\{(0, i)\}$, i.e., $t \hat{\ } i = t \cup \{(\text{dom}(t), i)\}$. We denote forcing orders in the form $(\mathbb{P}, \leq_{\mathbb{P}})$ and let $p \leq_{\mathbb{P}} q$ mean that q is *stronger* than p . We write $\lambda >_{\kappa}$ for the set of functions $f: \alpha \rightarrow \kappa$ for some $\alpha < \lambda$. The domain α of f is also called the length of f . The set of subsets of κ of size κ is denoted by $[\kappa]^\kappa$.

Definition 1.1. (1) \mathbb{Q}_κ^1 is the forcing $([\kappa]^\kappa, \supseteq)$.

- (2) \mathbb{Q}_κ^2 is the following version of κ -Miller forcing: Conditions are trees $T \subseteq {}^{\kappa > \kappa}$ that are κ *superperfect*: for each $s \in T$ there is $s \trianglelefteq t$ such that t is a κ -splitting node of T (short $t \in \text{spl}(T)$). A node $t \in T$ is called a κ -*splitting node* if

$$\text{set}_p(t) = \{i < \kappa : t \hat{\ } i \in T\}$$

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has size κ . We furthermore require that the limit of an increasing in the tree order sequence of length less than κ of κ -splitting nodes is a κ -splitting node if it has length less than κ .

For $p, q \in \mathbb{Q}_\kappa^2$ we write $p \leq_{\mathbb{Q}_\kappa^2} q$ if $q \subseteq p$. So subtrees are stronger conditions.

- (3) For $p \in \mathbb{Q}_\kappa^2$ and $\eta \in p$ we let $\text{suc}_p(\eta) = \{\eta' \in {}^{\kappa >} \kappa : (\exists i \in \kappa)(\eta' = \eta \hat{\ } \langle i \rangle \in p)\}$.
- (4) Let $\eta \in p \in \mathbb{Q}_\kappa^2$. We let $p^{(\eta)} = \{\nu \in p : \nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu\}$.
- (5) For $a, b \subseteq \kappa$ we write $a \subseteq_\kappa^* b$ if $|a \setminus b| < \kappa$.

Each of the two forcing orders \mathbb{P} has a weakest element, denoted by $0_{\mathbb{P}}$. Namely, \mathbb{Q}_κ^1 has as a weakest element $0_{\mathbb{Q}_\kappa^1} = \kappa$, and \mathbb{Q}_κ^2 has as a weakest element the full tree ${}^{\kappa >} \kappa$. We write $\mathbb{P} \Vdash \varphi$ if the weakest condition $0_{\mathbb{P}}$ forces φ .

2. RESULTS ABOUT \mathbb{Q}_κ^1

We will apply the following result for $\chi = 2^\kappa$.

Theorem 2.1. ([?, Theorem 0.5])

- (1) Under the assumption of an antichain of size χ in \mathbb{Q}_κ^1 , \mathbb{Q}_κ^1 collapses χ to \aleph_0 if $\aleph_0 < \text{cf}(\kappa) = \kappa$ or if $\aleph_0 < \text{cf}(\kappa) < 2^{\text{cf}(\kappa)} \leq \kappa$.
- (2) Under the assumption of an antichain of size χ in \mathbb{Q}_κ^1 , \mathbb{Q}_κ^1 collapses χ to \aleph_1 in the case of $\aleph_0 = \text{cf}(\kappa)$.

Definition 2.2. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called a κ -almost disjoint family if for $A \neq B \in \mathcal{A}$, $|A \cap B| < \kappa$. A κ -almost disjoint family of size at least κ that is maximal is called a κ -mad family.

Observation 2.3. If $2^{<\kappa} = \kappa$, there is a κ -mad family $\mathcal{A} \subseteq [\kappa]^\kappa$ of size 2^κ .

Proof. We let $f: {}^{\kappa >} 2 \rightarrow \kappa$ be an injection. We assign to each branch b of ${}^{\kappa >} 2$ a set $a_b = \{f(s) : s \in b\}$. Then we complete the resulting family $\{a_b : b \text{ branch of } {}^{\kappa >} 2\}$ to a maximal κ -almost disjoint family. \square

Observation 2.4. If \mathbb{Q}_κ^1 collapses 2^κ to ω , then there is a κ -mad family \mathcal{A} of size 2^κ .

Proof. \mathbb{Q}_κ^1 cannot have the 2^κ -c.c. Hence there is an antichain of size 2^κ . This is a κ -ad family, and we extend it to a κ -mad family. \square

For further use, we indicate the hypothesis for each technical step.

Lemma 2.5. Suppose that \mathbb{Q}_κ^1 collapses 2^κ to ω . Then there is a \mathbb{Q}_κ^1 -name $\tau: \aleph_0 \rightarrow 2^\kappa$ for a surjection, and there is a labelled tree $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho_\eta) : \eta \in {}^{\omega >} (2^\kappa) \rangle$ with the following properties

- (a) $a_\emptyset = \kappa$ and for any $\eta \in {}^{\omega >} (2^\kappa)$, $a_\eta \in [\kappa]^\kappa$.
- (b) $\eta_1 \triangleleft \eta_2$ implies $a_{\eta_1} \supseteq a_{\eta_2}$.

- (c) $n_\eta \in [\lg(\eta) + 1, \omega)$.
- (d) If $a \in [\kappa]^\kappa$ then there is some $\eta \in {}^{\omega>}2^\kappa$ such that $a \supseteq a_\eta$.
- (e) If $\eta \hat{\langle} \beta \rangle \in T$ then $a_{\eta \hat{\langle} \beta \rangle}$ forces $\mathcal{T} \upharpoonright n_\eta = \varrho_{\eta \hat{\langle} \beta \rangle}$ for some $\varrho_{\eta \hat{\langle} \beta \rangle} \in {}^{n_\eta}2^\kappa$, such that the $\varrho_{\eta \hat{\langle} \beta \rangle}, \beta \in 2^\kappa$, are pairwise different. Hence for any $\eta \in {}^{\omega>}2^\kappa$, the family $\{a_{\eta \hat{\langle} \alpha \rangle} : \alpha < 2^\kappa\}$ is a κ -ad family in $[a_\eta]^\kappa$.

Proof. Let \mathcal{T} be a \mathbb{Q}_κ^1 -name such that $\mathbb{Q}_\kappa^1 \Vdash \mathcal{T} : \aleph_0 \rightarrow 2^\kappa$ is onto. For $\alpha < 2^\kappa$ let AP_α be the set of objects \bar{m} satisfying

- (*)₁(1.1) $\bar{m} = (T, \bar{a}, \bar{n}, \bar{\varrho}) = (T_{\bar{m}}, \bar{a}_{\bar{m}}, \bar{n}_{\bar{m}}, \bar{\varrho}_{\bar{m}})$.
- (1.2) T is a subtree of $({}^{\omega>}2^\kappa, \triangleleft)$ of cardinality $\leq |\alpha| + \kappa$ and $\langle \rangle \in T$.
- (1.3) $\bar{a} = \langle a_\eta : \eta \in T \rangle$ fulfils $\eta \triangleleft \nu \rightarrow a_\nu \subseteq a_\eta$ and $a_\langle \rangle = \kappa$ and $a_\eta \in [\kappa]^\kappa$.
- (1.4) $\bar{n} = \langle n_\eta : \eta \in T \rangle$ fulfils $\text{dom}(\varrho_{\eta \hat{\langle} \beta \rangle}) = n_\eta > \lg(\eta)$ for any $\eta \hat{\langle} \beta \rangle \in T$.
- (1.5) If $\eta \hat{\langle} \beta \rangle \in T$, then $a_{\eta \hat{\langle} \beta \rangle}$ forces a value to $\mathcal{T} \upharpoonright n_\eta$ called $\varrho_{\eta \hat{\langle} \beta \rangle}$ and for $\beta \neq \gamma$ we have $\varrho_{\eta \hat{\langle} \beta \rangle} \neq \varrho_{\eta \hat{\langle} \gamma \rangle}$. Hence for any $\eta \hat{\langle} \beta \rangle, \eta \hat{\langle} \gamma \rangle \in T_{\bar{m}}$, $\beta \neq \gamma$ implies $a_{\eta \hat{\langle} \beta \rangle} \cap a_{\eta \hat{\langle} \gamma \rangle} \in [\kappa]^{<\kappa}$.
- (1.6) For $\eta \in T_{\bar{m}}$, we let

$$\text{Pos}(a_\eta, n_\eta) = \{\varrho \in {}^{n_\eta}2^\kappa : a_\eta \not\Vdash_{\mathbb{Q}_\kappa^1} \mathcal{T} \upharpoonright n_\eta \neq \varrho\},$$

and require that the latter has cardinality 2^κ .

In the next items we state some properties of AP_α that are derived from (*)₁.

- (*)₂ $AP = \bigcup\{AP_\alpha : \alpha < 2^\kappa\}$ is ordered naturally by \leq_{AP} , which means end extension.
- (*)₃ (a) AP_α is not empty and increasing in α .
- (b) For infinite α , AP_α is closed under unions of increasing sequences of length $< |\alpha|^+$.
- (*)₄ Let $\gamma < 2^\kappa$. If $\bar{m} \in AP_\gamma$ and $\eta \in T_{\bar{m}}$ and $\eta \hat{\langle} \alpha \rangle \notin T_{\bar{m}}$ then there is $\bar{m}' \in AP_\gamma$ such that $\bar{m} \leq_{AP} \bar{m}'$ and $T_{\bar{m}'} = T_{\bar{m}} \cup \{\eta \hat{\langle} \alpha \rangle\}$.

Proof: For $\eta \in T_{\bar{m}}$,

$$\mathcal{U} = \text{Pos}(a_\eta, n_\eta) = \{\varrho \in {}^{n_\eta}2^\kappa : a_\eta \not\Vdash_{\mathbb{Q}_\kappa^1} \mathcal{T} \upharpoonright n_\eta \neq \varrho\} \text{ has size } 2^\kappa,$$

whereas

$$\Lambda_\eta = \{\varrho_{\eta \hat{\langle} \beta \rangle} \upharpoonright n_\eta : \beta \in 2^\kappa \wedge \eta \hat{\langle} \beta \rangle \in T_{\bar{m}}\}$$

is of size $\leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Hence we can choose $\varrho_* \in \mathcal{U} \setminus \Lambda_\eta$ and $b_* \in [a_\eta]^\kappa$ such that $b_* \Vdash_{\mathbb{Q}_\kappa^1} \varrho_* = \mathcal{T} \upharpoonright n_\eta$. We let $\varrho_{\eta \hat{\langle} \alpha \rangle} = \varrho_*$. Since b_* forces a value of $\mathcal{T} \upharpoonright n_\eta$ that is incompatible with the one forced by $a_{\eta \hat{\langle} \beta \rangle}$ for any $\eta \hat{\langle} \beta \rangle \in T_{\bar{m}}$, the set b_* is κ -almost disjoint from $a_{\eta \hat{\langle} \beta \rangle}$ for any $\eta \hat{\langle} \beta \rangle \in T_{\bar{m}}$. We take $b_* = a_{\bar{m}', \eta \hat{\langle} \alpha \rangle} \subseteq a_{\bar{m}, \eta}$.

Since $\text{cf}(2^\kappa) > \aleph_0$ and since

$$|\{\text{range}(\varrho) : \varrho \in {}^{\omega>}2^\kappa \wedge b_* \not\Vdash_{\mathbb{Q}_\kappa^1} \mathcal{T} \upharpoonright n \neq \varrho\}| = 2^\kappa,$$

there is an n such that

$$\text{Pos}(b_*, n) = \{\varrho \in {}^n(2^\kappa) : b_* \not\Vdash_{\mathbb{Q}_\kappa^1} \mathcal{T} \upharpoonright n \neq \varrho\}$$

has cardinality 2^κ . We take the minimal one and let it be $n_{\eta \hat{\langle} \alpha \rangle}$.

(*)₅ If $\bar{m} \in AP_\alpha$ and $a \in [\kappa]^\kappa$ then there is some $\bar{m}' \geq \bar{m}$, such that there is $\eta \in T_{\bar{m}'}$ with $a_{\bar{m}', \eta} \subseteq a$.

Let

$$\mathcal{U}_a = \{\varrho \in {}^{\omega^>}(2^\kappa) : a \not\Vdash_{\mathbb{Q}_\kappa^1} \varrho \not\triangleleft \mathcal{T}\},$$

i.e.

$$\mathcal{U}_a = \{\varrho \in {}^{\omega^>}(2^\kappa) : (\exists b \geq_{\mathbb{Q}_\kappa^1} a)(b \Vdash_{\mathbb{Q}_\kappa^1} \varrho \triangleleft \mathcal{T})\}.$$

This set has cardinality 2^κ because $\mathbb{Q}_\kappa^1 \Vdash \mathcal{T} : \omega \rightarrow 2^\kappa$ is onto. We take n minimal such that

$$\mathcal{U}_{a,n} = \{\varrho \in {}^n(2^\kappa) : (\exists b \geq_{\mathbb{Q}_\kappa^1} a)(b \Vdash_{\mathbb{Q}_\kappa^1} \varrho \triangleleft \mathcal{T})\}$$

has size 2^κ . We let

$$\text{set}_n^+(\bar{m}) = \{\varrho_\eta : \eta \in T_{\bar{m}}, \text{lg}(\varrho_\eta) \geq n\}.$$

Clearly $|\text{set}_n^+(\bar{m})| \leq |T_{\bar{m}}| \leq |\gamma| + \kappa$. Thus we can take $\varrho_a \in \mathcal{U}_{a,n}$ that is incompatible with every element of $\text{set}_n^+(\bar{m})$. We take some $b_a \in [a]^\kappa$ such that $b_a \Vdash_{\mathbb{Q}_\kappa^1} \varrho_a \triangleleft \mathcal{T}$. The set

$$\Lambda_a = \{\eta \in T_{\bar{m}} : b_a \subseteq_\kappa^* a_\eta\}$$

is \triangleleft -linearly ordered by (*)₁ clauses 1.3 and 1.5 and $\langle \rangle \in \Lambda_a$. Since b_a does not pin down \mathcal{T} , Λ_a has a \triangleleft -maximal member η_a . Now we take $\alpha_* = \min\{\beta : \eta_a \hat{\langle} \beta \rangle \notin T_{\bar{m}}\}$. For any $\eta_a \hat{\langle} \beta \rangle \in T_{\bar{m}}$ we have $\varrho_{\eta_a \hat{\langle} \beta \rangle}$ and ϱ_a are incompatible, and hence $a_{\eta_a \hat{\langle} \beta \rangle} \cap b_a \in [\kappa]^{<\kappa}$. Now we choose $b_a^1 \in [b_a]^\kappa$ and ϱ_a^* such that $b_a^1 \Vdash_{\mathbb{Q}_\kappa^1} \varrho_a^* \triangleleft \mathcal{T}$ and $\text{lg}(\varrho_a^*) \geq n_{\bar{m}, \eta_a} > \text{lg}(\eta_a)$.

We let

$$\begin{aligned} T_{\bar{m}'} &= T_{\bar{m}} \cup \{\eta_a \hat{\langle} \alpha_* \rangle\}, \\ a_{\eta_a \hat{\langle} \alpha_* \rangle} &= b_a^1, \end{aligned}$$

We let $n_{\eta_a \hat{\langle} \alpha_* \rangle}$ be the minimal n such that $|\text{Pos}(b_a^1, n)| \geq 2^\kappa$. So (*)₅ holds.

Now we are ready to construct \mathcal{T} as in the statement of the lemma. We do this by recursion on $\alpha \leq 2^\kappa$. First we enumerate $[\kappa]^\kappa$ as $\langle c_\alpha : \alpha < 2^\kappa \rangle$, and we enumerate ${}^{\omega^>}(2^\kappa)$ as $\langle \eta_\alpha : \alpha < 2^\kappa \rangle$ such that $\eta_\alpha \triangleleft \eta_\beta$ implies $\alpha < \beta$. We choose an increasing sequence \bar{m}_α by induction on $\alpha < 2^\kappa$. We start with the tree $\{\langle \rangle\}$, $a_{\langle \rangle} = \kappa$, $\varrho_{\langle \rangle} = \emptyset$, $n_{\langle \rangle}$ be minimal such that $|\text{Pos}(\kappa, n)| = 2^\kappa$. In the odd successor steps we take $\bar{m}_{2\alpha+1} \geq_{AP} \bar{m}_\alpha$ so that $a_\eta \subseteq c_\alpha$ for some $\eta \in T_{2\alpha+1}$. This is done according to (*)₅. In the even successor steps we take $\bar{m}_{2\alpha+2} \geq_{AP} \bar{m}_{2\alpha+1}$ such that $\eta_\alpha \in T_{2\alpha+2}$. Since all initial segments of η_α appeared among the η_β , $\beta < \alpha$, $\bar{m}_{2\alpha+2}$ is found according to (*)₄. In the limit steps we take unions. Then \mathcal{T} that is given by the the last three

components of \bar{m}_{2^κ} has properties (a) to (e). \square

Since $\tau = \mathcal{T}[G]$ is not in \mathbf{V} , for any \mathcal{T} as in Lemma ?? no sequence of first components of a branch, i.e., no $\langle a_{f \upharpoonright n} : n \in \omega \rangle$, $f \in {}^\omega(2^\kappa) \cap \mathbf{V}$, has a \subseteq_κ^* -lower bound.

3. TRANSFER TO \mathbb{Q}_κ^2

In this section we use the tree \mathcal{T} from Lemma ?? for finding \mathbb{Q}_κ^2 -names.

Definition 3.1. Let μ, λ be cardinals. For $\nu, \nu' \in {}^{\lambda >} \mu$ we write $\nu \perp \nu'$ if $\nu \not\triangleleft \nu'$ and $\nu' \not\triangleleft \nu$.

Typical pairs (λ, μ) are $(\omega, 2^\kappa)$ and (κ, κ) .

An important tool for the analysis of \mathbb{Q}_κ^2 is the following particular kind of fusion sequence $\langle p_\alpha : \alpha < \kappa^{<\kappa} \rangle$ in \mathbb{Q}_κ^2 . Since we do not suppose $\kappa^{<\kappa} = \kappa$, a fusion sequence can be longer than κ . An important property is that for each $\nu \in {}^{\kappa >} \kappa$ there is at most one $\alpha < \kappa^{<\kappa}$ such that $\text{set}_{p_\alpha}(\nu) \supseteq \text{set}_{p_{\alpha+1}}(\nu)$.

Lemma 3.2. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that

$$(3.1) \quad \nu_\alpha \triangleleft \nu_\beta \rightarrow \alpha < \beta.$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

- (a) $p_0 \in \mathbb{Q}_\kappa^2$.
- (b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \in \text{sp}(p_\beta)$, then

$$c_\beta \in [\text{suc}_{p_\beta}(\nu_\beta)]^\kappa \text{ and}$$

$$p_\alpha = p_\beta(\nu_\beta, c_\beta) := \bigcup \{ p_\beta^{\langle \nu_\beta \hat{\ } \langle i \rangle \rangle} : i \in c_\beta \} \cup \bigcup \{ p_\beta^{\langle \eta \rangle} : \eta \not\triangleleft \nu_\beta \wedge \nu_\beta \not\triangleleft \eta \}$$

- (b2) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.

- (c) $p_\alpha = \bigcap \{ p_\beta : \beta < \alpha \}$ for limit $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}_\kappa^2$ and $\forall \beta < \lambda$, $p_\beta \leq_{\mathbb{Q}_\kappa^2} p_\lambda$.

Proof. We go by induction on λ . The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}_\kappa^2$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \supseteq t$ that is a splitting node in p_λ . We fix the smallest α such that $\nu_\alpha \supseteq_{p_0} t$ is a splitting node in p_0 . Then in p_0 there are no splitting nodes in $\{s : t \subseteq s \triangleleft \nu_\alpha\}$. Hence $\nu_\alpha \in \text{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle \nu^i : i < \gamma \rangle$ be an \triangleleft -increasing sequence of splitting nodes of p_λ with union $\nu \in \kappa^{<\kappa}$. Then ν is a splitting node of each p_α , $\alpha < \lambda$, and also in p_λ since $\langle \text{set}_{p_\alpha}(\nu) : \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ . \square

We need yet another type of fusion sequence.

Definition 3.3. Let $p \in \mathbb{Q}_\kappa^2$ and let $\nu \in \text{spl}(p)$.

- (1) Let $i \in \text{set}_p(\nu)$. We say η is the *shortest splitting node above $\nu \hat{\ } i$* in p and write $\eta = \text{next}_p(\nu \hat{\ } i)$ if η is the shortest splitting point in p such that $\eta \supseteq \nu \hat{\ } i$. Equality is allowed.
- (2) We say $F \subseteq p$ is the *front of next splitting nodes above ν* in p , if

$$F = \{\eta' \in \text{spl}(p) : \exists(\eta \in \text{suc}_p(\nu))(\eta' = \text{next}_p(\eta))\}.$$

Lemma 3.4. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ such that

$$(3.2) \quad \nu_\alpha \triangleleft \nu_\beta \rightarrow \alpha < \beta.$$

Let $\langle p_\alpha, \nu_\alpha, c_\alpha, F_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a sequence such that for any $\alpha \leq \lambda$ the following holds:

- (a) $p_0 \in \mathbb{Q}_\kappa^2$.
- (b1) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \in \text{sp}(p_\beta)$, then $c_\beta \in [\text{suc}_{p_\beta}(\nu_\beta)]^\kappa$, F_β contains for each $i \in c_\beta$ exactly one $\eta \in \text{spl}(p_\beta^{\langle \nu_\beta \hat{\ } i \rangle})$, and

$$p_\alpha = p_\beta(\nu_\beta, c_\beta, F_\beta) := \bigcup \{p_\beta^{(\eta)} : i \in c_\beta, \eta \in F_\beta\} \\ \cup \bigcup \{p_\beta^{(\eta)} : \eta \not\triangleleft \nu_\beta \wedge \nu_\beta \not\triangleleft \eta\}.$$

Note that this implies that F_β is the front of next splitting nodes of p_α above ν_β .

- (b2) If $\alpha = \beta + 1 < \kappa^{<\kappa}$ and $\nu_\beta \notin \text{spl}(p_\beta)$ then $p_\alpha = p_\beta$.

- (c) $p_\alpha = \bigcap \{p_\beta : \beta < \alpha\}$ for limit $\alpha \leq \kappa^{<\kappa}$.

Then for any $\lambda \leq \kappa^{<\kappa}$, $p_\lambda \in \mathbb{Q}_\kappa^2$ and $\forall \beta < \lambda$, $p_\beta \leq_{\mathbb{Q}_\kappa^2} p_\lambda$.

Proof. We go by induction on λ . The case $\lambda = 0$ and the successor steps are obvious. So we assume that $\lambda \leq \kappa^{<\kappa}$ is a limit ordinal and $p_\alpha \in \mathbb{Q}_\kappa^2$ for $\alpha < \lambda$. Since $\emptyset \in p_\lambda$, p_λ is not empty, and p_λ clearly is a tree. Let $t \in p_\lambda$. We show that there is $t' \supseteq t$ that is a splitting node in p_λ . We fix the smallest α such that $\nu_\alpha \supseteq_{p_0} t$ is a splitting node in p_0 . Then in p_0 there are no splitting nodes in $\{s : t \leq s \triangleleft \nu_\alpha\}$. Hence $\nu_\alpha \in \text{spl}(p_\beta)$ for any $\beta \in [0, \lambda]$.

Now we show that the limit of splitting nodes in p_λ is a splitting node. Let $\gamma < \lambda$ and let $\langle \nu^i : i < \gamma \rangle$ be an \triangleleft -increasing sequence of splitting nodes of p_λ with union $\nu \in \kappa^{<\kappa}$. Then ν is a splitting node of each p_α , $\alpha < \lambda$, and also in p_λ since $\langle \text{set}_{p_\alpha}(\nu) : \alpha < \lambda \rangle$ has at most two entries and their intersection has size κ . \square

In the special case $F_\beta = \{\nu_\beta \hat{\ } j : j \in c_\beta\}$, the construction of Lemma ?? coincides with the simpler construction from Lemma ??.

Definition 3.5. We assume \mathbb{Q}_κ^1 collapses 2^κ to ω . Let \mathcal{T} and $\mathcal{T} = \langle (a_\eta, n_\eta, \varrho) : \eta \in \omega^{>}(2^\kappa) \rangle$ be as in Lemma ???. Now let $Q_{\mathcal{T}}$ be the set of κ -Miller trees p such that for every $\nu \in \text{spl}(p)$ there is $\eta_{p,\nu} = \eta_\nu \in \omega^{>}(2^\kappa)$ such that

$$(3.3) \quad \text{set}_p(\nu) = \{\varepsilon \in \kappa : \nu \hat{\ } \varepsilon \in p\} = a_{\eta_\nu}.$$

By the properties of \mathcal{T} , the node $\eta_{p,\nu}$ is unique.

Lemma 3.6. *Assume that \mathbb{Q}_κ^1 collapses 2^κ to ω , let \mathcal{T} be chosen as in Lemma ??, and let $Q_{\mathcal{T}}$ be defined from \mathcal{T} as above. Then $Q_{\mathcal{T}}$ is dense in \mathbb{Q}_κ^2 .*

Proof. Let $p_0 = T \in \mathbb{Q}_\kappa^2$. Let $\langle \nu_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be an injective enumeration of $\kappa^{<\kappa}$ with property (??). We now define fusion sequence $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha \leq \kappa^\kappa \rangle$ according to the pattern in Lemma ?? in order to find $p_{\kappa^{<\kappa}} \geq T$ such that $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$.

Suppose that p_α and ν_α are given. If ν_α is not in p_α or is not a splitting node in p_α , then we let $p_{\alpha+1} = p_\alpha$. If $\nu_\alpha \in \text{spl}(p_\alpha)$, then according to Lemma ?? clause (d) there is $\eta \in \omega^{>(2^\kappa)}$ such that $\text{suc}_{p_\alpha}(\nu_\alpha) \supseteq a_\eta$. We choose such an η of minimal length and call it $\eta(\alpha)$.

Then we strengthen p_α to

$$(3.4) \quad p_{\alpha+1} = \bigcup \{ p_\alpha^{(\nu')} : \nu' = \nu_\alpha \hat{\ } i \wedge i \in a_{\eta(\alpha)} \} \cup \bigcup \{ p_\alpha^{(\eta)} : \eta \not\triangleleft \nu_\alpha \wedge \nu_\alpha \not\triangleleft \eta \}.$$

Now we have that

$$\eta_{p_{\alpha+1}, \nu_\alpha} = \eta(\alpha), c_\alpha = a_{\eta(\alpha)}.$$

For limit ordinals $\lambda \leq \kappa^{<\kappa}$, we let $p_\lambda = \bigcap \{ p_\beta : \beta < \lambda \}$. Since the sequence $\langle p_\alpha, \nu_\alpha, c_\alpha : \alpha \leq \kappa^{<\kappa} \rangle$ matches the pattern in Lemma ??, we have $p_{\kappa^{<\kappa}} \in \mathbb{Q}_\kappa^2$. By construction, for any $\alpha < \kappa^{<\kappa}$ for any $\delta \in [\alpha + 1, \kappa^{<\kappa})$, $\nu_\alpha \in \text{spl}(p_\delta)$ implies

$$\text{set}_{p_{\alpha+1}}(\nu_\alpha) = \text{set}_{p_\delta}(\nu_\alpha) = a_{\eta(\alpha)}.$$

Hence the condition $p = p_{\kappa^{<\kappa}}$ fulfils Equation (??) in its splitting node ν_α with witness $\eta_{p, \nu_\alpha} = \eta(\alpha)$. Since all nodes are enumerated, we have $p_{\kappa^{<\kappa}} \in Q_{\mathcal{T}}$. \square

We use only the inclusion $\text{set}_p(\nu) \subseteq a_{\eta_\nu}$ from Definition ??.

Definition 3.7. We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω and the \mathcal{T} is as in Lemma ?. For $T \in Q_{\mathcal{T}}$ and a splitting node ν of T we set $\varrho_{T,\nu} := \varrho_{\eta_{T,\nu}} \in \omega^{>(2^\kappa)}$. Recall $\eta_{T,\nu}$ is defined in Def. ??, and ϱ is a component of \mathcal{T} .

For $p \in Q_{\mathcal{T}}$, the relation $\nu \trianglelefteq \nu' \in p$ does neither imply $\eta_\nu \trianglelefteq \eta_{\nu'}$ nor $\varrho_\nu \trianglelefteq \varrho_{\nu'}$. However, $\eta_\nu \triangleleft \eta_{\nu'}$ implies $a_{\eta_\nu} \supset a_{\eta_{\nu'}}$ and $\varrho_\nu \triangleleft \varrho_{\nu'}$.

Observation 3.8. *We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω . Let $p_1, p_2 \in Q_{\mathcal{T}}$. If $p_1 \leq_{\mathbb{Q}_\kappa^2} p_2$ then for $\nu \in \text{spl}(p_2)$ we have $\nu \in \text{spl}(p_1)$ and $\varrho_{p_1,\nu} \trianglelefteq \varrho_{p_2,\nu}$.*

We introduce dense sets:

Definition 3.9. We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω . Let $n \in \omega$.

$$D_n = \{ p \in Q_{\mathcal{T}} : (\forall \nu \in \text{spl}(p)) (\text{lg}(\varrho_{p,\nu}) > n) \}.$$

D_n is open dense in $Q_{\mathcal{T}}$ and the intersection of the D_n is empty. The following technical lemma is the first step of a transformation of a \mathbb{Q}_{κ}^1 -name of a surjection from ω onto 2^κ into a \mathbb{Q}_{κ}^2 -name of such a surjection.

Lemma 3.10. *We assume that \mathbb{Q}_{κ}^1 collapses 2^κ to ω , $\text{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate \mathbb{Q}_{κ}^2 such that each Miller tree appears 2^κ times. There is $\langle (p_\alpha, n_\alpha, \bar{\gamma}_\alpha) : \alpha < 2^\kappa \rangle$ such that*

- (a) $n_\alpha < \omega$,
- (b) $p_\alpha \in D_{n_\alpha}$ and $p_\alpha \geq T_\alpha$.
- (c) If $\beta < \alpha$ and $n_\beta \geq n_\alpha$ then $p_\beta \perp p_\alpha$.
- (d) $\bar{\gamma}_\alpha = \langle \gamma_{\alpha,\nu} : \nu \in \text{spl}(p_\alpha) \rangle$.
- (e) $(\forall \nu \in \text{spl}(p_\alpha))(a_{\eta_{p_\alpha,\nu}} \Vdash_{\mathbb{Q}_{\kappa}^1} \gamma_{\alpha,\nu} \in \text{range}(\varrho_{p_\alpha,\nu}))$.
- (f) $\gamma_{\alpha,\nu} \in 2^\kappa \setminus W_{<\alpha,\nu}$ with

$$W_{<\alpha,\nu} = \bigcup \{ \text{range}(\varrho_{p_\beta,\nu}) : \beta < \alpha, \nu \in \text{spl}(p_\beta) \}.$$

Proof. Assume that $\langle (p_\beta, n_\beta, \bar{\gamma}_\beta) : \beta < \alpha \rangle$ has been defined and we are to define $(p_\alpha, n_\alpha, \bar{\gamma}_\alpha)$. Note that the p_β need not be increasing in strength.

- (\oplus)₁ The choice of the a_η in Lemma ?? and the choice $Q_{\mathcal{T}}$ and of $\eta_{p_\beta,\nu}$ for $\nu \in \text{spl}(p_\beta)$, $\beta < \alpha$, imply that the set $W_{<\alpha,\nu}$ is well defined and of cardinality $\leq |\alpha| + \aleph_0 < 2^\kappa$. Hence we can choose $\gamma_{\alpha,\nu} \in 2^\kappa \setminus W_{<\alpha,\nu}$.
- (\oplus)₂ With the fusion Lemma ?? we choose $q_\alpha \geq T_\alpha$, $q_\alpha \in Q_{\mathcal{T}}$, such that

$$(\forall \nu \in \text{spl}(q_\alpha))(a_{\eta_{q_\alpha,\nu}} \Vdash_{\mathbb{Q}_{\kappa}^1} \gamma_{\alpha,\nu} \in \text{range}(\varrho_{q_\alpha,\nu})).$$

- (\oplus)₃ Let $q \in \mathbb{Q}_{\kappa}^2$. For $n \in \omega$ and $\nu \in \text{spl}(q)$ we let

$$\mathcal{U}_{\alpha,\nu,n}(q) = \{ \beta < \alpha : n_\beta = n, \nu \in \text{spl}(p_\beta) \wedge |\text{set}_q(\nu) \cap \text{set}_{p_\beta}(\nu)| = \kappa \}.$$

$$\mathcal{U}_{\alpha,\nu}(q) = \bigcup \{ \mathcal{U}_{\alpha,\nu,n}(q) : n \in \omega \}.$$

- (\oplus)₄ (a) If $n \in \omega$ and $\nu \in \text{spl}(q_\alpha)$ then

$$\beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha) \rightarrow \varrho_{p_\beta,\nu} \trianglelefteq \varrho_{q_\alpha,\nu}.$$

This is seen as follows. We let $a = \text{set}_{p_\beta}(\nu) \cap \text{set}_{q_\alpha}(\nu)$. Since $\beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha)$, $a \in [\kappa]^\kappa$. Clearly $a \Vdash_{\mathbb{Q}_{\kappa}^1} \mathcal{T} \triangleright \varrho_{p_\beta,\nu}, \varrho_{q_\alpha,\nu}$. So either $\varrho_{p_\beta,\nu} \triangleleft \varrho_{q_\alpha,\nu}$ or $\varrho_{p_\beta,\nu} \supseteq \varrho_{q_\alpha,\nu}$. However, since $\gamma_{\alpha,\nu} \in \text{range}(\varrho_{q_\alpha,\nu}) \setminus W_{<\alpha,\nu}$, only $\varrho_{q_\alpha,\nu} \triangleright \varrho_{p_\beta,\nu}$ is possible.

- (b) So for $\nu \in \text{spl}(q_\alpha)$, the set $\{ \varrho_{p_\beta,\nu} : \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha) \}$ has at most $\text{lg}(\varrho_{q_\alpha,\nu})$ elements.
- (c) The assignment $\beta \mapsto \varrho_{p_\beta,\nu}$ is defined between $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ and $\{ \varrho_{p_\beta,\nu} : \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha) \}$. According to properties (e) and (f) in the induction hypothesis, the assignment is injective, and hence $|\mathcal{U}_{\alpha,\nu}(q_\alpha)| \leq \text{lg}(\varrho_{q_\alpha,\nu})$.
- (d) We state for further use that $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ is finite and for any $q \geq q_\alpha$, $\mathcal{U}_{\alpha,\nu}(q) \subseteq \mathcal{U}_{\alpha,\nu}(q_\alpha)$.

(\oplus)₅ We look at the cone above q_α and show:

$$(3.5) \quad \begin{aligned} & (\forall q \geq q_\alpha)(\forall \nu \in \text{spl}(q))(\exists r_{\alpha,\nu} \geq_{\mathbb{Q}_\kappa^2} q) \\ & (\exists c \in [\text{set}_q(\nu)]^\kappa)(\exists F \subseteq \{\eta \in \text{spl}(q) : \eta \triangleright \nu\}) \\ & (r_{\alpha,\nu} = q(\nu, c, F) \wedge (\forall \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha))(r_{\alpha,\nu}^{\langle \nu \rangle} \perp p_\beta^{\langle \nu \rangle} \vee p_\beta^{\langle \nu \rangle} \leq r_{\alpha,\nu}^{\langle \nu \rangle})). \end{aligned}$$

How do we find $r_{\alpha,\nu} = r_{\alpha,\nu}(q)$? Given $q \geq_{\mathbb{Q}_\kappa^2} q_\alpha$, $\nu \in \text{spl}(q)$ we enumerate $\mathcal{U}_{\alpha,\nu}(q_\alpha)$ as $\beta_0, \dots, \beta_{k-1}$. We let $r_0 = q$ and by induction on $i \leq k$ we define r_i , increasing in strength, with $\nu \in \text{spl}(r_i)$ and $c_i = \text{set}_{r_i}(\nu)$. Thus the c_i are \subseteq -decreasing sets of size κ . Given r_i , we distinguish cases:

First case: $\beta_i \notin \mathcal{U}_{\alpha,\nu}(r_i)$. Then there is $c_{i+1} \in [\text{set}_{r_i}(\nu)]^\kappa$, $c_{i+1} \cap \text{set}_{p_{\beta_i}}(\nu) = \emptyset$. We let $r_{i+1} = r_i(\nu, c_{i+1})$ and thus have $r_{i+1}^{\langle \nu \rangle} \perp p_{\beta_i}$.

Second case: $\beta_i \in \mathcal{U}_{\alpha,\nu}(r_i)$. We let

$$c_i = \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \wedge \langle j \rangle \rangle} \geq p_{\beta_i}^{\langle \nu \wedge \langle j \rangle \rangle}\} \cup \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \wedge \langle j \rangle \rangle} \not\geq p_{\beta_i}^{\langle \nu \wedge \langle j \rangle \rangle}\}.$$

If $c_{i,1} = \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \wedge \langle j \rangle \rangle} \geq p_{\beta_i}^{\langle \nu \wedge \langle j \rangle \rangle}\}$ has size κ , then we let $c_{i+1} = c_{i,1}$ and $r_{i+1} = r_i(\nu, c_{i+1})$ and thus get $r_{i+1}^{\langle \nu \rangle} \geq p_{\beta_i}$.

If $|c_{i,1}| < \kappa$, then $c_{i,2} = \{j \in \text{set}_{r_i}(\nu) : r_i^{\langle \nu \wedge \langle j \rangle \rangle} \not\geq p_{\beta_i}^{\langle \nu \wedge \langle j \rangle \rangle}\}$ has size κ , and we let $c_{i+1} = c_{i,2}$. For $j \in c_{i+1}$, $r_i^{\langle \nu \wedge \langle j \rangle \rangle} \not\geq p_{\beta_i}^{\langle \nu \wedge \langle j \rangle \rangle}$. Thus we can find a node in the $r_i^{\langle \nu \wedge \langle j \rangle \rangle} \setminus p_{\beta_i}^{\langle \nu \wedge \langle j \rangle \rangle}$ and above this node we find a splitting node of r_i . We take this latter splitting node into r_{i+1} as the direct successor splitting node to $\nu \wedge \langle j \rangle$. Doing so for every $j \in c_{i+1}$ we get $F_{\nu,i}$, a front strictly above ν in $r_{i+1} = r_i(\nu, c_{i+1}, F_{\nu,i})$. Again we get $r_{i+1}^{\langle \nu \rangle} \perp p_{\beta_i}$.

In the end we let $r_{\alpha,\nu} = r_k$. There is a front F that contains for each $j \in c_k$ the shortest splitting node of r_k above $\nu \wedge \langle j \rangle$. Thus we have $r_k = r_{\alpha,\nu} = q(\nu, c_k, F)$ and $r_{\alpha,\nu}$ fulfils (??).

(\oplus)₆ Now we use (\oplus)₅ iteratively along all $\nu \in \kappa^{<\kappa}$ to find a fusion sequence $\langle r_{\alpha,\nu}, \nu, c_\nu, F_\nu : \nu < \kappa^{<\kappa} \rangle$ with starting point $q_\alpha = r_{0,\nu_0}$. In this sequence, $r_{\alpha,\nu}$ is chosen as $r_{\alpha,\nu}(q)$ in \oplus_5 for $q = \bigcap_{\beta < \alpha} r_\beta$, if $\nu \in \text{spl}(q)$. If $\nu \notin \text{spl}(q)$, then $r_{\alpha,\nu} = q$. Then we apply the fusion Lemma ?? and get an upper bound r_α of $r_{\alpha,\nu}$, $\nu \in \kappa^{>\kappa}$. Note $r_\alpha^{\langle \nu \rangle} \perp p_\beta$ iff $r_\alpha^{\langle \nu \rangle} \perp p_\beta^{\langle \nu \rangle}$ and $r_\alpha^{\langle \nu \rangle} \geq p_\beta$ iff $r_\alpha^{\langle \nu \rangle} \geq p_\beta^{\langle \nu \rangle}$. Hence $r_\alpha \geq q_\alpha$ and

$$(\forall \nu \in \text{spl}(r_\alpha))(\forall \beta \in \mathcal{U}_{\alpha,\nu}(q_\alpha))(r_\alpha^{\langle \nu \rangle} \perp p_\beta \vee p_\beta \leq r_\alpha^{\langle \nu \rangle}).$$

(\oplus)₇ Finally we choose n_α and p_α . There are k and ν such that $n < \omega$ and $\nu \in \text{spl}(r_\alpha)$ such that $p_\alpha = r_\alpha^{\langle \nu \rangle}$ fulfils

$$(\forall \beta < \alpha)(n_\beta \geq k \rightarrow p_\alpha \perp p_\beta).$$

Proof of existence. By induction on $k \in \omega$ we try to find $\langle \nu_k, \beta_k : k \in \omega \rangle$ such that

- (a) $\nu_k \in \text{spl}(r_\alpha)$,
- (b) $\nu_k \triangleleft \nu_m$ for $k < m$,
- (c) $\beta_k < \alpha$ and $n_{\beta_k} \geq k$ and $r_\alpha^{\langle \nu_k \rangle} \geq p_{\beta_k}$.

If we succeed, then $\nu_* = \bigcup \{\nu_k : k \in \omega\} = \nu^* \in \text{spl}(r_\alpha)$ by Definition ?? (2). Here we use that $\text{cf}(\kappa) > \omega$. Hence

$$r_\alpha^{\langle \nu^* \rangle} \in Q_{\mathcal{T}} \cap \bigcap \{D_k : k < \omega\} \text{ and}$$

$a_{\eta_{r_\alpha, \nu^*}}$ determines in $\Vdash_{\mathbb{Q}_\kappa^1}$ for any $k < \omega$ the value of $\tau \upharpoonright k$.

This is a contradiction.

So there is a smallest k such that ν_k cannot be defined. We let $n_\alpha = k$. We let p_α be a strengthening of $r_\alpha^{\langle \nu_{k-1} \rangle}$ such that $p_\alpha \in D_{n_\alpha}$. For finding such a strengthening we again invoke the fusion Lemma ??.

We show that $p_\alpha \perp p_\beta$ for $\beta < \alpha$ with $n_\beta \geq k$. Otherwise, having arrived at $r_\alpha^{\langle \nu_{k-1} \rangle}$ we find some β_k, α such that $n_{\beta_k} \geq k$ and $r_\alpha^{\langle \nu_{k-1} \rangle}$ is compatible with p_{β_k} . Then we can prolong ν_{k-1} to a splitting node $\nu_k \in \text{spl}(p_{\beta_k}) \cap \text{spl}(r_\alpha)$. By the choice of r_α the latter implies that $r_\alpha^{\langle \nu_k \rangle} \geq p_{\beta_k}$. However, now we would have found ν_k, β_k as required in contradiction to the choice of k . □

Remark 3.11. Conditions (a) to (c) of Lemma ?? yield: For any $k < \omega$,

$$\{p_\alpha : n_\alpha \geq k\} \text{ is dense in } \mathbb{Q}_\kappa^2.$$

Proof. Let k and p be given. There is α_0 such that $T_{\alpha_0} \in D_0$ and $T_{\alpha_0} \geq_{\mathbb{Q}_\kappa^2} p$. Then $p_{\alpha_0} \geq T_{\alpha_0}$ and $n_{\alpha_0} > 0$. Then there is $\alpha_1 > \alpha_0$ such that $T_{\alpha_1} \geq_{\mathbb{Q}_\kappa^2} p_{\alpha_0}$. Then $p_{\alpha_1} \geq T_{\alpha_1}$ and hence by condition (c), $n_{\alpha_1} > n_{\alpha_0} \geq 0$. We can repeat the argument $k - 1$ times. □

Now we drop the component $\bar{\gamma}_\alpha$ from a sequence $\langle p_\alpha, n_\alpha, \bar{\gamma}_\alpha : \alpha < 2^\kappa \rangle$ given by Lemma ??. Then we get a sequence with properties (a), (b), and a weakening (c) with the property stated in the remark.

Lemma 3.12. *We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω , $\text{cf}(\kappa) > \omega$ and $2^{(2^{<\kappa})} = 2^\kappa$. Let $\langle T_\alpha : \alpha < 2^\kappa \rangle$ enumerate all Miller trees that such each tree appears 2^κ times. If $\langle (p_\alpha, n_\alpha) : \alpha < 2^\kappa \rangle$ are such that*

- (a) $n_\alpha < \omega$,
- (b) $p_\alpha \in D_{n_\alpha}$ and $p_\alpha \geq T_\alpha$,
- (c) if $\beta < \alpha$ and $n_\beta = n_\alpha$ then $p_\beta \perp p_\alpha$,
- (d) for any $k \in \omega$, $\{p_\alpha : n_\alpha \geq k\}$ is dense in \mathbb{Q}_κ^2 .

Then there is a \mathbb{Q}_κ^2 -name τ' for a surjection of ω onto 2^κ .

Proof. Let G be a \mathbb{Q}_κ^2 -generic filter over \mathbf{V} . We define $\tau(n)$, a \mathbb{Q}_κ^2 -name by $\tau(n)[G] = \alpha$ if $p_\alpha \in G$ and $n_\alpha = n$. The name τ is a name of a function by (c). By (d), the domain of τ is forced to be infinite. For any $p \in \mathbb{Q}_\kappa^2$ we let $U_p = \{\alpha : T_\alpha = p\}$. U_p is of size 2^κ , in particular for $\alpha \in 2^\kappa$ we have $|U_{p_\alpha}| = 2^\kappa$. Hence there is $f: 2^\kappa \rightarrow 2^\kappa$ such that for any $\alpha, \gamma \in 2^\kappa$ and $\exists \beta \in U_{p_\gamma}$ with $f(\beta) = \alpha$. We let $\tau'(n) = f(\tau(n))$. Next we show

$$\mathbb{Q}_\kappa^2 \Vdash \text{range}(\tau') = 2^\kappa.$$

Suppose $p \in \mathbb{Q}_\kappa^2$ and $\alpha < 2^\kappa$ are given. By construction the sequence $\{p_\beta : \beta < 2^\kappa\}$ is dense. Let $p \leq p_\gamma$. Then there is $\beta \in U_{p_\gamma}$, with $f(\beta) = \alpha$. However, $\beta \in U_{p_\gamma}$ means $T_\beta = p_\gamma \leq p_\beta$ by construction. By the definition of τ , $p_\beta \Vdash \tau(n_\beta) = \beta$, so $p_\beta \Vdash f(\tau(n_\beta)) = \alpha$. \square

So we can sum up:

Theorem 3.13. *We assume that \mathbb{Q}_κ^1 collapses 2^κ to ω and $\text{cf}(\kappa) > \omega$ and $2^{(\kappa^{<\kappa})} = 2^\kappa$. Then the forcing with \mathbb{Q}_κ^2 collapses 2^κ to \aleph_0 .*

4. κ -COHEN REALS AND THE LEVY COLLAPSE

Another vice of a κ -tree forcing is to add κ -Cohen reals. In this section we show that under the above conditions, \mathbb{Q}_2^κ adds Cohen reals and is equivalent to the Levy collapse of 2^κ to \aleph_0 .

Lemma 4.1. *If \mathbb{P} collapses 2^κ to \aleph_0 , $\text{cf}(\kappa) > \aleph_0$, and $2^{2^{<\kappa}} = 2^\kappa$, then \mathbb{Q}_2^κ adds a κ -Cohen real.*

Proof. Let G be \mathbb{Q}_2^κ -generic over \mathbf{V} . Let $f: \omega \rightarrow 2^{<\kappa}$ be a function in $\mathbf{V}[G]$, such that $(\forall \eta \in 2^{<\kappa})(\exists^\infty k f(k) = \eta)$. Such a function exists since $2^{<\kappa} \leq 2^\kappa$.

Since $2^{2^{<\kappa}} = 2^\kappa$, we can enumerate all antichains in $\mathbb{C}(\kappa)$ in $\alpha_* \leq 2^\kappa$ many steps. In $\mathbf{V}[G]$, α_* is countable. We list it as $\langle \alpha_n : n < \omega \rangle$. Now we choose $\eta_n \in \mathbb{C}(\kappa)^\mathbf{V}$ by induction on n in $\mathbf{V}[G]$: $\eta_0 = \emptyset$. Given η_n we choose k_n such that $f(k_n) = \eta_n$ and then we choose $\eta_{n+1} \supseteq \eta_n$, such that $\eta_{n+1} \in I_{\alpha_n}$. Then $\{\eta : (\exists n < \omega)(\eta \sqsubseteq f(k_n))\}$ is a $\mathbb{C}(\kappa)$ -generic filter over \mathbf{V} and it exists in $\mathbf{V}[G]$, since it is definable from $\{f(k_n) : n < \omega\}$. \square

Two forcings $\mathbb{P}_1, \mathbb{P}_2$ are said to be equivalent if their regular open algebras $\text{RO}(\mathbb{P}_i)$ coincide (for a definition of the regular open algebra of a poset, see, e.g., [?, Corollary 14.12]). Some forcings are characterised up to equivalence just by their size and their collapsing behaviour.

Definition 4.2. Let B be a Boolean algebra. We write $B^+ = B \setminus \{0\}$. A subset $D \subseteq B^+$ is called *dense* if $(\forall b \in B^+)(\exists d \in D)(d \leq b)$. The *density* of a Boolean algebra B is the least size of a dense subset of B . A Boolean algebra B has uniform density if for every $a \in B^+$, $B \upharpoonright a$ has the same density. The *density* of a forcing order $(\mathbb{P}, <)$ is the density of the regular open algebra $\text{RO}(\mathbb{P})$.

Lemma 4.3. [?, Lemma 26.7]. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto \aleph_0 , i.e.,*

$$0_Q \Vdash_Q |\check{\lambda}| = \aleph_0.$$

Then $\text{RO}(Q) = \text{Levy}(\aleph_0, \lambda)$.

Lemma 4.4. *If \mathbb{Q}_κ^1 collapses 2^κ to \aleph_0 , then \mathbb{Q}_κ^1 is equivalent of $\text{Levy}(\aleph_0, 2^\kappa)$.*

Proof. \mathbb{Q}_κ^1 has size 2^κ . Hence Lemma ?? yields $\text{RO}(\mathbb{Q}_\kappa^1) = \text{Levy}(\aleph_0, 2^\kappa)$. \square

Definition 4.5. A Boolean algebra is (θ, λ) -nowhere distributive if there are antichains $\bar{p}^\varepsilon = \langle p_\alpha^\varepsilon : \alpha < \alpha_\varepsilon \rangle$ of \mathbb{P} for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta$

$$|\{\alpha < \alpha_\varepsilon : p \not\leq p_\alpha^\varepsilon\}| \geq \lambda.$$

Lemma 4.6. [?, Theorem 1.15] *Let $\theta < \lambda$ be regular cardinals.*

(1) *Suppose that \mathbb{P} has the following properties (a) to (c).*

(a) *\mathbb{P} is a (θ, λ) -nowhere distributive forcing notion,*

(b) *\mathbb{P} has density λ ,*

(c) *in case $\theta > \aleph_0$, \mathbb{P} has a θ -complete subset S . The latter means:
 $(\forall B \in [S]^{<\theta})(\exists s \in S)(\forall b \in B)(b \leq_{\mathbb{P}} s)$.*

Then \mathbb{P} is equivalent to $\text{Levy}(\theta, \lambda)$.

(2) *Under (a) and (b) \mathbb{P} collapses λ to θ (and may or may not collapse \aleph_0).*

Proposition 4.7. *If there is a κ -mad family of size 2^κ the forcing \mathbb{Q}_κ^1 is $(\aleph_0, 2^\kappa)$ -nowhere distributive.*

Proof. Lemma ?? gives \mathcal{T} such that $\bar{p}^n = \{a_\eta : \eta \in {}^n(2^\kappa)\}$, $n \in \omega$, witnesses $(\aleph_0, 2^\kappa)$ -nowhere distributivity. \square

By Lemma ?? and Theorem ?? we get:

Proposition 4.8. *If \mathbb{Q}_κ^1 collapses 2^κ to \aleph_0 , $\text{cf}(\kappa) > \aleph$ and $2^{(\kappa < \kappa)} = 2^\kappa$ then \mathbb{Q}_κ^2 is equivalent to $\text{Levy}(\aleph_0, 2^\kappa)$.*

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