MORE FORCING FOR NO ULTRAFILTERS
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Abstract. We prove the consistency of "no α-ultrafilters" for $\alpha \geq 1$ a countable ordinal and no van-Dowen ultrafilter on $\mathbb{Q}$. This continues [She98b] where we prove the consistency of "there is no NWD (nowhere dense) ultrafilter on $\mathbb{N}$".

But we first deal with relatives of the forcing from [She98b]; of self interest in a self contained way.

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Anotated Content

§0 Introduction, pg. 3

[Revise the introduction of [She98b].]

§1 The Forcing $\mathbb{Q}^\ell_{\mathbf{i}}$ for $\ell = 1, 2, 3$, pg.5

[We repeat [She98b, §1] adding one more forcing to the $\ell = 3$ version for which $\leq^\otimes_n$ works.]

§2 A Creature Forcing, pg. 13

[We define and investigate the forcing notion $\mathbb{Q}^4_{\mathbf{i}}$.]

§3 Consistency of no α-ultrafilter and no van-Dowen-ultrafilter, pg. 24

§ 0. Introduction

In [She98b] we prove the consistency of "there is no NWD-ultrafilter on ω" (see below, non-principal, of course). This answers a question of van Douwen [vD81] which appears as question 31 of [Bau95]. Baumgartner [Bau95] considering this question, dealt more generally with J-ultrafilters where:

Definition 0.1. 1) An ultrafilter D, say on ω, is called a J-ultrafilter where J is an ideal on some set X (to which all singletons belong, to avoid trivialities) when for every function $f : \omega \longrightarrow X$ for some $A \in D$ we have $f''(A) \in J$.
2) The NWD-ultrafilters are the J-ultrafilters for $J = \{B \subseteq \mathbb{Q} : B$ is nowhere dense$\}$, ($\mathbb{Q}$ is the set of all rationals; we may use an equivalent version, see [She98b, 2.4]).
3) An ultrafilter D is called a J^1_α-ultrafilter for a (countable) ordinal $\alpha \geq 1$ when it is J^1_α-ultrafilter where $J^1_\alpha = \{A \subseteq \omega^\alpha : \mathrm{otp}(A) < \omega^\alpha\}$, where ω^α is an ordinal exponentiation.
3A) An ultrafilter D is called an α-ultrafilter when it is a J^0_α-ultrafilter where $J^0_\alpha = \{A \subseteq \alpha : \mathrm{otp}(A) < \alpha\}$.
4) A van-Dowen ultrafilter is one on $\mathbb{Q}$ such that the family of $A \subseteq \mathbb{Q}$ dense in themselves is dense in it.

The non-existence of NWD ultrafilters is also relevant for the consistency of "every (non-trivial) σ-centered forcing notion adds a Cohen real", see [BS01].

The most natural approach to a proof of the consistency of "there is no NWD-ultrafilter" was to generalize the proof of CON(there is no P-point) (see [She82, Ch.VI,§4] or [She98a, Ch.VI,§4]), but for some time I (and probably others) have not seen how.

We use in [She98b] an idea taken from [She92], which is to replace the given maximal ideal I on ω by a quotient; moreover, we allow ourselves to change the quotient. In fact, the forcing here is simpler than the one in [She92]. A related earlier work is Goldstern Shelah [GS90], it uses a "one real" version of $\mathbb{Q}^1_{\mathbf{i}}$ from §1.

We similarly may consider the consistency of "no J^1_α–ultrafilter" for non-zero $\alpha < \omega_1$ (see [Bau95] for discussion of α–ultrafilters). This question and the problems of preservation of ultrafilters and distinguishing existence properties of ultrafilters was promised in [She98b] to be dealt with in the subsequent work [S$^+$]; we try to deal here with the first and with van-Dowen ultrafilters; the second is still in preparation.

We still do not know whether we can have different answers to different such α's.

Discussion 0.2. In §1 we use $\mathbf{i} \in \mathrm{FP}_1$, a forcing parameter. We will have $\mathbb{Q}^\ell_{\mathbf{i}}$ for $\ell = 1, 2, 3$ and $\mathbb{Q}^4_{\mathbf{i}}$. Now $\ell = 1, 2$ are as in [She98b]; for $\ell = 1$, a kind of power of [GS90], for $\ell = 2$ a kind of power of [She92]. Now for $\ell = 1$, all n/E^p behave as in n, hence $\leq^\otimes_n$ does not work. For $\ell = 3$ we make $\leq^\otimes_n$ work but it is not good enough for our purpose (no α-ultrafilter). In §2 we combine forcing from §1 with creatures, see [RS99]. The $\mathbb{Q}_{\mathbf{i}}$ for $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$ from there combine the desired properties mentioned above.

How do we prove in [She98b] that by suitable iteration $\mathbf{q}$ of $\mathbb{Q}^1_{\mathbf{i}}$'s in $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$ there is no nowhere dense ultrafilter? In the end, i.e. in $\mathbf{V}^{\mathbb{P}_{\mathbf{q}}}$ toward proving "no nowhere dense ultrafilter" for a candidate $\underset{\sim}{D}$ we try $\langle \underset{\sim}{\eta}_n : n < \omega \rangle$, so toward contradiction $p_* \Vdash$ "$\underset{\sim}{X} \in \mathrm{fil}(I_{\mathbf{m}})$ satisfies $\{\underset{\sim}{\eta}_n : n \in \underset{\sim}{X}\}$ is nowhere dense". Without loss of generality above p_* we can read $\underset{\sim}{X}$ promptly, i.e. if $n \in A^{p_*}$ and $f : \{x^m_i : m \in A^{p_*} \cap (n+1), i < h(n)\} \Rightarrow \{-1, 1\}$ then $p^{[f]}$ read (i.e. forces a value to) $\underset{\sim}{X} \cap (n+1)$ and even p_* forces $u_n \cap \underset{\sim}{X} \neq \emptyset$ and moreover if $n_1 < n_2$ are from A then $\underset{\sim}{X} \cap [n_1, n_2] \neq \emptyset$ and moreover the members of u_n are not in $\cup\{m/E^p : m \in A^p \cap (n+1)$ and $u_n \subseteq \min(A^p \backslash (n+1))\}$. Then find q above p_* forcing density.

We continue this in our proof in §3.

§ 1. The basic forcing

In Definition 1.2 below we define the forcing notion $\mathbb{Q}^1_{\mathbf{i}}$ which will be the one used in the proof of the main result 3.10, 3.17. Among the other forcing notions defined below, $\mathbb{Q}^3_{\mathbf{i}}$ is the closest relative of $\mathbb{Q}^1_{\mathbf{i}}$. Various properties may be easier to check for some relatives but it is more complicated to define, anyhow unfortunately it does not do the job but we feel are of interest. In [She00] we have eventually used only $\mathbb{Q}^1_{\mathbf{i}}$.

Definition 1.1. Let I be an ideal on ω containing the family $[\omega]^{<\aleph_0}$ of finite subsets of ω.
1) We say that an equivalence relation E is an I–equivalence relation <u>when</u>:

(a) $\mathrm{dom}(E) \subseteq \omega$,
(b) $\omega \setminus \mathrm{dom}(E) \in I$,
(c) each E-equivalence class belongs to I.

2) For I-equivalence relations E_1, E_2 we write $E_1 \leq E_2$ if:

(i) $\mathrm{dom}(E_2) \subseteq \mathrm{dom}(E_1)$,
(ii) $E_1 \restriction \mathrm{dom}(E_2)$ refines E_2,
(iii) $\mathrm{dom}(E_2)$ is the union of a family of E_1-equivalence classes.

3) We say I is a P-c.c.c. ideal <u>when</u>:

(a) I is an ideal on ω containing all the finite subsets
(b) I is a P-filter, i.e. if $A_n \in I$ for $n < \omega$ then for some $A \in I$ we have $n < \omega \Rightarrow A_n \subseteq^* A$
(c) $\mathscr{P}(\omega)/I$ is a c.c.c. Boolean Algebra.

Definition 1.2. 1) Let FP_1 be the set of (forcing parameters) $\mathbf{i}$ which means it consists of:

(a) I, an ideal on ω to which all finite subsets of ω belong; let $D_{\mathbf{i}} = \mathrm{dual}(I)$
(b) $h : \omega \longrightarrow \omega$ be a non-decreasing function
(c) h goes to infinity (if $h(n) = n$ we may omit it).

2) For $\mathbf{i} \in \mathrm{FP}_1$ and $\ell = 1, 2, 3$ we define a forcing notion $\mathbb{Q}^\ell_{\mathbf{i}}$ intended to add $\langle y^n_i : i < h(n), n < \omega \rangle$ with $y^n_i \in \{-1, 1\}$. We use x^n_i as variables.
3) $p \in \mathbb{Q}^\ell_{\mathbf{i}}$ if and only if $p = (H, E, A) = (H^p, E^p, A^p)$ and:

(a) E is an I-equivalence relation, so on $\mathrm{dom}(E) \subseteq \omega$,
(b) $A = \{n \in \mathrm{dom}(E) : n = \min(n/E)\}$,
(c) if $\ell = 1$, <u>then</u> H is a function with range $\subseteq \{-1, 1\}$ and domain $B^p_1 = \{x^n_i : i < h(n), n \in \omega \setminus \mathrm{dom}(E)$ <u>or</u> $n \in \mathrm{dom}(E) \wedge i \in [h(\min(n/E)), h(n))\}$,
(d) if $\ell = 2$, <u>then</u>
 (α) H is a function with domain $\mathrm{dom}(H) = B^p_2 \cup B^p_3$, where
 $B^p_2 = \{x^m_i : m < \omega, A^p \cap (m+1) = \emptyset, i < h(m)\}$ and
 $B^p_3 = \{x^m_i : i < h(m) \wedge m \in \mathrm{dom}(E^p) \setminus A^p$ <u>or</u> $m \notin \mathrm{dom}(E^p)$ but $A^p \cap m \neq \emptyset, i < h(m)\}$,

(β) for $x_i^m \in B_3^p, H(x_i^m)$ is a function in the variables $\{x_j^n : (n,j) \in w_p(m,i)\}$ to $\{-1,1\}$, where $w_p(m) = w_p(m,i) = \{(\ell,j) : \ell \in A^p \cap m$ and $j < h(\ell)\}$; for $n \in A^p$ we stipulate $H^p(x_i^n) = x_i^n$

(γ) $H \restriction B_2^p$ is a function to $\{-1,1\}$.

(e) if $\ell = 2$ and $n \in \mathrm{Dom}(E^p), x_i^n \in B_3^p, n^* = \min(n/E^p) < n$ and $y_i^m \in \{-1,1\}$ for $m \in A^p \cap n \backslash \{n^*\}, i < h(m)$ and $z_j^n \in \{-1,1\}$ for $j < h(n^*)$, <u>then</u> for some $y_j^{n^*} \in \{-1,1\}$ for $j < h(n^*)$ we have $j < h(n^*) \Rightarrow z_j^n = (H^p(x_j^n))(\ldots, y_i^m, \ldots)_{(m,i)\in w_p(n,j)}$

(f) if $\ell = 3$ then H^p is a function from $B_0^p \cup B_1^p$ into $\{-1,1\}$ where (B_1^p is as defined above and) $B_0^p = \{x_i^n : n \in \mathrm{Dom}(E^p) \backslash A^p$ and $i < h(\min(n/E^p)\}$.

4) For $a \subseteq A^p$ we define $\mathscr{F}_a = \{f : f$ is a function with domain $\{x_i^n : i < h(n), n \in a\}$ such that $f(x_i^n) \in \{-1,1\}\}$.

When it cannot cause any confusion, or we mean "for $\ell = 1,2,3$", we may omit the superscript ℓ.

5) Defining functions like $H(x_i^m)$, $x_i^m \in B_3^p$ (when $\ell = 2$), we may allow to use dummy variables. In particular, if $H^p(x_i^m)$ is $-1,1$ we identify it with constant functions with this value.

6) We say that a function $f : \{x_i^n : i < h(n),\ n < \omega\} \longrightarrow \{-1,1\}$ satisfies a condition $p \in \mathbb{Q}_{\mathbf{i}}^\ell$ <u>when</u>:

(a) $f(x_i^n) = H^p(x_i^n)$ <u>when</u> one of the following occurs:

(α) $x_i^n \in B_1^p$ and $\ell = 1,3$

(β) $x_i^n \in B_2^p$ and $\ell = 2$,

(b) $f(x_i^n) = H^p(x_i^n)(\ldots, f(x_j^m), \ldots)_{(m,j)\in w_p(n,i)}$ <u>when</u> $\ell = 2$ and $x_i^n \in B_3^p$

(c) $f(x_i^n) = (f(x_i^{\min(n/E^p)}))$ <u>when</u> $\ell = 1$, $n \in \mathrm{dom}(E^p)$ and $i < h(\min(n/E^p))$

(d) $f(x_i^n) = (H^p(x_i^n))\cdot(f(x_i^{\min(n/E^p)}))$ when $\ell = 3, x_i^n \in B_0^p$, i.e. $n \in \mathrm{dom}(E^p)\backslash A^p$ and $i < h(\min(n/E^p))$, i.e. $x_i^n \in B_3^p$.

7) The partial order $\leq = \leq_{\mathbb{Q}_{\mathbf{i}}^\ell}$ is defined by $p \leq q$ if and only if:

(α) $E^p \leq E^q$, i.e.

- $\mathrm{dom}(E^q) \subseteq \mathrm{dom}(E^p)$
- if $n \in \mathrm{dom}(E^q)$ then $n/E^p \subseteq \mathrm{dom}(E^q)$
- $E^p \restriction \mathrm{dom}(E^q)$ refines E^q

(β) every function $f : \{x_i^n : i < h(n),\ n < \omega)\} \longrightarrow \{-1,1\}$ satisfying q satisfies p.

Proposition 1.3. $(\mathbb{Q}_{\mathbf{i}}^\ell, \leq_{\mathbb{Q}_{\mathbf{i}}^\ell})$ *is a partial order.*

Remark 1.4. We may reformulate the definition of the partial orders $\leq_{\mathbb{Q}_{\mathbf{i}}^\ell}$, making them perhaps more direct. Thus, in particular, if $p, q \in \mathbb{Q}_{\mathbf{i}}^1$ then $p \leq_{\mathbb{Q}_{\mathbf{i}}^1} q$ if and only if the demand (α) of 1.2(7) holds and

($*$) for each $x_i^n \in B_1^q$:

(i) if $x_i^n \in B_1^p$ then $H^q(x_i^n) = H^p(x_i^n)$,

(ii) if $n \in \mathrm{dom}(E^p) \setminus \mathrm{dom}(E^q), i < h(\min(n/E^p))$ and $n \notin A^p$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$,

(iii) if $n \in \mathrm{dom}(E^q) \setminus A^p, \min(n/E^p) > \min(n/E^q)$ and $h(\min(n/E^q)) \le i < h(\min(n/E^p))$ then $H^q(x^n_i) = H^q(x_i^{\min(n/E^p)})$.

The corresponding reformulation for the forcing notions $\mathbb{Q}^2_{\mathbf{i}}, \mathbb{Q}^3_{\mathbf{i}}$ is more complicated, but it should be clear too.

One may wonder why we have h in the definition of $\mathbb{Q}^\ell_{\mathbf{i}}$ and we do not fix that, e.g. $h(n) = n$. The reason is to be able to describe nicely what is the forcing notion $\mathbb{Q}^\ell_{\mathbf{i}}$ above a condition p like. The point is that $\mathbb{Q}^\ell_{\mathbf{i}} \restriction \{q : q \ge p\}$ is like $\mathbb{Q}^\ell_{\mathbf{i}}$ but we replace I by its quotient by E^p and we change the function h.
More precisely:

Proposition 1.5. *If $p \in \mathbb{Q}^\ell_{\mathbf{i}}$ and $A^p = \{n_k : k < \omega\}, n_k < n_{k+1}, h^* : \omega \longrightarrow \omega$ is $h^*(k) = h(n_k)$ and $I^* = \{B \subseteq \omega : \bigcup\limits_{k \in B} (n_k/E) \in I\}$ and $\mathbf{m}^* = (I^*, h^*)$ <u>then</u> $\mathbb{Q}^\ell_{\mathbf{i}} \restriction \{q : p \le_{\mathbb{Q}^\ell_{\mathbf{i}}} q\}$ is isomorphic to $\mathbb{Q}^\ell_{\mathbf{m}^*}$.*

Proof. Natural. $\square_{1.5}$

Definition 1.6. We define a $\mathbb{Q}_{\mathbf{i}}$-name $\bar{\underset{\sim}{\eta}} = \langle \underset{\sim}{\eta}_n : n < \omega \rangle$ by: $\underset{\sim}{\eta}_n$ is a sequence of length $h(n)$ of members of $\{-1, 1\}$ such that $\underset{\sim}{\eta}_n[\mathbf{G}_{\mathbb{Q}_{\mathbf{i}}}](i) = 1 \Leftrightarrow (\exists p \in \mathbf{G}_{\mathbb{Q}_{\mathbf{i}}})(H^p(x^n_i) = 1 \wedge n < \min(A^p))$. [Note that even if we omit "$n < \min(A^p)$" in all cases $\ell = 1, 2, 3$, if $H^p(x^n_i) = 1, x^n_i \in \mathrm{dom}(H^p)$ and $q \ge p$ then $H^q(x^n_i) = 1$; remember 1.2(2).]

Proposition 1.7. *1) If $n < \omega, p \in \mathbb{Q}_{\mathbf{i}}$ and $A^p \cap (n+1) = \emptyset$ <u>then</u> $p \Vdash$ "$\underset{\sim}{\eta}_n = \langle H^p(x^n_i) : i < h(n) \rangle$".*
2) For each $n < \omega$ the set $\{p \in \mathbb{Q}_{\mathbf{i}} : A^p \cap (n+1) = \emptyset\}$ is dense in $\mathbb{Q}_{\mathbf{i}}$.
3) If $p \in \mathbb{Q}_{\mathbf{i}}$ and $a \subseteq A^p$ is finite or at least $\bigcup\limits_{n \in a} (n/E^p) \in I$, and $f \in \mathscr{F}_a$ <u>then</u> for some unique q which we denote by $p^{[f]}$, we have:

(a) $p \le q \in \mathbb{Q}_{\mathbf{i}}$,
(b) $E^q = E^p \restriction \bigcup\{n/E^p : n \in A \setminus a\}$,
(c) *for $n \in a, i < h(n)$ we have $H^q(x^n_i)$ is $f(x^n_i)$.*

Proof. Straightforward. $\square_{1.7}$

Definition 1.8. 1) $p \le_n q$ (in $\mathbb{Q}_{\mathbf{i}}$) <u>iff</u> $p \le q$ and: if $k \in A^p$ and $|A^p \cap k| < n$ then $k \in A^q$.
2) $p \le^*_n q$ <u>iff</u> $p \le q$ and: if $k \in A^p$ and $|A^p \cap k| < n$ then $k \in A^q$ and $k/E^p = k/E^q$.
3) $p \le^\otimes_n q$ <u>iff</u> $p \le_{n+1} q$ and: $n > 0 \Rightarrow p \le^*_n q$ and $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$.
4) For a finite set $\mathbf{u} \subseteq \omega$ we let $\mathrm{var}(\mathbf{u}) := \{x^n_i : i < h(n), n \in \mathbf{u}\}$.

Proposition 1.9. *1) If $p \le q, \mathbf{u}$ is a finite initial segment of A^p and $A^q \cap \mathbf{u} = \emptyset$, <u>then</u> for some unique $f \in \mathscr{F}_{\mathbf{u}}$ we have $p \le p^{[f]} \le q$ (where $p^{[f]}$ is from 1.7(3)).*
2) If $p \in \mathbb{Q}^\ell_{\mathbf{i}}$ and $\mathbf{u}$ is a finite initial segment of A^p <u>then</u>:

$(*)_1$ *$f \in \mathscr{F}_{\mathbf{u}}$ implies $p \le p^{[f]}$ and $p^{[f]} \Vdash$ "$(\forall n \in \mathbf{u})(\forall i < h(n))(\underset{\sim}{\eta}_n(i) = f(x^n_i))$",*
$(*)_2$ *the set $\{p^{[f]} : f \in \mathscr{F}_{\mathbf{u}}\}$ is predense above p (in $\mathbb{Q}^\ell_{\mathbf{i}}$).*

3) $\le_n$ is a partial order on $\mathbb{Q}^\ell_{\mathbf{i}}$, and $p \le_{n+1} q \Rightarrow p \le_n q$. Similarly for $<^*_n$ and $<^\otimes_n$. Also

$(*)_1$ $p \le^{\otimes}_n q \Rightarrow p \le^*_n q \Rightarrow p \le_n q \Rightarrow p \le q$
$(*)_2$ $p \le^{\otimes}_n q \Rightarrow p \le_{n+1} q$.

4) If $p \in \mathbb{Q}^\ell_{\mathbf{i}}, \mathbf{u}$ is a finite initial segment of $A^p, |\mathbf{u}| = n$ and $f \in \mathscr{F}_{\mathbf{u}}$ and $p^{[f]} \le q \in \mathbb{Q}^\ell_{\mathbf{i}}$ <u>then</u> for some $r \in \mathbb{Q}^\ell_{\mathbf{i}}$ we have $p \le^*_n r \le q$ and $r^{[f]} = q$.
5) Let $\ell = 3$. If $p \in \mathbb{Q}^\ell_{\mathbf{i}}, \mathbf{u}$ is a finite initial segment of A^p, $|\mathbf{u}| = n+1$ and $f \in \mathscr{F}_{\mathbf{u}}$ and $p^{[f]} \le q$, <u>then</u> for some $r \in \mathbb{Q}^\ell_{\mathbf{i}}$ we have $p <^{\otimes}_n r \le q$ and $r^{[f]} = q$.

Proof. 1) Define $f \in \mathscr{F}_{\mathbf{u}}$ by: $f(x^n_i)$ is the value of $H^q(x^n_i)$; that is, if $\ell = 2$, it is a constant function by 1.2(3)(d)(α); if $\ell \ne 2$ this is just $H^q(x^n_i)$.
2) By 1.7(3) and 1.9(1).
3) Check.
4) First let us define the required condition r in the case $\ell = 1$. So we let $\mathrm{dom}(E^r) = \bigcup_{n\in\mathbf{u}} (n/E^p) \cup \mathrm{dom}(E^q), E^r = \{(n_1, n_2) : n_1 E^q n_2$ or for some $n \in \mathbf{u}$ we have: $\{n_1, n_2\} \subseteq (n/E^p)\}, A^r = \mathbf{u} \cup A^q$ (note that if $n_1 E^q n_2$ then $n_1 \notin \mathbf{u}$). Next, for $x^n_i \in B^r_1$ (where B^r_1 is given by 1.2(3)(c)) we define

$$H^r(x^n_i) = \begin{cases} H^q(x^n_i) & \text{if } n \notin \bigcup\limits_{k\in\mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^q), \\ H^p(x^n_i) & \text{if } n \in \bigcup\limits_{k\in\mathbf{u}} k/E^p \text{ and } x^n_i \in \mathrm{dom}(H^p). \end{cases}$$

It should be clear that $r = (H^r, E^r, A^r) \in \mathbb{Q}^\ell_{\mathbf{i}}$ is as required.

If $\ell = 2$ then we define r in a similar manner, but we have to be more careful defining the function H^r. Thus E^r and A^r are defined as above, B^r_2, B^r_3 and $w_r(m, i)$ for $x^m_i \in B^r_3$ are given by 1.2(3)(f) and 1.2(3)(d)(β). Note that $B^r_2 = B^p_2$ and $B^r_3 \subseteq B^p_3$.

Next we define:
if $x^m_i \in B^r_2$ then $H^r(x^m_i) = H^p(x^m_i)$,
if $x^m_i \in B^r_3$, $m \cap A^r \subseteq \mathbf{u}$ then $H^r(x^m_i) = H^p(x^m_i)$,
if $x^m_i \in B^r_3$ and $\min(\mathrm{dom}(E^q)) < m$ then
$H^r(x^m_i)(\ldots, x^k_j, \ldots)_{(k,j)\in w_r(m,i)} =$
$H^p(x^m_i)(x^k_j \cdot H^q(x^{k'}_{j'})(\ldots, x^{k''}_{j''}, \ldots)_{(k'',j'')\in w_q(k',j')}))_{(k,j)\in w_r(m,i),(k',j')\in w_p(m,i)\setminus w_r(m,i)}$.

Note that if $(k', j') \in w_p(m, i) \setminus w_r(m, i)$, $x^m_i \in B^r_3$ then $k' \in A^p \setminus (\mathbf{u} \cup A^q)$ and $w_q(k', j') \subseteq w_r(m, i)$.

For $\ell = 3$ similarly and in part (5) we say more.
5) Like the proof of (4). Let $n^* = \max(\mathbf{u})$. Put $\mathrm{dom}(E^r) = \mathrm{dom}(E^p)$ and declare that $n_1 E^r n_2$ if one of the following occurs:

(a) for some $n \in \mathbf{u} \setminus \{n^*\}$ we have $\{n_1, n_2\} \subseteq (n/E^p)$, or
(b) $n_1\, E^q\, n_2$ (so $n \in \mathbf{u} \Rightarrow \neg n E^p n_1$), or
(c) $\{n_1, n_2\} \subseteq B$, where $B := n^*/E^p \cup \bigcup\{m/E^p : m \in \mathrm{dom}(E^p) \setminus \mathrm{dom}(E^q), \min(m/E^p) > n^*\}$.

We let $A^r = \mathbf{u} \cup A^q$ (in fact A^r is defined from E^r). Finally the function H^r is defined exactly in the same manner as in (4) above:

(d) $H^r(x^m_j) = H^q(x^m_j)$ when $x^m_j \in \omega \setminus \mathrm{Dom}(E^p)$ or $n := \min(m/E^p) < m \wedge j \in [h(n), h(m))$
(e) $H^r(x^m_j) = H^p(x^m_j)$ if $n \in \cup\{m/E^p : m \in \mathbf{u}\}$

(f) $H^r(x^m_j) = f(x^{n^*}_j)H^q(x^m_j)$ if $m \in (n^*/E^r)\backslash(n^*/E^p)$.

$\square_{1.9}$

Corollary 1.10. *If $p \in \mathbb{Q}^\ell_{\mathbf{i}}, n < \omega$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^\ell_{\mathbf{i}}$-name of an ordinal, then there are $\mathbf{u}, q$ and $\bar{\alpha} = \langle \alpha_f : f \in \mathscr{F}_{\mathbf{u}}\rangle$ such that:*

(a) $p \le^*_n q \in \mathbb{Q}^\ell_{\mathbf{i}}$,
(b) $\mathbf{u} = \{\ell \in A^p : |\ell \cap A^p| < n\}$,
(c) *for* $f \in \mathscr{F}_{\mathbf{u}}$ *we have* $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$",
(d) $q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in \mathscr{F}_{\mathbf{u}}\}$" *(which is a finite set).*

Proof. Let $k = \prod_{\ell \in \mathbf{u}} 2^{h(\ell)}$. Let $\{f_\ell : \ell < k\}$ enumerate $\mathscr{F}_{\mathbf{u}}$. By induction on $\ell \le k$ define r_ℓ, α_{f_ℓ} such that:

- $r_0 = p$
- $r_\ell \le^*_n r_{\ell+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$
- $r^{[f_\ell]}_{\ell+1} \Vdash_{\mathbb{Q}^\ell_{\mathbf{i}}}$ "$\underset{\sim}{\tau} = \alpha_{f_\ell}$".

The induction step is by 1.9(4). Now $q = r_k$ and $\langle \alpha_f : f \in \mathscr{F}_{\mathbf{u}}\rangle$ are as required. $\square_{1.10}$

Corollary 1.11. *If $\ell = 3$ then in 1.10(a) we may require $p \le^\otimes_n q \in \mathbb{Q}^\ell_{\mathbf{i}}$.*

Proof. Similar: just use 1.9(5) instead of 1.9(4). $\square_{1.11}$

Definition 1.12. Let I be an ideal on ω containing $[\omega]^{<\aleph_0}$ and let E be an I-equivalence relation.
1) We define a game $GM_I(E)$ between two players. The game lasts ω moves. Both players choose I-equivalence relations, where those of player I are denoted by E^1_n and those of player II are denoted by E^2_n.

In the n-th move the first player chooses an I-equivalence relation E^1_n such that $E^1_0 = E, [n > 0 \Rightarrow E^2_{n-1} \le E^1_n]$, and the second player chooses an I-equivalence relation E^2_n such that $E^1_n \le E^2_n$. In the end, the second player wins if

$$\bigcup\{\mathrm{dom}(E^2_n) \setminus \mathrm{dom}(E^1_{n+1}) : n \in \omega\} \in I$$

(otherwise the first player wins).
2) For a countable elementary submodel N of $(\mathscr{H}(\chi), \in, <^*)$ such that $I, E \in N$ we define a game $GM^N_I(E)$ in a similar manner as $GM_I(E)$, but we demand additionally that the relations played by both players are from N (i.e. $E^1_n, E^2_n \in N$ for $n \in \omega$).

The following propositions 1.13,1.14 are needed for the case $\mathbb{Q}^2_{\mathbf{i}}$ but not for $\mathbb{Q}^1_{\mathbf{i}}$.

Proposition 1.13. *1) Assume that I is a maximal (non-principal) ideal on ω and E is an I-equivalence relation. Then the game $GM_I(E)$ is not determined. Moreover, for each countable $N \prec (\mathscr{H}(\chi), \in, <^*)$ such that $I, E \in N$ the game $GM^N_I(E)$ is not determined.*
2) For the conclusion of (1) it is enough to assume that $\mathscr{P}(\omega)/I \models$ ccc.

Proof. 1) As each player can imitate the other's strategy.
2) Easy, too, and will not be used in this paper. $\square_{1.13}$

Proposition 1.14. *1) Let $p \in \mathbb{Q}^\ell_{\mathbf{i}}$. Suppose that the first player has no winning strategy in $GM_I(E^p)$ where, of course, $I = I_{\mathbf{i}}$. Then in the following game, $\eth^\ell_{\mathbf{i}}$, Player I has no winning strategy:*

- (A) *in the n-th move, Player I chooses a $\mathbb{Q}^\ell_{\mathbf{i}}$-name, $\underset{\sim}{\tau}_n$ of an ordinal and Player II chooses $p_n, \mathbf{u}_n, w_n$ such that: w_n is a set of $\leq \prod_{\ell \in \mathbf{u}_n} 2^{h(\ell)}$ ordinals, $p \leq p_n \leq^*_n p_{n+1}$, $p_n \leq_{n+1} p_{n+1}$, $\mathbf{u}_n$ is a finite initial segment of A^{p_n} with n elements and $p_n \Vdash$ "$\underset{\sim}{\tau}_n \in w_n$", moreover $f \in \mathscr{F}_{\mathbf{u}_n} \Rightarrow p_n^{[f]}$ forces a value to $\underset{\sim}{\tau}_n$*
- (B) *In the end, the second player wins if for some $q \geq p$ we have $q \Vdash$ "$(\forall n \in \omega)(\underset{\sim}{\tau}_n \in w_n)$".*

2) The result of part (1) still holds when we let Player II choose $k_n < \omega$ and demand $|\mathbf{u}_n| \leq k_n$, and in the end Player II wins if $\liminf\langle k_n : n < \omega\rangle < \omega$ or there is q as above.
3) Let $p \in \mathbb{Q}^\ell_{\mathbf{i}}$ and let N be a countable elementary submodel of $(\mathscr{H}(\chi), \in, <^*)$ such that $p, I, h \in N$. If the first player has no winning strategy in $GM^N_I(E^p)$ then Player I has no winning strategy in the game like above but with restriction that $\underset{\sim}{\tau}_n, p_n \in N$.

Proof. 1) As in [She92, 1.11,p.436].

Let $\mathbf{St}_p$ be a strategy for Player I in the game $\eth^\ell_{\mathbf{i}}$ from 1.14. Our goal is to show that $\mathbf{St}_p$ cannot be a winning strategy. We shall define a strategy $\mathbf{St}$ for the first player in $GM_I(E^p)$ during which the first player, on a side, plays a play of the game $\eth^\ell_{\mathbf{i}}$ from 1.14, using $\mathbf{St}_p$, with $\langle p_\ell : \ell < \omega\rangle$ and he also chooses $\langle q_\ell : \ell < \omega\rangle$.

Then, as $\mathbf{St}$ cannot be a winning strategy in $GM_I(E^p)$, in some play in which the first player uses his strategy $\mathbf{St}$ he loses, and then $\langle p_\ell : \ell < \omega\rangle$ will have an upper bound which shows that $\mathbf{St}_p$ is not a winning strategy for player I, as required.

In the n-th move (so $E^1_\ell, E^2_\ell, q_\ell, p_\ell, \mathbf{u}_\ell, w_\ell$ for $\ell < n$ are defined), the first player in addition to choosing E^1_n chooses $q_n, p_n, \mathbf{u}_n$, such that:

- (a) $p = p_{-1} \leq q_0 = p_0$, $p_n \in \mathbb{Q}^\ell_{\mathbf{i}}, q_n \in \mathbb{Q}^\ell_{\mathbf{i}}$,
- (b) $p_n \leq^*_n p_{n+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$,
- (c) $\mathbf{u}_0$ is $\emptyset$,
- (d) $\mathbf{u}_{n+1} = \mathbf{u}_n \cup \{\min(A^{q_{n+1}} \setminus \mathbf{u}_n)\}$, so $|\mathbf{u}_{n+1}| = n+1$,
- (e) $E^1_0 = E^p$, $E^1_{n+1} = E^{p_n} \restriction (\mathrm{dom}(E^{p_n}) \setminus \bigcup_{i \in \mathbf{u}_n} i/E^{p_n})$,
- (f) q_n is defined as follows:
  - (f_0) if $n = 0$ then $E^{q_n} = E^2_0$,
  - (f_1) if $n > 0$ then $\mathrm{dom}(E^{q_n}) = \mathrm{dom}(E^{p_{n-1}})$ and $xE^{q_n}y$ if and only if either $xE^2_n y$, or for some $k \in \mathbf{u}_{n-1}$ we have $x, y \in k/E^{p_{n-1}}$ or $x, y \in (\mathrm{dom}(E^1_n) \setminus \mathrm{dom}(E^2_n)) \cup \min(\mathrm{dom}(E^2_n))/E^2_n$,
  - (f_2) H^{q_n} is such that $p_{n-1} \leq q_n$,
- (g) $p_n \leq^*_n q_{n+1} \leq^*_{n+1} p_{n+1}$, $p_n \leq_{n+1} q_{n+1}$ (so $p_n \leq_{n+1} p_{n+1}$),
- (h) if $f \in \mathscr{F}_{\mathbf{u}_n}$ then $p_n^{[f]}$ forces a value to $\underset{\sim}{\tau}_n$.

In the first move, when $n = 0$, the first player plays $E^1_0 = E^p$ (as the rules of the game require, according to (e)). The second player answers choosing an I-equivalence relation $E^2_0 \geq E^1_0$. Now, on a side, Player I starts to play the game of 1.14 using his strategy $\mathbf{St}_p$. The strategy instructs him to play a name $\underset{\sim}{\tau}_0$ of an ordinal. He defines q_0 by (f) (so $q_0 \in \mathbb{Q}^\ell_{\mathbf{i}}$ is a condition stronger than p and such that $E^{q_0} = E^2_0$) and chooses a condition $p_0 \geq q_0$ deciding the value of the name $\underset{\sim}{\tau}_0$, say p_0 forces $\underset{\sim}{\tau}_0 = \alpha$. He pretends that the second player answered (in the game of 1.14) by: p_0, $\mathbf{u}_0 = \emptyset$, $w_0 = \{\alpha\}$. Next, in the play of $GM_I(E^p)$, he plays $E^1_1 = E^{p_0}$ as declared in (e).

Now suppose that we are at the $(n+1)^{\text{th}}$ stage of the play of $GM_I(E^p)$, the first player has played E^1_{n+1} already and on a side he has played the play of the game 1.14 as defined by (a)–(h) and $\mathbf{St}_p$ (so in particular he has defined a condition p_n and $E^1_{n+1} = E^{p_n}{\restriction}(\mathrm{dom}(E^{p_n}) \setminus \bigcup\limits_{i\in\mathbf{u}_n} i/E^{p_n})$ and $\mathbf{u}_n$ is the set of the first n elements of A^{p_n}). The second player plays an I-equivalence relation $E^2_{n+1} \geq E^1_{n+1}$.

Now the first player chooses (on a side, pretending to play in the game of 1.14): a name $\underset{\sim}{\tau}_{n+1}$ given by the strategy $\mathbf{St}_p$, a condition $q_{n+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$ determined by (f) (check that (g) is satisfied), $\mathbf{u}_{n+1}$ as in (d) and a condition $p_{n+1} \in \mathbb{Q}^\ell_{\mathbf{i}}$ satisfying (g), (h) (the last exists by 1.10). Note that, by (g) and 1.9, the condition p_{n+1} determines a suitable set w_{n+1}. Thus, Player I pretends that his opponent in the game of 1.14 played $p_{n+1}, \mathbf{u}_{n+1}, w_{n+1}$ and he passes to the actual game $GM_I(E^p)$. Here he plays E^1_{n+2} defined by (e).

The strategy $\mathbf{St}$ described above cannot be a winning one by the assumptions of the theorem. Consequently, there is a play in $GM_I(E^p)$ in which Player I uses $\mathbf{St}$, but he looses. During the play he constructed a sequence $\langle (p_n, \mathbf{u}_n, w_n) : n \in \omega\rangle$ of legal moves of Player II in the game of 1.14 against the strategy $\mathbf{St}_p$. Let $E^q = \lim\limits_{n<\omega} E^{p_n}$ (i.e. $\mathrm{dom}(E^q) = \bigcap\limits_{n<\omega} \mathrm{dom}(E^{p_n})$, $x\, E^q\, y$ if and only if for every large enough n, $x\, E^{p_n}\, y$) and let $H^q(x^m_i)$ be $H^{p_n}(x^m_i)$ for any large enough n (it is eventually constant). It follows from the demand (g) that E^q-equivalence classes are in I. Moreover, $\mathrm{dom}(E^1_{n+1}) \setminus \mathrm{dom}(E^2_{n+1}) \subseteq k/E^q$, where k is the $(n+1)^{\text{th}}$ member of A^q.

Therefore $\omega \setminus \mathrm{dom}(E^q) = \omega \setminus \bigcap\limits_{n\in\omega} \mathrm{dom}(E^{p_n}) \subseteq \omega \setminus \mathrm{dom}(E^{p_0}) \cup \bigcup\{\mathrm{dom}(E^2_n) \setminus \mathrm{dom}(E^1_{n+1}) : n \in \omega\} \in I$ (remember, Player I lost in $GM_I(E^p)$). Now it should be clear that $q \in \mathbb{Q}^\ell_{\mathbf{i}}$ and it is stronger than every p_n (even $p_n \leq^*_n q$). Hence Player II wins the corresponding play of 1.14, showing that $\mathbf{St}_p$ is not a winning strategy.
2),3) The same proof. $\square_{1.14}$

Proposition 1.15. *If in 1.14 we assume $\ell = 3$ and demand $p_n \leq^{\otimes}_n p_{n+1}$ instead of $p_n \leq^*_n p_{n+1}$* then *Player II has a winning strategy.*

Proof. Using 1.11, the second player can find suitable conditions p_n (in the game of 1.14) such that $p_n \leq^{\otimes}_{n+1} p_{n+1}$. But note that the partial orders $\leq^{\otimes}_n$ have the fusion property, so the sequence $\langle p_n : n < \omega\rangle$ will have an upper bound in $\mathbb{Q}^3_{\mathbf{i}}$. $\square_{1.15}$

Remark 1.16. We could have used $<^{\otimes}_n$ also in [She92].

Definition 1.17. [See [She98a, Ch.VI,2.12,A-F].] 1) A forcing notion $\mathbb{P}$ has the PP-property <u>when</u>:

$\otimes^{\rm PP}$ for every $\eta \in {}^\omega\omega$ from $\mathbf{V}^{\mathbb{P}}$ and a strictly increasing $x \in {}^\omega\omega \cap \mathbf{V}$ there is a closed subtree $T \subseteq {}^{<\omega}\omega$ such that:

(α) $\eta \in \lim(T)$, i.e. $(\forall n < \omega)(\eta \restriction n \in T)$,

(β) $T \cap {}^n\omega$ is finite for each $n < \omega$,

(γ) for arbitrarily large n there are k, and $n < i(0) < j(0) < i(1) < j(1) < \ldots < i(k) < j(k) < \omega$ and for each $\ell \le k$, there are $m(\ell) < \omega$ and $\eta^{\ell,0}, \ldots, \eta^{\ell,m(\ell)} \in T \cap {}^{j(\ell)}\omega$ such that $j(\ell) > x(i(\ell) + m(\ell))$ and $(\forall \nu \in T \cap {}^{j(k)}\omega)(\exists \ell \le k)(\exists m \le m(\ell))(\eta^{\ell,m} \trianglelefteq \nu)$.

2) We say that a forcing notion $\mathbb{P}$ has the strong PP-property <u>when</u>:

$\oplus^{\rm sPP}$ for every function $g : \omega \longrightarrow \mathbf{V}$ from $\mathbf{V}^{\mathbb{P}}$ there exist a set $B \in [\omega]^{\aleph_0} \cap \mathbf{V}$ and a sequence $\langle w_n : n \in B\rangle \in \mathbf{V}$ such that for each $n \in B, |w_n| \le n$ and $g(n) \in w_n$.

Observation 1.18. *Of course, if a proper forcing notion has the strong* PP*-property <u>then</u> it has the* PP*-property.*

Conclusion 1.19. *Assume that for each $p \in \mathbb{Q}^\ell_{\mathbf{i}}$ and for each countable $N \prec (\mathscr{H}(\chi), \in, <^*)$ such that $p, I, h \in N$, the first player has no winning strategy in $GM^N_I(E^p)$ (e.g. if I is a maximal ideal).*

<u>Then</u>

(∗) *$\mathbb{Q}^\ell_{\mathbf{i}}$ is proper, α-proper, strongly α-proper for every $\alpha < \omega_1$, is ${}^\omega\omega$-bounding and it has the* PP*-property, even the strong* PP*-property.*

By [She98a, Ch.VI,2.12] we know

Theorem 1.20. *Suppose that $\langle \mathbb{P}_i, \mathbb{Q}_j : j < \alpha, i \le \alpha\rangle$ is a countable support iteration such that $\Vdash_{\mathbb{P}_j}$ "$\mathbb{Q}_j$ is proper and has the* PP*-property".*

<u>Then</u> $\mathbb{P}_\alpha$ has the PP*-property.*

§ 2. Creature forcing

We try to combine the good properties of the $\mathbb{Q}^\ell_{\mathbf{i}}$'s from §1 by putting a creature on finite intervals of A^p.

Definition 2.1. 1) Let FP_2 (forcing parameters) be the set of objects $\mathbf{i}$ consisting of (so $I = I_{\mathbf{i}} = I[\mathbf{i}]$, etc.):

- (a) I be an ideal on ω to which all finite subsets of ω belong and $D = \mathrm{dual}(I)$ its dual, a filter
- (b) let $h : \omega \to \omega$ be a non-decreasing function
- (c) h goes to infinity
- (d) $\bar{S} = \langle S_k : k < \omega\rangle$ is a partition of ω to intervals (each interval is finite non-empty)
- (e) $\min(S_{k+1}) = \max(S_k) + 1$ for every k
- (f) each $h \restriction S_k$ is constant and let $h' : \omega \to \omega$ be such that $n \in S_k \Rightarrow h'(k) = h(n)$
- (g) notation: let $E_{\mathbf{i}} = E_{\bar{S}} = \{(m, n) : (\exists k)[n, m \in S_k]\}$

2) Let FP_3 be the set of $\mathbf{i}$ consisting of:

- (A) as in part (1)
- $(\mathrm{B})_1$ the simple creature version:
  - (a) $\mathrm{CR}_n := \mathscr{P}^-(S_n) := \mathscr{P}(S_n)\backslash\{\emptyset\}$
  - (b) $\mathrm{nor}_n : \mathrm{CR}_n \to \mathbb{R}_{>0}$ is monotonically $\subseteq$-increasing
  - (c) $\langle \mathrm{nor}_n(S_n) : n < \omega\rangle$ goes to infinity
  - (d) we also let $\mathrm{val}_n : \mathrm{CR}_n \to \mathrm{CR}_n$ be the identity and $\Sigma_n(S) = \mathscr{P}^-(S)$ for $S \in \mathscr{P}^-(n)$, we stipulate $\mathrm{nor}_n(\emptyset) = -1$
- $(\mathrm{B})_2$ full creature version:
  - (a) $\langle\, \mathbf{CR}_n : n < \omega\rangle$ where $\mathbf{CR}_n = (\mathrm{CR}_n, \mathrm{val}_n, \mathrm{nor}_n, \Sigma_n)$
  - (b) the CR_n's are pairwise disjoint, each finite
  - (c) $\mathrm{val}_n(\mathfrak{c}) \in \mathscr{P}^-(S_n)$ for $n < \omega, \mathfrak{c} \in \mathrm{CR}_n$
  - (d) $\mathrm{nor}_n(\mathfrak{c}) \in \mathbb{R}_{>0}$ for $\mathfrak{c} \in \mathrm{CR}_n$
  - (e) if $\mathfrak{c} \in \mathrm{CR}_n$ then $\Sigma_n(\mathfrak{c}) \subseteq \mathrm{CR}_n$ and $\mathfrak{c} \in \Sigma_n(\mathfrak{c})$
  - (f) if $\mathfrak{d} \in \Sigma_n(\mathfrak{c})$ then $\mathrm{val}_n(\mathfrak{d}) \subseteq \mathrm{val}_n(\mathfrak{c})$ and $\Sigma_n(\mathfrak{d}) \subseteq \Sigma_n(\mathfrak{c})$ for $\mathfrak{c}, \mathfrak{d} \in \mathrm{CR}_n$.

3) We define FP_4 similarly to FP_3 but we add "$\mathbf{i}$ has the ultrafilter property", which means:

- $(\mathrm{B})_1$ (g) $\mathrm{nor}_n(S_1 \cup S_2) \leq \max\{\mathrm{nor}_n(S_1) + 1, \mathrm{nor}_n(S_2) + 1\}$ for $S_1, S_2 \in \mathscr{P}^-(n)$
- $(\mathrm{B})_2$ (g) for $\mathfrak{c} \in \mathrm{CR}_n$, if $\mathrm{val}_n(\mathfrak{c}) = S' \cup S''$, then for some $\mathfrak{d} \in \Sigma_n(\mathfrak{c})$ we have $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$ and $(\mathrm{val}(\mathfrak{d}) \subseteq S') \vee (\mathrm{val}(\mathfrak{d}) \subseteq S'')$.

4) We say that $\mathbf{i}$ has the strong ultrafilter property when in addition: ($\mathbb{Q}_{\mathbf{i}}$ is defined below):

- (h) if $p \in \mathbb{Q}_\iota$ and $B \in D/E$ (i.e. $B \in D$ and $(\forall n \in \mathrm{dom}(E^p))(B \cap (n/E^p) \in \{\emptyset, n/E^p\})$ then there is $q \in \mathbb{Q}_{\mathbf{i}}$ above p such that:

- $E^q = E^p{\restriction}\mathrm{dom}(E^q)$
- if $k \in w^q$ then $\mathrm{nor}(\mathfrak{c}_{q,k}) \supseteq \mathrm{nor}(c_{p,k}) - 1$.

5) For $\iota = \{1,2,3,4\}$:

(A) let $\mathrm{FP}^\iota_{\mathrm{uf}}$ be the set of $\mathbf{i} \in \mathrm{FP}_\iota$ such that $D_{\mathbf{i}} := \mathrm{dual}(I_{\mathbf{i}}) := \{\omega \backslash A : A \in I_{\mathbf{i}}\}$ is an ultrafilter on ω, necessarily non-principal

(B) let FP^ι_{cc} be the set of $\mathbf{i} \in \mathrm{FP}_\iota$ such that $D_{\mathbf{i}} = \mathrm{dual}(I_{\mathbf{i}})$ is a filter on ω such that the Boolean algebra $\mathscr{P}(\mathbb{N})/D_{\mathbf{i}}$ satisfies the c.c.c.

(C) let FP^ι_{nn} be the set of $\mathbf{i} \in \mathrm{FP}_\iota$; this is just so that we can write "for each $x \in \{\mathrm{uf}, \mathrm{cc}, \mathrm{nn}\}$ in FP^ι_x we have ... ".

6) If $(\forall n)(h'(n) = n)$ then we may omit h.
7) We say $\mathbf{i} \in \mathrm{FP}_3$ is fast <u>when</u>: if $\mathfrak{c} \in \mathrm{CR}_k, \mathrm{nor}(\mathfrak{c}) \geq 1$ then we can find $\bar{\mathfrak{d}} = \langle \mathfrak{d}_i : i < k\rangle$ such that:

(a) $\mathfrak{d}_i \in \Sigma_k(\mathfrak{c})$ and $\mathrm{nor}(\mathfrak{d}_i) \geq \mathrm{nor}(\mathfrak{c}) - 1$

(b) $\mathrm{val}(\mathfrak{c})\backslash\mathrm{val}(\mathfrak{d}_i)$ has at least $2^{h'(k)}$ members

(c) $\langle \mathrm{val}(\mathfrak{c})\backslash\mathrm{val}(\mathfrak{d}_\ell) : \ell < k\rangle$ are pairwise disjoint.

Remark 2.2. 1) The "$\mathbf{i}$ is fast" is used for the bounded game for $\mathbb{Q}_{\mathbf{i}}$, see 2.15 below assuming $\mathbf{i} \in \mathrm{FP}^3_{cc}$.
2) For $\mathbf{i} \in \mathrm{FP}_3$ alternatively to the assumptions on "$\mathbf{i}$ is fast" and $\mathscr{P}(\omega)/I_{\mathbf{i}}$ satisfies the c.c.c. used in 2.15 we can use:

(a) if $u_k \in [S_k]^{\leq k}$ for $k < \omega$ then $\cup\{u_k : k < \omega\} \in I_{\mathbf{i}}$

(b) if $\mathfrak{c} \in \mathrm{CR}_k, \mathrm{nor}(\mathfrak{c}) \geq 1$ <u>then</u> there is $\mathfrak{d} \in \Sigma_k(\mathfrak{c})$ such that $\mathrm{val}(\mathfrak{c})\backslash\mathrm{val}(\mathfrak{d})$ has $\geq 2^{h'(k)}$ elements and $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$.

3) Is it helpful to allow non-unary Σ (in Definition 2.1(2))?
4) The property "$\mathbf{i}$ is fast" is crucial: the ultrafilter property (and the stronger one) presently not but they are for defining families of ultrafilters.

Claim 2.3. *1) In Definition 2.1(2),(3), clause (B)$_1$ is a special case of clause (B)$_2$.*
2) In Definition 2.1, if $S_n = \{n\}$ for every n and clause (B)$_1$ holds with $\mathrm{nor}_n(\{n\}) = n$, <u>then</u> $\mathbf{i}$ essentially belongs to FP_1. Also every $\mathbf{i} \in \mathrm{FP}_1$ can be interpreted in this way.
3) Any $\mathbf{i} \in \mathrm{FP}_2$ is a special case of FP_3.

Proof. Read the definitions. $\square_{2.3}$

Definition 2.4. For $\mathbf{i} \in \mathrm{FP}_3$ we define the forcing notion $\mathbb{Q}_{\mathbf{i}} = \mathbb{Q}^1_{\mathbf{i}}$ and some auxiliary notions as follows:

(A) $p \in \mathbb{Q}_{\mathbf{i}}$ if and only if $p = (H, E, A, \bar{\mathfrak{c}}) = (H^p, E^p, A^p, \bar{\mathfrak{c}}^p)$ satisfies:

  (a) E is an I-equivalence relation, so on a set called $\mathrm{dom}(E)$ which belongs to $D_{\mathbf{i}}$ hence is $\subseteq \omega$,

  (b) $A = A^p := \{n \in \mathrm{dom}(E) : n = \min(n/E)\}$,

  (c) H is a function with range $\subseteq \{-1, 1\}$ and domain $B^p_1 = \{x^n_i : i < h(n) \wedge n \in (\omega \setminus \mathrm{dom}(E))$ or $n \in \mathrm{dom}(E) \wedge i \in [h(\min(n/E)), h(n))\}$,

  (d) (α) $\bar{\mathfrak{c}} = \langle \mathfrak{c}_k : k \in w\rangle$

(β) $w = w^p \subseteq \omega$ is infinite

(γ) $\mathfrak{c}_k \in \mathrm{CR}_k$

(δ) $A = \cup\{\mathrm{val}(\mathfrak{c}_k) : n \in w\}$

(e) $\infty = \limsup_{I[\mathbf{i}]} \langle \mathrm{nor}_n(\mathfrak{c}_n) : n \in w\rangle$ which means that for every $\mathscr{U} \in I_{\mathbf{i}}$, the set $\{\mathrm{nor}_k(\mathfrak{c}_{p,k}) : k \in w^p \backslash \mathscr{U}\}$ is unbounded (in ω).

(B) For $a \subseteq \omega$ we define $\mathscr{F}_a = \{f : f$ is a function with domain $\{x_i^n : i < h(n), n \in a\}$ such that $f(x_i^n) \in \{-1, 1\}$; let $a \triangleleft w^p$ mean that a is a finite initial segment of w^p; later in 2.9 we shall define $f_{p,a}$

(C) We say that a function $f \in \mathscr{F}_\omega$ satisfies a condition $p \in \mathbb{Q}_{\mathbf{i}}$ <u>when</u>:

(a) $f(x_i^n) = H^p(x_i^n)$ <u>when</u> $x_i^n \in B_1^p$

(b) $f(x_i^n) = (f(x_i^{\min(n/E^p)}))$ <u>when</u> $n \in \mathrm{dom}(E^p)$ and $i < h(\min(n/E^p))$.

(D) The partial order $\leq = \leq_{\mathbb{Q}^1_{\mathbf{i}}}$ is defined by $p \leq q$ if and only if:

(α) $E^p \leq E^q$, i.e.

- $\mathrm{dom}(E^q) \subseteq \mathrm{dom}(E^p)$
- if $n \in \mathrm{dom}(E^q)$ then $n/E^p \subseteq \mathrm{dom}(E^q)$
- $E^p \restriction \mathrm{dom}(E^q)$ refines E^q

(β) every function $f \in \mathscr{F}_\omega$ satisfying q satisfies p

(γ) $w^q \subseteq w^p$ and if $k \in w^q$, <u>then</u> $\mathfrak{c}_{q,k} \in \Sigma_k(\mathfrak{c}_{p,k})$.

Proposition 2.5. $(\mathbb{Q}_{\mathbf{i}}, \leq_{\mathbb{Q}_{\mathbf{i}}})$ *is a partial order.*

Proof. Easy. $\square_{2.5}$

Remark 2.6. 1) We may reformulate the definition of the partial order $\leq_{\mathbb{Q}_{\mathbf{i}}}$, making it perhaps more direct. Thus, if $p, q \in \mathbb{Q}_{\mathbf{i}}$ then $p \leq_{\mathbb{Q}_{\mathbf{i}}} q$ if and only if the demand (α) of 2.4(D) holds and

(β)* for each x_i^n:

(i) if $x_i^n \in B_1^p$ then $H^q(x_i^n) = H^p(x_i^n)$,

(ii) if $n \in \mathrm{dom}(E^p) \setminus \mathrm{dom}(E^q)$ and $i < h(\min(n/E^p))$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$,

(iii) if $n \in \mathrm{dom}(E^q) \setminus A^p, \min(n/E^p) > \min(n/E^q)$ and $h(\min(n/E^q)) \leq i < h(\min(n/E^p))$ then $H^q(x_i^n) = H^q(x_i^{\min(n/E^p)})$.

Remark 2.7. One may wonder why we have h in the definition of $\mathbb{Q}_{\mathbf{i}}$ and we do not fix that, e.g. $h(n) = n$. This is to be able to describe nicely what is the forcing notion $\mathbb{Q}_{\mathbf{i}}$ above a condition p. The point is that $\mathbb{Q}_{\mathbf{i}} \restriction \{q : q \geq p\}$ is like $\mathbb{Q}_{\mathbf{i}}$ but we replace I by its quotient by E^p and we change the function h.

More precisely:

Claim 2.8. *Assume* $\mathbf{i} \in \mathrm{FP}_3$ *and* $p \in \mathbb{Q}_{\mathbf{i}}$. <u>*Then*</u> $\mathbb{Q}_{\mathbf{j}}$ *is isomorphic to* $\mathbb{Q}_{\mathbf{i}} \restriction \{q : q$ *is* $\leq_{\mathbb{Q}_{\mathbf{i}}}$*-above* $p\}$ *and* $\mathbf{j} \in \mathrm{FP}_3$ *and* $\mathbf{j}$ *belongs to* $\mathrm{FP}_4/\mathrm{FP}^3_{\mathrm{uf}}/\mathrm{FP}^3_{cc}$ *when* $\mathbf{i}$ *does, provided that:*

$\boxplus$ $\mathbf{j}$ *is defined by: letting* $g_0 : w^p \to \omega$ *be increasing and onto* ω *and* $g_1 : A^p \to \omega$ *be increasing and onto* ω, *we have:*

(a) $h'_{\mathbf{j}}$ *is defined by if* $g_0(h) = \ell$ *so* $k \in w^p$ *then* $h'_{\mathbf{j}}(\ell) = h'_{\mathbf{i}}(\ell)$

(b) *if* $g_0(k) = \ell$ *then*
- $S_{\mathbf{j},\ell} = \{g_1(n) : n \in \mathrm{val}_{\mathbf{i}}(\mathfrak{c}_{p,k})\}$
- $\mathrm{CR}_{\mathbf{j},\ell} = \Sigma_{\mathbf{i}}(\mathfrak{c}_{p,k}), \Sigma_{\mathbf{j},\ell} = \Sigma_{\mathbf{i},k}{\restriction}\mathrm{CR}_{\mathbf{j},\ell}$ *and* $\mathrm{val}_{\mathbf{j}}(\mathfrak{c}) = \{g_1(n) : n \in \mathrm{val}_{\mathbf{i}}(\mathfrak{c}_{p,w})\}$

(c) $I_{\mathbf{j}} = \{C \subseteq w : \cup\{n/E^p : n \in A^p$ *and* $g_1(n) \in C\} \in I\}$.

Proof. Straightforward. $\square_{2.8}$

Definition 2.9. 1) We define a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\bar\eta} = \langle \underset{\sim}{\eta}_n : n < \omega\rangle$ by: $\underset{\sim}{\eta}_n$ is a sequence of length $h(n)$ of members of $\{-1,1\}$ such that $\underset{\sim}{\eta}_n[\underset{\sim}{G}_{\mathbb{Q}_{\mathbf{i}}}](i) = 1 \Leftrightarrow (\exists p \in \underset{\sim}{G}_{\mathbb{Q}_{\mathbf{i}}})(H^p(x^n_i) = 1 \wedge x^n_i \in B^p_1))$.
2) For $p \in \mathbb{Q}_{\mathbf{i}}$ and $a \subseteq w^p$ let $\mathscr{F}_{p,a} = \{f{\restriction}\{x^n_i : n \in \cup\{(m/E_{\mathbf{i}}) : (\exists k \in a) m \in S_k\} \cap A^p$ and $i < h(n)\} : f$ satisfies $p\}$ equivalently $\{f : f$ is a function from $\{x^n_i : n \in A^p$ and $n \in S_k$ for some $k \in a$ and $i < h(n)\}$ into $\{1,-1\}$.
3) Let (for $\mathbf{i} \in \mathrm{FP}_3$ and $p \in \mathbb{Q}_{\mathbf{i}}$)

- $\mathrm{set}_n(A^p) = \{\ell \in A^p$: the set $A^p \cap ((\ell+1)/E_{\mathbf{i}})$ has $n+1$ members$\}$ recalling 2.1(1)(g)
- $\mathrm{set}_{<n}(A^p) = \cup\{\mathrm{set}_m(A^p) : m < n\}$
- $\mathrm{set}_{\le n}(A^p) = \mathrm{set}_{<(n+1)}(A^p)$.

4) We say $f \in \mathscr{F}_{p,a}$ is p-rich <u>when</u> $a \subseteq w^p$ has a last element and $\{\langle f(x^n_i) : i < h'(\max(a)\rangle : n \in S_{\max(a)} \cap A^p\}$ is equal to ${}^{h'(\max(a))}2$, that is, all possibilities occur.
5) $\mathscr{F}_{p,<n} = \mathscr{F}_{\mathrm{set}_{<n}(A^p)}$.

Proposition 2.10. *1) If $n < \omega, A^p \cap (n+1) = \emptyset$ <u>then</u> $p \Vdash$ "$\underset{\sim}{\eta}_n = \langle H^p(x^n_i) : i < h(n)\rangle$".*
2) For each $n < \omega$ the set $\{p \in \mathbb{Q}_{\mathbf{i}} : A^p \cap (n+1) = \emptyset\}$ is dense open in $\mathbb{Q}_{\mathbf{i}}$.
3) If $p \in \mathbb{Q}_{\mathbf{i}}$ and $a \subseteq w^p$ is finite or at least $b = \cup\{n/E_p : n \in (\bigcup_{k\in a} S_k) \cap A^p\} \in I$, and $f \in \mathscr{F}_{p,b}$ <u>then</u> for some unique q which we denote by $p^{[f]}$, we have:

(a) $p \le q \in \mathbb{Q}_{\mathbf{i}}$,
(b) $E^q = E^p \restriction \bigcup\{n/E^p : n \in A^p \setminus b\}$,
(c) *for $n \in b, i < h(n)$ we have $H^q(x^n_i)$ is $f(x^n_i)$*
(d) $k \in w^q \backslash a \Rightarrow \mathfrak{c}_{q,k} = \mathfrak{c}_{p,k}$.

4) For every $p \in \mathbb{Q}_{\mathbf{i}}$ there is $q \in \mathbb{Q}_{\mathbf{i}}$ above p such that:

$\oplus_q$ *the sequence $\langle \mathrm{nor}_k(\mathfrak{c}_{q,k}) : k \in w^q\rangle$ is increasing and $k \in w^p \Rightarrow \mathrm{nor}(\mathfrak{c}^p_k) > k + |\mathscr{F}_{p,w^p\cap k}|$ and $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$.*

5) Moreover, if $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} \subseteq w^p$ is a finite initial segment of w^p and $\langle \mathrm{nor}(\mathfrak{c}_{p,\ell}) : \ell \in \mathbf{u}\rangle$ is increasing and $k \in \mathbf{u} \Rightarrow \mathrm{nor}(\mathfrak{c}^p_k) > k + |\mathscr{F}_{p,\mathbf{u}\cap k}|$, <u>then</u> for some $q \in \mathbb{Q}^1_{\mathbf{i}}$ we have $p \le^{\otimes}_{|\mathbf{u}|} q$, see Definition 2.11(3) below and $\oplus_q$ above holds.

Proof. Straightforward. $\square_{2.10}$

Definition 2.11. 0) For $p \in \mathbb{Q}_{\mathbf{i}}$ and $n < \omega$ let $\mathbf{k}_p(n) = \mathbf{k}(n,p)$ be the minimal k (actually unique k) such that:

(a) $k \in w^p$
(b) $|k \cap w^p| = n$

1) $p \le_n q$ (in $\mathbb{Q}_{\mathbf{i}}$) <u>iff</u>:

(a) $p \le q$
(b) if $k \in w^p \cap \mathbf{k}_p(n)$ then $k \in w^q$ and $\mathfrak{c}_{q,k} = \mathfrak{c}_{p,k}$.

2) $p \le_n^* q$ <u>iff</u> $p \le_n q$ and:

$(*)$ if $k \in w^p \cap \mathbf{k}_p(n)$, <u>then</u> not only $k \in w^q, \mathfrak{c}_{q,k} = \mathfrak{c}_{p,k}$ but also $m \in \mathrm{val}(\mathfrak{c}_{q,k}) \Rightarrow m/E^p = m/E^q$.

3) $p \le_n^{\otimes} q$ <u>iff</u> $p \le_{n+1} q$ and: $n > 0 \Rightarrow p \le_n^* q$ and $\mathrm{dom}(E^q) = \mathrm{dom}(E^p)$.

Proposition 2.12. *1) If $p \le q, \mathbf{u}$ is an initial segment of w^p and $w^q \cap \mathbf{u} = \emptyset$, <u>then</u> for some unique $f \in \mathscr{F}_{p,\mathbf{u}}$ we have $p \le p^{[f]} \le q$ (where $p^{[f]}$ is defined in 2.10(3)).*
2) If $p \in \mathbb{Q}_{\mathbf{i}}$ and $\mathbf{u} \lhd w^p$, i.e. is a finite initial segment of w^p <u>then</u>:

$(*)_1$ *$f \in \mathscr{F}_{p,\mathbf{u}}$ implies $p \le p^{[f]}$ and $p^{[f]} \Vdash$ "$(\forall n \in \mathbf{u})(\forall i < h(n))(\underset{\sim}{\eta}_n(i) = f(x_i^n))$",*

$(*)_2$ *the set $\{p^{[f]} : f \in \mathscr{F}_{p,\mathbf{u}}\}$ is predense above p (in $\mathbb{Q}_{\mathbf{i}}$).*

3) $\le_n$ is a partial order on $\mathbb{Q}_{\mathbf{i}}$, and $p \le_{n+1} q \Rightarrow p \le_n q$. Similarly for $<_n^*$ and $<_n^{\otimes}$. Also

$(*)_1$ *$p \le_n^{\otimes} q \Rightarrow p \le_n^* q \Rightarrow p \le_n q \Rightarrow p \le q$*
$(*)_2$ *$p \le_n^{\otimes} q \Rightarrow p \le_{n+1} q$.*

4) If $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} = w^p \cap \mathbf{k}_p(n)$ and $f \in \mathscr{F}_{p,\mathbf{u}}$ and $p^{[f]} \le q \in \mathbb{Q}_{\mathbf{i}}$ <u>then</u> for some $r \in \mathbb{Q}_{\mathbf{i}}$ we have $p \le_n^* r \le q$ and $r^{[f]} = q$.
5) If $p \in \mathbb{Q}_{\mathbf{i}}, \mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$ and $f \in \mathscr{F}_{p,\mathbf{u}}$ is rich (see Definition 2.9(4)) and $p^{[f]} \le q$, <u>then</u> for some $r \in \mathbb{Q}_{\mathbf{i}}^{\ell}$ we have $p <_n^{\otimes} r \le q$ and $r^{[f]} = q$.

Proof. 1) Define $f : \{x_i^n : i < h(n) \text{ and } n \in \mathbf{u}\} \to \{-1, 1\}$ by: $f(x_i^n)$ is the value of $H^q(x_i^n)$.
2) By 2.10(3) for $(*)_1$ and direct inspection for $(*)_2$.
3) Check.
4) We define $r \in \mathbb{Q}_{\mathbf{i}}$ by: $w^r = \mathbf{u} \cup w^q, \mathrm{dom}(E^r) = \cup\{(n/E^p) : n \in S_k \cap A^p$ for some $k \in \mathbf{u}\} \cup \mathrm{dom}(E^q), E^r = \{(n_1, n_2) : n_1 E^q n_2$ or some $k \in \mathbf{u}$ satisfies $\min(n_1/E^p) = \min(n_2/E^p) \in S_k \cap A^p\}, A^r = \cup\{S_k \cap A^p : k \in \mathbf{u}\} \cup A^q$

Next, we define $\mathfrak{c}_{r,k}$ for $k \in w^r$ by:

- $\mathfrak{c}_{r,k} = \mathfrak{c}_{p,k}$ if $k \in \mathbf{u}$
- $\mathfrak{c}_{r,k} = \mathfrak{c}_{q,k}$ if $k \in w^q$.

Lastly, for $x_i^n \in B_1^r$ (where B_1^r is defined in 1.2(1)(e)) we define

$$H^r(x_i^n) = \begin{cases} H^q(x_i^n) & \text{if } n \notin \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x_i^n \in \mathrm{dom}(H^q), \\ H^p(x_i^n) & \text{if } n \in \bigcup\limits_{k \in \mathbf{u}} k/E^p \text{ and } x_i^n \in \mathrm{dom}(H^p). \end{cases}$$

It should be clear that $r = (H^r, E^r, A^r, \bar{\mathfrak{c}}^r) \in \mathbb{Q}_{\mathbf{i}}$ is as required.
5) We choose $\mathbf{n}(\eta) \in \mathrm{val}(\mathfrak{c}_{p,\max(\mathbf{u})})$ for $\eta \in {}^{h'(\max(\mathbf{u}))}2$ such that $\eta = \langle f(x_i^{\mathbf{n}(\eta)}) : i < h(\max(\mathbf{u}))\rangle$; note that there is such $\mathbf{n}(\eta) \in \mathrm{val}(\mathfrak{c}_{p,n})$ because f is p-rich, see Definition 2.9(4). Now we define r.

Put $\mathrm{dom}(E^r) = \mathrm{dom}(E^p)$ and declare that X is an E^r-equivalence class <u>iff</u> (at least) one of the following occurs:

(a) for some $m \in \mathrm{val}(\mathfrak{c}^p_\ell)$ and $\ell \in \mathbf{u} \setminus \{\max(\mathbf{u})\}$ we have $X = (m/E^p)$
(b) $X = m/E^q$ for some $m \in A^q$
(c) $X = m/E^p$ for some $m \in \mathrm{val}(\mathfrak{c}_{p,\mathbf{k}_p(n)})$ which $\notin \{\mathbf{n}(\eta) : \eta \in {}^{h'(\max(\mathbf{u}))}2\}$
(d) for some $\eta \in {}^{h'(\max(\mathbf{u}))}2, X$ is equal to $X_\eta := (\mathbf{n}(\eta)/E^p) \cup \bigcup\{m/E^p : m \in A^p \backslash \mathbf{u}$ and $m \notin \mathrm{Dom}(E^q)$ and $\langle H^q(x^n_i) : i < h'(\max(\mathbf{u}))\rangle = \eta\}$.

We let $A^r = \mathbf{u} \cup A^q$ (in fact A^r is defined from E^r). Finally the function H^r is defined naturally:

$(*)$ $H^r(x^m_j) = H^q(x^m_j)$ when $m \in \omega \backslash \mathrm{Dom}(E^r)$ or $m \in \mathrm{Dom}(E^r) \wedge m' := \min(m/E^r) < m \wedge j \in [h(m'), h(m))$.

The reader may wonder: how come H^p does not appear in the definition of H^p? The answer is that $H^p \subseteq H^q$. $\square_{2.12}$

Corollary 2.13. *If $p \in \mathbb{Q}_\mathbf{i}, n < \omega$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}_\mathbf{i}$-name of an ordinal, <u>then</u> there are $\mathbf{u}, q$ and $\bar{\alpha} = \langle \alpha_f : f \in \mathscr{F}_{p,\mathbf{u}}\rangle$ such that:*

(a) $p \le^*_n q \in \mathbb{Q}_\mathbf{i}$,
(b) $\mathbf{u} = w^p \cap \mathbf{k}_p(n)$
(c) *for $f \in \mathscr{F}_{p,\mathbf{u}}$ we have $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$",*
(d) *$q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in \mathscr{F}_\mathbf{u}\}$" (which is a finite set).*

Proof. Let $k = \prod\limits_{\ell \in \mathbf{u}} 2^{h(\ell)\cdot|S_\ell|}$. Let $\{f_\ell : \ell < k\}$ enumerate $\mathscr{F}_\mathbf{u}$. By induction on $\ell \le k$ define r_ℓ, α_{f_ℓ} such that:

$$r_0 = p, r_\ell \le^*_n r_{\ell+1} \in \mathbb{Q}_\mathbf{i}, r^{[f_\ell]}_{\ell+1} \Vdash_{\mathbb{Q}_\mathbf{i}} \text{“}\underset{\sim}{\tau} = \alpha_{f_\ell}\text{”}.$$

The induction step is by 2.12(4). Notice that $q^{[f_\ell]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_{f_\ell}$" since $r^{[f_\ell]}_{\ell+1} \le q^{[f_\ell]}$. Now $q = r_k$ and $\langle \alpha_f : f \in \mathscr{F}_\mathbf{u}\rangle$ are as required. $\square_{2.13}$

Corollary 2.14. *As in 2.13 but replacing (a)-(d) there by:*

(a) $p \le^\otimes_n q \in \mathbb{Q}_\mathbf{i}$
(b) $\mathbf{u} = w^p \cap \mathbf{k}_p(n+1)$
(c) *if $f \in \mathscr{F}_{p,\mathbf{u}}$ is p-rich then $q^{[f]} \Vdash$ "$\underset{\sim}{\tau} = \alpha_f$", see 2.10(3)*
(d) *$q \Vdash_{\mathbb{Q}_\mathbf{i}}$ "if $\{\underset{\sim}{\eta}_n : n \in \mathrm{val}(\mathfrak{c}_{p,\mathbf{k}_n(p)})\} = {}^{\mathbf{k}(n,p)}2$ <u>then</u> $q \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_f : f \in \mathscr{F}_{p,\mathbf{u}}$ is p-rich$\}$.*

Proof. Similarly to the proof of 2.13 using 2.12(5) instead of 2.12(4). $\square_{2.14}$

Claim 2.15. *1) Assume $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{cc}}$ is fast (see part (7) of 2.1; alternatively see 2.2(2) and $\mathbf{i} \in \mathrm{FP}_3$). The COM player has a winning strategy in the bounding game $\partial^{\mathrm{bd}}_{\mathbb{Q}_\mathbf{i},p}$ for $p \in \mathbb{Q}_\mathbf{i}$ recalling:*

(a) *a play last ω-moves*
(b) *in the n-th move*
- INC *chooses a $\mathbb{Q}_\mathbf{i}$-name $\underset{\sim}{\tau}_n$ of an ordinal*

- *then the* COM *player chooses a finite set $\mathscr{U}_n$ of ordinals*

(c) *in the end the* COM *player wins* <u>*iff*</u> *there is $q \in \mathbb{Q}_{\mathbf{i}}$ above p forcing $\underset{\sim}{\tau}_n \in \mathscr{U}_n$ for every n.*

2) This is true even for the game $\partial^{\rm be}_{\mathbb{Q}_{\mathbf{i}},p}$ defined similarly but we change clause (b) to:

(b)$'$ *in the n-th move*
- *first, the* COM *player chooses $m^\bullet_n$*
- *second, the* INC *player chooses a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\tau}_n$ of an ordinal*
- *third, the* COM *player chooses a set $\mathscr{U}_n$ of $\le m^\bullet_n$ ordinals*

3) This is true even for the game $\partial_{\mathbb{Q}_{\mathbf{i}},p}$ defined similarly but we change clause (b) to:

(b)$''$ *in the n-th move*
- (α) *first, the* COM *player chooses $m^\bullet_n$*
- (β) *second, the* INC *player chooses $\ell^\bullet_n$*
- (*gamma*) *third, they play a subgame with $\ell^\bullet_n$ moves in the ℓ-th move*
  - $\bullet_1$ *the INC player chooses a $\mathbb{Q}_{\mathbf{i}}$-name $\underset{\sim}{\tau}_{n,\ell}$ of an ordinal*
  - $\bullet_2$ *then, the* COM *player chooses a set $\mathscr{U}_{n,\ell}$ of $\le m^\bullet_n$ ordinals.*

4) Moreover, $\mathbb{Q}_{\mathbf{i}}$ is strongly bounding (see [She, 4.1] *or 2.26 below).*

Proof. 1), 2) Follows by part (3).
3) Without loss of generality let $p_0 = p(0) = p$ be as is q in 2.10(4),(5). Now on the side, COM chooses in the n-th move also $\langle p_{n,\ell} : \ell \le \ell^\bullet_n\rangle, p_{n+1} = p(n+1), k_n = k(n)$ such that:

$(*)^1_n$
- (a) $p_n \in \mathbb{Q}_{\mathbf{i}}$ is above p and $\langle \text{nor}(\mathfrak{c}_{p,n,k}) : k \in w^p\rangle$ is increasing
- (b) if $n = m+1$ then $p_m <^{\otimes}_{k_n} p_n$ and $k_m < k_n$
- (c) $p_{n+1} \Vdash$ "if $\langle \underset{\sim}{\eta}_m : m \in S_{k(n)} \cap A^{p_n}\rangle$ is p_n-rich then $\ell < \ell^\bullet_n \Rightarrow \underset{\sim}{\tau}_{n,\ell} \in \mathscr{U}_{n,\ell}$"
- (d) $p_n = p_{n,0} \le^{\otimes}_n p_{n,1} \le^{\otimes}_n \ldots <^{\otimes}_n p_{n,\ell^\bullet_n} = p_{n+1}$
- (e) $p_{n,\ell+1} \Vdash$ "$\underset{\sim}{\tau}_{n,\ell} \in \mathscr{U}_{n,\ell}$"
- (f) p_n is as q is in 2.10(4),(5)
- (g) $h_{p_n}(\ell) = k_\ell$ for $\ell < n$
- (h) $p_0 = p$ and $k_0 = \min(w^p)$.

Why is it possible? That is, why is it a legal strategy for COM?

In the n-th move so p_n is well defined, let $k^\bullet_n = \sup\{\text{nor}(\mathfrak{c}_{p,\ell}) : \ell \in w^{p(n)} \cap \mathbf{k}_{p_n}(k_i + 1)$ for every $i < n\} + 1$. Let k_n be $\min(w^{p_0}) = \min(w^p)$ when $n = 0$, otherwise let $k = k_n > \mathbf{k}_{p_n}(n)$ be from w^{p_n} such that $k \ge k_n \wedge (k \in w^{p_n} \Rightarrow \text{nor}(\mathfrak{c}_{p_n,k}) \ge k^\bullet_n + 2$ and $k_n > \sup\{k_\ell : \ell < n\}$.

Let $p^\bullet_n$ be such that $p_n \le^{\otimes}_n p^\bullet_n, w^{p^\bullet_n} \cap k_n = \{k_\ell : \ell < n\}$ and $w^{p^\bullet_n} \backslash k_n = w^{p_n} \backslash k_n$ and $\mathfrak{c}_{p^\bullet_n,k} = \mathfrak{c}_{p_n,k}$ for $k \in w^{p^\bullet_n}$, clearly exists. Let $\mathbf{u} = \{k_\ell : \ell \le n\}$ so $\mathbf{u} \lhd w^{p^\bullet_n}$ and for parts (2),(3) let $m^\bullet_n = |\mathscr{F}_{p_n,\mathbf{u}_n}|$ be the move of COM. For parts (1),(2) after INC choose $\underset{\sim}{\tau}_n$ let $p_{n+1} \in \mathbb{Q}_{\mathbf{i}}$ be as in 2.14 for the triple $(p^\bullet_n, k_n, \underset{\sim}{\tau}_n)$. For part

(3), p_n, k_n are well defined and we choose $p_{n,0} = p^\bullet_n$ and $(p_{n,\ell+1}, \mathscr{U}_{n,\ell})$ such that $p_{n,\ell} \le^\otimes_n p_{n,\ell+1}$ and $p_{n,\ell+1} \Vdash_{\mathbb{Q}_\mathbf{i}}$ "$\mathcal{I}_{n,\ell} \in \mathscr{U}_{n,\ell}$" by 2.14.

Why this is a winning strategy? So assume $\langle (k_n, p_n) : n < \omega\rangle$ and (for part (1),(2)) $\langle (\mathcal{I}_n, \mathscr{U}_n) : n < \omega\rangle$ and (for parts (2),(3)) $\langle m_n : n < \omega\rangle$ and (for part (3)) $\langle \mathcal{I}_{n,\ell}, \mathscr{U}_{n,\ell} : \ell < \ell^\bullet_n\rangle$ were chosen. Let $q = \lim\langle p_n : n < \omega\rangle \in \mathbb{Q}_\mathbf{i}$ be naturally defined. Clearly $p, p_n \le_{\mathbb{Q}_\mathbf{i}} q$ for $n < \omega$.

Now we use "$\mathbf{i}$ is fast", consider $\mathfrak{c}_{p_n,k_n} = \mathfrak{c}_{q,k_n}$ and choose $m_{n,\eta,\iota} \in \mathrm{val}(\mathfrak{c}_{p_n,k_n})$ for $\eta \in {}^{h'(k(n))}2, \iota < k_n$ and $\mathfrak{d}^\iota_n \in \Sigma_n(\mathfrak{c}_{p_n,k_n})$ such that:

- $\mathrm{nor}(\mathfrak{d}^\iota_n) \ge n+1$
- $\mathrm{val}(\mathfrak{d}^\iota_n)$ is disjoint to $\{m_{n,\eta,\iota} : \eta \in {}^{h'(k)}2\}$
- for each $\iota < k_n$ the sequence $\langle m_{n,\eta,\iota} : \eta \in {}^{h'(k(n))}2\rangle$ is without repetition
- $\langle \mathrm{val}(\mathfrak{c}^\iota_n)\backslash\mathrm{val}(\mathfrak{d}^\iota_n) : \iota < k_n\rangle$ are pairwise disjoint.

For $\nu \in \prod\limits_n k_n$, let $A^\bullet_\nu = \mathrm{Dom}(E^q)\backslash(\cup\{m/E^q\text{: for some } n, m \in \mathrm{val}(\mathfrak{d}^{\nu(n)}_n)\})$. Clearly we can find $\Lambda \subseteq \prod\limits_n k_n$ of cardinality $2^{\aleph_0}$ such that $\nu \ne \rho \in \Lambda \Rightarrow |\{n : \nu(n) = \rho(n)\}| < \aleph_0$.

Also $\langle \mathrm{Dom}(q)\backslash A^\bullet_\nu : \nu \in \Lambda\rangle$ has pairwise finite intersection so as $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$ or just $\mathbf{i} \in \mathrm{FP}^3_{cc}$, for some $\nu \in A, A^\bullet_\nu = \omega \mod I_\mathbf{i}$.

Now we can define r as desired:

$(*)$
- (a) $\mathrm{dom}(E^r) = A^\bullet_\nu$
- (b) $E^r = E^q{\restriction}A^\bullet_\nu$
- (c) $w^r = w^q$
- (d) $\mathfrak{c}_{r,m}$ is
  - $\mathfrak{d}^{\nu(k)}_{k_n}$ if $m = k_n, n > 0$
  - $\mathfrak{c}_{q,m}$ if otherwise
- (e) H^r extends H^q
- (f) if $n < \omega, k = k_n, \eta \in {}^{h'(k(n))}2, m = m_{n,\eta,\nu(n)}$ and $\ell \in m/E^q$ <u>then</u> $\langle H^r(x^\ell_i) : i < h(k_n)\rangle = \eta$.

Now check. $\square_{2.15}$

Conclusion 2.16. *1) If $\mathbf{i} \in \mathrm{FP}^3_{cc}$ is fast (see 2.1(7), alternatively use 2.2(2)), <u>then</u> $\mathbb{Q}_\mathbf{i}$ is bounding (i.e. $\Vdash$ "every $f \in {}^\omega\omega$ is $\le g$ for some $g \in ({}^\omega\omega)^\mathbf{V}$"). Hence this holds, in particular, whenever $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}, \mathbf{i}$ is fast.*
2) Moreover, $\mathbb{Q}_\mathbf{i}$ has the PP*-property (even the strong one) see* [She98a, Ch.VI,2.12] *or Definition 2.19 below.*
3) Each of the properties from part (1) and (2) is preserved by CS *iteration.*

Proof. 1) By 2.15(1).
2) By 2.15(2).
3) For bounding by [She98a, Ch.V], for the $\mathbb{P}$-property by [She98a, Ch.VI,2.12A-F] $\square_{2.16}$

Claim 2.17. *Let an ideal $I \supseteq [\omega]^{<\aleph_0}$ on ω be given.*

1) For any function $h' : \mathbb{N} \to \mathbb{N}$ going to infinity, if $\bar{n} = \langle n_k : k < \omega \rangle$ satisfies $n_0 = 0$ and $n_{k+1} - n_k > 2^{h'(k)}k$, <u>then</u> letting $S_k = [n_k, n_{k+1})$, there is $\mathbf{i} \in \mathrm{FP}_4$ which is fast.
2) If $\mathrm{dual}(I)$ is an ultrafilter <u>then</u> some $\mathbf{i}$ as above has the uf-property, i.e. belongs to FP_4.
3) With stronger bound on n_{k+1}, we can demand that every $(\mathrm{CR}_k, \Sigma_k)$ has bigness (see [RS99]) which means: if $\mathfrak{c} \in \mathrm{CR}_n, \mathrm{val}(\mathfrak{c}) = u_1 \cap u_2$ then for $\mathfrak{d} \in \Sigma_n(\mathfrak{c})$ and $\iota \in \{1, 2\}$ we have $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$ and $\mathrm{val}(\mathfrak{d}) \subseteq u_\iota$.

Proof. As, ignoring the numerical bounds but are not important here, part (3) implies part (1),(2) we do elaborate in their proof.
1) We use 2.1(2)(B)$_1$ and we define $\mathrm{nor}_k : \mathscr{P}^-(S_k)$ by $\mathrm{nor}_k(X) = \lfloor |X|/2^{h'(k)} \rfloor$. Now check.
2) We use 2.1(2)(B)$_1$ choosing:

(∗) if $X \subseteq S_k$ then $\mathrm{nor}_k(X) = \lfloor \log_2(\mathrm{nor}_{k,0}(X)) \rfloor$ where $\mathrm{nor}_{k,0}(X) = \lfloor |X|/2^{h'(k)} \rfloor$, the use of $\log_2$ is to help prove "the uf property". Easy to check.

3) For $k < \omega$ we define $\mathbf{CR}_n$ as follows:

$(*)_1$ (a) $\mathfrak{c} \in \mathrm{CR}_k$ <u>iff</u> $\mathfrak{c} \subseteq {}^{S(k)}2$ is not empty
(b) for $\mathfrak{c} \in \mathrm{CR}_k$ let $\mathrm{val}(\mathfrak{c}) = \{n \in S_k$: for some $\eta, \nu \in \mathfrak{c}$ we have $\eta(n) \neq \nu(n)\}$
(c) for $\mathfrak{c} \in \mathrm{CR}_k$ let $\mathrm{nor}_k(\mathfrak{c}) = \frac{1}{(k+1) \cdot 2^{h'(k)}} \log_2(\log_2(|\mathfrak{c}|))$
(d) $\Sigma_n(\mathfrak{c}) = \{\mathfrak{d} \in \mathrm{CR}_n : \mathfrak{d} \subseteq \mathfrak{c}\}$

$(*)_2$ $\mathbf{CR}_k$ is as required in Definition 2.1 - check.

[Why? Easy.]

$(*)_3$ $\mathbf{CR}_k$ has bigness.

[Why? Obvious by the definitions.]

$(*)_4$ $\mathbf{CR}_k$ has the uf-property.

[Why? Assume $\mathfrak{c} \in \mathrm{CR}_k, \mathrm{nor}_k(\mathfrak{c}) \geq 1$. For $S \subseteq S_k$ let $m_{\mathfrak{c},S} = \max\{|\{\rho \in \mathfrak{c} : \rho \supseteq \nu\}| : \nu \in {}^{S(k) \setminus S}\{-1, 1\}\}$ hence $S = u \cup v \Rightarrow |\mathfrak{c}| \leq |m_{\mathfrak{c},u}| \times |m_{\mathfrak{c},v}|$. So if $h : S \to \{0, 1\}$, then for some $\iota < 2$ we have $|\mathfrak{c}_\iota| \geq \sqrt{\mathfrak{c}}$ where $\mathfrak{c}_\iota = \{\rho \in {}^{S(k)}2 : \rho \supseteq \nu_\iota\}$ where $v_\iota : \{n \in S_k : h(n) = \iota\} \to \{-1, 1\}$ is chosen such that $|\mathfrak{c}_\iota|$ is maximal. Now compute.]

$(*)_5$ $\mathbf{CR}_k$ is fast.

[Why? Assume $\mathfrak{c} \in \mathrm{CR}_k$ and $\mathrm{nor}_k(\mathfrak{c}) \geq 1$.
Now we try to choose $(n_\ell, \iota_\ell, \mathfrak{c}_\ell)$ by induction on $\ell < m = k \cdot 2^{h'(k)}$

$(*)_{5.1}$ (a) $\mathfrak{c}_\ell = \{\eta \in \mathfrak{c}$: if $k < \ell$ then $\eta(n_k) = \iota_\ell\}$
(b) $n_\ell \in \mathrm{val}_k(\mathfrak{c}) \backslash \{n_j : j < \ell\}$
(c) $|\mathfrak{c}_\ell| \geq |\mathfrak{c}| \cdot 2^{-\ell}$.

Now as $\mathrm{nor}_k(\mathfrak{c}) \geq 1$, clearly $|\mathfrak{c}| \geq 2^m$ hence $\mathrm{val}_k(\mathfrak{c}) \geq m$, so we can carry the induction. For $\iota < k$ we let $\mathfrak{d}_\iota = \mathfrak{d} = \mathfrak{c}_{k-h'(k)}$ and for each $\ell < k$ let $\langle m_{\eta,\ell} : \eta \in {}^{h'(k)}2\rangle$ list $\{n_j : j \in [2^{h'(k)}\ell, 2^{h'(k)}(\ell+1))\}$.

Lastly, $\mathrm{nor}_k(\mathfrak{d}_\iota) = \mathrm{nor}_k(\mathfrak{c}_m) \geq \frac{1}{(k+1)\cdot 2^{h'(k)}} \log_2(\log_2(|\mathfrak{c}| \cdot 2^{-m}) \geq \mathrm{nor}_k(\mathfrak{c}) - 1$.

$(*)_6$ if $\mathrm{nor}_k(\mathfrak{c}) \geq 1$ then for some partition u_1, u_2 of S_k and $\mathfrak{c}_1, \mathfrak{c}_2 \in \Sigma(\mathfrak{c})$, we have $\mathrm{val}(\mathfrak{c}_\iota) \subseteq u_\iota, \mathrm{nor}_k(\mathfrak{c}_\iota) \geq \mathrm{nor}_k(\mathfrak{c}) - 1$.

[Why? We can find a maximal $u \subseteq S_k$ such that $|\mathscr{P}_{\mathfrak{c},u}| \leq \sqrt{|\mathfrak{c}|}$, so $u \subsetneq S_k$. Let $n \in S_k \backslash u$ hence $|\mathscr{P}_{\mathfrak{c},u}| \leq \sqrt{|\mathfrak{c}|} < |\mathscr{P}_{\mathfrak{c},u\cup\{n\}}| \leq |\mathscr{P}_{\mathfrak{c},u}| \cdot 2 \leq 2\sqrt{|\mathfrak{c}|}$, so

- $|\mathscr{P}_{\mathfrak{c},u}| \in [\sqrt{|\mathfrak{c}|}, 2\sqrt{|\mathfrak{c}|})$.

Let $v = S\backslash u$, similarly

- $|\mathscr{P}_{\mathfrak{c},v}| \geq [\sqrt{\mathfrak{c}}.]$

$\square_{2.17}$

§ 2(A). **Further Comments.**

We deal with "On $\mathbb{Q}_{\mathbf{i}}$ and the PP-property".

Remark 2.18. We could have used $<^{\otimes}_n$ also in [She92].

Definition 2.19. [See [She98a, Ch.VI,2.12,A-F].] 1) A forcing notion $\mathbb{P}$ has the PP-property <u>when</u>:

$\otimes^{\mathrm{PP}}$ for every $\eta \in {}^\omega\omega$ from $\mathbf{V}^{\mathbb{P}}$ and a strictly increasing $x \in {}^\omega\omega \cap \mathbf{V}$ there is a closed subtree $T \subseteq {}^{<\omega}\omega \cap \mathbf{V}$ such that:
- (α) $\eta \in \lim(T)$, i.e. $(\forall n < \omega)(\eta \restriction n \in T)$,
- (β) $T \cap {}^n\omega$ is finite for each $n < \omega$,
- (γ) for arbitrarily large n there are k, and $n < i(0) < j(0) < i(1) < j(1) < \ldots < i(k) < j(k) < \omega$ and for each $\ell \leq k$, there are $m(\ell) < \omega$ and $\eta^{\ell,0}, \ldots, \eta^{\ell,m(\ell)} \in T \cap {}^{j(\ell)}\omega$ such that $j(\ell) > x(i(\ell) + m(\ell))$ and $(\forall \nu \in T \cap {}^{j(k)}\omega)(\exists \ell \leq k)(\exists m \leq m(\ell))(\eta^{\ell,m} \trianglelefteq \nu)$.

2) We say that a forcing notion $\mathbb{P}$ has the strong PP-property <u>when</u>:

$\oplus^{\mathrm{sPP}}$ for every function $g : \omega \longrightarrow \mathbf{V}$ from $\mathbf{V}^{\mathbb{P}}$ there exist a set $B \in [\omega]^{\aleph_0} \cap \mathbf{V}$ and a sequence $\langle w_n : n \in B\rangle \in \mathbf{V}$ such that for each $n \in B, |w_n| \leq n$ and $g(n) \in w_n$.

Observation 2.20. *Of course, if a proper forcing notion has the strong* PP-*property <u>then</u> it has the* PP-*property.*

Conclusion 2.21. *Assume that for each $p \in \mathbb{Q}_{\mathbf{i}}$ the first player has no winning strategy in $\partial^{\mathrm{sb}}_p(\mathbb{Q}_{\mathbf{i}})$, see 2.26 below (e.g. if I is a maximal ideal).*

<u>Then</u>

$(*)$ *$\mathbb{Q}_{\mathbf{i}}$ is proper, α-proper, strongly α-proper for every $\alpha < \omega_1$, is ${}^\omega\omega$-bounding and it has the* PP-*property, even the strong* PP-*property.*

By [She98a, Ch.VI,2.12] we know

Theorem 2.22. *Suppose that $\langle \mathbb{P}_i, \mathbb{Q}_j : j < \alpha, i \le \alpha \rangle$ is a countable support iteration such that $\Vdash_{\mathbb{P}_j}$ "$\mathbb{Q}_j$ is proper and has the* PP*-property".*

Then $\mathbb{P}_\alpha$ has the PP*-property.*

Definition 2.23. For $\mathbf{i} \in \mathrm{FP}_3$ we define the forcing notion $\mathbb{Q}^2_{\mathbf{i}}$ as follows (compare with Definition 2.4):

(A) $p \in \mathbb{Q}^2_{\mathbf{i}}$ iff $p = (H, E, A, \bar{\mathfrak{c}})$ satisifes (a),(b),(d),(e) as in Definition 2.4
 (c)$'$ H is a function from ω to $\{-1, 1\}$ such that $H \restriction A^p$ is constantly 1

(B) as in Definition 2.4

(C) a function $f : \{x^n_i : n < \omega, i < h(n)\} \to \{-1, 1\}$ satisfies a condition $p \in \mathbb{Q}^2_{\mathbf{i}}$ when:
 (a) if $n \in \omega \backslash \mathrm{Dom}(E^p)$ and $i < h(n)$ then $f(x^n_i) = H^p(x^n_i)$
 (b) if $n \in \mathrm{Dom}(E^p)$ but $i \in [h(\min(n/E^p), h(n))$ then $f(x^n_i) = H(x^n_i)$
 (c) if $n \in \omega \backslash \mathrm{Dom}(E^p)$ and $h(n) \le i < h(\min(n/E^p))$ then $f(x^n_i) = f(x^{\min(n/E^p)}_i) \cdot H^p(x^{\min(n/E^p)}_i)$, yes the product

(D) $\mathbb{Q}^2_{\mathbf{i}} \models$ "$p \le q$" iff:
 (a) $p, q \in \mathbb{Q}^2_{\mathbf{i}}$
 (b) $E^p \le E^q$
 (c) if $f : \{x^n_i : n < \omega$ and $i < h(n)\} \to \{-1, 1\}$ and f satisfies q then f satisfies p.

Definition 2.24. We repeat Definition 2.9 for $\mathbb{Q}^2_{\mathbf{i}}$.

Definition 2.25. We repeat Definition 2.11 for $\mathbb{Q}^2_{\mathbf{i}}$.

Definition 2.26. 1) For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define $\partial_{\mathrm{sb}} = \partial^{\mathrm{sb}}_p = \partial^{\mathrm{sb}}_p(\mathbb{Q}) = \partial^{\mathrm{sb}}_{\mathbb{Q},p}$, the strong bounding game between the null player NU and the bounding player BND as follows:

(a) a play last ω moves and

(b) in the n-th move:
 (α) first the NU player gives a (non-empty) tree $\mathscr{T}_n$ with ω levels and no maximal node and a $\mathbb{Q}$-name $\underset{\sim}{\mathscr{F}}_n$ of a function with domain $\mathscr{T}_n$ such that $\eta \in \mathscr{T}_n \Rightarrow p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathscr{F}}_n(\eta) \in \mathrm{suc}_{\mathscr{T}_n}(\eta)$"
 (β) then BND player chooses $\eta_n \in \mathscr{T}_n$

(c) in the end, the BND player wins the play $\langle \mathscr{T}_n, \eta_n : n < \omega \rangle$ iff there is $q \in \mathbb{Q}$ above p forcing that "$(\forall n < \omega)(\exists k < \mathrm{level}(\eta_n))(\underset{\sim}{\mathscr{F}}_n(\eta_n \restriction k) \le_{\mathscr{T}_n} \eta_n \wedge k$ is even)" where $\eta_n \restriction k$ is the unique $\nu \le_{\mathscr{T}_n} \eta_n$ of level k.

2) Omitting p means NU chooses it in his first move.
3) A forcing notion $\mathbb{Q}$ is strongly bounding if for every condition $p \in \mathbb{Q}$ player BND has a winning strategy in the game $\partial^{\mathrm{sb}}_{\mathbb{Q},p}$.

§ 3. On no α-ultrafilter

Observation 3.1. *For an ultrafilter D on $\mathbb{N}$ and $\alpha, 2 \leq \alpha, \alpha < \omega_1$ the following conditions are equivalent:*

- (A) *D is an α-ultrafilter, see Definition 0.1, i.e. J^1_α-ultrafilter where*
  - $J^1_\alpha = \{A \subseteq \omega^\alpha : \mathrm{otp}(A) < \omega^\alpha\}$
- (B) *D is an $\mathrm{id}_{\mathscr{T}}$-ultrafilter for some (equivalent all) $\mathscr{T} \in \mathbf{T}_1$ such that $\mathrm{dp}^*(\mathscr{T}) = \alpha$, see Definition 3.3 below.*

Recall (this justifies the restriction to $\omega^\alpha, \alpha \geq 2$):

Observation 3.2. *Assume α, β are countable non-zero ordinals and D is a non-principal ultrafilter on ω.*
1) If $\beta < \alpha + \beta$ then D is not an $(\alpha+\beta)$-ultrafilter.
2) If $\alpha = \omega$ then D is not an α-ultrafilter.
3) If $\alpha < \beta$ and D is $\omega^{\alpha+\beta}$ ultrafilter, <u>then</u> D is ω^α-ultrafilter.
4) If the ultrafilter D on ω is a P-point and $\alpha \geq 2$, <u>then</u> D is an α-ultrafilter.

Proof. E.G.
3) By 3.4(3) below. $\square_{3.2}$

Definition 3.3. 1) For $\nu \in {}^{\omega>}\omega$ let $\mathbf{T}^1_\nu$ be the set of $\mathscr{T}$ such that $\mathscr{T}$ is a well founded subtree of $(({}^{\omega>}\omega)_{[\geq \nu]}, \triangleleft)$, so downward closed (up to the root) where $({}^{\omega>}\omega)_{[\geq \nu]} = \{\eta \in {}^{\omega>}\omega : \nu \trianglelefteq \eta\}$, let $\mathrm{rt}(\mathscr{T}) = \nu$ called the root of $\mathscr{T}$.
2) Let $\mathbf{T}_1 = \cup\{\mathbf{T}^1_\nu : \nu \in {}^{\omega>}\omega\}$.
3) For $\mathscr{T} \in \mathbf{T}_1$ let $\mathrm{dp}^*(\mathscr{T}) = \mathrm{dp}^*_{\mathscr{T}}(\mathrm{rt}(\mathscr{T}))$ where for $\eta \in \mathscr{T}, \mathrm{dp}^*_{\mathscr{T}}(\eta)$ is the minimal ordinal α such that:

- $\alpha \geq \sup\{\mathrm{dp}^*_{\mathscr{T}}(\rho) : \rho \in \mathrm{suc}_{\mathscr{T}}(\eta)\}$
- $\alpha \geq \limsup \langle \mathrm{dp}^*_{\mathscr{T}}(\rho) + 1 : \rho \in \mathrm{suc}_{\mathscr{T}}(\eta)\rangle$ <u>when</u> $\mathrm{suc}_{\mathscr{T}}(\eta)$ is infinite.

4) Let $\mathbf{T}_2$ be the set of $\mathscr{T} \in \mathbf{T}_1$ such that:

- $\{\eta, \nu\} \subseteq \mathscr{T} \wedge \eta \triangleleft \nu \Rightarrow_{\mathscr{T}} \mathrm{dp}^*(\nu) < \mathrm{dp}^*_{\mathscr{T}}(\eta)$ moreover
- $\eta \in \mathscr{T} \wedge \mathrm{suc}_{\mathscr{T}}(\eta) \neq \emptyset \Rightarrow \mathrm{suc}_{\mathscr{T}}(\eta) = \{\eta\hat{\ }\langle n\rangle : n < \omega\}$.

5) For $\mathscr{T} \in \mathbf{T}_1, \nu \in \mathscr{T}$ let $\mathscr{T}_{[\geq \nu]} = \{\eta \in \mathscr{T} : \nu \trianglelefteq \eta\}$, hence $\mathscr{T}_{[\geq \nu]} \in \mathbf{T}^1_\nu, \mathrm{dp}^*(\mathscr{T}_{[\geq \nu]}) = \mathrm{dp}^*_{\mathscr{T}}(\nu)$ and $\mathscr{T} \in \mathbf{T}_2 \Rightarrow \mathscr{T}_{[\geq \nu]} \in \mathbf{T}_2$.
6) For $\mathscr{T} \in \mathbf{T}_1$ and $A \subseteq \mathscr{T}$ let:

- $\mathscr{T}_{[A]} = \{\eta \in \mathscr{T} : (\exists \nu \in A)(\eta \trianglelefteq \nu)\}$.

7) For $\mathscr{T} \in \mathbf{T}_1$ and antichain A of $\mathscr{T}$ let:

- $\mathrm{id}_{\mathscr{T}}(A) = \{B \subseteq A : \mathrm{dp}^*(\mathscr{T}_{[B]}) < \mathrm{dp}^*(\mathscr{T}_{[A]})\}$.

8) For $\mathscr{T} \in \mathbf{T}_1$ let $\mathrm{id}_{\mathscr{T}} = \mathrm{id}_{\mathscr{T}}(\max(\mathscr{T}))$.

Observation 3.4. *1) For every countable ordinal γ there is $\mathscr{T}$ such that:*

- (a) $\mathscr{T} \in \mathbf{T}^1_{\langle\rangle}$
- (b) $\mathrm{dp}^*(\mathscr{T}) = \gamma$
- (c) *if $\eta \in \mathscr{T} \setminus \max(\mathscr{T})$, <u>then</u>:*

(α) $\mathrm{suc}_{\mathscr{T}}(\eta) = \{\eta\hat{\ }\langle n\rangle : n < \omega\}$
(β) *if* $\mathrm{dp}^*_{\mathscr{T}}(\eta) = \alpha + 1$ *then* $\mathrm{dp}^*_{\mathscr{T}}(\eta\hat{\ }\langle n\rangle) = \alpha$ *for* $n < \omega$
(γ) *if* $\mathrm{dp}^*_{\mathscr{T}}(\eta)$ *is a limit ordinal, then* $\langle \mathrm{dp}^*_{\mathscr{T}}(\eta\hat{\ }\langle n\rangle) : n < \omega\rangle$ *is increasing with limit* $\mathrm{dp}^*_{\mathscr{T}}(\eta)$

(d) *(follows)* $\mathscr{T} \in \mathbf{T}_2$.

2) If $\mathscr{T}_1 \in \mathbf{T}_1$ *and* $\alpha = \mathrm{dp}^*(\mathscr{T}_1)$ *then there is* $\mathscr{T}_2$ *such that:*

(a) $\mathscr{T}_2 \in \mathbf{T}_1$
(b) $\mathscr{T}_2 \subseteq \mathscr{T}_1$ *and* $\mathrm{rt}(\mathscr{T}_2) = \mathrm{rt}(\mathscr{T}_1)$
(c) *if* $\eta \in \mathscr{T}_2$ *then* $\mathrm{dp}^*_{\mathscr{T}_2}(\eta) = \mathrm{dp}^*_{\mathscr{T}_1}(\eta)$
(d) *exactly one of the following occurs for each* $\eta \in \mathscr{T}_2$*:*
- $\eta \in \max(\mathscr{T}_1), \eta \in \max(\mathscr{T}_2)$
- $\mathrm{dp}^*_{\mathscr{T}_1}(\eta) = \mathrm{dp}^*_{\mathscr{T}_1}(\nu)$ *for some* $\nu \in \mathrm{suc}_{\mathscr{T}_1}(\eta)$ *and* $\mathrm{suc}_{\mathscr{T}_2}(\eta)$ *is a singleton*
- $\mathrm{suc}_{\mathscr{T}_2}(\eta)$ *is infinite and* $\mathrm{suc}_{\mathscr{T}_2}(\eta) = \{\nu \in \mathrm{suc}_{\mathscr{T}_1}(\eta) : \mathrm{dp}^*_{\mathscr{T}_2}(\nu) < \mathrm{dp}^*_{\mathscr{T}_1}(\eta)\}$ *and so the set* $\mathrm{suc}_{\mathscr{T}_1}(\eta)\backslash\mathrm{suc}_{\mathscr{T}_2}(\eta)$ *is finite*

(e) *if* $\nu \in \mathscr{T}_2$ *then* $\max((\mathscr{T}_2)_{[\geq\nu]}) = \max((\mathscr{T}_1)_{[\geq\nu]})$ *modulo* $\mathrm{id}_{(\mathscr{T}_1)_{[\geq\nu]}}$.

3) If $\mathscr{T}_* \in \mathbf{T}_1, \mathrm{dp}^*(\mathscr{T}) = \alpha, \alpha$ *is a countable ordinal,* $\gamma = \omega^\alpha$ *ordinal exponentiation,* *<u>then</u>* $\mathrm{id}_{\mathscr{T}_*}$ *is isomorphic to* J^1_γ.
4) If $\mathscr{T} \in \mathbf{T}_1, A \subseteq \max(\mathscr{T})$ *and* $n < \omega$ *and* $B = \{\eta \in \max(\mathscr{T})$*: for some* $\nu \in A$ *we have* $\eta = \nu \wedge \ell g(\nu) \leq n$ *<u>or</u>* $(\ell g(\eta) = \ell g(\nu) \geq n) \wedge (\eta{\restriction}[\eta, \ell g(\eta)) = \nu{\restriction}[n, \ell g(\nu)))$*, <u>then</u>* $B \in \mathrm{id}(\mathscr{T})$.

Proof. E.g.
3) Without loss of generality $\mathscr{T}_*$ is like $\mathscr{T}_2$ in part (2). Let $<_{\mathrm{lex}}$ be the lexicographic order on $\mathscr{T}_*$, and we prove by induction on $\beta \leq \alpha$

$(*)_\beta$ if $\mathscr{T} \in \mathbf{T}^1_{\mathrm{rt}(\mathscr{T}_*)}$ and $\eta \in \mathscr{T}$ and $\mathrm{dp}^*_{\mathscr{T}}(\eta) = \beta$ then
(a) $\omega^\beta \leq \mathrm{otp}(\max(\mathscr{T}_{[\geq\eta]}), <_{\mathrm{lex}}) < \omega^{\beta+1}$
(b) if β is a limit ordinal or $\mathscr{T} \subseteq \mathscr{T}_*$ then $\mathrm{otp}(\max(\mathscr{T}_{[\geq\eta]})) = \omega^\beta$.

$\square_{3.4}$

Question 3.5. Define a family of ideals J for which the proof works.

Definition 3.6. Let $\mathrm{FP}_{3,\alpha}$ for $\alpha \in [1, \omega_1)$ be the set of $\mathbf{j}$ which consists of:

(A) (a) $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$ or at least $\mathbf{i} \in \mathrm{FP}^3_{cc}$
(b) $\mathbf{i}$ is fast
(c) we may write $\mathbb{Q}_{\mathbf{j}}$ instead of $\mathbb{Q}_{\mathbf{i}}$, etc.

(B) $\mathscr{T}$ as in 3.4(1) and $\bar{s} = \langle s_n : n < \omega\rangle$, s_n a finite subset of $\max(\mathscr{T})$, increasing with n such that $\bigcup\limits_n s_n = \max(\mathscr{T})$; note that $\mathscr{T}$ is uniquely determined by $\mathbf{j}$, so let $\mathscr{T}_{\mathbf{j}} = \mathscr{T}$.

Remark 3.7. Should we use the $\langle s_n : n < \omega\rangle, s_{\mathbf{j},n}, \mathfrak{c}, \max(\mathscr{T}_{\mathbf{j}})$ increasing with n with union all, to define the norm? Presently, no need.

For our purpose (in order to apply $\mathbb{Q}_{\mathbf{i}}$ from §2) we code members of $\mathscr{T}$'s as follows:

Choice 3.8. 1) We choose a one-to-one function cd from ${}^{\omega>}\{-1,1\}$ onto ${}^{\omega>}\omega$ such that:

- $(*)$ if $\eta_1 \in {}^{\omega>}\{-1,1\}$ and $\rho_1 = \mathrm{cd}(\eta_1) \triangleleft \rho_2 \in {}^{\omega>}\omega$, <u>then</u> for some η_2 as have
  - $\eta_1 \triangleleft \eta_2 \in {}^{\omega>}\{-1,1\}$
  - $\rho_2 \triangleleft \mathrm{cd}(\eta_2)$.

2) For $\eta \in \mathrm{dom}(\mathrm{cd})$ let $\mathbf{k}[\eta] = \ell g(\mathrm{cd}(\eta))$.
3) For $\mathscr{T} \in \mathbf{T}_1$ and $\eta \in {}^{\omega>}\omega$, let $\varrho_{\mathscr{T}}[\eta] = \varrho[\eta, \mathscr{T}]$ be the $\nu \in \max(\mathscr{T})$ such that $\nu \trianglelefteq \mathrm{cd}(\eta)$ if there is one and undefined otherwise.

Main Lemma 3.9. *There is no $\alpha_\bullet$-ultrafilter in $\mathbf{V}^{\mathbb{P}}$ extending $D_{\mathbf{i}}$ <u>when</u>:*

- $(*)$ (a) $\alpha_\bullet \in [2, \omega_1)$
  - (b) $\mathbf{i} \in \mathrm{FP}_{3,\alpha_\bullet}$ *so $D_{\mathbf{i}}$ is a non-principal ultrafilter on ω*
  - (c) $\mathbb{Q}_0 = \mathbb{Q}^1_{\mathbf{i}}$ *where* $\mathbf{i} \in \mathrm{FP}^3_{\mathrm{uf}}$ *is fast, see 2.1(7)*
  - (d) $\underset{\sim}{\mathbb{Q}}_1$ *is a $\mathbb{Q}_0$-name of proper bounding forcing, (instead proper we may demand* CH+ *forcing with $\underset{\sim}{\mathbb{Q}}_1$ preserves $\aleph_1$)*
  - (e) $\mathbb{P} = \mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$.

Proof. Toward contradiction assume:

$\oplus$ $p = (p_0, p_1) \in \mathbb{P}$ forces $\underset{\sim}{D}$ is an ultrafilter extending $D_{\mathbf{i}}$ which is an $\alpha_\bullet$-ultrafilter.

Now

$(*)_1$ $\mathbb{P}$ is a bounding forcing notion.

[Why? First, $\mathbb{Q}_0$ is by clause (c) recalling 2.16. Second, also $\underset{\sim}{\mathbb{Q}}_1$ is by clause (d). Together we are done.]

Now

$(*)_2$ any member of $\mathrm{id}(\mathscr{T}_{\mathbf{i}})$ from $\mathbf{V}^{\mathbb{P}}$ is included in another member which is from $\mathbf{V}$ where $\mathscr{T}_{\mathbf{i}}$ is from 3.6(B).

[Why? We use just that $\mathbb{P}$ is bounding. Let $p \in \mathbb{P}, p \Vdash$ "$\underset{\sim}{A} \in \mathrm{id}(\mathscr{T}_{\mathbf{i}})$ is non-empty", let $\underset{\sim}{B} = \{\eta \in \mathscr{T}_{\mathbf{i}} : (\exists \nu \in \underset{\sim}{A})(\eta \trianglelefteq \nu)\}$ so $p \Vdash$ "$\underset{\sim}{B}$ is a non-empty subtree of $\mathscr{T}_{\mathbf{i}}$", without loss of generality $p \Vdash$ "$\mathrm{dp}^*_{\mathscr{T}_{\mathbf{i}}}(\underset{\sim}{B}) = \alpha$" so $\alpha < \alpha_\bullet$.

Define the $\mathbb{P}$-name $\mathrm{dp}^\bullet_{\underset{\sim}{B}} : \mathscr{T}_{\mathbf{i}} \to (\alpha+1) \cup \{-1\}$ by: $\mathrm{dp}^\bullet_{\underset{\sim}{B}}(\eta) = \mathrm{dp}^*_{\underset{\sim}{B}}(\eta)$ if $\eta \in \underset{\sim}{B}$ and -1 otherwise. So there is $q \in \mathbb{P}$ above p and functions $g_1 \in \mathbf{V}$ from $\mathscr{T}$ into $[(\alpha+1) \cup \{-1\}]^{<\aleph_0}$ and $g_0 : \mathscr{T}_{\mathbf{i}} \mapsto \omega$ such that for every $\eta \in \mathscr{T}_{\mathbf{i}}$

- $\bullet_1$ $q \Vdash$ "$\mathrm{dp}^\bullet_{\underset{\sim}{B}}(\eta) \in g_1(\eta)$"
- $\bullet_2$ if $\eta \in \mathscr{T}_{\mathbf{i}} \backslash \max(\mathscr{T}_{\mathbf{i}})$ then $q \Vdash$ "if $\{\eta : \eta\hat{\ }\langle n\rangle \in \underset{\sim}{B}\}$ is finite then it is included in $[0, g_0(\eta)) \cup \{-1\}$".

We define a function g_2 from $\mathscr{T}_{\mathbf{i}}$ into $(\alpha+1) \cup \{-1\}$ by defining $g_2 \restriction \{\eta \in \mathscr{T}_{\mathbf{i}} : \ell g(\eta) = n\}$ by induction on n and proving:

- (a) it is well defined
- (b) $\nu \triangleleft \eta \Rightarrow g_2(\nu) \geq g_2(\eta) \vee g_2(\nu) = -1 = g_2(\eta)$
- (c) $g_2(\eta) = -1 \Rightarrow q \Vdash$ "$\eta \notin \underset{\sim}{B}$"

(d) if $g_2(\eta) = \beta \geq 0$ then $q \nVdash$ "$\mathrm{dp}^\bullet_{\underset{\sim}{B}}(\eta) \neq \beta$" and $q \Vdash$ "$\mathrm{dp}^\bullet_{\underset{\sim}{B}}(\eta) \leq \beta$"

(e) if $\eta = \nu\hat{\ }\langle k\rangle \in \mathscr{T}_{\mathbf{i}}$ and $g_2(\eta) = g_2(\nu)$ then $k < g_0(\nu)$.

Let $\beta_\bullet = \alpha + 1$.

In stage n, for any relevant η let $g_2(\eta) = \max\{\beta \in g_1(\eta) : \beta < \beta_\bullet$ and $q \nVdash$ "$\mathrm{dp}^*_{\underset{\sim}{B}}(\eta) \neq \beta$" and if $\eta = \nu\hat{\ }\langle k\rangle$ then $\beta \leq g_2(\nu)$ and $k \geq g_0(\nu) \Rightarrow \beta < g_2(\eta)\}$. Why is it well defined? If $n = 0$ then $\eta = \langle\rangle$ and clearly $g_1(\eta)$ is necessarily non-empty, in fact, $\alpha \in g_1(\langle\rangle) \cap \beta_\bullet$ and $q \Vdash$ "$\mathrm{dp}^*_{\underset{\sim}{B}}(\eta) = \alpha$". If $n = m+1$, let $\beta = g_2(\eta{\restriction}m)$, so there is $q_1 \in \mathbb{P}$ satisfying $q_1 \geq q$ and forcing $\mathrm{dp}^\bullet_{\underset{\sim}{B}}(\eta{\restriction}m) = \beta$ and without loss of generality forcing a value to $\mathrm{dp}^\bullet_{\underset{\sim}{B}}(\eta)$, say γ. Necessarily $\gamma \in g_1(\eta)$ and we can prove the other demands, so $g_2(\eta)$ is well defined.

Now let $B_2 = \{\eta \in \mathscr{T} : g_2(\eta) \geq 0\}$. Now easily $g_2(\langle\rangle) = \alpha, \mathrm{dp}^*(B_2 \cap \mathscr{T}_{\geq\eta}) \leq g_2(\eta)$ for every $\eta \in \mathscr{T}$ (by induction on $g_2(\eta)$), hence $B_2 \in \mathrm{id}(\mathscr{T}_{\mathbf{i}})$. So we are done proving $(*)_2$.]

Recalling Definition 3.8(3):

$(*)_3$ there is a $\mathbb{P}$-name $\underset{\sim}{\tau}$ such that $p \Vdash$ "$\underset{\sim}{\tau} \in \underset{\sim}{D}$ and $\underset{\sim}{A}^0 \in \mathrm{id}(\mathscr{T}_{\mathbf{i}})$ and if $\underset{\sim}{A}^1 \in \underset{\sim}{D}$ then $\underset{\sim}{\tau} \subseteq A^1$" where:

- $\underset{\sim}{A}^1 = \{n : \varrho_{\mathscr{T}}[\underset{\sim}{\eta}_n]$ is well defined$\}$
- $\underset{\sim}{A}^0 = \{\varrho_{\mathscr{T}}[\underset{\sim}{\eta}_n] : n \in \underset{\sim}{\tau}$ satisfies $\varrho_{\mathscr{T}_{\mathbf{i}}}[\underset{\sim}{\eta}_n]$ is well defined$\}$.

[Why? Consider the function $\underset{\sim}{f}$, i.e. the $\mathbb{P}$-name $\underset{\sim}{f}$ defined by:

$\bullet_1$ $\underset{\sim}{f}(n) = \varrho$ <u>iff</u> $\varrho_{\mathscr{T}}[\underset{\sim}{\eta}_n]$ is well defined and equal to ϱ.

Clearly

$\bullet_2$ $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}$ is a partial function from ω into $\max(\mathscr{T})$"

$\bullet_3$ $\mathrm{dom}(\underset{\sim}{f}) = \underset{\sim}{A}^1$.

We now finish by $\oplus$.]

$(*)_4$ there are $q = (q_0, q_1), B$ such that:

(a) $q \in \mathbb{P}$ is above p

(b) $B \in \mathrm{id}(\mathscr{T}_{\mathbf{i}})$ so $B \in \mathbf{V}$

(c) $q \Vdash$ "$\underset{\sim}{A}^0 \subseteq B$".

[Why? By $(*)_2$.]

$(*)_5$ Now by induction on $\ell < \omega$ we choose $k(\ell) > \ell$ and $\nu_\ell \in {}^{[\ell,k(\ell))}\{-1,1\}$ such that for every $\eta \in {}^{\ell}\{-1,1\}, \varrho_{\mathscr{T}}[\eta\hat{\ }\nu_\ell]$ is well defined and $\notin B$.

[Why can we? By 3.4(4).]

Hence

$(*)_6$ we choose $n_\ell \in w^{q_0}$ by induction on ℓ such that $n_{\ell+1} > k(h(n_\ell))$, now without loss of generality $\mathscr{Y} := \cup\{n/E^q : n \in [n_{2\ell+1}, n_{2\ell+2}) \cap w^q$ for some $\ell < \omega\} \in D_{\mathbf{i}}$. For each $\ell < \omega$ let g_ℓ be a function from $([n_{2\ell+1}, n_{2\ell+2}) \cap A^q) \cup \mathrm{val}(\mathfrak{c}^q_{n_{2\ell}})$ into $\mathrm{val}(\mathfrak{c}^q_{n_{2\ell}})$ which is the identity on $\mathrm{val}(\mathfrak{c}^q_{n_{2\ell+1}})$.

Now we define $r \in \mathbb{Q}^1_{\mathbf{i}}$ by:

$(*)_7$ (a) $\mathrm{Dom}(E^r) = \cup\{n/E^q : n \in \mathrm{val}(\mathfrak{c}^q_{2\ell})$ for some $\ell < \omega\} \cup \{m/E^q : m \in [n_{2\ell+1}, n_{2\ell+2}) \cap A^q$ for some $\ell < \omega\}$

(b) $W^r = \{n_{2\ell} : \ell < \omega\}$

(c) $(m_1, m_2) \in E^r$ iff for some ℓ we have $g_\ell(\min(m_1/E^q)) = g_\ell(\min(m_2/E^q))$

(d) H^r extends H^q

(e) if $n \in [n_{2\ell+1}, n_{2\ell+2}) \cap A^p$ then for every $m \in n/E^q_1$ the sequence $\langle H^r(x^n_\ell) : \ell \in [h'(n_{2\ell}), h[(m)] \rangle$ extends $\nu_{(n_{2\ell})}$.

So

$(*)_8$ (a) $q_0 \le r \in \mathbb{Q}^1_{\mathbf{i}}$

(b) the set $\cup\{m/E^r_1$: for some $\ell, m = n_{2\ell}$ and if $n \in A^q \cap [n_{2\ell+1}, n_{2\ell+1})$ then $\langle H^r(x^n_i) : i \in [n, h(n)) \rangle$ is extended by no member of $B\}$ is included in $\mathscr{Y} \setminus \cup \{n/E^q : n < n_{2\ell}\}$.

[Why? Should be clear.]

So we are done. $\square_{3.9}$

Conclusion 3.10. *1) For some forcing notion $\mathbb{P}$, in $\mathbf{V}^{\mathbb{P}}$ there is no α-ultrafilter on ω for any ordinal $\alpha \in [1, \omega_1)$.*
2) $\mathbb{P}$ is as in part (1) above <u>when</u>:

$(*)$ (a) $\mathbf{V} \models$ "$2^{\aleph_0} = \aleph_1$ *and* $\diamondsuit_{\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}}$"

(b) $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \omega_2, \beta < \omega_2 \rangle$ *is a* CS *iteration, each* $\mathbb{Q}_\beta$ *is proper bounding forcing of cardinality* $\aleph_1$ *and* $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\omega_2}$

(c) *if* $\underset{\sim}{D}$ *is a* $\mathbb{P}_{\mathbf{q}}$*-name of a non-principal ultrafilter on* ω*, <u>then</u> for some* $\delta < \kappa, \mathrm{cf}(\delta) = \aleph_1, \underset{\sim}{D}_\delta = \underset{\sim}{D} {\restriction} \mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}_\delta]}$ *is a* $\mathbb{P}_\delta$*-name,* $\underset{\sim}{\mathbb{Q}}_\delta = \mathbb{Q}^1_{\underset{\sim}{\mathbf{i}}(\delta)}, D_{\underset{\sim}{\mathbf{i}}(\delta)} = \underset{\sim}{D}_\delta$ *and* $\underset{\sim}{\mathbf{i}}(\delta) \in \mathrm{FP}_3$ *is fast.*

3) $\mathbb{P}$ is as in part (2) above <u>when</u>:

$(*)$ (a) $\mathbf{V} \models$ "$2^{\aleph_0} = \aleph_1$ *and* $\diamondsuit_S$ *where* $S \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_1\}$[1] *so* $\kappa = \mathrm{cf}(\kappa) = \kappa^{<\kappa}, S$ *stationary in* κ

(b) $\bar{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \le \kappa \rangle$ *is* $\lessdot$*-increasing sequence of forcing notions each bounding,* κ*-c.c. and* $\alpha < \kappa \Rightarrow |\mathbb{P}_\alpha| < \kappa$

(c) *if* $\underset{\sim}{D}$ *is a* $\mathbb{P}_\kappa$*-name of a non-principal ultrafilter on* ω*, <u>then</u> for some* $\delta < \kappa, \mathrm{cf}(\delta) = \aleph_1, \underset{\sim}{D}_\delta = \underset{\sim}{D} {\restriction} \mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}_\delta]}$ *is a* $\mathbb{P}_\delta$*-name,* $\underset{\sim}{\mathbb{Q}}_\delta = \mathbb{Q}_{\mathbf{i}(\delta)}, D_{\underset{\sim}{\mathbf{i}}(\delta)} = \underset{\sim}{D}_\delta$ *and* $\underset{\sim}{\mathbf{i}}(\delta) \in \mathrm{FP}_{3,\alpha}$ *is fast.*

Remark 3.11. As in [She98b, §2] but easier because of 2.15, 2.16.

Proof. 1) By a preliminary forcing without loss of generality $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$ and $\diamondsuit_{\{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_1\}}$ holds. Hence clearly there are $\mathbf{q}, \mathbb{P}_{\mathbf{q}}$ as in $(*)$ of part (2) by [She98a] so we are done by part (2).
2) By [She98a] forcing by $\mathbb{P}$ collapse no cardinal, $\mathbb{P}$ has cardinality $\aleph_2$ and $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} = \aleph_2$" and by 2.16, for each $\alpha < \omega_2, \mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\alpha+1}$ is bounding (and $\underset{\sim}{\mathbb{Q}}_\alpha$, of course). So if $p \Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$\underset{\sim}{D}$ is a non-principal ultrafilter on $\mathbb{N}$", let δ be as in clause (c) then we can apply 3.9 with $\mathbf{V}^{\mathbb{P}_\delta}, \underset{\sim}{\mathbb{Q}}_\delta, \mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\delta+1}$ here standing for $\mathbf{V}, \mathbb{Q}_0, \underset{\sim}{\mathbb{Q}}_1$ there.
3) Also straightforward. $\square_{3.10}$

[1]Can use $S \subseteq \{\delta < \kappa : \Vdash_{\mathbb{P}}$ "$\mathrm{cf}(\delta) = \aleph_1$"$\}$.

Discussion 3.12. 1) We may replace in the Main Claim $\mathrm{id}^{\mathbf{V}}_{\mathscr{T}}, (\mathrm{id}_{\mathscr{T}})^{\mathbf{V}[\mathbb{P}]}$ by the ideals $J_1, \underset{\sim}{J}_2, J_1$ an ideal on a countable set and $\underset{\sim}{J}_2$ the ideal it generates in $\mathbf{V}^{\mathbb{P}}$.

Remark 3.13. 1) We have freedom in choosing α when $\alpha < \omega_2, \mathrm{cf}(\alpha) \leq \aleph_0$ or $\mathrm{cf}(\alpha) = \aleph_1$ but the diamond does not give $\underset{\sim}{D}_\delta$ as requried. Of course, $\mathbb{Q}_\alpha$ has still to be bounding, proper and of cardinality $\leq \aleph_1$.
2) Note that the forcing $\mathbb{P}$ above works also for "no P-point" if we demand $\mathbb{P}$ has the PP-property; and for no "nowhere dense ultrafilter on ω", when every nowhere dense $A \subseteq {}^{\omega>}\omega$ from $\mathbf{V}^{\mathbb{P}}$ is included in one from $\mathbf{V}$ see [She98a, Ch.VI,§2]; so "each $\mathbb{Q}_\alpha$ does" suffice; see [She98b].

Problem 3.14. Can we distinguish between the properties "there is an α-ultrafilter" for distinct ordinals $\alpha \in [1, \omega_1)$?

Definition 3.15. We say $\mathscr{D}$ is a van-Dowen ultrafilter when $\mathscr{D}$ is an ultrafilter on $\mathbb{Q}$, the rationals such that every member contains a closed infinite set with no isolated points.

Recall that van-Dowen [vD81] asked:

Question 3.16. Can we force to have no van-Dowen ultrafilters?
It seems that with a proof like 3.9, yes.

Claim 3.17. *1) In 3.10 we can add: in $\mathbf{V}^{\mathbb{P}}$ there is no van-Dowen ultrafilter provided that $\Vdash_{\mathbb{P}}$ "every nowhere dense $A \subseteq {}^{\omega>}\omega$ is included in an old one such B".*
2) In 3.9 we can add "in $\mathbf{V}^{\mathbb{P}}$ there is no van-Dowen ultrafilter D and one-to-one function π from $\mathbb{Q}$ into ω such that $\pi(D)$ extending $D_{\mathbf{i}}$.

Proof. 1) By part (2).
2) First:

$(*)_1$ (a) let $\mathbf{c}$ be the following function from ${}^{\omega>}\{-1,1\}$ into $\mathbb{Q} : \mathbf{c}(\eta) = \Sigma\{\eta(\ell)/2^\ell : \ell < \ell g(\eta)\}$
(b) note that $\mathbb{Q}' = \mathrm{Rang}(\mathbf{c})$ is a subset of $\mathbb{Q}$ dense in itself, hence is order isomorphic to $\mathbb{Q}$.

$(*)_2$ in $\mathbf{V}^{\mathbb{P}}$ for $A \subseteq \mathbb{Q}'$
(a) we define $\mathrm{dp}_A : \mathbb{Q}' \to \mathrm{Ord} \cup \{\infty\}$ by defining when $\mathrm{dp}_A(a) \geq \alpha$ by induction on the ordinal $\alpha : \mathrm{dp}_A(a) \geq \alpha$ iff one of the following cases holds:
- $\bullet_1$ $\alpha = 0$
- $\bullet_2$ $\alpha = 1$ and for every open interval I around a, we have $I \cap A \neq \emptyset$
- $\bullet_3$ $\alpha \geq 2$ and for any $\beta < \alpha$ and open interval I around a, there is $b \in A \cap I \backslash \{a\}$ such that $\mathrm{dp}_A(b) \geq \beta$

(b) let g_A be a function with domain $\mathbb{Q}'$ such that if $a \in A$ then $g_A(a)$ is an open interval around a such that
- $\bullet_1$ if $\mathrm{dp}_A(a) = 0$ then $I \cap A = \emptyset$
- $\bullet_2$ if $\mathrm{dp}_A(a) \geq 1$ then if possible (equivalently if $\mathrm{dp}_A(a) < \infty$), for no $b \in A \cap I \backslash \{a\}$ do we have $\mathrm{dp}_A(b) \geq \mathrm{dp}_A(a)$

(c) we say A is scattered if $\infty \notin \mathrm{Rang}(\mathrm{dp}_A)$ equivalently $\mathrm{Rang}(\mathrm{dp}_A)$ is a countable ordinal (call it γ_A) equivalently A has no (infinite) subset dense in itself

$(*)_3$ without loss of generality $g_A \in \mathbf{V}$.

[Why? Because $\mathbb{P}$ is bounding.]

$(*)_4$ there is a function $\mathbf{d} : \mathbb{Q} \to \gamma_A$ from $\mathbf{V}$ such that:

(a) $\mathrm{dp}_A(a) \leq \mathbf{d}(a)$ for $a \in \mathbb{Q}$

(b) if $a \in \mathbb{Q}$ and $b \in g_A(a) \backslash \{a\}$ then $\mathbf{d}(b) < \mathbf{d}(a)$ or $\mathbf{d}(a) = 0 \wedge \mathbf{d}(b) = \gamma$.

[Why? As in the proof of 3.9.]

We continue as in the proof of 3.9. $\square_{3.17}$

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