BLACK BOXES
SH309

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Abstract. We shall deal comprehensively with Black Boxes, the intention being that provably in ZFC we havea sequence of guesses of extra structure on small subsets, the guesses are pairwise with quite little interaction, are far but together are "dense". We first deal with the simplest case, were the existence comes from winning a game by just writing down the opponent's moves. We show how it help when instead orders we have trees with boundedly many levels, having freedom in the last. After this we quite systematically look at existence of black boxes, and make connection to non-saturation of natural ideals and diamonds on them.

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§ 0. Introduction

The non-structure theorems we have discussed in [Shef] rests usually on some freedom on finite sequences and on a kind of order. When our freedom is related to infinite sequences, and to trees, our work is sometimes harder. In particular, we may consider, for $\lambda \geq \chi, \chi$ regular, and $\varphi = \varphi(\bar{x}_0, \ldots, \bar{x}_\alpha, \ldots)_{\alpha<\chi}$ in a vocabulary τ:

(∗) For any $I \subseteq {}^{\chi \geq}\lambda$ we have a τ–model M_I and sequences $\bar{a}_\eta$ (for $\eta \in {}^{\chi >}\lambda$), where
$$[\eta \triangleleft \nu \;\Rightarrow\; \bar{a}_\eta \neq \bar{a}_\nu], \qquad \ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)}),$$
such that for $\eta \in {}^{\chi}\lambda$ we have:
$$M_I \models \varphi(\ldots, \bar{a}_{\eta \restriction \alpha}, \ldots)_{\alpha<\chi} \text{ if and only if } \eta \in I.$$

(Usually, M_I is to some extend "simply defined" from I). Of course, if we do not ask more from M_I, we can get nowhere: we certainly restrict its cardinality and/or usually demand it is φ–representable (see Definition [Shef, 2.4] clauses (c),(d)) in (a variant of) $\mathscr{M}_{\mu,\kappa}(I)$ (for suitable μ, κ). Certainly for T unsuperstable we have such a formula φ:

$$\varphi(\ldots, \bar{a}_{\eta\restriction n}, \ldots) = (\exists \bar{x}) \bigwedge_n \varphi_n(\bar{x}, \bar{a}_{\eta \restriction n}).$$

There are many natural examples.

Formulated in terms of the existence of I for which our favorite "anti-isomorphism" player has a winning strategy, we prove this in 1969/70 (in proofs of lower bounds of $\mathbb{I}(\lambda, T_1, T)$, T unsuperstable), but it was shortly superseded. However, eventually the method was used in one of the cases in [She78b, Ch.VIII,§2]: for strong limit singular [She78b, Ch.VIII,2.6]. It was developed in [She84a], [She84b] for constructing Abelian groups with prescribed endomorphism groups. See further a representation of one of the results here in Eklof-Mekler [EM90], [EM02] a version which was developed for a proof of the existence of Abelian (torsion free $\aleph_1$–free) group G with

$$G^{***} = G^* \oplus A \qquad (G^* := \mathrm{Hom}(G, \mathbf{Z}))$$

in a work by Mekler and Shelah. A preliminary version of this paper appeared in [She87, Ch.III,§4,§5] but §3 here was just almost ready, and §4 on partitions of stationary sets and $\diamondsuit_D$ was written up as a letter to Foreman in the late nineties answering his question on what I know on this.

§ 1. The Easy Black Box and an Easy Application

In this section we do not try to get the strongest results, just provide some examples (i.e., we do not present the results when $\lambda = \lambda^\chi$ is replaced by $\lambda = \lambda^{<\chi}$). By the proof of [She78b, Ch.VIII,2.5] (see later for a complete proof):

Theorem 1.1. *Suppose that*

(∗) (a) $\lambda = \lambda^\chi$

(b) *τ a vocabulary $\varphi = \varphi(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_\alpha \ldots)_{\alpha<\chi}$ is a formula in $\mathscr{L}(\tau)$ for some logic $\mathscr{L}$*

(c)$_{\tau,\varphi}$ *For any $I \subseteq {}^{\chi\geq}\lambda$ we have a τ–model M_I and sequences $\bar{a}_\eta$ (for $\eta \in {}^{\chi>}\lambda$), where*

$$[\eta \triangleleft \nu \Rightarrow \bar{a}_\eta \neq \bar{a}_\nu], \qquad \ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\lg(\eta)}),$$

such that for $\eta \in {}^\chi\lambda$ we have:

$$M_I \models \varphi(\ldots, \bar{a}_{\eta\restriction\alpha}, \ldots)_{\alpha<\chi} \text{ if and only if } \eta \in I$$

(c) *$\|M_I\| = \lambda$ for every I satisfying ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\leq}\lambda$, and $\ell g(\bar{a}_\eta) \leq \chi$ or just $\lambda^{\ell g(\bar{a}_\eta)} = \lambda$.*

<u>Then</u> (using ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$):
1) There is no model M of cardinality λ into which every M_I can be $(\pm\varphi)$–embedded (i.e., by a function preserving φ and $\neg\varphi$).
2) For any M_i (for $i < \lambda$), $\|M_i\| \leq \lambda$, for some I satisfying ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$, the model M_I cannot be $(\pm\varphi)$–embedded into any M_i.

Example 1.2. Consider the class of Boolean algebras and the formula

$$\varphi(\ldots, x_n, \ldots) := (\bigcup_n x_n) = 1$$

(i.e., there is no $x \neq 0$ such that $x \cap x_n = 0$ for each n).
For ${}^{\omega>}\lambda \subseteq I \subseteq {}^{\omega\geq}\lambda$, let M_I be the Boolean algebra generated freely by x_η (for $\eta \in I$) except the relations: for $\eta \in I$, if $n < \ell g(\eta) = \omega$ then $x_\eta \cap x_{\eta\restriction n} = 0$.

So $\|M_I\| = |I| \in [\lambda, \lambda^{\aleph_0}]$ and in M_I for $\eta \in {}^\omega\lambda$ we have: $M_I \models (\bigcup_n x_{\eta\restriction n}) = 1$ if and only if $\eta \notin I$ (work a little in Boolean algebras).

So

Conclusion 1.3. *If $\lambda = \lambda^{\aleph_0}$, <u>then</u> there is no Boolean algebra $\mathbf{B}$ of cardinality λ universal under σ–embeddings (i.e., ones preserving countable unions).*

Remark 1.4. This is from [She78b, Ch.VIII,Ex.2.5,pg.464].

Proof of the Theorem 1.1. First we recall the simple black box (and a variant) in 1.5, 1.6 below:

The Simple B.B. Lemma 1.5. *There are functions f_η (for $\eta \in {}^\chi\lambda$) such that:*

(i) $\mathrm{Dom}(f_\eta) = \{\eta \restriction \alpha : \alpha < \chi\}$,

(ii) $\mathrm{Rang}(f_\eta) \subseteq \lambda$, (*iii*) *if* $f : {}^{\chi>}\lambda \longrightarrow \lambda$, *then for some* $\eta \in {}^{\chi}\lambda$ *we have* $f_\eta \subseteq f$.

Proof. For $\eta \in {}^{\chi}\lambda$ let f_η be the function (with domain $\{\eta{\restriction}\alpha : \alpha < \chi\}$) such that:

$$f_\eta(\eta{\restriction}\alpha) = \eta(\alpha).$$

So $\langle f_\eta : \eta \in {}^{\chi}\lambda\rangle$ is well defined. Properties (i), (ii) are straightforward, so let us prove (iii). Let $f : {}^{\chi>}\lambda \longrightarrow \lambda$. We define $\eta_\alpha = \langle \beta_i : i < \alpha\rangle$ by induction on α.

For $\alpha = 0$ or α limit — no problem.

For $\alpha + 1$: let β_α be the ordinal such that $\beta_\alpha = f(\eta_\alpha)$.

So $\eta =: \langle \beta_i : i < \chi\rangle$ is as required. $\square_{1.5}$

Fact 1.6. In 1.5:

(*a*) we can replace the range of f, f_η by any fixed set of power λ,

(*b*) we can replace the domains of f, f_η by $\{\bar{a}_\eta : \eta \in {}^{\chi>}\lambda\}$, $\{\bar{a}_{\eta{\restriction}\alpha} : \alpha < \chi\}$, respectively, as long as

$$\alpha < \beta < \chi \wedge \eta \in {}^{\chi}\lambda \quad \Rightarrow \quad \bar{a}_{\eta{\restriction}\alpha} \neq \bar{a}_{\eta\lceil\beta}.$$

Remark 1.7. We can present it as a game. (As in the book [She78b, Ch.VIII,2.5]).

Continuation of the Proof of Theorem 1.1.

It suffices to prove 1.1(2). Without loss of generality $\langle |M_i| : i < \lambda\rangle$ are pairwise disjoint. Now we use 1.6; for the domain we use $\langle \bar{a}_\eta : \eta \in {}^{\chi>}\lambda\rangle$ from the assumption of 1.1, and for the range: $\bigcup_{i<\lambda} {}^{\chi\geq}|M_i|$ (it has cardinality $\leq \lambda$ as $\|M_i\| \leq \lambda = \lambda^\chi$).

We define

$$I = ({}^{\chi>}\lambda) \cup \{\eta \in {}^{\chi}\lambda : \begin{array}{l}\text{for some } i < \lambda, \mathrm{Rang}(f_\eta) \text{ is a set of sequences} \\ \text{from } |M_i| \text{ and } M_i \models \neg\varphi(\ldots, f_\eta(\bar{a}_{\eta{\restriction}\alpha}), \ldots)_{\alpha<\chi}\}.\end{array}$$

Look at M_I. It suffices to show:

$\otimes$ for $i < \lambda$ there is no $(\pm\varphi)$–embedding of M_I into M_i.

Why does $\otimes$ hold?

If $f : M_I \longrightarrow M_i$ is a $(\pm\varphi)$–embedding, then by Fact 1.6, for some $\eta \in {}^{\chi}\lambda$ we have

$$f{\restriction}\{\bar{a}_{\eta{\restriction}\alpha} : \alpha < \kappa\} = f_\eta.$$

By the choice of f,

$$M_I \models \varphi\left[\ldots, \bar{a}_{\eta\lceil\alpha}, \ldots\right]_{\alpha<\chi} \iff M_i \models \varphi\left[\ldots, f(\bar{a}_{\eta{\restriction}\alpha}), \ldots\right]_{\alpha<\chi},$$

but by the choice of I and M_I we have

$$M_I \models \varphi\left[\ldots, \bar{a}_{\eta\lceil\alpha}, \ldots\right]_{\alpha<\chi} \iff M_i \models \neg\varphi\left[\ldots, f_\eta(\bar{a}_{\eta{\restriction}\alpha}), \ldots\right]_{\alpha<\chi}.$$

A contradiction, as by the choice of η,

$$\bigwedge_{\alpha<\chi} f(\bar{a}_{\eta{\restriction}\alpha}) = f_\eta(\bar{a}_{\eta{\restriction}\alpha}).$$

Discussion 1.8. We may be interested whether in 1.1, when $\lambda^+ < 2^\lambda$, we may

(a) allow in (1) $\|M\| = \lambda^+$, and/or
(b) get $\geq \lambda^{++}$ non-isomorphic models of the form M_I, assuming $2^\lambda > \lambda^+$.

The following lemma shows that we cannot prove those better statements in ZFC, though (see 1.11) in some universes of set theory we can. So this require (elementary) knowledge of forcing, but is not used later. It is here just to justify the limitations of what we can prove and the reader can skip it.

Lemma 1.9. *Suppose that in the universe* $\mathbf{V}$ *we have* $\kappa < \lambda = \mathrm{cf}(\lambda) = \lambda^{<\lambda}$ *and* $(\forall \lambda_1 < \lambda)[\lambda_1^\kappa < \lambda]$ *and* $\lambda < \mu = \mu^\lambda$.
Then for some notion forcing $\mathbb{P}$*:*

(a) *$\mathbb{P}$ is λ–complete and satisfies the λ^+–c.c., and $|\mathbb{P}| = \mu, \Vdash_{\mathbb{P}}$ "$2^\lambda = \mu$" (so forcing with $\mathbb{P}$ collapses no cardinals, changes no cofinalities, adds no new sequences of ordinals of length $< \lambda$, and $\Vdash_{\mathbb{P}}$ "$\lambda^{<\lambda} = \lambda$").*
(b) *We can find φ, M_I (for ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\geq}\lambda$) as in $(*)$ of **??**, so with $\|M_I\| = \lambda$(τ–models with $|\tau| = \kappa$ for simplicity) such that:*
   - $\oplus$ *there are up to isomorphism exactly λ^+ models of the form M_I (${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\lambda\geq}\lambda$).*
(c) *In (b), there is a model M such that $\|M\| = \lambda^+$ and every model M_I can be $(\pm\varphi)$–embedded into M.*

Remark 1.10. 1) Essentially M_I is $(I^+, \lhd)$, the addition of level predicates is immaterial, where I^+ extends I "nicely" so that we can let $a_\eta = \eta$ for $\eta \in I$.
2) Clearly clause (c) also shows that weakening $\|M\| = \lambda$, even when $\lambda^+ < 2^\lambda$ may make 1.1 false.

Proof. Let $\tau = \{R_\zeta : \zeta \leq \kappa\} \cup \{<\}$ with R_ζ being a monadic predicate, and $<$ being a binary predicate. For a set I, ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\geq}\lambda$ let N_I be the τ-model:

$$|N_I| = I, R_\zeta^{N_I} = I \cap {}^\zeta\lambda, \quad <^{N_I} = \{(\eta, \nu) : \eta \in I, \nu \in I, \eta \lhd \nu\},$$

and

$$\varphi(\ldots, x_\zeta, \ldots)_{\zeta<\kappa} = \bigwedge_{\zeta<\xi<\kappa} (x_\zeta < x_\xi \ \& \ R_\zeta(x_\zeta)) \wedge (\exists y)[R_\kappa(y) \ \& \ \bigwedge_{\zeta<\kappa} x_\zeta < y].$$

Now we define the forcing notion $\mathbb{P}$. It is $\mathbb{P}_{\lambda^+}$, where

$$\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \lambda^+, \ j < \lambda^+ \rangle$$

is an iteration with support $< \lambda$, of λ–complete forcing notions, where $\underset{\sim}{\mathbb{Q}}_j$ is defined as follows.
For $j = 0$ we add μ many Cohen subsets to λ:

$$\mathbb{Q}_0 = \{f : f \text{ is a partial function from } \mu \text{ to } \{0,1\}, \ |\mathrm{Dom}(f)| < \lambda\},$$

the order is the inclusion.

For $j > 0$, we define $\underset{\sim}{\mathbb{Q}}_j$ in $\mathbf{V}^{\mathbb{P}_j}$. Let $\langle I(j,\alpha) : \alpha < \alpha(j)\rangle$ list all sets $I \in \mathbf{V}^{\mathbb{P}_j}$, ${}^{\kappa >}\lambda \subseteq I \subseteq {}^{\kappa\geq}\lambda$ (note that the interpretation of ${}^{\kappa\geq}\lambda$ does not change from $\mathbf{V}$ to $\mathbf{V}^{\mathbb{P}_j}$ as $\kappa < \lambda$ but the family of such I-s increases).

Now

$$\underset{\sim}{\mathbb{Q}}_j = \Big\{\bar f : \begin{array}{l} f = \langle f_\alpha : \alpha < \alpha(j)\rangle,\ f_\alpha \text{ is a partial isomorphism} \\ \text{ from } N_{I(j,\alpha)} \text{ into } N_{({}^{\kappa\geq}\lambda)}, \\ w(\bar f) := \{\alpha : f_\alpha \neq \emptyset\} \text{ has cardinality } < \lambda, \\ \mathrm{Dom}(f_\alpha) \text{ has the form } \bigcup\limits_{\beta<\gamma} {}^{\kappa\geq}\beta \cap N_{I(j,\alpha)} \text{ for some } \gamma < \lambda; \\ \text{ and if } \alpha_1, \alpha_2 < \alpha(j) \text{ and } \eta_1, \eta_2 \in {}^{\kappa}\lambda, \text{ and for every } \zeta < \kappa, \\ f_{\alpha_1}(\eta_1{\restriction}\zeta), f_{\alpha_2}(\eta_2{\restriction}\zeta) \text{ are defined and equal, then} \\ \eta_1 \in I(j,\alpha_1) \iff \eta_2 \in I(j,\alpha_2)\Big\}. \end{array}$$

The order is:

$$\bar f^1 \leq \bar f^2 \quad \text{if and only if} \quad \begin{array}{l} (\forall\alpha < \alpha(j))(f^1_\alpha \subseteq f^2_\alpha) \text{ and} \\ \text{ for all } \alpha < \beta < \alpha(j), f^1_\alpha \neq \emptyset \wedge f^1_\beta \neq \emptyset \text{ implies} \\ \mathrm{Rang}(f^2_\alpha) \cap \mathrm{Rang}(f^2_\beta) = \mathrm{Rang}(f^1_\alpha) \cap \mathrm{Rang}(f^1_\beta). \end{array}$$

Then, $\underset{\sim}{\mathbb{Q}}_j$ is λ–complete and it satisfies the version of λ^+–c.c. from [She78a] (see more [She00]), hence each $\mathbb{P}_j$ satisfies the λ^+–c.c. (by [She78a]).

Now the $\mathbb{P}_{j+1}$-name $\underset{\sim}{I}_j$, (interpreting it in $\mathbf{V}^{\mathbb{P}_{j+1}}$ we get $I^* e_j$) is:

$$I^*_j = {}^{\kappa>}\lambda \cup \{\eta \in {}^{\kappa}\lambda : \begin{array}{l} \text{for some } \bar f \in \underset{\sim}{G}_{\mathbb{Q}_j}, \alpha < \alpha(j) \text{ and } \nu \in N_{I(j,\alpha)}, \\ \ell g(\nu) = \kappa \text{ and } f_\alpha(\nu) = \eta\}. \end{array}$$

This defines also $f^j_\alpha : I(j,\alpha) \longrightarrow I^*_j$, which is forced to be a $(\pm\varphi)$–embedding and also just an embedding.

So now we shall define for every I, ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\geq}\lambda$, a τ–model M_I: clearly I belongs to some $\mathbf{V}^{\mathbb{P}_j}$. Let $j = j(I)$ be the first such j, and let $\alpha = \alpha(I)$ be such that $I = I(j,\alpha)$. Let $M_{I(j,\alpha)} = N_{I^*_j}$ (and $a_\rho = f^j_\alpha(\rho)$ for $\rho \in I(j,\alpha)$).

We leave the details to the reader. $\square_{1.9}$

On the other hand, consistently we may easily have a better result.

Lemma 1.11. *Suppose that, in the universe* $\mathbf{V}$,

$$\lambda = \mathrm{cf}(\lambda) = \lambda^\kappa = \lambda^{<\lambda}, \quad \lambda < \mu = \mu^\lambda.$$

For some forcing notion $\mathbb{P}$*:*

(*a*) *as in 1.9*

(*b*) *in* $\mathbf{V}^{\mathbb{P}}$*, assume that* φ *and the function* $I \mapsto (M_I, \langle \bar a^I_\eta : \eta \in {}^{\kappa>}\lambda\rangle)$ *are as required in clauses (a),(b),(c) of* $(*)$ *of 1.1),* $\zeta(*) < \mu$*, and* N_ζ *(for* $\zeta < \zeta(*)$*) is a model in the relevant vocabulary,* $\sum\limits_{\zeta<\zeta(*)} \|N_\zeta\|^\kappa < \mu$ *(if the vocabulary is of cardinality* $< \lambda$ *and each predicate or relation symbol has finite arity, then requiring just* $\Sigma\{|N_\zeta\| : \zeta < \zeta(*)\} < \mu$ *suffices). Then for some* I*, the model* M_I *cannot be* $(\pm\varphi)$*–embedded into any* N_ζ*.*

(c) *Assume* $\mu_1 = \mathrm{cf}(\mu_1), \lambda < \mu_1 \leq \mu$ *and* $\mathbf{V} \models (\forall \chi < \mu_1)[\chi^\lambda < \mu_1]$. *Then in* $\mathbf{V}^{\mathbb{P}}$, *if* $\langle M_{I_i} : i < \mu_1 \rangle$ *are pairwise non-isomorphic,* ${}^{\kappa>}\lambda \subseteq I_i \subseteq {}^{\kappa\geq}\lambda$, *and* $M_{I_i}, \bar{a}^i_\eta$ $(\eta \in I_i)$ *are as in* $(*)$ *of* **??**), *then for some* $i \neq j, M_{I_i}$ *is not embeddable into* M_{I_j}.

(d) *In* $\mathbf{V}^{\mathbb{P}}$ *we can find a sequence* $\langle I_\zeta : \zeta < \mu \rangle$ *(so* ${}^{\kappa>}\lambda \subseteq I_\zeta \subseteq {}^{\kappa\geq}\lambda$*) such that the* M_{I_ζ}*'s satisfy that no one is* $(\pm\varphi)$*–embeddable into another.*

Proof. $\mathbb{P}$ is $\mathbb{Q}_0$ from the proof of 1.9. Let $\mathbf{F}$ be the generic function that is $\cup\{f : f \in \underset{\sim}{G}_{\mathbb{Q}_0}\}$, clearly it is a function from μ to $\{0,1\}$. Now clause (a) is trivial.

Next, concerning clause (b), we are given $\langle N_\zeta : \zeta < \zeta(*) \rangle$. Clearly for some $A \in \mathbf{V}$ of size smaller than $\mu, A \subseteq \mu$, to compute the isomorphism types of N_ζ (for $\zeta < \zeta(*)$) it is enough to know $\mathbf{F}{\restriction}A$. We can force by $\{f \in \mathbb{Q}_0 : \mathrm{Dom}(f) \subseteq A\}$, then $\mathbf{f} \restriction B$ for any $B \subseteq \lambda \setminus A$ of cardinality λ, (from $\mathbf{V}$) gives us an I as required.

To prove clause (c) use Δ–system argument for the names of various M_I's.

The proof of (d) is like that of (c). $\square_{1.11}$

§ 2. An Application for many models in λ

Discussion 2.1. Next we consider the following:

Assume λ is regular, $(\forall \mu < \lambda)[\mu^{<\chi} < \lambda]$. Let $\mathscr{U}_\alpha \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$ for $\alpha < \lambda$ be pairwise disjoint stationary sets.

For $A \subseteq \lambda$, let

$$\mathscr{U}_A = \bigcup_{i \in A} \mathscr{U}_i.$$

We want to define I_A such that ${}^{\chi>}\lambda \subseteq I_A \subseteq {}^{\chi\geq}\lambda$ and

$$A \nsubseteq B \Rightarrow M_{I_A} \ncong M_{I_B}.$$

We choose $\langle\langle M^i_{I_A} : i < \lambda\rangle : A \subseteq \lambda\rangle$ with $M_{I_A} = \bigcup_{i<\lambda} M^i_{I_A}$, $\|M^i_{I_A}\| < \lambda$, $M^i_{I_A}$ increasing continuous.

Of course, we have to strengthen the restrictions on M_I. For $\eta \in I_A \cap {}^\chi\lambda$, let $\delta(\eta) =: \bigcup\{\eta(i) + 1 : i < \chi\}$, we are specially interested in η such that η is strictly increasing converging to some $\delta(\eta) \in \mathscr{U}_A$; we shall put only such η's in I_A. The decision whether $\eta \in I_A$ will be done by induction on $\delta(\eta)$ for all sets A. Arriving to η, we assume we know quite a lot on the isomorphism $f : M_{I_A} \to M_{I_B}$, specially we know

$$f \restriction \bigcup_{\alpha<\chi} \bar{a}_{\eta\restriction\alpha},$$

which we are trying to "kill", and we can assume $\delta(\eta) \notin \mathscr{U}_B$ and δ belongs to a thin enough club of λ and using all this information we can "compute" what to do. Note: though this is the typical case, we do not always follow it.

Notation 2.2. 1) For an ordinal α and a regular $\theta \geq \aleph_0$, let $\mathscr{H}_{<\theta}(\alpha)$ be the smallest set Y such that:

- (i) $i \in Y$ for $i < \alpha$,
- (ii) $x \in Y$ for $x \subseteq Y$ of cardinality $< \theta$.

2) We can agree that $\mathscr{M}_{\lambda,\theta}(\alpha)$ from [Shef, §2] is interpretable in $(\mathscr{H}_{<\theta}(\alpha), \in)$ when $\alpha \geq \lambda$, and in particular its universe is a definable subset of $\mathscr{H}_{<\theta}(\alpha)$, and also R is, where:

$$\begin{aligned} R = \{(\sigma^*, \langle t_i : i < \gamma_x\rangle, x) : \quad & x \in \mathscr{M}_{\lambda,\theta}({}^{\theta>}\alpha), \\ & \sigma^* \text{ is a } \tau_{\lambda,\kappa} - \text{term and } \ \theta \leq \lambda \leq \alpha, x = \sigma^*(\langle t_i : i < \gamma_x\rangle)\}. \end{aligned}$$

Similarly $\mathscr{M}_{\lambda,\theta}(I)$, where $I \subseteq {}^{\kappa>}\lambda$ is interpretable in $(\mathscr{H}_{<\chi}(\lambda^*), \in)$ if $\lambda \leq \lambda^*, \theta \leq \chi, \kappa \leq \chi$.

The main theorem of this section is:

Theorem 2.3. $\dot{I}\dot{E}_{\pm\varphi}(\lambda, K) = 2^\lambda$, *provided that:*

- (a) $\lambda = \lambda^\chi$,
- (b) $\varphi = \varphi(\ldots, \bar{x}_\alpha, \ldots)_{\alpha<\chi}$ *is a formula in the vocabulary* τ_K,

(c) *for every I such that ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$ we have a model $M_I \in K_\lambda$ and a function f_I, and $\bar{a}_\eta \in {}^{\chi\geq}|M_I|$ for $\eta \in {}^{\chi>}\lambda$ with $\ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)})$ such that:*

(α) *for $\eta \in {}^{\chi}\lambda$ we have $M_I \models \varphi(\dots, \bar{a}_{\eta\restriction\alpha}, \dots)$ if and only if $\eta \in I$,*

(β) *$f_I : M_I \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$, where $\mu \leq \lambda$, $\kappa = \chi^+$, and:*

(d) *for I, ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$ and $\bar{b}_\alpha \in M_I, \ell g(\bar{x}_\alpha) = \ell g(\bar{b}_\alpha)$ for $\alpha < \chi$, $f_I(\bar{b}_\alpha) = \bar{\sigma}_\alpha(\bar{t}_\alpha)$ we have: the truth value of $M_I \models \varphi[\dots, \bar{b}_\alpha, \dots]_{\alpha<\chi}$ can be computed from $\langle \bar{\sigma}_\alpha : \alpha < \chi \rangle$, $\langle \bar{t}_\alpha : \alpha < \chi \rangle$ (not just its q.f. type in I) and the truth values of statements of the form*

$$(\exists \nu \in I \cap {}^{\chi}\lambda)[\bigwedge_{i<\chi} \nu\restriction\epsilon_i = \bar{t}_{\beta_i}(\gamma_i)\restriction\epsilon_i]$$

for $\alpha_i, \beta_i, \gamma_i, \epsilon_i < \chi$ (i.e., in a way not depending on I, f_I) [we can weaken this].

We shall first prove 2.3 under stronger assumptions.

Fact 2.4. Suppose

(∗) $\lambda = \lambda^{2^\chi}$, (so $\mathrm{cf}(\lambda) > \chi$) and $\chi \geq \kappa$.

Then there are $\{(M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ such that:

(*i*) for every model M with universe $\mathscr{H}_{<\chi^+}(\lambda)$ such that $|\tau(M)| \leq \chi$ (and, e.g., $\tau \subseteq \mathscr{H}_{<\chi^+}(\lambda)$), for some α we have $M^\alpha \prec M$,

(*ii*) $\eta^\alpha \in {}^{\chi}\lambda$, $(\forall i < \chi)[\eta^\alpha\restriction i \in M^\alpha]$, $\eta^\alpha \notin M^\alpha$, and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \eta^\beta$,

(*iii*) for every $\beta < \alpha(*)$ we have: $\{\eta^\alpha\restriction i : i < \chi\} \nsubseteq M^\beta$,

(*iv*) for $\beta < \alpha$ if $\{\eta^\beta\restriction i : i < \chi\} \subseteq M^\alpha$, then $|M^\beta| \subseteq |M^\alpha|$,

(*v*) $\|M^\alpha\| = \chi$.

Proof. By 3.20 + 3.21 below with $\lambda, 2^\chi, \chi$ here standing for $\lambda, \chi(*), \theta$ there.

Proof of 2.3 from the Conclusion of 2.4.

Without loss of generality the universe of M_I is λ in 2.3.

We shall define for every $A \subseteq \lambda$ a set $I[A]$ satisfying ${}^{\chi>}\lambda \subseteq I[A] \subseteq {}^{\chi\geq}\lambda$, moreover

$$I[A] \setminus {}^{\chi>}\lambda \subseteq \{\eta^\alpha : \alpha < \alpha(*)\}.$$

For $\alpha < \alpha(*)$, let $\mathscr{U}_\alpha = \{\eta \in {}^{\chi}\lambda : \{\eta\restriction i : i < \chi\} \subseteq M^\alpha\}$. We shall define by induction on α, for every $A \subseteq \lambda$ the set $I[A] \cap \mathscr{U}_\alpha$ so that on the one hand those restrictions are compatible (so that we can define I[A] in the end, for each $A \subseteq \lambda$), and on the other hand they guarantee the non $(\pm\varphi)$–embeddability.

For each α: (essentially we decide whether $\eta^\alpha \in I[A]$ assuming M^α "guesses" rightly a function $g : M_{I_1} \longrightarrow M_{I_2}$ ($I_\ell = I[A_\ell]$), and $A_\ell \cap M^\alpha$ for $\ell = 1, 2$, and we make our decision to prevent this)

Case I: there are distinct subsets A_1, A_2 of λ and I_1, I_2 satisfying ${}^{\chi>}\lambda \subseteq I_\ell \subseteq {}^{\chi\geq}\lambda$, and a $(\pm\varphi)$–embedding g of M_{I_1} into M_{I_2} and

$$M^\alpha \prec (\mathscr{H}_{<\chi^+}(\lambda), \in, R, A_1, A_2, I_1, I_2, M_{I_1}, M_{I_2}, f_{I_1}, f_{I_2}, g),$$

where

$$R = \{\{(0, \sigma_x, x), (1+i, t^x_i, x)\} : i < i_x \text{ and } x \text{ has the form } \sigma_x(\langle t^x_i : i < i_x\rangle)\}$$

(we choose for each x a unique such term σ), and $I_2 \cap \mathscr{U}_\alpha \subseteq I_2 \cap (\bigcup_{\beta<\alpha} \mathscr{U}_\beta)$, and I_1, I_2 satisfy the restrictions we already have imposed on $I[A_1]$, $I[A_2]$, respectively for each $\beta < \alpha$. Computing according to clause (d) of 2.3 the truth value for $M_{I_2} \models \varphi[\dots, f(\bar{a}_{\eta^\alpha \restriction i}), \dots]_{i<\chi}$ we get $\mathbf{t}^\alpha$.

<u>Then</u> we restrict:

(i) if $B \subseteq \lambda$, $B \cap |M^\alpha| = A_2 \cap |M^\alpha|$, then $I[B] \cap (\mathscr{U}^\alpha \setminus \bigcup_{\beta<\alpha} \mathscr{U}^\beta) = \emptyset$,

(ii) if $B \subseteq \lambda$, $B \cap |M^\alpha| = A_1 \cap |M^\alpha|$ and $\mathbf{t}^\alpha$ is true, then

$$I[B] \cap (\mathscr{U}^\alpha \setminus \bigcup_{\beta<\alpha} \mathscr{U}^\beta) = \emptyset,$$

or just

$$\eta^\alpha \notin I[B]$$

(iii) if $B \subseteq \lambda$, $B \cap |M^\alpha| = A_1 \cap |M^\alpha|$ and $\mathbf{t}^\alpha$ is false, then

$$I[B] \cap (\mathscr{U}^\alpha \setminus \bigcup_{\beta<\alpha} \mathscr{U}^\beta) = \{\eta^\alpha\}$$

or just

$$\eta^\alpha \in I[B]$$

<u>Case II:</u> quad Not I.
No restriction is imposed.

The point is the two facts below which should be clear. $\square_{2.4}$

Fact 2.5. The choice of A_1, A_2, I_1, I_2, g is immaterial (any two candidates lead to the same decision).

Proof. Use clause (d) of 2.3. $\square_{2.5}$

Fact 2.6. $M_{I[A]}$ (for $A \subseteq \lambda$) are pairwise non-isomorphic. Moreover, for $A \neq B$ (subsets of λ) there is no $(\pm\varphi)$–embedding of $M_{I[A]}$ into $M_{I[B]}$.

Proof. By the choice of the I[A]'s and (i) of 2.4. $\square_{2.6}$

* * *

Still the assumption of 2.4 is too strong: it does not cover all the desirable cases, though it cover many of them. However, a statement weaker than the conclusion of 2.4 holds under weaker cardinality restrictions and the proof of 2.3 above works using it, thus we will finish the proof of 2.3.

Fact 2.7. Suppose $\lambda = \lambda^\chi$.
<u>Then</u> there are $\{(M^\alpha, A^\alpha_1, A^\alpha_2, \eta^\alpha) : \alpha < \alpha(*)\}$ such that:

$(*)$ (i) for every model M with universe $\mathscr{H}_{<\chi^+}(\lambda)$ such that $|\tau(M)| \leq \chi$ and $\tau(M) \subseteq \mathscr{H}_{<\chi^*}(\lambda)$ (arity of relations and functions finite) and sets $A_1 \neq A_2 \subseteq \lambda$, for some $\alpha < \alpha(*)$ we have $(M^\alpha, A_1^\alpha, A_2^\alpha) \prec (M, A_1, A_2)$,

(ii) $\eta^\alpha \in {}^\chi\lambda, \{\eta^\alpha {\restriction} i : i < \chi\} \subseteq |M^\alpha|$, $\eta^\alpha \notin M^\alpha$, and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \eta^\beta$,

(iii) for every $\beta < \alpha(*)$, if $\{\eta^\alpha {\restriction} i : i < \chi\} \subseteq M^\beta$, then $\alpha < \beta + 2^\chi$, and $\alpha + 2^\chi = \beta + 2^\chi$ implies $A_1^\alpha \cap |M^\alpha| \neq A_2^\beta \cap |M^\alpha|$,

(iv) for every $\beta < \alpha$ if $\{\eta^\beta {\restriction} i : i < \chi\} \subseteq M^\alpha$, then $|M^\beta| \subseteq |M^\alpha|$,

(v) $\|M^\alpha\| = \chi$.

Proof. See 3.46.

Proof of 2.3: Should be clear, We act as in the proof of 2.3 from the conclusion of 2.4 but now we have to use the "or just" version in (ii),(iii) there, $\square_{2.7}$

Conclusion 2.8. *1) If $T \subseteq T_1$ are complete first order theories, T in the vocabulary $\tau, \kappa = \mathrm{cf}(\kappa) < \kappa(T)$, hence T unsuperstable and $\lambda = \lambda^{\aleph_0} \geq |T_1|$, then $\dot{\mathbb{I}}_\tau(\lambda, T_1) = 2^\lambda$ ($\dot{\mathbb{I}}_\tau$ — see Definition* [Shef, 1.2(2)]*).*
2) Assume $\kappa = \mathrm{cf}(\kappa)$, Φ is proper and almost nice for K^κ_{tr}, see [Shef, 1.7], *$\bar{\sigma}^i$ $(i \leq \kappa)$ finite sequence of terms, $\tau \subseteq \tau_\Phi, \varphi_i(\bar{x}, \bar{y})$ first order in $\mathscr{L}[\tau]$ and for $\nu \in {}^i\lambda$, $\eta \in {}^\kappa\lambda$, $\nu \triangleleft \eta$ we have $\mathrm{EM}({}^\kappa\lambda, \Phi) \models \varphi_i(\bar{\sigma}^\kappa_i(x_\eta), \bar{\sigma}^{i+1}(x_{\eta\char`^\langle\alpha\rangle}))$ holds <u>if and only if</u> $\alpha = \eta(i)$. <u>Then</u>*

$$2^\lambda = |\{EM_\tau(S, \Phi)/\cong : {}^{\kappa>}\lambda \subseteq S \subseteq {}^{\kappa\geq}\lambda\}|.$$

Proof. 1) By [Shef, 1.10] there is a template Φ proper for K^κ_{tr}, as required in part (2).
2) By 2.3. $\square_{2.8}$

Discussion 2.9. What about Theorem 2.3 in the case we assume only $\lambda = \lambda^{<\chi}$? There is some information in [She78b, Ch.VIII,§2].

Of course, concerning unsuperstable T, that is 2.8, more is done there: the assumption is just $\lambda > |T|$.

Claim 2.10. *In 2.3, we can restrict ourselves to I such that $I^0_{\lambda,\chi} \subseteq I \subseteq {}^{\chi\geq}\lambda$, where*

$$I^0_{\lambda,\chi} = {}^{\chi>}\lambda \cup \{\eta \in {}^\chi\lambda : \eta(i) = 0 \text{ for every } i < \chi \text{ large enough}\}.$$

Proof. By renaming. $\square_{2.10}$

§ 3. Black Boxes

We try to give comprehensive treatment of black boxes, not few of them are useful in some contexts and some parts are redone here, as explained in §0,§1.

Note that "omitting countable types" is a very useful device for building models of cardinality $\aleph_0$ and $\aleph_1$. The generalization to models of higher cardinality, λ or λ^+, usually requires us to increase the cardinality of the types to λ, and even so we may encounter problems (see [Shee] and background there). Note we do not look mainly at the omitting type theorem *per se*, but its applications.

Jensen defined square and proved existence in **L**: in Facts 3.1 — 3.8, we deal with related just weaker principles which can be proved in ZFC. E.g., for λ regular $> \aleph_1$, $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ is the union of λ sets, each has square (as defined there). You can skip them in first reading, particularly 3.1 (and later take references on belief).

Then we deal with black boxes. In 3.12 we give the simplest case: λ regular $> \aleph_0$, $\lambda = \lambda^{<\chi(*)}$; really $\lambda^{<\theta} = \lambda^{<\chi(*)}$ is almost the same. In 3.12 we also assume "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is a good stationary set". In 3.16 we weaken this demand such that enough sets S as required exists (provably in ZFC!). The strength of the cardinality hypothesis ($\lambda = \lambda^{<\chi(*)}$, $\lambda^{<\theta} = \lambda^{<\chi(*)}$, $\lambda^{\theta} = \lambda^{<\chi(*)}$) vary the conclusion. In 3.14 – 3.17 we prepare the ground for replacing "λ regular" by "$\mathrm{cf}(\lambda) \geq \chi(*)$", which is done in 3.18.

As we noted in §2, it is much nicer to deal with $(\bar{M}^\beta, \eta^\beta)$, this is the first time we deal with η^β, i.e., for no $\alpha < \beta$,

$$\{\eta^\beta \restriction i : i < \theta\} \subseteq \bigcup_{i<\theta} M_i^\alpha.$$

In 3.20, 3.21 (parallel to 3.12, 3.18, respectively) we guarantee this, at the price of strengthening $\lambda^{<\theta} = \lambda^{<\chi(*)}$ to

$$\lambda^{<\theta} = \lambda^{\chi(1)}, \chi(1) = \chi(*) + (< \chi(*))^\theta.$$

Later, in 3.46, we draw the conclusion necessary for section 2 (in its proof the function h, which may look redundant, plays the major role). This (as well as 3.20, 3.21) exemplifies how those principles are self propagating — better ones follow from the old variant (possibly with other parameters).

In 3.22 — 3.27 we deal with the black boxes when θ (the length of the game) is $\aleph_0$. We use a generalization of the Δ–system lemma for trees and partition theorems on trees (see Rubin-Shelah [RS87, §4], [She82, Ch.XI] = [She98, Ch.XI],[Shed, 1.10=L1.7],[Shed, 1.16=L1.15] and here the proof of 3.24; see history there, and 3.6). We get several versions of the black box — as the cardinality restriction becomes more severe, we get a stronger principle.

It would be better if we can use for a strong limit $\kappa > \aleph_0 = \mathrm{cf}(\kappa)$,

$$\kappa^{\aleph_0} = \sup\{\lambda : \text{ for some } \kappa_n < \kappa \text{ and uniform ultrafilter } D \text{ on } \omega, \ \mathrm{cf}(\prod_{n<\omega} \kappa_n / D) = \lambda\}.$$

We know this for the uncountable cofinality case (see [She86b] or [She94b]), but then there are other obstacles. Now [She94a] gives a partial remedy, but lately by [She94c] there are many such cardinals.

In 3.41, 3.42 we deal with the case $\mathrm{cf}(\lambda) \leq \theta$. Note that $\mathrm{cf}(\lambda^{<\chi(*)}) \geq \chi(*)$ is always true, so you may wonder why wouldn't we replace λ by $\lambda^{<\chi(*)}$? This is true in quite many applications, but is not true, for example, when we want to construct structures with density character λ.

Several times, we use results quoted from [Shea, §2], but no vicious circle. Also, several times we quote results on pcf quoting [Shed, §3]. We end with various remarks and exercises.

Fact 3.1. 1) If $\mu^\chi = \mu < \lambda \leq 2^\mu, \chi$ and λ are regular uncountable cardinals, and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$ is a stationary set, <u>then</u> there are a stationary set $W \subseteq \chi$ and functions $h_a, h_b : \lambda \longrightarrow \mu$ and $\langle S_\zeta : 0 < \zeta < \lambda \rangle$ such that:

- (a) $S_\zeta \subseteq S$ is stationary,
- (b) $\xi \neq \zeta \Rightarrow S_\xi \cap S_\zeta = \emptyset$,
- (c) if $\delta \in S_\xi$, then for some increasing continuous sequence $\langle \alpha_i : i < \chi \rangle$ we have $\delta = \bigcup\limits_{i<\chi} \alpha_i, h_b(\alpha_i) = i, h_a(\alpha_i) \in \{\xi, 0\}$, and the set $\{i < \chi : h_a(\alpha_i) = \xi\}$ is stationary, in fact is W.

2) If in (1), a sequence $\langle C_\delta : \delta < \lambda, \mathrm{cf}(\delta) \leq \chi \rangle$ satisfying

$$(\forall \alpha \in C_\delta)[\alpha \text{ limit } \Rightarrow \ \alpha = \sup(\alpha \cap C_\delta)]$$

is given, C_δ is closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$, <u>then</u> in the conclusion we can get also $S^*, \langle C^*_\delta : \delta \in S^* \rangle$ such that (a), (b), (c) hold, and

- $(c)'$ in (c) we add $C_\delta = \{\alpha_i : i < \chi\}$,
- (d) $\bigcup\limits_{0<\xi<\lambda} S_\xi \subseteq S^* \subseteq \bigcup\limits_{0<\xi<\lambda} S_\xi \cup \{\delta < \lambda : \mathrm{cf}(\delta) < \chi\}$,
- (e) W is a $(> \aleph_0)$-closed, stationary in cofinality $\aleph_0$, subset of χ, which means:
  - (i) if $i < \chi$ is a limit ordinal, $i = \sup(i \cap W)$ has cofinality $> \aleph_0$ then $i \in W$,
  - (ii) $\{i \in W : \mathrm{cf}(i) = \aleph_0\}$ is a stationary[1] subset of χ,
- (f) for $\delta \in \bigcup\limits_{0<\xi<\lambda} S_\xi$ we have

$$C^*_\delta = \{\alpha \in C_\delta : \mathrm{otp}(\alpha \cap C_\delta) = \sup(W \cap \mathrm{otp}(\alpha \cap C_\delta))\}$$

- (g) C^*_δ is a club of δ included in C_δ for $\delta \in S^*$, and if $\delta(1) \in C^*_\delta$, $\delta \in S^*, \delta \in \bigcup\limits_{0<\zeta<\lambda} S_\zeta, \delta(1) = \sup(\delta(1) \cap C^*_\delta)$ and $\mathrm{cf}(\delta(1)) > \aleph_0$ <u>then</u> $C^*_{\delta(1)} \subseteq C^*_\delta$,
- (h) if C is a closed unbounded subset of λ, and $0 < \xi < \lambda$ <u>then</u> the set $\{\delta \in S_\xi : C^*_\delta \subseteq C\}$ is stationary.

Proof. 1) We can find $\{\langle h^1_\xi, h^2_\xi \rangle : \xi < \mu\}$ such that:

- (a) for every ξ we have $h^1_\xi : \lambda \longrightarrow \mu$ and $h^2_\xi : \lambda \longrightarrow \mu$,
- (b) if $A \subseteq \lambda$, $|A| \leq \chi$, and $h^1, h^2 : A \longrightarrow \mu$, then for some ξ, $h^1_\xi \restriction A = h^1$, and $h^2_\xi \restriction A = h^2$.

[1] we can ask $\notin I$ if I is any normal ideal on $\{i < \chi : \mathrm{cf}(i) = \aleph_0\}$

This holds by Engelking-Karlowicz [EK65] (see for example [She90a, AP]).

For $\delta < \lambda$ let C_δ be a closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$. Now for each $\xi < \mu$ and a stationary $a \subseteq \chi$ ask whether for every $i < \lambda$ for some $j < \lambda$ we have

$(*)^{\xi,a}_{i,j}$ the following subset of λ is stationary:

$$S^{\xi,a}_{i,j} = \{\delta \in S : \begin{array}{l} \text{(i) if } \alpha \in C_\delta, \mathrm{otp}(\alpha \cap C_\delta) \notin a \text{ then } h^1_\xi(\alpha) = 0, \\ \text{(ii) if } \alpha \in C_\delta, \mathrm{otp}(\alpha \cap C_\delta) \in a \text{ then the } h^1_\xi(\alpha)\text{-th} \\ \text{member of } C_\alpha \text{ belongs to } [i,j), \\ \text{(iii) if } \alpha \in C_\delta \text{ then } h^2_\xi(\alpha) = \mathrm{otp}(\alpha \cap C_\delta)\} \end{array}$$

$\square_{3.1}$

Subfact 3.2. For some $\xi < \mu$ and a stationary set $a \subseteq \chi$, for every $i < \lambda$ for some $j \in (i, \lambda)$, the statement $(*)^{\xi,a}_{i,j}$ holds.

Proof. If not, then for every $\xi < \mu$ and a stationary $a \subseteq \chi$, for some $i = i(\xi, a) < \lambda$, for every $j < \lambda$, $j > i(\xi, a)$, there is a closed unbounded subset $C(\xi, a, i, j)$ of λ disjoint from $S^{\xi,a}_{i,j}$.

Let

$$i(*) = \bigcup \{i(\xi, a) + \omega : \xi < \mu \text{ and } a \subseteq \chi \text{ is stationary}\}.$$

Clearly $i(*) < \lambda$.

For $i(*) \leq j < \lambda$ let $C(j) = \bigcap \{C(\xi, a, i(\xi, a), j) : \ a \subseteq \chi$ is stationary and $\xi < \mu\} \cap (i(*) + \omega, \lambda)$, clearly it is a closed unbounded subset of λ.

Let

$$C^* = \{\delta < \lambda : \delta > i(*) \text{ and } (\forall j < \delta)[\delta \in C(j)]\}.$$

So C^* is a closed unbounded subset of λ, too. Let C^+ be the set of accumulation points of C^*. Choose $\delta(*) \in C^+ \cap S$, and we shall define

$$h^1 : C_{\delta(*)} \longrightarrow \mu, \quad h^2 : C_{\delta(*)} \longrightarrow \mu.$$

For $\alpha \in C_{\delta(*)}$ let $h^0(\alpha)$ be:

$$\mathrm{Min}\{\gamma < \chi : \gamma > 0 \text{ and the } \gamma\text{–th member of } C_\alpha \text{ is } > i(*)\}$$

if $\alpha = \sup(C_{\delta(*)} \cap \alpha) > i(*)$, and zero otherwise. Clearly the set

$$\{\alpha \in C_{\delta(*)} : \ h^0(\alpha) = 0\}$$

is not stationary. Now we can define $g : C_{\delta(*)} \longrightarrow \delta(*)$ by:

$$g(\alpha) \text{ is the } h^0(\alpha)\text{–th member of } C_\alpha.$$

Note that g is pressing down and $\{\alpha \in C_{\delta(*)} : g(\alpha) \leq i(*)\}$ is not stationary. So (by the variant of Fodor's Lemma speaking on an ordinal of uncountable cofinality) for some $j < \sup(C_{\delta(*)}) = \delta(*)$ the set

$$a := \{\alpha \in C_{\delta(*)} \cap C^* : i(*) < g(\alpha) < j\}$$

is a stationary subset of $\delta(*)$, and let $h^1 : C_{\delta(*)} \longrightarrow \mu$ be

$$h^1(\alpha) = \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{otp}(\alpha \cap C^*_\delta) \notin a, \\ h^0(\alpha) & \text{if } \mathrm{otp}(\alpha \cap C^*_\delta) \in a. \end{array} \right\}.$$

Let $h^2 : C_{\delta(*)} \to \mu$ be $h^2(\alpha) = \mathrm{otp}(\alpha \cap C_{\delta(*)})$. By the choice of $\langle (h^1_\xi, h^2_\xi) : \xi < \mu \rangle$, for some ξ, we have $h^1_\xi \restriction C_{\delta(*)} = h^1$ and $h^2_\xi \restriction C_{\delta(*)} = h^2$. Easily, $\delta(*) \in S^{\xi,a}_{i,j}$ which is disjoint to $C(\xi, a, i(*), j)$, a contradiction to $\delta(*) \in C^*$ by the definition of $C(j)$ and C^*.

So we have proved the subfact 3.2. $\square_{3.2}$

Having chosen ξ, a we define by induction on $\zeta < \lambda$ an ordinal $i(\zeta) < \lambda$ such that $\langle i(\zeta) : \zeta < \lambda \rangle$ is increasing continuous, $i(0) = 0$, and $(*)^{\xi,a}_{i(\zeta),i(\zeta+1)}$ holds.

Now, for $\alpha < \lambda$ we define $h_a(\alpha)$ as follows: it is ζ if $h^1_\xi(\alpha) > 0$ and the $h^1_\xi(\alpha)$-th member of C_α belongs to $[i(1+\zeta), i(1+\zeta+1))$, and it is zero otherwise. Lastly, let $h_b(\alpha) =: h^2_\xi(\alpha)$ and $W = a$ and

$$S_\zeta =: \big\{ \delta \in S : \begin{array}{ll} \text{(i)} & \text{for } \alpha \in C_\delta,\ \mathrm{otp}(\alpha \cap C_\delta) = h_b(\alpha), \\ \text{(ii)} & \text{for } \alpha \in C_\delta,\ h_b(i) \in a \ \Rightarrow\ h_a(\alpha) = \zeta, \\ \text{(iii)} & \text{for } \alpha \in C_\delta,\ h_b(i) \notin a \ \Rightarrow h_a(i) = 0 \quad \big\}. \end{array}$$

Now, it is easy to check that a, h_a, h_b, and $\langle S_\zeta : 0 < \zeta < \lambda \rangle$ are as required.

2) In the proof of 3.1(1) we shall now consider only sets $a \subseteq \chi$ which satisfy the demand in clause (e) of 3.1(2) on W [i.e., in the definition of $C(j)$ during the proof of Subfact 3.2 this makes a difference]. Also in $(*)^{\xi,a}_{i,j}$ in the definition of $S^{\xi,a}_{i,j}$ we change (iii) to:

$(iii)'$ if $\alpha \in C_\delta$, $h^2_\xi(\alpha)$ codes the isomorphism type of, for example,

$$(C_\delta \cup \bigcup_{\beta \in C_\delta} C_\beta, <, \alpha, C_\delta, \{\langle i, \beta \rangle : i \in C_\beta\}).$$

In the end, having chosen ξ, a we can define C^*_δ and S^* in the natural way.

Fact 3.3. 1) If λ is regular $> 2^\kappa, \kappa$ regular, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary and for $\delta \in S, C^0_\delta$, is a club of δ of order type κ $(= \mathrm{cf}(\delta))$, then we can find a club c^* of κ (see 3.4(1)) such that letting for $\delta \in S, C_\delta = C_\delta[c^*] := \{\alpha \in C^0_\delta : \mathrm{otp}(C^0_\delta \cap \alpha) \in c^*\}$, it is a club of δ and

- $(*)$ for every club $C \subseteq \lambda$ we have:
  - (a) if $\kappa > \aleph_0$, $\{\delta \in S : C_\delta \subseteq C\}$ is stationary,
  - (b) if $\kappa = \aleph_0$, then the set

    $\{\delta \in S : (\forall \alpha, \beta)[\alpha < \beta \wedge \alpha \in C_\delta \wedge \beta \in C_\delta \ \Rightarrow\ (\alpha, \beta) \cap C \neq \emptyset]\}$

    is stationary.

2) If λ is a regular cardinal $> 2^\kappa$, then we can find $\langle \langle C^\zeta_\delta : \delta \in S_\zeta \rangle : \zeta < 2^\kappa \rangle$ such that:

- (a) $\bigcup \{S_\zeta : \zeta < 2^\kappa\} = \{\delta < \lambda : \aleph_0 < \mathrm{cf}(\delta) \leq \kappa\}$,
- (b) C^ζ_δ is a club of δ of order type $\mathrm{cf}(\delta)$,

(c) if $\alpha \in S_\zeta$, $\mathrm{cf}(\alpha) > \theta > \aleph_0$, then

$$\{\beta \in C^\zeta_\alpha : \mathrm{cf}(\beta) = \theta,\ \beta \in S_\zeta \text{ and } C^\zeta_\beta \subseteq C^\zeta_\alpha\}$$

is a stationary subset of α.

3) If λ is regular, $2^\mu \geq \lambda > \mu^\kappa$, then we can find $\langle\langle C^\zeta_\delta : \delta \in S_\zeta\rangle : \zeta < \mu\rangle$ such that:

(a) $\bigcup\{S_\zeta : \zeta < 2^\kappa\} = \{\delta < \lambda : \aleph_0 < \mathrm{cf}(\delta) \leq \kappa\}$,
(b) C^ζ_δ is a club of δ of order type $\mathrm{cf}(\delta)$,
(c) if $\alpha \in S_\zeta$, $\beta \in C^\zeta_\alpha, \mathrm{cf}(\beta) > \aleph_0$, then $\beta \in S_\zeta$ and $C^\zeta_\beta \subseteq C^\zeta_\alpha$,
(d) moreover, if α, $\beta \in S_\zeta$, $\beta \in C^\zeta_\alpha$, then

$$\{(\mathrm{otp}(\gamma \cap C^\zeta_\beta), \mathrm{otp}(\gamma \cap C^\zeta_\alpha)) : \gamma \in C_\beta\}$$

depends on $(\mathrm{otp}(\beta \cap C_\alpha), \mathrm{otp}(C_\alpha))$ only.

4) We can replace in (1)(a) and (b) of $(*)$ "stationary" by "$\notin I$" for any normal ideal I on λ.

Remark 3.4. 1) A club C of δ where $\mathrm{cf}(\delta) = \aleph_0$ means here just an unbounded subset of δ.
2) In 3.3(1) instead of 2^κ, the cardinal

$$\mathrm{Min}\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\kappa\kappa \ \&\ (\forall g \in {}^\kappa\kappa)(\exists f \in F)(\forall \alpha < \kappa)[g(\alpha) < f(\alpha)]\}$$

suffices.
3) In (b) above, it is equivalent to ask

$$\{\delta \in S : (\forall \alpha, \beta)[\alpha < \beta \wedge \alpha \in C_\delta \wedge \beta \in C_\delta \ \Rightarrow\ \mathrm{otp}((\alpha, \beta) \cap C) > \alpha]\}$$

is stationary.

Proof. 1) If 3.3(1) fails, for each club c^* of κ there is a club $C[c^*]$ of λ exemplifying its failure. So $C^+ = \bigcap\{C[c^*] : c^* \subseteq \kappa \text{ a club}\}$ is a club of λ. Choose $\delta \in S$ which is an accumulation point of C^+ and get contradiction easily.
2) Let $\lambda = \mathrm{cf}(\lambda) > 2^\kappa$, C_α be a club of α of order type $\mathrm{cf}(\alpha)$, for each limit $\alpha < \lambda$. Without loss of generality

$$\beta \in C_\alpha \ \&\ \beta > \sup(\beta \cap C_\alpha) \quad \Rightarrow \quad \beta \text{ is a successor ordinal.}$$

For any sequence $\bar{c} = \langle c_\theta : \aleph_0 < \theta = \mathrm{cf}(\theta) \leq \kappa\rangle$ such that each c_θ is a club of θ, for $\delta \in S^* = \{\alpha < \lambda : \aleph_0 < \mathrm{cf}(\alpha) \leq \kappa\}$ we let:

$$C^{\bar{c}}_\delta = \{\alpha \in C_\delta : \mathrm{otp}(C_\delta \cap \alpha) \in c_{\mathrm{cf}(\delta)}\}.$$

Now we define $S_{\bar{c}}$, by defining by induction on $\delta < \lambda$, the set $S_{\bar{c}} \cap \delta$; the only problem is to define whether $\alpha \in S_{\bar{c}}$ knowing $S_{\bar{c}} \cap \delta$; now

$\alpha \in S_{\bar{c}}$ <u>if and only if</u> (i) $\aleph_0 < \mathrm{cf}(\alpha) \leq \kappa$,
(ii) if $\aleph_0 < \theta = \mathrm{cf}(\theta) < \mathrm{cf}(\alpha)$ then the set $\{\beta \in C^{\bar{c}}_\alpha : \mathrm{cf}(\beta) = \theta,\ \beta \in S_{\bar{c}} \cap \alpha\}$ is stationary in α.

Let $\langle \bar{c}^\zeta : \zeta < 2^\kappa \rangle$ list the possible sequences $\bar{c}$, and let $S_\zeta = S_{\bar{c}^\zeta}$ and $C^\zeta_\delta = C^{\bar{c}^\zeta}_\delta$. To finish, note that for each $\delta < \lambda$ satisfying $\aleph_0 < \mathrm{cf}(\delta) \le \kappa$, for some ζ, $\delta \in S_\zeta$.
3) Combine the proof of (2) and of 3.1.
4) Similarly. $\square_{3.4}$

We may remark

Fact 3.5. Suppose that λ is a regular cardinal $> 2^\kappa$, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, a set

$$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$$

is stationary, and I is a normal ideal on λ and $S \notin I$. If I is λ^+–saturated (i.e., in the Boolean algebra $\mathscr{P}(\lambda)/I$, there is no family of λ^+ pairwise disjoint elements), then we can find $\langle C_\delta : \delta \in S\rangle$, C_δ a club of δ of order type $\mathrm{cf}(\delta)$, such that:

(∗) for every club C of λ we have $\{\delta \in S : C_\delta \setminus C$ is unbounded in $\delta\} \in I$.

Proof. For $\delta \in S$, let C'_δ be a club of δ of order type $\mathrm{cf}(\delta)$. Call $\bar{C} = \langle C_\delta : \delta \in S^*\rangle$ (where $S^* \subseteq \lambda$ stationary, $S^* \notin I$, C_δ a club of δ) I-large if: for every club C of λ the set

$$\{\delta < \lambda : \delta \in S^* \text{ and } C_\delta \setminus C \text{ is bounded in } \delta\}$$

does not belong to I.
We call $\bar{C}$ I-full if above $\{\delta \in S^* : C_\delta \setminus C$ unbounded in $\delta\} \in I$.
3.3(4), for every stationary $S' \subseteq S$, $S' \notin I$ there is a club c^* of κ such that $\langle C'_\delta[c^*] : \delta \in S'\rangle$ is I–large.
Now note:

(∗) if $\langle C_\delta : \delta \in S'\rangle$ is I-large, $S' \subseteq S$, then for some $S'' \subseteq S'$, $S'' \notin I$, $\langle C_\delta : \delta \in S''\rangle$ is I–full (hence $S'' \notin I$).

[Proof of (∗): Choose by induction on $\alpha < \lambda^+$, a club C^α of λ such that:

(a) for $\beta < \alpha$, $C^\alpha \setminus C^\beta$ is bounded in λ,
(b) if $\beta = \alpha + 1$ then $A_\beta \setminus A_\alpha \in I^+$, where
$$A_\gamma =: \{\delta \in S' : C_\delta \setminus C^\gamma \text{ is unbounded in } \delta\}.$$

As clearly

$$\beta < \alpha \quad \Rightarrow \quad A_\beta \setminus A_\alpha \text{ is bounded in } \lambda$$

(by (a) and the definition of A_α, A_β) and as I is λ^+–saturated, clearly for some α we cannot define C^α. This cannot be true for $\alpha = 0$ or a limit α, so necessarily $\alpha = \beta + 1$. Now $S' \setminus A_\beta$ is not in I as $\bar{C}$ was assumed to be I–large. Check that $S'' =: S' \setminus A_\beta$ is as required.]
Using repeatedly 3.3(4) and (∗) we get the conclusion. $\square_{3.5}$

Claim 3.6. *Suppose $\lambda = \mu^+$, $\mu = \mu^\chi$, χ is a regular cardinal and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$ is stationary. Then we can find $S^*, \langle C_\delta : \delta \in S^*\rangle$ and $\langle S_\xi : \xi < \lambda\rangle$ such that:*

(a) $\bigcup_{\zeta < \mu} S_\zeta \subseteq S^* \subseteq S \cup \{\delta < \lambda : \mathrm{cf}(\delta) < \chi\}$,

(b) $S_\zeta \cap S$ *is a stationary subset of* λ *for each* $\zeta < \mu$,

(c) *for* $\alpha \in S^*, C_\alpha$ *is a closed subset of* α *of order type* $\leq \chi$, *if* $\alpha \in S^*$ *is a limit then* C_α *is unbounded in* α *(so is a club of* α*),*

(d) $\langle C_\alpha : \alpha \in S_\zeta \rangle$ *is a square on* S_ζ, *i.e., (*S_ζ *is stationary in* $\sup(S_\zeta)$ *and):*

  (i) C_α *is a closed subset of* α, *unbounded if* α *is limit,*

  (ii) *if* $\alpha \in S_\zeta, \alpha(1) \in C_\alpha$ *then* $\alpha(1) \in S_\zeta$ *and* $C_{\alpha(1)} = C_\alpha \cap \alpha(1)$,

(e) *for each club* C *of* λ *and* $\zeta < \mu$, *for some* $\delta \in S_\zeta$, $C_\delta \subseteq C$.

Proof. Similar to the proof of 3.1 (or see [She86c]). $\square_{3.6}$

We shall use in 3.27

Claim 3.7. *Suppose* $\lambda = \mu^+$, γ *a limit ordinal of cofinality* χ,

$$h : \gamma \longrightarrow \{\theta : \theta = 1 \text{ or } \theta = \mathrm{cf}(\theta) \leq \mu\},$$

$\mu = \mu^{|\gamma|}$, *and* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$ *is stationary.* *Then* *we can find* $S^*, \langle C_\delta : \delta \in S^* \rangle$ *and* $\langle S_\zeta : \zeta < \lambda \rangle$ *such that:*

(a) $\bigcup_{\zeta<\lambda} S_\zeta \subseteq S^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \leq \chi\}$,

(b) $S_\zeta \cap S$ *is stationary for each* $\zeta < \lambda$,

(c) *for* $\delta \in S^*$,

  (i) C_δ *is a club of* δ *of order type* $\leq \gamma$ *and*

  (ii) $\mathrm{otp}(C_\delta) = \gamma$ *iff* $\delta \in S \cap S^*$,

  (iii) $\alpha \in C_\delta \wedge \sup(C_\delta \cap \alpha) < \alpha \quad \Rightarrow \quad \alpha$ *has cofinality* $h[\mathrm{otp}(C_\delta \cap \alpha)]$,

(d) *if* $\delta \in S_\zeta, \delta(1)$ *a limit ordinal* $\in C_\delta$ *then* $\delta(1) \in S_\zeta$ *and* $C_{\delta(1)} = C_\delta \cap \delta(1)$,

(e) *for each club* C *of* λ *and* $\zeta < \lambda$ *for some* $\delta \in S_\zeta$, $C_\delta \subseteq C$.

Proof. Like 3.6. $\square_{3.7}$

Claim 3.8. *1) Suppose* λ *is regular* $> \aleph_1$, *then* $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ *is a good stationary subset of* λ^+ *(i.e., it is in* $\check{I}[\lambda^+]$, *see* [Shed, 3.4=Lcd1.1] *or* [Sheb, 0.6,0.7] *or 3.9(2) below).*
2) Suppose λ *is regular* $> \aleph_1$. *Then* *we can find* $\langle S_\zeta : \zeta < \lambda \rangle$ *such that:*

(a) $\bigcup_{\zeta<\lambda} S_\zeta = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda\}$,

(b) *on each* S_ζ *there is a square (see 3.6 clause (d)), say it is* $\langle C^\zeta_\alpha : \alpha \in S_\zeta \rangle$ *with* $|C^\zeta_\delta| < \lambda$,

(c) *if* $\delta(*) < \lambda$, *and* $\kappa = \mathrm{cf}(\kappa) < \lambda$, *then: for some* $\zeta < \lambda$ *for every club* C *of* λ^+, *for some accumulation point* δ *of* $C, \mathrm{cf}(\delta) = \kappa$ *and* $\mathrm{otp}(C^\zeta_\delta \cap C)$ *is divisible by* $\delta(*)$,

(d) *if* $\mathrm{cf}(\delta(*)) = \kappa$, *we can add in* **(c)***'s conclusion:*

$$C^\zeta_\delta \subseteq C \text{ and } \mathrm{otp}(C^\zeta_\delta) = \delta(*).$$

Remark 3.9. 1) For $\lambda = \aleph_1$ the conclusion of 3.8(1), (2)(a),(b) becomes totally trivial; but for $\delta < \omega_1$, it means something if we add: $\{\alpha \in S_\zeta : \mathrm{otp}(C^\zeta_\alpha) = \delta\}$ is stationary and for every club C of λ the set $\{\alpha \in S_\delta : \mathrm{otp}(C^\zeta_\alpha) = \delta, C^\zeta_\alpha \subseteq C\}$ is stationary. So 3.8(2)(c,d) are not so trivial, but still true. Their proofs are similar so we leave them to the reader (used only in [Shea, 2.7]).
2) Recall that for a regular uncountable cardinal μ, the family $\check{I}[\mu]$ of good subsets of μ is the family of $S \subseteq \mu$ such that there are a sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ and a club $C \subseteq \mu$ satisfying: $a_\alpha \subseteq \alpha$ of order type $< \alpha$ when λ is a successor cardinal, $\beta \in a_\alpha \;\Rightarrow\; a_\beta = a_\alpha \cap \beta$ and

$$(\forall \delta \in S \cap C)(\sup(a_\delta) = \delta \;\&\; \mathrm{otp}(a_\alpha) = \mathrm{cf}(\delta)).$$

We may say that the sequence $\bar{a}$ as above exemplifies that S is good; if $C = \mu$ we say "explicitly exemplifies".

Proof. Appears also in detail in [She91] (originally proved for this work but as its appearance was delayed we put it there, too). Of course,
1) follows from (2).
2) Let $S = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda\}$. For each $\alpha \in S$ choose $\bar{A}^\alpha$ such that:

- (α) $\bar{A}^\alpha = \langle A^\alpha_i : i < \lambda\rangle$ is an increasing continuous sequence of subsets of α of cardinality $< \lambda$, such that $\bigcup\limits_{i<\lambda} A^\alpha_i = \alpha \cap S$,
- (β) if $\beta \in A^\alpha_i \cup \{\alpha\}$, β is a limit ordinal and $\mathrm{cf}(\beta) < \lambda$ (the last actually follows), then $\beta = \sup(A^\alpha_i \cap \beta)$,
- (γ) if $\beta \in A^\alpha_i \cup \{\alpha\}$ is limit and $\aleph_0 < \mathrm{cf}(\beta) < \lambda$ then A^α_i contains a club of β,
- (δ) $0 \in A^\alpha_i$ and $\beta \in S \;\&\; \beta + 1 \in A^\alpha_i \cup \{\alpha\} \;\Rightarrow\; \beta \in A^\alpha_i$,
- (ε) the closure of A^α_i in α (in the order topology) is included in A^α_{i+1}.

There are no problems with choosing $\bar{A}^\alpha$ as required.
We define B^α_i (for $i < \lambda$, $\alpha \in S$) by induction on α as follows:

$$B^\alpha_i = \begin{cases} \mathrm{closure}(A^\alpha_i) \cap \alpha & \text{if } \mathrm{cf}(\alpha) \neq \aleph_1, \\ \bigcap\{\bigcup\limits_{\beta \in C} B^\beta_i : C \text{ a club of } \alpha\} & \text{if } \mathrm{cf}(\alpha) = \aleph_1. \end{cases}$$

For $\zeta < \lambda$ we let:

$$S_\zeta = \{\alpha \in S : \quad \alpha \text{ satisfies} \quad \begin{array}{l} \text{(i) } B^\alpha_\zeta \text{ is a closed subset of } \alpha, \\ \text{(ii) if } \beta \in B^\alpha_\zeta, \text{ then } B^\beta_\zeta = B^\alpha_\zeta \cap \beta \text{ and} \\ \text{(iii) if } \alpha \text{ is limit, then } \alpha = \sup(B^\alpha_\zeta)\ \} \end{array}$$

and for $\alpha \in S_\zeta$ let $C^\zeta_\alpha = B^\alpha_\zeta$.
Now, demand (b) holds by the choice of S_ζ. To prove clause (a) we shall show that for any $\alpha \in S$, for some $\zeta < \lambda$, $\alpha \in S_\zeta$; moreover we shall prove

$(*)^0_\alpha$ $E_\alpha := \{\zeta < \lambda$: if $\mathrm{cf}(\zeta) = \aleph_1$ then $\alpha \in S_\zeta\}$ contains a club of λ.

For $\alpha \in S$ define $E^0_\alpha = \{\zeta < \lambda$: if $\mathrm{cf}(\zeta) = \aleph_1$ then $B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \alpha\}$. We prove by induction on $\alpha \in S$ that $E_\alpha \cap E^0_\alpha$ contains a club of λ and we then choose such a club E^1_α.

Arriving to α, let

$$E = \{\zeta < \lambda : \text{ if } \beta \in A^\alpha_\zeta \text{ then } \zeta \in E^1_\beta \text{ and } A^\beta_\zeta = A^\alpha_\zeta \cap \beta\}.$$

Clearly E is a club of λ. Let $\zeta \in E$, $\mathrm{cf}(\zeta) = \aleph_1$, and we shall prove that $\alpha \in S_\zeta \cap E_\alpha \cap E^0_\alpha$, this clearly suffices. By the choice of ζ (and the definition of E) we have: if β belongs to A^α_ζ then $A^\beta_\zeta = A^\alpha_\zeta \cap A$ and $B^\beta_\zeta = \mathrm{closure}(A^\beta_\zeta) \cap \beta$, so

$(*)_1$ $\beta \in A^\alpha_\zeta \ \Rightarrow\ B^\beta_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \beta$.

Let us check the three conditions for "$\alpha \in S_\zeta$" this will suffice for clause (a) of the claim.

Clause (i): B^α_ζ is a closed subset of α.

If $\mathrm{cf}(\alpha) \neq \aleph_1$ then $B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \alpha$, hence necessarily it is a closed subset of α.

If $\mathrm{cf}(\alpha) = \aleph_1$ then $B^\alpha_\zeta = \bigcap\{\bigcup\limits_{\beta \in C} B^\beta_\zeta : C \text{ is a club of } \beta\}$. Now, for any club C of β, $C \cap A^\alpha_\zeta$ is a club of α (see clause (γ) above). By $(*)_1$ above,

$$\bigcup_{\beta \in C} B^\beta_\zeta \supseteq \bigcup_{\beta \in C \cap A^\alpha_\zeta} B^\beta_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \beta.$$

Note that we have gotten

$(*)_2$ $\alpha \in E^0_\zeta$.

[Why? If $\mathrm{cf}(\alpha) = \aleph_1$ see above, if $\mathrm{cf}(\alpha) \neq \aleph_1$ this is trivial.]

noindent Clause (ii): If $\beta \in B^\alpha_\zeta$ then $B^\beta_\zeta = B^\alpha_\zeta \cap \beta$.

We know that $B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \alpha$, by $(*)_2$ above. If $\beta \in A^\alpha_\zeta$ then (by $(*)_1$) we have $B^\beta_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \beta$, so we are done. So assume $\beta \notin A^\alpha_\zeta$. Then, by clause (ϵ) necessarily

$$\varepsilon < \zeta \quad \Rightarrow \quad \beta > \sup(A^\alpha_\varepsilon \cap \beta) \text{ and } \sup(A^\alpha_\varepsilon \cap \beta) \in A^\alpha_{\varepsilon+1} \subseteq A^\alpha_\zeta.$$

But $\beta \in B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta)$ by $(*)_2$, hence together A^α_ζ contains a club of β and $\mathrm{cf}(\beta) = \mathrm{cf}(\zeta)$, but $\mathrm{cf}(\zeta) = \aleph_1$, so $\mathrm{cf}(\beta) = \aleph_1$. Now, as in the proof of clause (i), we get $B^\beta_\zeta = \bigcup\{B^\gamma_\zeta : \gamma \in A^\alpha_\zeta \cap \beta\}$, so by the induction hypothesis we are done.

Clause (iii): If α is limit then $\alpha = \sup(A^\alpha_i)$.

By clause (γ) we know A^α_ζ is unbounded in α, but $A^\alpha_\zeta \subseteq B^\alpha_\zeta$ (by $(*)_2$) and we are done.

So we have finished proving $(*)^0_\alpha$ by induction on α hence clause (a) of the claim.

For proving (c) of 3.8(2), note that above, if α is limit, C is a club of α, $C \subseteq S$, and $|C| < \lambda$, then for every i large enough, $C \subseteq A^\alpha_i$, and even $C \subseteq B^\alpha_i$.

Now assume that the conclusion of (c) fails (for fixed $\delta(*)$ and κ). Then for each $\zeta < \lambda$ we have a club E^0_ζ exemplifying it. Now, $E^0 =: \bigcap\limits_{\zeta < \lambda} E^0_\zeta$ is a club of λ^+, hence for some $\delta \in E^0$, $\mathrm{otp}(E^0 \cap \delta)$ is divisible by $\delta(*)$ and $\mathrm{cf}(\delta) = \kappa$. Choose an unbounded in δ set $e \subseteq E^0 \cap \delta$ of order type divisible by $\delta(*)$. Then, for a final segment of $\zeta < \lambda$ we have $e \cap \delta \subseteq C^\zeta_\delta$.

Note that for any set C_1 of ordinals, $\mathrm{otp}(C_1)$ is divisible by $\delta(*)$ if C_1 has an unbounded subset of order type divisible by $\delta(*)$, so we get a contradiction because by $(*)^0_{\delta(*))}$ for some $\zeta \in E_{\delta(*)}$ (so $\delta(*) \in S_\zeta$) by $E^0_\zeta \cap C^\zeta_\delta \supseteq E^0 \cap \delta \supseteq e$, $\sup(e) = \delta$ and e has order type divisible by $\delta(*)$.

We are left with clause (d) of 3.8(2). Fix $\kappa, \delta(*)$ and ζ as above, we may add $\leq \lambda$ new sequences of the form $\langle C_\alpha : \alpha \in S_\zeta \rangle$ as long as each is a square. First assume that for every γ, $\beta < \lambda$, such that $\mathrm{cf}(\beta) = \kappa = \mathrm{cf}(\gamma)$, γ divisible by $\delta(*)$ we have

$(*)^3_{\beta,\gamma}$ there is a club $E_{\beta,\gamma}$ of λ^+ such that for no $\delta \in S_\zeta$ do we have $\mathrm{otp}(C^\zeta_\delta) = \beta$ and $\mathrm{otp}(C^\zeta_\delta \cap E_{\beta,\gamma}) = \gamma$,

then let

$$E =: \bigcap \{E_{\beta,\gamma} : \gamma < \lambda,\ \beta < \lambda,\ \mathrm{cf}(\beta) = \kappa = \mathrm{cf}(\gamma),\ \gamma \text{ divisible by } \delta(*)\}.$$

Applying part (c) we get a contradiction.

So for some γ, $\beta < \lambda$, $\mathrm{cf}(\beta) = \kappa = \mathrm{cf}(\gamma)$, γ divisible by $\delta(*)$ and $(*)^3_{\beta,\gamma}$ fails. Also there is a club E^* of λ^+ such that for every club $E \subseteq E^*$ for some $\delta \in S_\zeta$, $\mathrm{otp}(C^\zeta_\delta) = \beta$, $\mathrm{otp}(C^\zeta_\delta \cap E) = \gamma$ and $C^\zeta_\delta \cap E = C^\zeta_\gamma \cap E^*$ (by 3.10 below). Let $e \subseteq \gamma = \sup(e)$ be closed and such that $\mathrm{otp}(e) = \delta(*)$ and

$$\epsilon \in e \text{ is limit } \Rightarrow \epsilon = \sup(e \cap \epsilon).$$

We define ${}^*C^\zeta_\delta$ (for $\delta \in S_\zeta$) as follows: if $\delta \notin E^*$

$${}^*C^\zeta_\delta := C^\zeta_\delta \setminus (\max(\delta \cap E^*) + 1),$$

if $\delta \in E^*, \mathrm{otp}(C^\zeta_\delta \cap E^*) \in e \cup \{\gamma\}$ then

$${}^*C^\zeta_\delta = \{\alpha \in C^\zeta_\delta \cap E^* : \mathrm{otp}(\alpha \cap C^\zeta_\delta \cap E^*) \in e\}$$

and if $\delta \in E^*, \mathrm{otp}(C^\zeta_\delta \cap E^*) \notin e \cup \{\gamma\}$ let

$${}^*C^\zeta_\delta = C^\zeta_\delta \setminus \Big(\max\{\alpha : \mathrm{otp}(C^\zeta_\delta \cap E^* \cap \alpha) \in e \cup \{\gamma\}\} + 1\Big).$$

One easily checks that (d) and square hold for $\langle {}^*C^\zeta_\delta : \delta \in S_\zeta \rangle$. So, we just have to add $\langle {}^*C^\zeta_\delta : \delta \in S_\zeta \rangle$ to $\{\langle C^\zeta_\delta : \delta \in S_\zeta \rangle : \zeta < \lambda\}$ for any $\zeta, \delta(*), \kappa$ (for which we choose ζ and E^*). $\square_{3.9}$

Claim 3.10. *1) Assume that $\aleph_0 < \kappa = \mathrm{cf}(\kappa)$, $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary, C_δ is a club of δ (for $\delta \in S$), and $(\forall \delta \in S)(|C_\delta| = \kappa)$, or at least $\sup_{\delta \in S} |C_\delta|^+ < \lambda$. Then for some club $E^* \subseteq \lambda$, for every club $E \subseteq E^*$, the set $\{\delta \in S^* : C_\delta \cap E^* \subseteq E\}$ is stationary, where*

$$S^* := \{\delta \in S : \delta \in \mathrm{acc}(E^*)\}.$$

2) Assume that $\kappa = \mathrm{cf}(\kappa), \kappa^+ < \lambda = \mathrm{cf}(\lambda), S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary, C_δ is a club of δ (for $\delta \in S$), $\sup_{\delta \in S} |C_\delta|^+ < \lambda$, I_δ is an ideal on C_δ including the *bounded subsets, and for every club E of λ for stationarily many $\delta \in S$, $C_\delta \cap E \notin I_\delta$ (or $C_\delta \setminus E \in I_\delta$).*

Then for some club E^* of λ, for every club $E \subseteq E^*$ of λ the set $\{\delta \in S^* : C_\delta \cap E^* \subseteq E\}$ is stationary, where

$$S^* := \{\delta \in S : \begin{array}{l} \delta \in \mathrm{acc}(E^*), \delta = \sup(C_\delta \cap E^*) \text{ and} \\ C_\delta \cap E^* \notin I_\delta \text{ (or } C_\delta \setminus E^* \in I_\delta)\}. \end{array}$$

Remark 3.11. This also was written in [She94d].

Proof. 1) If not, choose by induction on $i < \mu =: \sup_{\delta \in S}(|C_\delta|^+)$ a club $E^*_i \subseteq \lambda$, decreasing with i, E^*_{i+1} exemplifies that E^*_i is not as required, i.e.,

$$\{\delta \in S^*(E^*_i) : C_\delta \cap E^*_i \subseteq E^*_{i+1}\} = \emptyset.$$

Now, $\mathrm{acc}(\bigcap_{i<\mu} E^*_i)$ is a club of λ, so there is $\delta \in S \cap \mathrm{acc}\big(\bigcap_{i<\mu} E^*_i\big)$. The sequence $\langle C_\delta \cap E^*_i : i < \mu\rangle$ is necessarily strictly decreasing, and we get an easy contradiction.
2) Similarly. $\square_{3.10}$

* * *

Now we turn to the main issue: black boxes.

Lemma 3.12. *Suppose that λ, θ and $\chi(*)$ are regular cardinals and $\lambda^\theta = \lambda^{<\chi(*)}, \theta < \chi(*) \le \lambda$, and a set $S \subseteq \{\delta < \lambda : \mathrm{cf}(\lambda) = \theta\}$ is stationary and in $\check{I}[\lambda]$ (if $\theta = \aleph_0$ this holds trivially; see* [Shed, 3.4=Lcd1.1] *or* [Sheb, 0.6,0.7] *or just 3.9(2)).*

Then we can find

$$\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$$

(pedantically, $\mathbf{W}$ is a sequence) and functions $\dot{\zeta} : \alpha(*) \longrightarrow S$, and $h : \alpha(*) \longrightarrow \lambda$ such that:

- (a0) *$h(\alpha)$ depends on $\dot{\zeta}(\alpha)$ only, and $\dot{\zeta}$ is non-decreasing function (but not necessarily strictly increasing)*
- (a1) *We have*
  - (α)) *$\bar{M}^\alpha = \langle M^\alpha_i : i \le \theta\rangle$ is an increasing continuous chain, ($\tau(M^\alpha_i)$, the vocabulary, may be increasing),*
  - (β) *each M^α_i is an expansion of a submodel of $(\mathscr{H}_{<\chi(*)}(\lambda), \in, <)$ belonging to $\mathscr{H}_{<\chi(*)}(\lambda)$ [so necessarily has cardinality $< \chi(*)$, of course the order mean the order on the ordinals and, for transparency, the vocabulary belongs to $\mathscr{H}_{<\chi(*)}(\chi(*))$],*
  - (γ) *$M^\alpha_i \cap \chi(*)$ is an ordinal, $[\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha_i]$, and $M^\alpha_i \in \mathscr{H}_{<\chi(*)}(\eta^\alpha(i))$,*
  - (δ) *$M^\alpha_i \cap \lambda \subseteq \eta^\alpha(i)$,*
  - (ε) *$\langle M^\alpha_j : j \le i\rangle \in M^\alpha_{i+1}$,*
  - (ζ) *$\eta^\alpha \in {}^\theta\lambda$ is increasing with limit $\dot{\zeta}(\alpha) \in S, \eta^\alpha{\restriction}(i+1) \in M^\alpha_{i+1}$.*

(a2) *In the following game, $\partial(\theta, \lambda, \chi(*), \mathbf{W}, h)$, player I has no winning strategy. A play lasts θ moves, in the i-th move player I chooses a model $M_i \in \mathscr{H}_{<\chi(*)}(\lambda)$, and then player II chooses $\gamma_i < \lambda$. In the first move player I also chooses $\beta < \lambda$. In the end player II wins the play if $(\alpha) \Rightarrow (\beta)$ where*
 (α) *the pair $(\langle M_i : i < \theta\rangle, \langle \gamma_i : i < \theta\rangle)$ satisfies the relevant demands on the pair* [2] *and M_i expand a submodel of $(\mathscr{H}_{<\chi(*)}(\lambda), \in, <)$ $(\bar{M}^i \restriction \theta, \eta^\alpha)$ in clause (a1)*
 (β) *for some $\alpha < \alpha(*), \eta^\alpha = \langle \gamma_i : i < \theta\rangle, M_i = M^\alpha_i$ (for $i < \theta$) and $h(\alpha) = \beta$.*

(b0) *$\eta^\alpha \neq \eta^\beta$ for $\alpha \neq \beta$,*

(b1) *if $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta_\theta$ then $\alpha < \beta + (<\chi(*))^\theta$, see below, and $\dot{\zeta}(\alpha) \leq \dot{\zeta}(\beta)$,*

(b2) *if also $\lambda^{<\theta} = \lambda^{<\chi(*)}$, then for every $\alpha < \alpha(*)$ and $i < \theta$, there is $j < \theta$ such that: $\eta^\alpha \restriction j \in M^\beta_\theta$ implies $M^\alpha_i \in M^\beta_\theta$ (hence $M^\alpha_i \subseteq M^\beta_\theta$),*

(b3) *if $\lambda = \lambda^{<\chi(*)}$ and $\eta^\alpha \restriction (i+1) \in M^\beta_j$ then $M^\alpha_i \in M^\beta_j$ (and hence $x \in M^\alpha_i \Rightarrow x \in M^\beta_j$) and*

$$[\eta^\alpha \restriction i \neq \eta^\beta \restriction i \quad \Rightarrow \quad \eta^\alpha(i) \neq \eta^\beta(i)].$$

Remark 3.13. 1) If $\mathbf{W}$ (with $\dot{\zeta}, h, \lambda, \theta, \chi(*)$) satisfies (a0), (a1), (a2), (b0), (b1) we call it a barrier.
2) Remember, $(<\chi)^\theta =: \sum\limits_{\mu<\chi} \mu^\theta$.
3) The existence of a good stationary set $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ follows, for example, from $\lambda = \lambda^{<\theta}$ (see [Shed, 3.4=Lcd1.1] or [Sheb, 0.6,0.7]) and from "λ is the successor of a regular cardinal and $\lambda > \theta^{+}$". But see 3.16(1),(2),(3).
4) Compare the proof below with [She84b, Lemma 1.13,pg.49] and [She81].

Proof. First assume $\lambda = \lambda^{<\chi(*)}$.

Let $\langle S_\gamma : \gamma < \lambda\rangle$ be a sequence of pairwise disjoint stationary subsets of $S, S = \bigcup\limits_{\gamma<\lambda} S_\gamma$ and without loss of generality $\gamma < \mathrm{Min}(S_\gamma)$. We define $h^* : S \longrightarrow \lambda$ by $h^*(\alpha) =$ "the unique γ such that $\alpha \in S_\gamma$", and below we shall let $h(\alpha) := h^*(\dot{\zeta}(\alpha))$.

Let $\mathrm{cd} = \mathrm{cd}_{\lambda,\chi(*)}$ be a one-to-one function from $\mathscr{H}_{<\chi(*)}(\lambda)$ onto λ such that: $\mathrm{cd}(\langle\alpha, \beta\rangle)$ is an ordinal $> \alpha, \beta$, but $< |\alpha + \beta|^+$ or $< \omega$, and $x \in \mathscr{H}_{<\chi(*)}(\mathrm{cd}(x))$ for every relevant x. For $\xi \in S$ let:

$$\begin{aligned} \mathbf{W}_\xi := \{(\bar{M}, \eta) : \quad & \text{the pair } (\bar{M}, \eta) \text{ satisfies (a1) of 3.12, } \sup\{\eta(i) : i < \theta\} = \xi, \\ & \text{and for every } i < \theta \text{ for some } y \in \mathscr{H}_{<\chi(*)}(\lambda), \\ & \eta(i) = \mathrm{cd}(\langle \bar{M} \restriction i, \eta \restriction i, y\rangle)\}. \end{aligned}$$

So (a0), (a1), (b0), (b3) (hence (b2)) should be clear.

We can choose $\langle(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\rangle$ an enumeration of $\bigcup\limits_{\xi\in S} \mathbf{W}^0_\xi$ to satisfy (b1) (and $\dot{\zeta}(\alpha) = \sup \mathrm{rang}(\eta^\alpha)$, of course) because:

---

[2] so $\langle M_j : j \leq i\rangle$ is an increasing continuous chain, $M_i \cap \chi(*)$ an ordinal, $\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M_i, \langle M_\epsilon : \epsilon \leq j\rangle \in M_{j+1}$ and $\langle \gamma_\epsilon : \epsilon \leq j\rangle \in M_{j+1}$ for $j < i, M_i \in \mathscr{H}_{<\chi(*)}(\gamma_i)$ and $\langle \gamma_i : j \leq i\rangle \in M_{i+1}$

$(*)$ if $(\bar{M}^*, \eta^*) \in \bigcup_\xi \mathbf{W}^0_\xi$, then

$$|\{\eta \in {}^\theta\lambda : \{\eta{\restriction}i : i < \theta\} \subseteq M^*_\theta\}| \leq \|M^*_\theta\|^\theta \leq (< \chi(*))^\theta.$$

This, in fact, defines the function $\dot{\zeta}$ as follows: we have $\dot{\zeta}(\alpha) = \xi$ if and only if $(\bar{M}^\alpha, \eta^\alpha) \in \mathbf{W}^0_\xi$.

We are left with proving (a2). Let G be a strategy for player I.

Let $\langle C_\delta : \delta < \lambda \rangle$ exemplify "S is a good stationary subset of λ", see 3.9(2), and let $R = \{(i, \alpha) : i \in C_\alpha, \alpha < \lambda\}$.

Let $\langle \mathscr{A}_i : i < \lambda \rangle$ be a representation of the model

$$\mathscr{A} = (\mathscr{H}_{<\chi(*)}(\lambda), \in, G, R, \mathrm{cd}),$$

i.e. it is increasing continuous, $\|\mathscr{A}_i\| < \lambda$, and $\bigcup_i \mathscr{A}_i = \mathscr{A}$; without loss of generality $\mathscr{A}_i \prec \mathscr{A}$ and $|\mathscr{A}_i| \cap \lambda$ is an ordinal for $i < \lambda$.

Let G "tell" player I to choose $\beta^* < \lambda$ in his first move. So there is $\delta \in S_{\beta^*}$ (hence $\delta > \beta^*$) such that $|\mathscr{A}_\delta| \cap \lambda = \delta$. Now, necessarily $C_\delta \cap \alpha \in \mathscr{A}_\delta$ for $\alpha < \delta$. Let $\{\alpha_i : i < \mathrm{cf}(\delta)\}$ list C_δ in increasing order.

Lastly, by induction on i, we choose $M_i, \eta(i)$ as follows:

$$\eta(i) = \mathrm{cd}(\langle \langle M_j : j \leq i \rangle, \langle \eta(j) : j < i \rangle, \langle \alpha_j : j < i \rangle \rangle),$$

and M_i is what the strategy G "tells" player I to choose in his i-th move if player II have chosen $\langle \eta(j) : j < i \rangle$ so far.

Now, for each $i < \theta$ the sequences $\langle M_j : j \leq i \rangle$, $\langle \eta(j) : j < i \rangle$ are definable in $\mathscr{A}_\delta$ with $\langle \alpha_j : j \leq i \rangle$ as the only parameter, hence they belong to $\mathscr{A}_\delta$. So $\sup\{\eta(j) : j < \theta\} \leq \delta$; however, by the choice of $\eta(i)$ (and cd), $\eta(i) \geq \sup\{\alpha_j : j < i\}$ and hence $\sup\{\eta(j) : j < \theta\}$ is necessarily δ. Now check.

The case $\lambda < \lambda^{<\theta} = \lambda^{<\chi(*)}$ is similar. For a set $A \subseteq \theta$ of cardinality θ we let $\mathrm{cd}^A = \mathrm{cd}^A_{\lambda,\chi(*)}$ be a one-to-one function from $\mathscr{H}_{<\chi(*)}(\lambda)$ onto A_λ where:

$$A_\lambda = \{h : h \text{ is a function from } A \text{ to } \lambda\}.$$

We strengthen (b2) to

$(b2)'$ let $A_i := \{\mathrm{cd}(i, j) : j < \theta\}$ for $i \in [1, \theta)$ and $A_0 := \theta \setminus \cup\{A_{1+i} : i < \theta\}$ so $\langle A_i : i < \theta \rangle$ is a sequence of pairwise disjoint subsets of θ each of cardinality θ with $\min(A_i) \geq i$ and we have
  $(*)$ $\eta^\alpha{\restriction}A_i = \mathrm{cd}^{A_i}((\bar{M}^\alpha{\restriction}i, \eta^\alpha{\restriction}i)$.

$\square_{3.13}$

* * *

What can we do when S is not good? As we say in 3.13(3), in many cases a good S exists, note that for singular λ we will not have one.

The following rectifies the situation in the other cases (but is interesting mainly for λ singular). We shall, for a regular cardinal λ, remove this assumption in 3.16(1)–(3), while 3.17 helps for singular λ. (This is carried in 3.18).

Definition 3.14. Let ∂ be an ordinal and for $\alpha < \partial$ let κ_α be a regular uncountable cardinal, $S_\alpha \subseteq \{\delta < \kappa_\alpha : \mathrm{cf}(\delta) = \theta\}$ be a stationary set. Assume θ, χ are regular cardinals such that for every $\alpha < \partial$ we have $\theta < \chi \le \kappa_\alpha$. Let $\bar{S} = \langle S_\alpha : \alpha < \partial \rangle$, $\bar{\kappa} = \langle \kappa_\alpha : \alpha < \partial \rangle$. If $\partial = 1$ we may write S_0, κ_0.

We say that $\bar{S}$ is good for $(\bar{\kappa}, \theta, \chi)$ when: for every large enough μ and model $\mathscr{A}$ expanding $(\mathscr{H}_{<\chi}(\mu), \in), |\tau(\mathscr{A})| \le \aleph_0$, there are M_i for $i < \theta$ such that:

- $M_i \prec \mathscr{A}$ and $\bar{S} \in M_i$
- $\langle M_j : j \le i \rangle \in M_{i+1}, \|M_i\| < \chi, M_i \cap \chi \in \chi, \chi = \chi_1^+ \Rightarrow \chi_1 + 1 \subseteq M_i$, and
- $\alpha < \partial, \alpha \in \bigcup\limits_{j<\theta} M_j$ implies that $\sup[\kappa_\alpha \cap (\bigcup\limits_{j<\theta} M_j)]$ belongs to S_α.

If $\partial = 1$, we may write S_0, κ_0 instead $\bar{S}, \bar{\kappa}$. If $\partial < \chi$ then we can demand $\partial \subseteq M_0$.

Definition 3.15. For regular uncountable cardinal λ and regular $\theta < \lambda$ let $\check{J}_\theta[\lambda]$ be the family of subsets S of λ such that $(\{\delta \in S : \mathrm{cf}(\delta) = \theta\}$ is not good for $(\lambda, \lambda, \theta)$.

Claim 3.16. *Assume $\theta = \mathrm{cf}(\theta) < \chi = \mathrm{cf}(\chi) \le \kappa = \mathrm{cf}(\kappa)$.*
1) Then $\{\delta < \kappa : \mathrm{cf}(\delta) = \theta\}$ is good for (κ, θ, χ), i.e. is not in $\check{J}_\theta[\lambda]$.
2) Any $S \subseteq \kappa$ good for (κ, θ, χ) is the union of κ pairwise disjoint such sets.
3) In 3.12 it suffices to assume that S is good for (λ, θ, χ).
4) $\check{J}_\theta[\lambda]$ is a normal ideal on λ and there is no stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ which belongs to $\check{J}_\theta[\lambda] \cap \check{I}[\lambda]$.
5) In Definition 3.14, any $\mu > \lambda^{<\chi}$ is O.K.; and we can preassign $x \in \mathscr{H}_{<\chi}(\mu)$ and demand $x \in M_i$.
6) In 3.12 we can replace the assumption "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary and in $\check{I}[\lambda]$" by "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary not in $\check{J}_\theta[\lambda]$" (which holds for $S = \{\delta < \kappa : \mathrm{cf}(\delta) = \theta\}$).

Proof. 1) Straightforward (play the game).
2) Similar to the proof of 3.1.
3) Obvious.
4) Easy.
5) Easy.
6) Follows. $\square_{3.16}$

Claim 3.17. *Assume that $\bar{\kappa}, \theta, \chi$ are as in 3.14 with $|\partial| \le \chi$.*
1) Then the sequence $\langle \{\delta < \kappa_i : \mathrm{cf}(\delta) = \theta\} : i < \partial \rangle$ is good for $(\bar{\kappa}, \theta, \chi)$.
2) If $\partial_1 < \partial$ and $\langle S_i : i < \partial_1 \rangle$ is good for $(\bar{\kappa}{\restriction}\partial_1, \theta, \chi)$ then

$$\langle S_i : i < \partial_1 \rangle \hat{\ } \langle \{\delta < \kappa_i : \mathrm{cf}(\delta) = \theta\} : \partial_1 \le i < \partial \rangle$$

is good for $(\bar{\kappa}, \theta, \chi)$.
3) If $\langle S_i : i < \partial_1 \rangle$ is good for $(\bar{\kappa}, \theta, \chi)$ and $i(*) < \partial$, then we can partition $S_{i(*)}$ to pairwise disjoint sets $\langle S_{i(*),\epsilon} : \epsilon < \kappa_i \rangle$ such that for each $\epsilon < \kappa_i$, the sequence

$$\langle S_i : i < i(*) \rangle \hat{\ } \langle S_{i(*),\epsilon} \rangle \hat{\ } \langle \{\delta : \delta < \kappa_i, \mathrm{cf}(\delta) = \theta\} : i(*) < i < \partial \rangle$$

is good for $(\bar{\kappa}, \theta, \chi)$.
4) $\bar{S}$ good for $(\bar{\kappa}, \theta, \chi)$ implies that S_i is a stationary subset of κ_i for each $i < \mathrm{lg}(\bar{\kappa})$.

Proof. Like 3.16 [in 3.17(3) we choose for $\delta \in S_{i(*)}$, a club C_δ of δ of order type $\mathrm{cf}(\delta)$; for $j < \theta$, $\epsilon < \kappa_{i(\alpha)}$, let $S^j_{i(*),\epsilon} = \{\delta \in S_{i(*)} : \epsilon$ is the j-th member of $C_\delta\}$; for some j and unbounded $A \subseteq \kappa_{i(*)}$, $\langle S^j_{i(*),\epsilon} : \epsilon \in A\rangle$ are as required]. $\square_{3.17}$

Now we remove from 3.12 (and subsequently 3.20) the hypothesis "λ is regular" when $\mathrm{cf}(\lambda) \geq \chi(*)$.

Lemma 3.18. *Suppose $\lambda^\theta = \lambda^{<\chi(*)}$, λ is singular, θ and $\chi(*)$ are regular, $\theta < \chi(*)$ and $\mathrm{cf}(\lambda) \geq \chi(*)$. Suppose further that $\lambda = \sum\limits_{i<\mathrm{cf}(\lambda)} \mu_i$, each μ_i is regular $> \chi(*) + \theta^+$. Then we can find $\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ and functions $\dot{\zeta} : \alpha(*) \longrightarrow \mathrm{cf}(\lambda), \dot{\xi} : \alpha(*) \longrightarrow \lambda$, and $h : \alpha(*) \longrightarrow \lambda$, and $\{\mu'_i : i < \mathrm{cf}(\lambda)\}$ such that ($\{\mu'_i : i < \mathrm{cf}(\lambda)\} = \{\mu_i : i < \mathrm{cf}(\lambda)\}$ and):*

(a0) *$h(\alpha)$ depends only on $\langle\dot{\zeta}(\alpha)$,*
dot$\xi(\alpha)\rangle$,

$$[\alpha < \beta \ \Rightarrow\ \dot{\zeta}(\alpha) \leq \dot{\zeta}(\beta)], [\alpha < \beta \wedge \dot{\zeta}(\alpha) = \dot{\zeta}(\beta) \ \Rightarrow\ \dot{\xi}(\alpha) \leq \dot{\xi}(\beta)],$$

and $\dot{\xi}(\alpha) < \mu'_{\dot{\zeta}(\alpha)}$

(a1) *as in 3.12 except that: $\langle\eta^\alpha(3i) : i < \theta\rangle$ is strictly increasing with limit $\dot{\zeta}(\alpha)$ and $\langle\eta^\alpha(3i+1) : i < \theta\rangle$ is strictly increasing with limit $\dot{\xi}(\alpha)$ for $i < \theta$,*

$$\sup(|M^\alpha_i| \cap \mu'_{\zeta(\alpha)}) < \dot{\xi}(\alpha) = \sup(|M^\alpha_\theta| \cap \mu'_{\dot{\zeta}(\alpha)})$$

and for every $i < \theta$,

$$\sup(|M^\alpha_i| \cap \mathrm{cf}(\lambda)) < \dot{\zeta}(\alpha) = \sup(|M^\alpha_\theta| \cap \mathrm{cf}(\lambda)),$$

(a2) *as in 3.12*

(b0), (b1), (b2) *as in 3.12 but in clause (b3) we demand $i = 2 \mod 3$.*

Remark 3.19. To make it similar to 3.12, we can fix S^a, S^a_i, S^b_i, $S^b_{i,a}$, μ'_i as in the first paragraph of the proof below.

Proof. First, by 3.16 [(1) + (2)], we can find pairwise disjoint $S^a_i \subseteq \mathrm{cf}(\lambda)$ for $i < \mathrm{cf}(\lambda)$, each good for $(\mathrm{cf}(\lambda), \theta, \chi(*))$ (and $\alpha \in S^a_i \ \Rightarrow\ \alpha > i$ & $\mathrm{cf}(\alpha) = \theta$), and let $S^a = \bigcup\limits_{i<\mathrm{cf}(\lambda)} S^a_i$. We define $\mu'_i \in \{\mu_j : j < i\}$ such that for each $i < \mathrm{cf}(\lambda) : [j \in S^a_i \Rightarrow \mu'_j = \mu_i]$.

Then for each i, by 3.17 parts (2) (3) (with $1, 2, S_0, \kappa_0, \kappa_1$ standing for σ_1, σ, S^a_i, $\mathrm{cf}(\lambda)$, μ'_i), we can find pairwise disjoint subsets $\langle S^b_{i,\alpha} : \alpha < \mu'_i\rangle$ of $\{\delta < \mu'_i : \mathrm{cf}(\delta) = \theta\}$ such that for each $\alpha < \mu'_\alpha$, $(S^a_i, S^b_{i,\alpha})$ is good for $(\langle\mathrm{cf}(\lambda), \mu'_i\rangle, \theta, \chi)$. Let $S^b_i = \bigcup\{S^b_{i,\alpha} : \alpha < \mu'_i\}$.

Let cd be as in 3.12's proof coding only for ordinals $i = 2 \mod 3$, and for $\zeta \in S^a_i$, $\xi \in S^a_{i,j}$ let

$$\begin{aligned}\mathbf{W}^0_{\zeta,\xi} = \{(\bar{M}, \eta) :\ & \bar{M} \text{ satisfies (a1) } \zeta = \sup\{\eta(3i) : i < \theta\},\\ & \xi = \sup\{\eta(3i+1) : i < \theta\} \text{ and}\\ & \text{for each } i < \theta, \text{ for some } y \in \mathscr{H}_{<\chi(*)}(\lambda),\\ & \eta(3i+2) = \mathrm{cd}(\langle M_j : j \leq 3i+1\rangle, \eta{\restriction}(3i+1), y)\}.\end{aligned}$$

The rest is as in 3.12's proof. $\square_{3.18}$

* * *

The following Lemma improves 3.12 when λ satisfies a stronger requirement making the distinct $(\bar{M}^\alpha, \eta^\alpha)$ interact less. Lemmas 3.20 + 3.18 were used in the proof of 2.4 (and 2.3).

Lemma 3.20. *In 3.12, if $\lambda = \lambda^{\chi(*)}, \chi(*)^\theta = \chi(*)$, then we can strengthen clause (b1) to*

(b1)$^+$ *if $\alpha \neq \beta$ and $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta$ then $\alpha < \beta$ and $x \in M^\alpha_\theta \Rightarrow x \in M^\beta_\theta$.*

Proof. Apply 3.12 (actually, its proof) but using λ, $\chi(*)^+$, θ, instead of $\lambda, \chi(*), \theta$; and get $\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*))\}$, and the functions $\dot{\zeta}, h$.

Let cd be as in the proof of 3.12. Let $<^*$ be some well ordering of $\mathscr{H}_{<\chi(*)}(\lambda)$, and let $\mathscr{U}$ be the set of ordinals $\alpha < \alpha(*)$ such that for $i < \theta$, M^α_i has the form $(N^\alpha_i, \in^\alpha_i, <^\alpha)$ and $(|N^\alpha_i|, \in^\alpha_i, <^\alpha) \prec (\mathscr{H}_{<\chi(*)}(\lambda), \in, <^*)$.

Let $\alpha \in \mathscr{U}$, by induction on $\epsilon < \chi(*)$ we define $M^{\epsilon,\alpha}_i, \eta^{\epsilon,\alpha}$ as follows:

- (A) $\eta^{\epsilon,\alpha}(i)$ is $\mathrm{cd}(\langle \eta^\alpha(i), \epsilon\rangle)$, (which is an ordinal $< \lambda$ but $> \eta^\alpha(i)$ and $> \epsilon$)
- (B) $M^{\epsilon,\alpha}_i \prec N^\alpha_i$ is the Skolem Hull of $\{\eta^{\epsilon,\alpha} \restriction (j+1) : j < i\}$ inside N^α_i, using as Skolem functions the choice of the $<^*$–first element and making $M^{\epsilon,\alpha}_i \cap \chi(*)$ an ordinal [if we want we can use $\eta^{\epsilon,\alpha}$ such that it fits the definition in the proof of 3.12].

Note that $\chi(*) = \chi^+ \;\Rightarrow\; \chi+1 \subseteq M^\alpha_i$ and $M^{\epsilon,\alpha}_i$ is definable in $M^{\epsilon,\alpha}_{i+1}$ as $M^{\epsilon,\alpha}_i \in M^{\epsilon,\alpha}_{i+1}$ (by the definition of $\mathbf{W}^0_\xi$ in the proof of 3.12). Similarly, $\langle M^{\epsilon,\alpha}_j : j \leq i\rangle$ is definable in M^α_{i+1}. It is easy to check that the pair $(\bar{M}^{\epsilon,\alpha}, \eta^{\epsilon,\alpha})$ satisfies condition (a1) of 3.12.

Next we choose by induction on $\alpha \in \mathscr{U}, \epsilon(\alpha) < \chi(*)$ as follows:

- (C) $\epsilon(\alpha)$ is the first $\epsilon < \chi(*)$ such that: if $\beta < \alpha$ but $\beta + \chi(*) > \alpha$ then:
  - $(*)$ $\{\eta^{\alpha,\epsilon} \restriction j : j < \theta\} \nsubseteq M^{\beta,\epsilon(\beta)}_\theta$.

This is possible and easy, as for $(*)$ it suffices to have for each suitable β, $\epsilon \notin M^{\beta,\epsilon(\beta)}_\theta$, so each β "disqualifies" $< \chi(*)$ ordinals as candidates for $\epsilon(\alpha)$, and there are $< \chi(*)$ such β's, and $\chi(*)$ is by the assumptions (see 3.12) regular.

Now

$$\mathbf{W}' = \{(\bar{N}^{\alpha,\epsilon(\alpha)}, \eta^{\alpha,\epsilon(\alpha)}) : \alpha \in \mathscr{U}\}, \dot{\zeta} \restriction \mathscr{U}, h \restriction \mathscr{U}$$

are as required except that we should replace $\mathscr{U}$ by an ordinal (and adjust ζ, h accordingly). In the end replace N^α_i by $N^\alpha_i \cap \mathscr{H}_{<\chi(*)}(\lambda)$. $\square_{3.20}$

Claim 3.21. *If in 3.18 we add "$\lambda = \lambda^{\chi(*)^\theta}$" (or the condition from 3.20) ?? θ can replace (b1) by*

(b1)$^+$ *if $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta_\theta$ then $\alpha \leq \beta$.*

Proof. The same as the proof of 3.20 combined with the proof of 3.18. $\square_{3.21}$

* * *

Next we turn to the case (of black boxes with) $\theta = \aleph_0$. We shall deal with several cases.

Lemma 3.22. *Suppose that*

(∗) λ *is a regular cardinal,* $\theta = \aleph_0$, $\mu = \mu^{<\chi(*)} < \lambda \le 2^\mu$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ *is stationary and* $\aleph_0 < \chi(*) = \mathrm{cf}(\chi(*))$.

Then we can find

$$\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$$

and functions

$$\dot{\zeta} : \alpha(*) \longrightarrow S \text{ and } h : \alpha(*) \longrightarrow \lambda$$

such that:

$(a0)-(a2)$ *as in 3.12,*

$(b0)-(b2)$ *as in 3.12, and even*

$(b1)^*$ $\alpha \ne \beta, \{\eta^\alpha{\restriction}n : n < \omega\} \subseteq M^\beta_\omega$ *implies* $\alpha < \beta$ *and even* $\dot{\zeta}(\alpha) < \dot{\zeta}(\beta)$,

$(c1)$ *if* $\dot{\zeta}(\alpha) = \dot{\zeta}(\beta)$ *then* $|M^\alpha_\omega| \cap \mu = |M^\beta_\omega| \cap \mu$ *and there is an isomorphism* $h_{\alpha,\beta}$ *from* M^α_ω *onto* M^β_ω, *mapping* $\eta^\alpha(n)$ *to* $\eta^\beta(n)$, *and* M^α_n *to* M^β_n *for* $n < \omega$, *and* $h_{\alpha,\beta}{\restriction}(|M^\alpha_\omega| \cap |M^\beta_\omega|)$ *is the identity,*

$(c2)$ *there is* $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ *an* ω*–sequence converging to* δ, $0 \notin C_\delta$, *and letting* $\langle \gamma^\delta_n : n < \omega\rangle$ *enumerate* $\{0\} \cup C_\delta$ *we have, when* $\dot{\zeta}(\alpha) = \delta$:

  (i) $\lambda \cap |M^\alpha_n| \subseteq \gamma^\delta_{n+1}$ *but* $\lambda \cap |M^\alpha_n|$ *is not a subset of* γ^δ_n, *(hence* $M^\alpha_n \cap [\gamma^\delta_n, \gamma^\delta_{n+1}) \ne \emptyset$*);*

  (ii) $C_\delta \cap |M^\alpha_\omega| = \emptyset$;

  (iii) *if* $\dot{\zeta}(\beta) = \delta$ *too then, for each* n, $h_{\alpha,\beta}$ *maps* $|M^\alpha_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1})$ *onto* $|M^\beta_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}]$;

  (iv) *if* $\dot{\zeta}(\beta) = \delta = \dot{\zeta}(\alpha)$ *and* $\lambda = \lambda^{<\chi(*)}$, *then* $|M^\alpha_\omega| \cap \gamma^\delta_1 = |M^\beta_\omega| \cap \gamma^\delta_1$.

Remark 3.23. 1) We use $\lambda \le 2^\mu$ only to get "$h_{\alpha,\beta}{\restriction}(|M^\alpha_\omega| \cap |M^\beta_\omega|) = \mathrm{id}$" in condition **(c1)**.
2) Below we quote "guessing of clubs" that is clause (ii) in the proof, without this we just get a somewhat weaker conclusion.

Proof. Let S be the disjoint union of stationary

$$S_{\alpha,\beta,\gamma} \quad (\alpha < \mu, \beta < \lambda, \gamma < \lambda).$$

For each α, β, γ let $\langle C_\delta : \delta \in S_{\alpha,\beta,\gamma}\rangle$ satisfy

⊠ (i) C_δ is an unbounded subset of δ of order type ω, and

  (ii) for every club C of λ, for stationarily many $\delta \in S_{\alpha,\beta,\gamma}$ we have $C_\delta \subseteq C$

  (iii) $0 \notin C_\delta$

(exists by [Shea, 2.2] or [She94d]).

Let $\mathbf{W}^*$ be the family of quadruples $(\delta, \bar{M}, \eta, C)$ such that:

(α) $(\bar{M}, \eta)$ satisfies the requirement **(a1)** (so $\bar{M} = \langle M_n : n < \omega \rangle$);

(β) $0 \notin C$, and letting $\{\gamma_n : n < \omega\}$ enumerate in increasing order $C \cup \{0\}$ we have $\lambda \cap M_n$ is a subset of γ_{n+1} but not of γ_n, and $\bigcup_{n<\omega} \gamma_n = \delta$ and $C \cap (\bigcup_n M_n) = \emptyset$;

(γ) $\bigcup_n |M_n| \subseteq \mathscr{H}_{<\chi(*)}(\mu + \mu)$;

(δ) in $\tau(M_n)$ there are a two place relation R and a one place function cd (not necessarily $\mathrm{cd}\restriction M_n = \mathrm{cd}^{M_n}$, similarly for R, see below recall that as usual, $\tau(M_n)) \in \mathscr{H}_{<\chi(*)}(\chi(*))$ for transparency.)

As $\mu^{<\chi(*)} = \mu$ clearly $|\mathbf{W}^*| = \mu$, so let

$$\mathbf{W}^* = \{(\delta^j, \langle M_{j,n} : n < \omega\rangle, \eta_j, C^j) : j < \mu\}.$$

If $\lambda = \lambda^{<\chi(*)}$ let $\{N_\beta : \beta < \lambda\}$ list the models $N \in \mathscr{H}_{<\chi(*)}(\lambda)$ with $\tau(N) \in \mathscr{H}_{<\chi(*)}(\chi(*))$.

Also, let $\langle A_\alpha : \alpha < \lambda\rangle$ be a sequence of pairwise distinct subsets of μ, and define the two place relation R on λ by

$$[\gamma_1 \; R \; \gamma_2 \Leftrightarrow \gamma_1 < \mu \;\&\; \gamma_1 \in A_{\gamma_2}].$$

Lastly, for $\delta \in S_{\alpha,\beta,\gamma}$ let

$$\begin{aligned}\mathbf{W}^0_\delta := \{(\bar{M}, \eta) : \;& \bar{M} = \langle M_n : n < \omega\rangle,\ \eta \in {}^\omega\lambda,\ \text{satisfy } \mathbf{(a1)}, \text{so} \\ & \quad \eta \text{ is increasing with limit } \delta \text{ and there is an isomorphism} \\ & h \text{ from } \bigcup_{n<\omega} M_n \text{ onto } \bigcup_{n<\omega} M_{\alpha,n},\ \text{mapping } \eta(n) \text{ to } \eta^\alpha(n) \text{ and} \\ & M_n \text{ onto } M_{\alpha,n},\ \text{preserving } \in, R, \mathrm{cd}(x) = y \text{ and their negations; (for } R \text{ and cd} : \\ & \text{in } \bigcup_{n<\omega} M_n \text{ we mean the standard cd over } \bigcup_{n<\omega} M_{\alpha,n} \text{ as in } (\delta) \text{ above); and} \\ & (\forall \epsilon < \lambda)[\epsilon \in \bigcup_n M_n \;\Rightarrow\; \mathrm{otp}(C_\delta \cap \epsilon) = \mathrm{otp}(C^\alpha \cap h(\epsilon))]. \\ & \text{Also, if } \lambda = \lambda^{<\chi(*)} \text{ then} \\ & N_\beta = (\bigcup_n M_n)\restriction\{x \in \bigcup_n M_n : \mathrm{cd}(x) < \mathrm{Min}(C_\delta)\}\}.\end{aligned}$$

We proceed as in the proof of 3.12 after $\mathbf{W}^0_\delta$ was defined (only $\dot{\zeta}(\alpha) = \delta \in S_{\alpha_1,\beta_1,\gamma_1} \Rightarrow h(\alpha) = \gamma_1$).

Suppose G is a winning strategy for player I. So suppose that if player II has chosen $\eta(0), \eta(1), \ldots, \eta(n-1)$, player I will choose M_η. So $|M_\eta|$ is a subset of $\mathscr{H}_{<\chi(*)}(\lambda)$ of cardinality $< \chi(*)$ and $\mathrm{Rang}(\eta) \subseteq M_\eta$. For $\eta \in {}^\omega\lambda$ we define $M_\eta = \bigcup_{\ell<\omega} M_{\eta\restriction\ell}$.

Let $\mathscr{T}_n$ be the set of $\eta \in {}^n\lambda$ such that M_η is well defined; so $\cup\{\mathscr{T}_n : n < \omega\}$ is a subtree of $({}^{\omega>}\lambda, \triangleleft)$ with each node having λ immediate successors.

We can find a function $\mathbf{c}_n$ from $\mathscr{T}_n$ into μ such that $\mathbf{c}_n(\eta) = \mathbf{c}_n(\nu)$ iff there is an isomorphism h from M_η onto M_ν mapping $M_{\eta\restriction k}$ onto $M_{\nu\restriction k}$ for every $k < n$. By [Shed, 1.10=L1.7] or the proof of 3.24 below, there is $\mathscr{T}$ such that

$\mathscr{T} \subseteq {}^{\omega>}\lambda$, $\quad \mathscr{T}$ is closed under initial segments,
$\langle\rangle \in \mathscr{T}$, $\quad [\eta \in \mathscr{T} \Rightarrow (\exists^\lambda \alpha)[\eta\hat{\ }\langle\alpha\rangle \in \mathscr{T}]]$,
$\mathbf{c}_n \restriction (\mathscr{T} \cap \mathscr{T}_n)$ is constant.

It follows that fixing any $\nu_* \in \lim(\mathscr{T})$ we can find $\langle h_\eta ; \eta \in \mathscr{T}\rangle$ such that h_η is an isomorphism from $M_{\nu_* \restriction \ell g(\eta)}$ onto M_η increasing with η.

Note that above all those isomorphisms are unique as the interpretation of $\in$ satisfies comprehension. Also clause (c1) follows from the use of R.
The rest should be clear. $\square_{3.22}$

Lemma 3.24. *Let S, λ, μ, θ be as in $(*)$ of 3.22 and in addition:*

$$\aleph_0 \leq \kappa = \mathrm{cf}(\kappa) < \chi(*), \qquad (\forall \chi < \chi(*))[\chi^{<\chi(*} < \chi(*)].$$

Then we can find $\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ and functions $\dot{\zeta} : \alpha(*) \longrightarrow S$ and $h : \alpha(*) \longrightarrow \lambda$ such that:

$(a0), (b0), (b2)$ *as in 3.22 (i.e. as in 3.12),*

$(b1)^*, (c1), (c2)$ *as in 3.22,*

$(a1)^*$ *as* **(a1)** *in 3.12 except that we omit "$\langle M_j : j \leq i\rangle \in M_{i+1}$" and add: $[a \subseteq |M_i|$ & $|a| < \kappa \Rightarrow a \in M_i]$ and for $i < j$, $M_i \cap \lambda$ is an initial segment of $M_j \cap \lambda$,*

$(a2)^*$ *for every expansion $\mathscr{A}$ of $(\mathscr{H}_{<\chi(*)}(\lambda), \in, <)$ by $\chi < \chi(*)$ relations, for some $\alpha < \alpha(*)$, for every n, $M^\alpha_n \prec \mathscr{A}$ in fact, for stationarily many $\zeta \in S$, there is such α satisfying $\dot{\zeta}(\alpha) = \zeta$.*

Remark 3.25. We can retain $(a1)^*$ and add $a \subseteq M_i \wedge |a| < \kappa \Rightarrow a \in M_i$.

Proof. Similar to 3.22, use the proof of [She86a], but for completeness we give details.

We choose $\langle S_{\alpha,\beta,\gamma} : \alpha < \mu, \beta < \lambda, \gamma < \lambda\rangle$ as there. The main point is that defining $\mathbf{W}^*$ we have one additional demand:

(ϵ) if $n < \omega$ and $u \subseteq M_n$ has cardinality $< \kappa$ then $u \in M_n$.

We then define $\mathbf{W}^0_\delta$ and $\langle N_\alpha : \alpha < \lambda\rangle$ as there.

This gives the changed demand in $(a1)^*$, but it give extra work in verifying the demand $(a2)^*$.

So let a model $\mathscr{A}$ and cardinal $\chi = \chi^{<\kappa} < \chi(*)$ as there be given; as usual, $\tau(\mathscr{A}) \in \mathscr{H}_{<\chi(*)}(\chi(*))$ and $\mathscr{A}$ expand $(\mathscr{H}_{<\chi(*)}(\lambda), \in, <)$. For every $\mathbf{x} = (\delta_{\mathbf{x}}, \bar{M}_{\mathbf{x}}, \eta_{\mathbf{x}}, C_{\mathbf{x}}) \in \mathbf{W}^*$ we define a family $\mathscr{F}_{\mathbf{x}}$, a function $n : \mathscr{F} \to \omega$ and a function $\mathrm{rank}_{\mathbf{x}}$ from $\mathscr{F}_{\mathbf{x}}$ into $\mathrm{Ord} \cup \{\infty\}$ as follows:

(α) $\mathscr{F}_{\mathbf{x}} = \Cup\{\mathscr{F}_{\mathbf{x},\mathbf{n}} : n < \omega\}$
(β) $\mathscr{F}_{\mathbf{x},n} = \{f : f$ is an elementary embedding of $M_{\mathbf{x},n}$ into $\mathscr{A}\}$
(γ) $n(f) = k$ if and only if $f \in \mathscr{F}_{\mathbf{x},k}$
(δ) $\mathrm{rank}(f) = \cup\{\epsilon + 1$: for every $\alpha < \lambda$ there is $g \in \mathscr{F}_{\mathbf{x},n(f)}$ extending f, such that $\beta = \mathrm{rank}_{\mathbf{x}}(g)$ and $\mathrm{Rang}(g) \cap \alpha = \mathrm{Rang}(f) \cap \lambda\}$.

Now

Case 1: for no $\mathbf{x} \in \mathbf{W}^*$ and $f \in \mathscr{F}_{\mathbf{x},0}$ do we have $\mathrm{rank}_{\mathbf{x}}(f) = \infty$

For every $\mathbf{x} \in \mathbf{W}^*$ and $f \in \mathscr{F}_{\mathbf{x}}$ let $\beta(f, \mathbf{x})$ be the first ordinal $\alpha < \lambda$ such that if $\mathrm{rank}_{\mathbf{x}}(f) = \epsilon$ then there is no $g \in \mathscr{F}_{\mathbf{x},n(f)+1}$ extending f with $\mathrm{rank}_{\mathbf{x}}(g) = \epsilon$ and $\mathrm{Rang}(g) \cap \alpha = \mathrm{Rang}(f) \cap \lambda$.

Next let $\langle \mathscr{A}_i : i < \lambda \rangle$ be an increasing continuous sequence of elementary submodels of $\mathscr{A}$, each of cardinality $< \lambda$ such that $\langle \mathscr{A}_j : j \leq i \rangle \in \mathscr{A}_{i+1}$.

Easily the set $E = \{i < \lambda : \mathscr{A}_i \cap \lambda = i > \mu\}$ is a club of λ.

Choose by induction on $n < \omega$ an ordinal i_n increasing with n such that $i_n \in E$ is of cofinality κ, possible as $2^{<\kappa} < \lambda$ as $\kappa < \chi(*)$ and $\alpha < \lambda \rightarrow |\alpha|^{<\chi(*)} < \lambda$ hence $\mathscr{A}_{i_n}$ is an elementary submodel of $\mathscr{A}$ of cardinality $< \lambda$.

Choose $M \prec \mathscr{A}$ of cardinality χ, including $\{i_n : n < \omega\}$ such that every $u \subseteq M$ of cardinality $< \kappa$ belongs to M.

Note that, if $u \subseteq \mathscr{A}_{i_n}$ has cardinality $< \kappa$ then $u \in \mathscr{A}_{i_n}$ because $i_n \in E$ and $\mathrm{cf}(i_n) = \kappa$.

Let M^*_n be $\mathscr{A} \restriction (\mathscr{A}_{i_n} \cap M)$, easily $M^*_n \in \mathscr{A}_{i_n}$, so $[u \subseteq M^*_n \wedge |u| < \kappa \Rightarrow u \in M^*_n]$. We can find $\mathbf{x} \in \mathbf{W}$, and isomorphism f_n from $M_{\mathbf{x},n}$ onto M^*_n increasing with n. Now clearly $\mathbf{x} \in \mathscr{A}_{i_n}$, (why? as $s\mathbf{W}^* \in \mathscr{A}_{i_n}$ and $|\mathbf{W}^*| \leq \mu$ and $\mu + 1 \subseteq \mathscr{A}_{i_n}$). Also $f_n \in \mathscr{F}_{\mathbf{x},n}$ and $f_n \in \mathscr{A}_{i_n}$, (as $M^*_n, M_{\mathbf{x},n} \in \mathscr{A}_{i_n}$) and the uniqueness of f_n as those models expand a submodel of $(\mathscr{H}_{<\chi(*)}(\lambda), \in, <)$ and necessarily are transitive over the ordinals). Similarly by the choice of $\mathbf{x}$, we have $f_n \subseteq f_{n+1}$. So $\langle \mathrm{rank}_{\mathbf{x}}(f_n) : n < \omega \rangle$ is constantly ∞ as otherwise we get an infinite decreasing sequence of ordinals.

But this contradict our case assumption.

Case 2: Not case 1

So we choose $\mathbf{x} \in \mathbf{W}^*$ and $f \in \mathscr{F}_{\mathbf{x},0}$ such that $\mathrm{rank}_{\mathbf{x}}(f) = \infty$.

We easily get the desired contradiction and even a Δ-system tree of models. How? Let $\langle \eta_\alpha : \alpha < \lambda \rangle$ list ${}^{\omega>}\lambda$ such that $\eta_\alpha \triangleleft \eta_\beta$ implies $\alpha < \beta$.

Now we choose a pair $(f_{\eta_\alpha}, \gamma_\alpha)$ by induction on $\alpha < \lambda$ such that

- (i) $f_{\eta_\alpha} \in \mathscr{F}_{\mathbf{x},\ell g(\eta_\alpha)}$
- (ii) $\gamma_\alpha = \sup \cup \{\lambda \cap \mathrm{Rang}(f_{\eta_\beta}) : \beta < \alpha\}$
- (iii) if $\eta_\beta \triangleleft \eta_\alpha$ and $\ell g(\eta_\alpha) = (\ell g(\eta_\beta) + 1$ then $\gamma_\alpha \cap \mathrm{Rang}(f_{\eta_\alpha}) = \lambda \cap \mathrm{Rang}(f_{\eta_\beta})$.

There is no problem to carry the induction. This finishes the proof. $\square_{3.25}$

Lemma 3.26. *1) In 3.24 if in addition $\lambda = \mu^+$ then we can add:*

- $(c3)$ *if $\dot\zeta(\alpha) = \dot\zeta(\beta)$, then $|M^\alpha_\omega| \cap |M^\beta_\omega| \cap \lambda$ is an initial segment of $|M^\alpha_\omega| \cap \lambda$ and of $|M^\beta_\omega| \cap \lambda$, so when $\alpha \neq \beta$ it is a bounded subset of $\dot\zeta(\alpha)$.*

2) In 3.24 (and 3.26), when $\kappa > \aleph_0$ then it follows that:

- $(c4)^*$ *if $\alpha \neq \beta$ and $\{\eta^\alpha \restriction n : n < \omega\} \subseteq M^\beta_\omega$ then $\bar M^\alpha, \bar\eta^\alpha \in M^\beta_\omega$.*

3) Assume $\lambda = \mu^+$ and $\mu = \mu^\kappa$ and $S \subseteq \{\delta : \delta < \lambda, \mathrm{cf}(\delta) = \aleph_0\}$ is a stationary subset of λ and $\langle C_\delta : \delta \in S \rangle$ guess clubs (and C_δ is an unbounded subset of δ of order type ω, of course).

Then we can find $\langle \bar N_\eta : \eta \in \Gamma \rangle$ such that:

- (a) *$\Gamma = \cup \{\Gamma_\delta : \delta \in S\}$ where $\Gamma_\delta \subseteq \{\eta : \eta$ in an increasing ω-sequence of ordinals $< \delta$ with limit $\delta\}$ and $\delta(\eta) = \delta$ when $\eta \in \Gamma_\delta, \delta \in S$*
- (b) *$\bar N_\eta$ is $\langle N_{\eta,n} : n \leq \omega \rangle$ in $\prec$-increasing, and we let $N_\eta = N_{\eta,\omega}$*

(c) *each* N_η *is a model of cardinality* κ *with vocabulary* $\subseteq \mathscr{H}(\kappa^+)$ *for notational simplicity, and universe* $\subseteq \delta := \delta(\eta)$ *and* $N_{\eta,n} = N_\eta \restriction \gamma^\delta_n$ *where* γ^δ_n *is the* n*-the member of* C_δ *and* $N_\eta \cap (\gamma^\delta_n, \gamma^\delta_{n+1}) \neq \emptyset$

(d) *for every distinct* $\eta, \nu \in \Gamma_\delta$ *where* $\delta \in S$*, for some* $n < \omega$ *we have* $N_\eta \cap N_\nu = N_{\eta,n} = N_{\nu,n}$

(e) *for every* $\eta, \nu \in \Gamma_\delta$ *the models* N_η, N_ν *are isomorphic, moreover there is such isomorphism* f *which preserve the order of the ordinals and maps* $N_{\eta,n}$ *onto* $N_{\nu,n}$

(f) *if* $\mathscr{A}$ *is a model with universe* λ *and vocabulary* $\subseteq \mathscr{H}(\kappa^+)$ *then for stationarily many* $\delta \in S$ *for some* $\eta \in \Gamma_\delta \subseteq \Gamma$ *we have* $N_\eta \prec \mathscr{A}$*. Moreover, if* $\kappa^\partial = \kappa$ *and* h *is a one to one function from* ${}^\partial\lambda$ *into* λ *then we can add: if* $\rho \in {}^\partial(N_{\eta,n})$ *then* $h(\rho) \in N_{\eta,n}$*.*

Proof. 1) Let g^0, g^1 be two place functions from $\lambda \times \lambda$ to λ such that for $\alpha \in [\mu, \lambda]$: $\langle g^0(\alpha, i) : i < \mu \rangle$ enumerate $\{j : j < \mu\}$ without repetitions, and $g^1(\alpha, g^0(\alpha, i)) = i$ for $i < \lambda$.

Now we can restrict ourselves to $\bar{M}^\alpha$ such that each M^α_i (for $i \leq \omega$) is closed under g^0, g^1. Then **(c3)** follows immediately from

$$[\dot{\zeta}(\alpha) = \dot{\zeta}(\beta) \Rightarrow |M^\alpha_\omega| \cap \mu = |M^\beta_\omega| \cap \mu]$$

(required in **(c1)**).
2) Should be clear.
3) This just rephrase what we have proved above. $\square_{3.26}$

Lemma 3.27. *Suppose that* $\lambda = \mu^+$, $\mu = \kappa^{\aleph_0} = 2^\kappa > 2^{\aleph_0}$, $\mathrm{cf}(\kappa) = \aleph_0$ *and* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ *is stationary,* $\theta = \aleph_0$, $\aleph_0 < \chi(*) = \mathrm{cf}(\chi(*)) < \kappa$. *Then we can find* $\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ *and functions*

$$\dot{\zeta} : \alpha(*) \longrightarrow S, \qquad h : \alpha(*) \longrightarrow \lambda$$

and $\langle C_\delta : \delta \in S \rangle$ *with* $\langle \gamma^\delta_n : n < \omega \rangle$ *listing* C_δ *in increasing order such that:*

$(a0) - (a1)$ *as in 3.12,*
$(a2)^*$ *as in 3.24,*
$(b0) - (b2)$ *as in 3.12 and even*
$(b1)^*$ $\alpha \neq \beta, \{\eta^\alpha \restriction n : n < \omega\} \subseteq M^\beta_\omega$ *implies* $\alpha < \beta$ *and even* $\dot{\zeta}(\alpha) < \dot{\zeta}(\beta)$,
$(c1) - (c3)$ *as in 3.22 + 3.26(1),*
$(c4)$ *if* $\dot{\zeta}(\alpha) = \dot{\zeta}(\beta) = \delta$ *but* $\alpha \neq \beta$ *then for some* $n_0 \geq 1$*, there are no* $n > n_0$ *and* $\alpha_1 \leq \beta_2 \leq \alpha_3$ *satisfying:*

$$\begin{aligned} \alpha_1 &\in |M^\alpha_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}), \\ \beta_2 &\in |M^\beta_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}), \\ \alpha_3 &\in |M^\alpha_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}), \end{aligned}$$

i.e., either $\sup([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|) < \min([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|)$
or $\sup([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|) < \min([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|)$*;*

$(c5)$ *if* $\Upsilon < \kappa$ *and there is* $B \subseteq {}^\omega\kappa$, $|B| = \kappa^{\aleph_0}$ *which contains no perfect set with density* Υ *(holds trivially if* κ *is strong limit), then also* $\{\eta^\alpha : \alpha < \alpha(*)\}$ *does not contain such a set. (See 3.28).*

Proof. We repeat the proof of 3.22 with some changes.

Let $\langle S_{\alpha,\beta,\gamma} : \alpha < \mu,\ \beta < \lambda,\ \gamma < \lambda\rangle$ be pairwise disjoint stationary subsets of S. Let g^0, g^1 be as in the proof of 3.26. By 3.7 there is a sequence $\langle C_\delta : \delta \in S\rangle$ such that:

(*i*) C_δ is a club of δ of order type κ, not $\omega!$, $0 \notin C_\delta$,

(*ii*) for $\alpha < \mu$, $\beta < \lambda$, $\gamma < \lambda$, for every club C of λ, the set

$$\{\delta \in S_{\alpha,\beta,\gamma} : C_\delta \subseteq C\}$$

is stationary.

We then define $\mathbf{W}^*, (\delta^j, \langle M_{j,n} : n < \omega\rangle, \eta_j, C^j)$ for $j < \mu$,A_α for $\alpha < \lambda$, and R as in the proof of 3.22.

Now, for $\delta \in S_{\alpha,\beta,\gamma}$ let $\mathbf{W}^1_\delta$ be the collection of all systems $\langle M_\rho, \eta_\rho : \rho \in {}^{\omega>}\kappa\rangle$ such that:

(*i*) η_ρ is an increasing sequence of ordinals of length $\lg(\rho)$,

(*ii*) $\mathrm{otp}\,(C_\delta \cap \eta_\rho(\ell)) = 1 + \rho(\ell)$ for $\ell < \lg(\rho)$,

(*iii*) there are isomorphisms $\langle h_\rho : \rho \in {}^{\omega>}\kappa\rangle$ such that h_ρ maps M_ρ onto $M_{\alpha,\lg(\rho)}$ preserving $\in, R$, $\mathrm{cd}(x) = y$, $g^0(x_1, x_2) = y$, $g^1(x_1, x_2) = y$ (and their negations),

(*iv*) if $\rho \triangleleft \nu$ then $h_\rho \subseteq h_\nu$, $M_\rho \prec M_\sigma$, $M_\rho \in M_\nu$,

(*v*) $M_\rho \cap C_\delta = \emptyset$, and $M_\rho \cap \lambda \subseteq \bigcup\limits_\ell [\gamma_{\rho(\ell)}, \gamma_{\rho(\ell)+1})$, where γ_ζ is the ζ-th member of C_δ,

(*vi*) if $\rho \in {}^{\omega>}\kappa$, $\ell < \lg(\rho)$, γ is the $(1 + \rho(\ell))$-th member of C_δ then $M_\ell \cap \gamma$ depends only on $\rho{\restriction}\ell$, and $M_\rho{\restriction}\gamma \prec M_\rho$,

(*vii*) $N_\beta = M_{\langle\rangle}$.

Now clearly $|\mathbf{W}^1_\delta| \le \mu$, so let $\mathbf{W}^1_\delta = \{\langle (M^j_\rho, \eta^j_\rho) : \rho \in {}^{\omega>}\kappa\rangle : j < \mu\}$. Let $\langle \rho_j : j < \mu\rangle$ be a list of distinct members of ${}^\omega\kappa$; for **(c5)** — choose as there.

Let

$$M^j_\ell = \bigcup_{\ell<\omega} M^j_{\rho_j\restriction\ell}, \quad \eta^j = \langle \eta^j_{\rho_j\restriction(\ell+1)}(\ell+1) : \ell \le \omega\rangle.$$

Now,

$$\{\langle M^j_\ell : \ell < \omega\rangle : j < \mu\}$$

is as required in **(c4)**. Also **(c5)** is straightforward (as taking union for all δ's change little), (of course, we are omitting δ's where we get unreasonable pairs).

The rest is as before. $\square_{3.27}$

Remark 3.28. The existence of B as in **(c5)** is proved, for some Υ for all strong limit κ of cofinality $\aleph_0$ in [She94b, Ch II,6.9,pg.104], really stronger conclusions hold. If 2^κ is regular and belongs to $\{\mathrm{cf}(\Pi\kappa_n/D) : D$ an ultrafilter on $\omega, \kappa_n < \kappa\}$ or 2^κ is singular and is the supremum of this set, then it exists for $\Upsilon = (2^{\aleph_0})^+$. Now, if above we replace D by the filter of co-bounded subsets of ω, then we get it even for $\Upsilon = \aleph_0$; by [Shec, Part D] the requirement holds, e.g., for $\beth_\delta$ for a club of $\delta < \omega_1$.

Moreover, under this assumption on κ we can demand (essentially, this is expanded in 3.33)

$(c4)^*$ we strengthen clause (c4) to: if $\dot\zeta(\alpha) = \dot\zeta(\beta) = \delta$ but $\alpha \neq \beta$ then for some $n_0 \geq 1$, we have either for every $n \in [n_1, \omega)$ we have $\sup([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|) < \min([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|)$ or for every $n \in [n_1, \omega)$ we have $\sup([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega| < \min([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|)$.

Lemma 3.29. *We can combine 3.27 with 3.24.*

Proof. Left to the reader. $\square_{3.29}$

Lemma 3.30. *Suppose* $\aleph_0 = \theta < \chi(*) = \mathrm{cf}(\chi(*))$ *and:* $\lambda^{\aleph_0} = \lambda^{<\chi(*)}$, $\chi(*) \leq \lambda$ *and:* $\lambda = \lambda_1^+$, *and* $(*)_{\lambda_1}$ *(see below) holds.*
Then

$(*)_\lambda$ *we can find* $\mathbf{W} = \{(\bar M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ *and functions* $\dot\zeta : \alpha(*) \longrightarrow S$ *and* $h : \alpha(*) \longrightarrow \lambda$ *such that:*

$(a0) - (a2)$ *as in 3.12,*

$(b0) - (b2)$ *as in 3.12, and even*

$(c3)$ *if* $\dot\zeta(\alpha) = \dot\zeta(\beta)$ *then* $|M_\alpha| \cap |M_\beta|$ *is a bounded subset of* $\dot\zeta(\alpha)$.

Proof. Left to the reader. $\square_{3.30}$

Lemma 3.31. *Suppose that* λ *is a strongly inaccessible uncountable cardinal,*

$$\mathrm{cf}(\lambda) \geq \chi(*) = \mathrm{cf}(\chi(*)) > \theta = \aleph_0,$$

and let $S \subseteq \lambda$ *consist of strong limit singular cardinals of cofinality* $\aleph_0$ *and be stationary.* Then *we can find* $\mathbf{W} = \{(\bar M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ *and functions* $\dot\zeta : \alpha(*) \longrightarrow S$ *and* $h : \alpha(*) \longrightarrow \lambda$ *such that:*

$(a0) - (a2)$ *of 3.12 (except that* $h(\alpha)$ *depends not only on* $\dot\zeta(\alpha)$*),*

$(b0), (b3)$ *of 3.12,*

$(b1)^+$ *of 3.20,*

$(c3)^-$ *if* $\dot\zeta(\alpha) = \delta = \dot\zeta(\beta)$ *then* $|M^\alpha_\omega| \cap |M^\beta_\omega| \cap \delta$ *is a bounded subset of* δ.

Remark 3.32. 1) See [She75b] for essentially a use of a weaker version.
2) We can generalize 3.24.

Proof. See the proof of [Shea, 1.10(3)] but there $\sup(N_{\langle\rangle} \cap \lambda) < \delta$. $\square_{3.31}$

Lemma 3.33. *1) Suppose that* $\lambda = \mu^+$, $\mu = \kappa^\theta = 2^\kappa$, $\theta < \mathrm{cf}(\chi(*)) = \chi(*) < \kappa$, κ *is strong limit,* $\kappa > \mathrm{cf}(\kappa) = \theta > \aleph_0$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ *is stationary.*
Then *we can find* $\mathbf{W} = \{(\bar M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ *(actually, a sequence), functions* $\dot\zeta : \alpha(*) \longrightarrow S$ *and* $h : \alpha(*) \longrightarrow \lambda$ *and* $\langle C_\delta : \delta \in S\rangle$ *such that:*

$(a1) - (a2)$ *as in 3.12,*

$(b0)$ $\eta^\alpha \neq \eta^\beta$ *for* $\alpha \neq \beta$,

$(b1)$ *if* $\{\eta^\alpha{\restriction} i : i < \theta\} \subseteq M^\beta_\theta$ *and* $\alpha \neq \beta$ *then* $\alpha < \beta$ *and even* $\dot\zeta(\alpha) < \dot\zeta(\beta)$,

$(b2)$ *if* $\eta^\alpha{\restriction}(j+1) \in M^\beta_\theta$ *then* $M^\alpha_j \in M^\beta_\theta$, (c2) *$\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a club of δ of order type θ, and every club of λ contains C_δ for stationarily many $\delta \in S$,*

(c3) *if $\delta \in S$, $C_\delta = \{\gamma_{\delta,i} : i < \theta\}$ is the increasing enumeration, $\alpha < \alpha^*$ satisfies $\dot\zeta(\alpha) = \delta$, then there is $\langle\langle\gamma^-_{\alpha,i}, \gamma^+_{\alpha,i}\rangle : i < \theta \mathrm{odd}\rangle$ such that $\gamma^-_{\alpha,i} \in M^\alpha_i$, $M^\alpha_i \cap \lambda \subseteq \gamma^+_{\alpha,i}$, $\gamma_{\delta,i} < \gamma^-_{\alpha,i} < \gamma^+_{\alpha,i} < \gamma_{\delta,i+1}$ and*

  $(*)$ *if $\dot\zeta(\alpha) = \dot\zeta(\beta), \alpha < \beta$ then for every large enough odd $i < \theta$, $\gamma^+_{\alpha,i} < \gamma^-_{\beta,i}$ (hence $[\gamma^-_{\alpha,i}, \gamma^+_{\alpha,i}) \cap [\gamma^-_{\beta,i}, \gamma^+_{\beta,i}) = \emptyset$) and $[\gamma^-_{\beta,i}, \gamma^+_{\beta,i}) \cap M^\alpha_\theta = \emptyset$.*

2) In part (1), assume $\theta = \aleph_0$ and $\mathrm{pp}(\kappa) =^+ 2^\kappa$. Then the conclusion holds; moreover, **(c3)** *(from 3.26).*

Remark 3.34. The assumption $\mathrm{pp}(\kappa) = 2^\kappa$ holds, for example, for $\kappa = \beth_\delta$ for a club of $\delta < \omega$ (see [**?**, §5]).

Proof. 1) By 3.6 we can find $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a club of δ, of order type κ such that for any club C of λ for stationarily many $\delta \in S$, we have: $C_\delta \subseteq C$.

First Case: assume $\mu(= 2^\kappa)$ is regular.

By [She94b, Ch.II,5.9], we can find an increasing sequence $\langle\kappa_i : i < \theta\rangle$ of regular cardinals $> \chi(*)$ such that $\kappa = \sum_{i<\theta} \kappa_i$, and $\prod_{i<\theta} \kappa_i / J^{\mathrm{bd}}_\theta$ has true cofinality μ, and let $\langle f_\epsilon : \epsilon < \mu\rangle$ exemplify this, which means:

$$\epsilon < \zeta < \mu \quad \Rightarrow \quad f_\epsilon < f_\zeta \mod J^{\mathrm{bd}}_\theta,$$

and for every $f \in \prod_{i<\theta} \kappa_i$, for some $\epsilon < \mu$ we have $f < f_\epsilon \mod J^{\mathrm{bd}}_\theta$. We may assume that if ϵ is limit and $\bar{f}\restriction\epsilon$ has $<_{J^{\mathrm{bd}}_\theta}$–l.u.b., then f_ϵ is a $<_{J^{\mathrm{bd}}_\theta}$–l.u.b., and we know that if $\mathrm{cf}(\epsilon) > 2^\theta$ then this holds, and that without loss of generality $\bigwedge_{i<\theta} \mathrm{cf}(f_\epsilon(i)) = \mathrm{cf}(\epsilon)$. Without loss of generality $\kappa_i > f_\epsilon(i) > \bigcup_{j<i} \kappa_j$.

We shall define W later. Let St be a strategy for player I. By the choice of $\bar{C}$, for some $\delta \in S$, for every $\alpha \in C_\delta$ of cofinality $> \theta$, $\mathscr{H}_{<\chi(*)}(\alpha)$ is closed under the strategy St. Let $C_\delta = \{\alpha_i : i < \kappa\}$ be increasing continuous. For each $\epsilon < \mu$ we choose a play of the game, player I using St, $\langle M^\epsilon_j, \eta^\epsilon_j : j < \theta\rangle$ such that:

$$\begin{aligned}&\langle M^\epsilon_j : j \le j_1\rangle \in \mathscr{H}_{<\chi(*)}(\alpha_{f_\epsilon(j_1)+1}),\\ &\eta^\epsilon_\gamma = \langle \mathrm{cd}(\alpha_{f_\epsilon(i)}, \langle M^\epsilon_i : i \le j\rangle) : j < \gamma\rangle, \qquad \text{and}\\ &\eta^\epsilon_{j+1} \in M^\epsilon_{j+1}.\end{aligned}$$

Then let $g_\epsilon \in \prod_{i<\theta} \kappa_i$ be:

$$g_\epsilon(i) = \sup(\kappa_i \cap \bigcup_{j<\theta} M^\epsilon_j),$$

so for some $\beta_\epsilon \in (\epsilon, \mu)$, we have $g_\epsilon < f_{\beta_\epsilon} \mod J^{\mathrm{bd}}_\theta$.

On the other hand, if $\mathrm{cf}(\epsilon) = (2^\theta)^+$, without loss of generality, $\mathrm{cf}\big(f_\epsilon(i)\big) = \mathrm{cf}(\epsilon)$ for every $i < \theta$ (see [She94b, Ch.II,§1]), so there is $\gamma_\epsilon < \epsilon$ such that

$$h_\epsilon < f_{\gamma_\epsilon} \mod J^{\mathrm{bd}}_\theta \qquad \text{where } h_\epsilon(i) = \sup(f_\epsilon(i) \cap \bigcup_{j<\theta} M^\epsilon_j).$$

So for some $\gamma(*) < \mu$ we have:

$$S_\delta[\mathrm{St}] = \{\epsilon < \mu : \mathrm{cf}(\epsilon) = (2^\theta)^+, \text{ and } \gamma_\epsilon = \gamma(*)\} \text{ is stationary.}$$

Now, for each $\delta \in S$ we can consider the set $\mathbf{C}_\delta$ of all possible such $\langle \bar{M}^\epsilon, \eta^\epsilon : \epsilon < \mu\rangle$, where $\bar{M}^\epsilon = \langle M^\epsilon_j : j < i\rangle$, η^ϵ_θ are as above (letting St vary on all strategies of player I for which $\alpha \in C_\delta$ & $\mathrm{cf}(\alpha) > \theta \quad \Rightarrow \quad \mathscr{H}_{<\chi(*)}(\alpha)$ is closed under St).

A better way to write the members of $\mathbf{C}_\delta$ is $\langle\langle \bar{M}^\epsilon_j, \eta^\epsilon_j : j < \theta\rangle : \epsilon < \mu\rangle$, but for $j < \theta, f_{\epsilon(1)}\restriction j = f_{\epsilon(2)}\restriction j \quad \Rightarrow \quad \bar{M}^{\epsilon(1)}_j = M^{\epsilon(2)}_j$ & $\eta^{\epsilon(1)}_j = \eta^{\epsilon(2)}_j$; actually it is a function from $\{f_\epsilon \restriction j : \epsilon < \mu, j < \theta\}$ to $\mathscr{H}_{<\chi(*)}(\delta)$. But the domain has power κ, the range has power $|\delta| \le \mu$. So $|\mathbf{C}_\delta| \le \mu^\kappa = (2^\kappa)^\kappa = 2^\kappa = \mu$.

So we can well order $\mathbf{C}_\delta$ in a sequence of length μ, and choose by induction on $\epsilon < \mu$ a representative of each for $\mathbf{W}$ satisfying the requirements.

<u>Second case</u>: assume μ is singular.

So let $\mu = \sum\limits_{\xi < \mathrm{cf}(\mu)} \mu_\xi$, μ_ξ regular, without loss of generality $\mu_\xi > (\sum\{\mu_\epsilon : \epsilon < \xi\})^+ + (\mathrm{cf}(\mu))^+$. We know that $\mathrm{cf}(\mu) > \kappa$, and again by [She94b, Ch.VIII,§1] there are $\langle \kappa_{\xi,i} : i < \theta\rangle$, $\langle \kappa_i : i < \theta\rangle$ such that:

$$\mathrm{tcf}(\prod_{i<\theta} \kappa_{\xi,i}/J^{\mathrm{bd}}_\theta) = \mu_\xi, \qquad \mathrm{tcf}(\prod_{i<\theta} \kappa_i/J^{\mathrm{bd}}_\theta) = \mathrm{cf}(\mu),$$

$$\kappa^a_i < \kappa_{\xi i} < \kappa^b_i, \qquad \kappa^a_i < \kappa_i < \kappa^b_i \qquad \text{and} \qquad i < j \quad \Rightarrow \quad \kappa^b_i < \kappa^a_j$$

(we can even get $\kappa^a_i > \prod\limits_{j<i} \kappa^b_j$ as we can uniformize on ξ).

Let $\langle f^\xi_\epsilon : \epsilon < \mu_\xi\rangle$, $\langle f_\epsilon : \epsilon < \mathrm{cf}(\mu)\rangle$ witness the true cofinalities. Now, for every $f \in \prod\limits_{i<\theta} \kappa_i$ (for simplicity such that $f(i) > \sum\limits_{j<i} \kappa_j$, $\bigwedge\limits_i \mathrm{cf}(f(i)) = (2^\theta)^+$) and ξ we can repeat the previous argument for $\langle f + f^\xi_\epsilon : \epsilon < \mu_\epsilon\rangle$. After "cleaning inside", replacing by a subset of power μ_ξ we find a common bound below $\prod\limits_{i<\theta} \kappa_i$ and below $\prod f$, and we can uniformize on ξ.

Thus we apply on every f_ϵ, $\mathrm{cf}(\epsilon) = (2^\theta)^+$ and use the same argument on the bound we have just gotten.

2) Should be clear. $\square_{3.33}$

Similarly to 3.22 with ω^2 for θ, (not a cardinal!) we have

Claim 3.35. *Suppose that*

(∗) *λ is a regular cardinal, $\theta = \aleph_0$, $\mu = \mu^{<\chi(*)} < \lambda \le 2^\mu$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ is stationary and $\aleph_0 < \chi(*) = \mathrm{cf}(\chi(*))$.*

<u>Then</u> we can find

$$\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$$

and functions

$$\dot{\zeta} : \alpha(*) \longrightarrow S \ \text{ and } \ h : \alpha(*) \longrightarrow \lambda$$

such that:

(a0) *like 3.12,*

(a1) $\bar{M}^\alpha = \langle M^\alpha_i : i \le \omega^2\rangle$ *is an increasing continuous elementary chain* ($\tau(M^\alpha_i)$, *the vocabulary, may be increasing too and belongs to* $\mathscr{H}_{<\chi(*)}(\chi(*))$*, each* M^α_i *is a model belonging to* $\mathscr{H}_{<\chi(*)}(\lambda)$ *[so necessarily has cardinality* $< \chi(*)$*],* $M^\alpha_i \cap \chi(*)$ *is an ordinal,* $[\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha_i]$, $\eta^\alpha \in {}^{\omega^2}\lambda$ *is increasing with limit* $\dot\zeta(\alpha) \in S$, $\eta^\alpha \restriction i \in M^\alpha_{i+1}$, M^α_i *belongs to* $\mathscr{H}_{<\chi(*)}(\eta^\alpha(i))$ *and* $\langle M^\alpha_i : i \le j\rangle$ *belongs to* M^α_{j+1},

(a2) *like 3.12 (with* ω^2 *instead* θ*),*

(b0), (b1), (b2) *as in 3.12,*

(b1)* *as in 3.22,*

(c1) *if* $\dot\zeta(\alpha) = \dot\zeta(\beta)$ *then* $M^\alpha_{\omega^2} \cap \mu = M^\beta_{\omega^2} \cap \mu$ *and there is an isomorphism* $h_{\alpha,\beta}$ *from* $M^\alpha_{\omega^2}$ *onto* $M^\beta_{\omega^2}$ *mapping* $\eta^\alpha(i)$ *to* $\eta^\beta(i), M^\alpha_i$ *to* M^β_i *for* $i < \omega^2, h_{\alpha,\beta}\restriction(|M^\alpha_{\omega^2}| \cap |M^\beta_{\omega^2}|)$ *is the identity,*

(c2) *as in 3.22 using* $\langle M^\alpha_{\omega n} : n < \omega\rangle$,

(c3) *as in 3.26 assuming* $\lambda = \mu^+$,

(c4) $\eta^\alpha(i) > \sup(|M^\alpha_i| \cap \lambda)$ *(so* $\sup(|M^\alpha_{\omega(n+1)}| \cap \lambda) = \bigcup_\ell \eta^\alpha(\omega n + \ell)$*).*

Proof. We use $\langle \bar{M}^{\alpha,0} : \alpha < \alpha(*)\rangle$ which we got in 3.22. Now for each α we look at $\bigcup_{n<\omega} M^{\alpha,0}_n$ as an elementary submodel of $(\mathscr{H}_{<\chi(*)}(\lambda), \in)$ with a function St (intended as strategy for player I, in the play for (a2) above).

Play in $\bigcup_{n<\omega} M^{\alpha,0}_n$ and get

$$\begin{gathered}\langle M^\alpha_i, \eta^\alpha(i) : i < \omega n\rangle \in M^{\alpha,0}_n,\\ \sup\{\eta^\alpha(i) : i < \omega n\} \in M^{\alpha,0}_{n+1},\\ \eta^\alpha(\omega n) > \sup(M^{\alpha,0}_n \cap \lambda).\end{gathered}$$

$\square_{3.35}$

Lemma 3.36. *Assume that* $\lambda \ge \chi(*) > \theta$ *are regular cardinals,* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ *is a stationary set,* $\lambda^{<\chi(*)} = \lambda$*, and the conclusion of 3.33 holds for them.* Then *it holds also for* λ^+ *instead of* λ.

Proof. By [Shea, 2.10(2)] or see [She94d], we know

(*) there are $\langle C_\delta : \delta < \lambda^+$ and $\mathrm{cf}(\delta) = \theta\rangle, \langle e_\alpha : \alpha < \lambda^+\rangle$ such that:

(i) C_δ is a club of δ of order type θ, $\alpha \in C_\delta$ & $\alpha > \sup(C_\delta \cap \alpha) \Rightarrow \mathrm{cf}(\alpha) = \lambda$,

(ii) e_α is a club of α of order type $\mathrm{cf}(\alpha)$, $e_\alpha = \{\beta^\alpha_i : i < \mathrm{cf}(\alpha)\}$ (increasing continuous),

(iii) if E is a club of λ^+ then for stationarily many $\delta < \lambda^+$, $\mathrm{cf}(\delta) = \theta$, $C_\delta \subseteq E$ and the set

$$\{i < \lambda : \text{ for every } \alpha \in C_\delta,\ \mathrm{cf}(\alpha) = \lambda \ \Rightarrow\ \beta^\alpha_{i+1} \in E\}$$

is unbounded in λ.

Now copying the black box of λ on each $\delta < \lambda^+$, $\mathrm{cf}(\delta) = \theta$, we can finish easily.

$\square_{3.36}$

Lemma 3.37. *If λ, μ, κ, θ, $\chi(*)$, S are as in 3.33, and*

$$\alpha < \chi(*) \quad \Rightarrow \quad |\alpha|^\theta < \chi(*)$$

then there is a stationary $S^* \subseteq \{A \subseteq \lambda : |A| < \chi(*)\}$ and a one-to-one function cd *from S^* to λ such that:*

$$A \in S^* \ \&\ B \in S^* \ \&\ A \neq B \ \&\ A \subset B \quad \Rightarrow \quad \mathrm{cd}(A) \in B.$$

Remark 3.38. This gives another positive instance to a problem of Zwicker. (See [She86a].)

Proof. Similar to the proof of 3.33 only choose

$$\mathrm{cd} : \{A : A \subseteq \lambda \text{ and } |A| < \chi(*)\} \longrightarrow \lambda$$

one-to-one, and then define

$$S^* \cap \{A : A \subseteq \alpha, |A| < \chi(*)\}$$

by induction on α. $\square_{3.37}$

Problem 3.39. 1) Can we prove in ZFC that for some regular $\lambda > \theta$

$(*)_{\lambda,\theta,\chi(*)}$ we can define for $\alpha \in S^\lambda_\theta = \{\delta < \lambda : \aleph_0 \leq \mathrm{cf}(\delta) = \theta\}$ a model M_α with a countable vocabulary and universe an unbounded subset of α of power $< \chi(*)$, $M_\delta \cap \chi(*)$ is an ordinal such that: for every model M with countable vocabulary and universe λ, for some (equivalently: stationarily many) $\delta \in S^\lambda_\kappa$, $M_\delta \subseteq M$.

2) The same dealing with relational vocabularies only (we call it $(*)^{\mathrm{rel}}_{\lambda,\theta,\kappa}$).

Remark 3.40. Note that by 3.8 if $(*)_{\lambda,\theta,\kappa}$, $\mu = \mathrm{cf}(\mu) > \lambda$ then $(*)_{\mu^+,\theta,\kappa}$.

* * *

Now (3.41—3.45) we return to black boxes for singular λ, i.e., we deal with the case $\mathrm{cf}(\lambda) \leq \theta$.

Lemma 3.41. *Suppose that $\lambda^\theta = \lambda^{<\chi(*)}$, λ is a singular cardinal, θ is regular, and $\chi(*)$ is regular $> \theta$.*

Assume further

- (α) $\mathrm{cf}(\lambda) \leq \theta$,
- (β) $\lambda = \sum_{i \in w} \mu_i$, $|w| \leq \theta$, $w \subseteq \theta^+$ *(usually $w = \mathrm{cf}(\lambda)$) and $[i < j \ \Rightarrow\ \mu_i < \mu_j]$, and each μ_i is regular $< \lambda$ and*
$$\mathrm{cf}(\lambda) > \aleph_0 \ \wedge\ \mathrm{cf}(\lambda) = \theta \quad \Rightarrow \quad w = \mathrm{cf}(\lambda),$$
- (γ) *$\mu > \lambda$, μ is a regular cardinal, D is a uniform filter on w (so $\{\alpha \in w : \alpha > \beta\} \in D$ for each $\beta \in w$), μ is the true cofinality of $\prod_{i \in w} (\mu_i, <)/D$ (see* [Shed, 3.6(2)=Lc18] *or* [She94b]*),*

(δ) *$\bar f = \langle f_i/D : i < \mu\rangle$ exemplifies "the true cofinality of $\prod_i(\mu_i,<)/D$ is μ", i.e.,*

$$\begin{array}{lcl} \alpha < \beta < \lambda & \Rightarrow & f_\alpha/D < f_\beta/D, \\ f \in \prod_i \mu_i & \Rightarrow & \bigvee_\alpha f/D < f_\alpha/D, \end{array}$$

(ϵ) *$S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \theta\}$ is good for $(\mu, \theta, \chi(*))$, and*

(ζ) *if $\theta > \mathrm{cf}(\lambda)$, $\delta \in S$, then for some $A_\delta \in D$ and unbounded $B_\delta \subseteq \delta$ we have*

$$\alpha \in B_\delta \ \wedge \ \beta \in B_\delta \ \wedge \ \alpha < \beta \ \wedge \ i \in A_\delta \quad \Rightarrow \quad f_\alpha(i) < f_\beta(i),$$

i.e., $\langle f_\alpha \restriction A_\delta : \alpha \in B_\delta\rangle$ is $<$–increasing.

Then we can find $\mathbf{W} = \{(\bar M^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ (pedantically a sequence) and functions $\dot\zeta$ from $\alpha(*)$ to S and h from $\alpha(*)$ to μ such that:

$(a0), (a1), (a2)$ *as in 3.12 except that we replace $(*)$ of (a1) by*

$(*)'$ *(i)* $\eta^\alpha \in {}^\theta\lambda$,

(ii) *if $i < \mathrm{cf}(\lambda)$ then $\sup(\mu_i \cap \mathrm{Rang}(\eta^\alpha)) = \sup(\mu_i \cap M^\alpha_\theta)$, and*

(iii) *if $\xi < \dot\zeta(\alpha)$ then $f_\xi/E < \langle \sup(\mu_i \cap M^\alpha_\theta) : i < \mathrm{cf}(\lambda)\rangle/E \leq f_{\dot\zeta(\alpha)}/E$,*

$(b0) - (b3)$ *as in 3.12.*

Proof. For $A \subseteq \theta$ of cardinality θ let $\mathrm{cd}^A_{\lambda,\chi(*)} : \mathscr{H}_{<\chi(*)}(\lambda) \longrightarrow {}^A\lambda$ be one-to-one, and $G : \lambda \longrightarrow \lambda$ be such that for γ divisible by $|\gamma|$, $\alpha < \gamma \leq \lambda$ ($\mu \geq \aleph_0$), the set $\{\beta < \gamma : G(\beta) = \alpha\}$ is unbounded in γ and of order type γ. Let $\bar A = \langle A_i : i < \theta\rangle$ be a sequence of pairwise disjoint subsets of θ each of cardinality θ.
Let for $\delta \in S$

$$\begin{aligned} \mathbf{W}^0_\delta = \{(\bar M, \eta) : \quad & \bar M, \eta \text{ satisfy } \mathbf{(a1)}, \text{ and for some } y \in \mathscr{H}_{<\chi(*)}(\lambda), \\ & \text{for every } i < \theta \text{ we have} \\ & \langle G(\eta(i)) : i \in A_j\rangle = \mathrm{cd}^A_{\lambda,\chi(*)}(\langle \bar M \restriction j, \eta\restriction j, y\rangle)\}, \end{aligned}$$

The rest is as before. $\square_{3.41}$

Claim 3.42. *Suppose that $\lambda^\theta = \lambda^{<\chi(*)}$, λ is singular, $\theta, \chi(*)$ are regular, $\chi(*) > \theta$.*
1) If $(\forall \alpha < \lambda)[|\alpha|^{<\chi(*)} < \lambda]$ then by $\lambda^\theta = \lambda^{<\chi(*)}$ we know that either $\mathrm{cf}(\lambda) \geq \chi(*)$ (and so lemma 3.18 applies) or $\mathrm{cf}(\lambda) \leq \theta$.
2) We can find regular μ_i ($i < \mathrm{cf}(\lambda)$) increasing with i, $\lambda = \sum_{i<\mathrm{cf}(\lambda)} \mu_i$.
3) For $\langle \mu_i : i \in w\rangle$ as in 3.41(β) we can find $D, \mu, \bar f$ as in 3.41(γ),(δ), D the co-bounded filter plus one unbounded subset of ω.
4) For $\langle \mu_i : i \in w\rangle$, $D, \mu, \bar f$ as in (β), (γ), (δ) of 3.41 we can find μ and pairwise disjoint $S \subseteq \mu$ as required in (ϵ), (δ) of 3.41 provided that $\theta > \mathrm{cf}(\lambda) \Rightarrow 2^\theta < \mu$ [equivalently $< \lambda$].
5) If $\mathrm{cf}(\lambda) > \aleph_0$, $(\forall \alpha < \lambda)[|\alpha|^{\mathrm{cf}(\lambda)} < \lambda]$, $\lambda < \mu = \mathrm{cf}(\mu) \leq \lambda^{\mathrm{cf}(\lambda)}$ then we can find $\langle \mu_i : i < \mathrm{cf}(\lambda)\rangle$, and the co-bounded filter D on $\mathrm{cf}(\lambda)$ as required in (β), (γ) of 3.31.

Proof. Now 1),2),3) are trivial, for (5) see [She90b, §9]. As for 4), we should recall [She90b, §5] actually say: $\square_{3.42}$

Fact 3.43. If $\langle \mu_i : i \in w \rangle, \bar{f}, D$ are as in 3.41, then

$$S = \{\delta < \mu : \ \mathrm{cf}(\delta) = \theta \text{ and there are } A_\delta \in D, \ \text{ and unbounded } B_\delta \subseteq \delta \\ \text{such that } [\alpha \in B_\delta \wedge \beta \in B_\delta \wedge \alpha < \beta \wedge i \in A_\delta f_\alpha(i) < f_\beta(i)]\}.$$

is good for $(\mu, \theta, \chi(*))$.

Lemma 3.44. *Let $\chi(1) = \chi(*) + (< \chi(*))^\theta$.*
In 3.41, if $\lambda^\theta = \lambda^{\chi(1)}$, we can strengthen **(b1)** *to* **(b1)**$^+$ *(of 3.20).*

Proof. Combine proofs of 3.41, 3.20. $\square_{3.44}$

Lemma 3.45. $\frac{3.17}{3.11} \times 3.29$ *and* $\frac{3.19}{3.11} \times 3.37$ *hold (we need also the parallel to 3.33).*

Proof. Left to the reader. $\square_{3.45}$

* * *

Now we draw some conclusions.
The first, 3.46, gives what we need in 2.7 (so 2.3).

Conclusion 3.46. *Suppose* $\lambda^\theta = \lambda^{<\chi(*)}$, $\mathrm{cf}(\lambda) \geq \chi(*) + \theta^+$, $\theta = \mathrm{cf}(\theta) < \chi(*) = \mathrm{cf}(\chi(*))$. *Then we can find*

$$\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}, M_i^\alpha = (N_i^\alpha, A_i^\alpha, B_i^\alpha),$$

where

$$A_i^\alpha \subseteq \lambda \cap |N_i^\alpha|, B_i^\alpha \subseteq \lambda \cap |N_i^\alpha|, A_i^\alpha \neq B_i^\alpha,$$

and functions $\dot{\zeta}, h$ such that:

- $(a0), (a1)$ *as in 3.12;*
- $(a2)$ *as in 3.12 except that in the game, player I can choose M_i, only as above;*
- $(b0), (b1), (b2)$ *as in 3.12;*
- $(b1)''$ *if $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta$ but $\alpha < \beta$ (so $\beta < \alpha + (< \chi(*))^\theta$ then*

$$A_\theta^\alpha \cap (|M_\theta^\alpha| \cap |M_\theta^\beta|) \neq B_\theta^\beta \cap (|M_\theta^\alpha| \cap |M_\theta^\beta|), \\ B_\theta^\alpha \cap (|M_\theta^\alpha| \cap |M_\theta^\beta|) \neq A_\theta^\beta \cap (|M_\theta^\alpha| \cap |M_\theta^\beta|).$$

Proof. First assume λ is regular, and $\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}, \dot{\zeta}, h$ be as in the conclusion of 3.12. Let $w = \{\mathrm{cd}(\alpha, \beta) : \alpha, \beta < \lambda\}$, and $G_1, G_2 : w \longrightarrow \lambda$ be such that for $\alpha \in E$, $\alpha = \mathrm{cd}(G_1(\alpha), G_2(\alpha))$.
Let

$$Y = \{\alpha < \alpha(*) : \ \bar{M}_i^\alpha \text{ has the form } (N_i^\alpha, A_i^\alpha, B_i^\alpha), \\ A_i^\alpha, B_i^\alpha \text{ distinct subsets of } \lambda \cap |N_i^\alpha| \text{ (equivalently, monadic relations)}, h(\alpha) \in E, \text{ and} \\ G_2(h(\alpha)) = \min\{\gamma : \gamma \in A_i^\alpha \setminus B_i^\alpha \text{ or } \gamma \in B_i^\alpha \setminus A_i^\alpha\}\}.$$

Now we let

$$\mathbf{W}^* = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha \in Y\}, \dot{\zeta}^* = \dot{\zeta} \restriction Y, h^* = G_1 \circ h.$$

They exemplify that 3.46 holds.

What if λ is singular? Still $\mathrm{cf}(\lambda) \geq \chi(*) + \theta^*$ and we can just use 3.18 instead 3.12. $\square_{3.46}$

Claim 3.47. *1) In 3.12, if $\lambda = \lambda^{<\chi(*)}$, we can let $h : S \longrightarrow \mathscr{H}_{<\chi(*)}(\lambda)$ be onto; generally we can still make* $\mathrm{Rang}(h)$ *be* $\subseteq A$, *whenever* $|A| = \lambda$.
2) In 3.12, by its proof, whenever $S' \subseteq S$ is stationary, and $\bigwedge\limits_{\zeta}(h^{-1}(\zeta) \cap S'$ stationary) then $\{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*), \dot{\zeta}(\alpha) \in S'\}$ satisfies the same conclusion.
3) For any unbounded $a \subseteq \theta$ we can let player I choose also $\eta(i)$ for $i \in \theta \setminus a$, without changing our conclusions.
4) Similar statements hold for the parallel claims.
5) It is natural to have $\chi(*) = \chi^+$.

Proof. Straightforward. $\square_{3.47}$

Fact 3.48. We can make the following changes in **(a1)**, **(a2)** of 3.12 (and in all similar lemmas here) getting equivalent statements:

- $(*)$ $M_i^\alpha \in \mathscr{H}_{<\chi(*)}(\lambda+\lambda)$; in the game, for some arbitrary $\lambda^* \geq \lambda$ (but fix during the game) player I chooses the $M_i^\alpha \in \mathscr{H}(\lambda^*)$ (of cardinality $< \chi(*)$), and in the end instead "$\bigwedge\limits_i M_i = M_i^\alpha$" we have "there is an isomorphism from M_θ onto M_θ^α taking M_i onto M_i^α and is the identity on $M_\theta \cap \mathscr{H}_{<\chi(*)}(\lambda)$ and maps $|M_\theta| \setminus \mathscr{H}(\lambda)$ into $\mathscr{H}_{<\chi(*)}(\lambda+\lambda) \setminus \mathscr{H}_{<\chi(*)}(\lambda)$ and preserves $\in$ and $\notin$ and "being an ordinal" and "not being an ordinal".

Exercise 3.49. If D is a normal fine filter on $\mathscr{P}(\mu)$, λ is regular, $\lambda \leq \mu$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary, moreover:

$(*)_{D,S}$ $\{a \in \mathscr{P}(\mu) : \sup(a \cap \lambda) \in S\} \neq \emptyset \mod D$

then we can partition S to λ stationary disjoint subsets each satisfying $(*)$.

[Hint: like the proof of 3.3.]

Notation 3.50. 1) Let κ be an uncountable regular cardinal. We let $\mathrm{seq}^\alpha_{<\kappa}(\mathscr{A})$, where $\mathscr{A}$ is an expansion of a submodel of some $\mathscr{H}_{\leq\mu}(\lambda)$ with $|\tau(\mathscr{A})| \leq \chi$, be the set of sequences $\langle M_i : i < \alpha\rangle$, which are increasing continuous, $M_i \prec \mathscr{A}$, $\|M_i\| < \kappa$, $M_i \cap \kappa \in \kappa$, $\kappa = \kappa_1^+ \;\Rightarrow\; \kappa_1 + 1 \subseteq M_i$, $\langle M_j : j \leq i\rangle \in M_{i+1}$. (If $\alpha = \delta$ is limit, $M_\delta =: \bigcup\limits_{i<\delta} M_i$).
2) If $\kappa = \kappa_1^+$, we may write $\leq \kappa_1$ instead $< \kappa$.

We repeat the definition of filters introduced in [She75a, Definition 3.2].

Definition 3.51. 1) $\mathscr{E}^\theta_{<\kappa}(A)$ is the following filter on $[A]^{<\kappa}$: $Y \in \mathscr{E}^\theta_{<\kappa}(A)$ if and only if for (every) χ large enough, for some $x \in \mathscr{H}(\chi)$ the set $\{(\bigcup\limits_{i<\theta} M_i) \cap A : \langle M_i : i < \theta\rangle \in \mathrm{seq}^\theta_{<\kappa}(\mathscr{H}(\chi), \in, x)\}$ is included in Y.

Exercise 3.52. Let λ, κ, θ, and $Y \subseteq [\lambda]^{<\kappa}$ be given. Then

$$(\mathbf{a}) \quad \Rightarrow \quad (\mathbf{b}) \quad \Rightarrow (\mathbf{c}),$$

where

(a) for some $\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}, \dot{\zeta}, h$ satisfy 3.12,

$$Y = \{M^\alpha_\theta \cap \lambda : \alpha < \alpha(*)\},$$

and

(∗) $\alpha \neq \beta \wedge \bigwedge_{i<\theta} \eta^\alpha_i \in M^\beta_\theta \Rightarrow \alpha < \beta$.

(b) $\diamondsuit_{E^\theta_{<\kappa}(\lambda)}$ holds.

(c) Like (a) without (∗).

Exercise 3.53. If $\lambda^{2^\kappa} = \lambda$, $\theta \leq \kappa$ then $\diamondsuit_{E^\theta_{<\kappa}}$ (main case: $\kappa = \theta$).

Exercise 3.54. If $\lambda = \mu^+$, $\lambda^\kappa = \lambda$, $\theta = \aleph_0$, $\kappa = \kappa^\theta$, then there is a coding set with diamond (see [She86a]).

Exercise 3.55. Suppose that $\mathrm{cf}(\lambda) > \aleph_0$, $2^\lambda = \lambda^{\mathrm{cf}(\lambda)}$, $\chi(*) \geq \theta > \mathrm{cf}(\lambda)$, $(\forall \alpha < \lambda)[|\alpha|^{\chi(*)} < \lambda], \mathfrak{C}$ is a model expanding $(\mathscr{H}_{<\chi(*)}(\lambda), \in), |\tau(\mathfrak{C})| \leq \aleph_0$. Then we can find $\{\bar{M}^\alpha : \alpha < \alpha(*)\}$ such that:

(i) $\bar{M}^\alpha = \langle M^\alpha_i : i < \sigma \rangle$, $M^\alpha_i \in \mathscr{H}_{<\chi(*)}(\lambda)$, $M^\alpha_i \cap \chi(*)$ is an ordinal, $M^\alpha_i \restriction \tau(\mathfrak{C}) \prec \mathfrak{C}$, $[i < j \Rightarrow M^\alpha_i \prec M^\alpha_j]$, $\langle M^\alpha_j : j \leq i \rangle \in M^\alpha_{i+1}$,

(ii) if f_n is a k_n–place function from λ to $\mathscr{H}_{<\chi(*)}(\lambda)$ then for some α, $M^\alpha_\sigma \prec (\mathfrak{C}, f_n)_{n<\omega}$.

Exercise 3.56. Suppose $\theta = \mathrm{cf}(\mu) < \mu$, $(\forall \alpha < \mu)[|\alpha|^\theta < \mu]$, $2^\mu = \mu^\theta$ and $\lambda = (2^\mu)^+$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$. Let $\mu = \sum_{i<\theta} \mu_i$, μ_i regular strictly increasing, and $\mathrm{cf}(\prod \mu_i / E) = 2^\mu$. Then we can find

$$\mathbf{W} = \{(\bar{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}, \quad \dot{\zeta} : \alpha(*) \longrightarrow S, \quad h : \alpha(*) \longrightarrow \lambda$$

such that:

(∗) for $\delta \in S$ there is a club C_δ of δ of order type θ such that

$$\alpha \in C_\delta \wedge \mathrm{otp}(\alpha \cap C_\delta) = \gamma + 1 \quad \Rightarrow \quad \mathrm{cf}(\alpha) = \mu_\gamma.$$

Remark 3.57. We do not know if the existence of a Black Box for λ^+ with h one-to-one follows from ZFC (of course it is a consequence of $\diamondsuit$). On the other hand, it is difficult to get rid of such a Black Box (i.e., prove the consistency of non-existence).

If $\lambda = \lambda^{<\lambda}$ then we have $h : S \longrightarrow \lambda$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ such that C_δ is a club of δ, $\mathrm{otp}(C_\delta) = \mathrm{cf}(\delta)$ and

$$\forall \text{ club } (C \subseteq \alpha \in C_\delta)$$

$$[\mathrm{cf}(\alpha) > \aleph_0 \ \wedge \ \min_{C' \text{ club of } C'_\alpha} \sup(h \restriction C') = \mathrm{otp}(C \cap \alpha)].$$

This is hard to get rid of, (i.e., hard to find a forcing notion making it no longer a black box, without collapsing too many cardinals); compare with Mekler-Shelah [MS89].

§ 4. On Partitions to stationary sets

We present some results on the club filter on $[\kappa]^{\aleph_0}$ and $[\kappa]^\theta$ and some relatives, and $\diamondsuit$ (see Definition [Shed, 4.6=Ld12] or 4.4(2) below). There are overlaps of the claims hence redundant parts which still have some interest.

Claim 4.1. *Assume κ is a cardinal $> \aleph_1$, <u>then</u> $[\kappa]^{\aleph_0}$ can be partitioned to $\kappa^{\aleph_0}$ (pairwise disjoint) stationary sets.*

Proof. Follows by 4.2 below, [in details, let τ be the vocabulary $\{c_n : n < \omega\}$ where each c_n is an individual constant. By 4.2 below there is a sequence $\bar{M} = \langle M_u : u \in [\kappa]^{\aleph_0}\rangle$ of τ-models, with M_u having universe u such that $\bar{M}$ is a diamond sequence.

For each $\eta \in {}^\omega\lambda$ let $\mathscr{S}_\eta$ be the set $u \in [\kappa]^{\aleph_0}$ such that for every $n < \omega$ we have $c_n^{M_u} = \eta(n)\}$.

By the choice of $\bar{M}$ necessarily each set $\mathscr{S}_\eta$ is a stationary subset of $[\kappa]^{\aleph_0}$, and trivially those sets are pairwise disjoint.] $\square_{4.1}$

Claim 4.2. *Let $\kappa > \aleph_1$. Then we have diamond on $[\kappa]^{\aleph_0}$ (modulo the filter of clubs on it, see 4.4(2) or* [Shed, 4.6=Ld12]*), and we can find $A_\alpha \subseteq [\kappa]^{\aleph_0}$ for $\alpha < \lambda := 2^{\kappa^{\aleph_0}}$ such that each is stationary but the intersection of any two is not.*

Proof. The existence of the A_α-s for $\alpha < \lambda$ follows from the other result. Let τ be a countable vocabulary, $\tau_1 = \tau \cup \{<\}$. First we prove it when $\kappa = \aleph_2 \in [\aleph_2, 2^{\aleph_0})$. Let $\omega \setminus \{0\}$ be the disjoint union of s_n for $n < \omega$, each s_n is infinite with the first element $> n + 3$ when $n > 0$. Let $\langle C_\delta : \delta \in S^2_0\rangle$ be club guessing, where $S^2_0 = \{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\}$, such that $C_\delta \subseteq \delta = \sup(C_\delta)$ has order type ω.

Let $\langle(\mathfrak{A}^\zeta, \bar{\alpha}^\zeta) : \zeta < 2^{\aleph_0}\rangle$ list without repetitions the pairs $(\mathfrak{A}, \bar{\alpha}), \mathfrak{A}$ a model with vocabulary τ_1 and universe a limit countable ordinal and $\bar{\alpha} = \langle\alpha_n : n < \omega\rangle$ an increasing sequence of ordinals with limit $\sup(\mathfrak{A})$ and $\mathfrak{A}{\restriction}\alpha_n \prec \mathfrak{A}$. Let E_n be the following equivalence relation relation on $2^{\aleph_0} : \epsilon E_n \zeta$ iff $(\mathfrak{A}^\epsilon{\restriction}\alpha^\epsilon_n, \bar{\alpha}^\epsilon{\restriction}n)$ is isomorphic to $(\mathfrak{A}^\zeta{\restriction}\alpha^\zeta_n, \bar{\alpha}^\zeta \restriction n)$ which means: there is an isomorphism f from $\mathfrak{A} \restriction \alpha^\epsilon_n$ onto $\mathfrak{A}^\zeta \restriction \alpha^\zeta_n$ which maps $\mathfrak{A}^\epsilon \restriction \alpha^\epsilon_k$ onto $\mathfrak{A}^\zeta \restriction \alpha^\zeta_k$ for $k < n$ and is an order preserving function (for the ordinals, alternatively we restrict ourselves to the case $<$ is interpreted as a well ordering).

We can find subsets t^ζ of ω such that:

- $(*)$ for $\zeta, \epsilon < 2^{\aleph_0}$ we have $t^\zeta \cap s_n = t^\epsilon \cap s_n$ iff $\mathfrak{A}^\zeta{\restriction}\alpha^\zeta_n = \mathfrak{A}^\epsilon \restriction \alpha^\epsilon_n$ and $\alpha^\zeta_k = \alpha^\epsilon_k$ for $k \le n$. Also $t^\zeta \cap s_n$ is infinite and $\epsilon \ne \zeta \Rightarrow \aleph_0 > |t^\epsilon \cap t^\zeta|$ for simplicity (so $t^\zeta \cap s_n$ depend just on ζ/E_n, in fact code it).

For $\zeta < 2^{\aleph_0}$ let

$$\mathscr{S}_\zeta := \{a \in [\kappa]^{\aleph_0} : t_\zeta = \{|C_{\sup}(a) \cap \beta| : \beta \in a\}\},$$

and let

$$\mathscr{S}'_\zeta = \{a \in \mathscr{S}_t : \mathrm{otp}(a) = \mathrm{otp}(\mathfrak{A}^\zeta)\},$$

and for $a \in \mathscr{S}'_\zeta$ let N_a be the model isomorphic to $\mathfrak{A}^\zeta$ by the function f_a, where $\mathrm{Dom}(f_a) = a, f_a(\gamma) = \mathrm{otp}(\gamma \cap a)$.

Let $\mathscr{S}$ be the union of $\mathscr{S}'_\zeta$ for $\zeta < 2^{\aleph_0}$. Clearly $\zeta \ne \xi \ \Rightarrow \ \mathscr{S}_\zeta \cap \mathscr{S}_\xi = \emptyset$, and so $\mathscr{S}'_\zeta \cap \mathscr{S}'_\xi = \emptyset$. Hence N_a is well defined for $a \in \mathscr{S}$.

Let K_n be the set of pairs $(\mathfrak{A}, \bar{\alpha})$ such that $\mathfrak{A}$ is a tau_1-model with universe a countable subset of κ with no last member, and $\bar{\alpha}$ is an increasing sequence of ordinals $< \kappa$ of length n such that $\alpha_k < \sup(\mathfrak{A})$ and $[\alpha_k, \alpha_{k+1}) \cap \mathfrak{A} \neq \emptyset$ and $\mathfrak{A} \restriction \alpha_k \prec \mathfrak{A}$. So clearly there is a function $\mathrm{cd}_n : K_n \to \mathscr{P}(s_n)$ such that: if $\zeta < 2^{\aleph_0}$ then $\mathrm{cd}_n(\mathfrak{A}, \bar{\alpha}) = t^\zeta \cap s_n$ iff the pairs $(\mathfrak{A}, \bar{\alpha}), ((\mathfrak{A}^\zeta, \bar{\alpha}^\zeta \restriction n))$ are isomorphic.

Let M be a τ_1–model with universe κ. Now (see [Shed, 1.16=L1.15], or history in the introduction of §3, and the proof of 3.24) we can find a full subtree $\mathscr{T}$ of ${}^{\omega>}(\aleph_2)$ (i.e., it is non-empty, closed under initial segments and each member has $\aleph_2$ immediate successors) and elementary submodels N_η of M for $\eta \in \mathscr{T}$ such that:

- (a) $\mathrm{Rang}(\eta) \subseteq N_\eta$,
- (b) if η is an initial segment of ρ then N_η is a submodel N_ρ, moreover $N_\eta \cap \aleph_2$ is an initial segment of $N_\rho \cap \aleph_2$.

Now let E be the set of $\delta < \aleph_2$ satisfying: if $\rho \in \mathscr{T}$ and $\rho \in {}^{\omega>}\delta$ then $N_\rho \cap \aleph_2$ is a bounded subset of δ, and δ is a limit ordinal. Let E_1 be the set of $\delta \in E$ such that if $\rho \in \mathscr{T} \cap {}^{\omega>}\delta$ then for every $\beta < \delta$ there is γ such that $\beta < \gamma < \delta$ and $\eta\char`^\langle\gamma\rangle \in \mathscr{T}$. So by the choice of $\langle C_\delta : \delta \in S\rangle$ for some $\delta \in S$ we have $C_\delta \subset E_1$.

Let $\langle \alpha_{\delta,k} : k < \omega\rangle$ list C_δ in increasing order.

Now we choose by induction on n a triple $(\eta_n, s^*_n, \alpha_n, k_n)$ such that:

$(*)$ (a) $\eta_n \in \mathscr{T}$ has length n (so η_0 is necessarily $\langle\rangle$)

(b) if $n = m+1$ then η_n is a successor of η_m

(c) s^*_n is $\mathrm{cd}_n((N_{\eta_n}, \langle \alpha_\ell : \ell < n\rangle))$ if the pair $(N_{\eta_n}, \langle \alpha_\ell : \ell < n\rangle)$ belongs to K_n and is s_n otherwise; actually it is so,

(d) $\alpha_n = \sup(N_{\eta_n}) + 1$

(e) $k_n = \min\{k : N_{\eta_n} \subseteq \alpha_{\delta,k}\}$ and $k_0 = 0$ and $\bar{n}[0, k_n] \subseteq \bigcup\limits_{\ell<n} s_\ell \cup \{0\}$

(f) if $n = m+1$ and $k_m < k_n$ <u>then</u>

- (α) $\min(N_{\eta_n} \setminus N_{\eta_m}) > \alpha_{\delta,k_n-1}$
- (β) (k_m, k_n) is disjoint to $\bigcup\limits_{\ell<n} s^*_\ell$
- (δ) $k_n \in \cup\{s^*_\ell : \ell < n\}$
- (ϵ) k_n is minimal under those restrictions.

(g) if $n = m+1$ and $k_n = k_m$ then we cannot find $k \in (k_m, \omega)$ satisfying $(\beta), (\gamma)$ of clause (f).

There is no problem to carry the induction. In the end let $\eta = \bigcup\limits_n \eta_n \in \lim(i\mathscr{T})$, so we get a τ_1-model $N_\eta =: \cup\{N_{\eta_n} : n < \omega\}$, and an increasing sequence $\langle \alpha_n : n < \omega\rangle$ of ordinals with limit $\sup(\mathfrak{A})$. Now by the choice of $\langle(\mathfrak{A}^\zeta, \bar{\alpha}^\zeta) : \zeta < 2^{\aleph_0}\rangle$ clearly for some ζ we have $(N_\eta, \bar{\alpha}), (\mathfrak{A}^\zeta, \bar{\alpha}^\zeta)$ are isomorphic, so necessarily $(N_\eta \restriction \alpha_n, \bar{\alpha} \restriction n)$ belongs to K_n and necessarily $\mathrm{cd}_n(N_\eta, \langle \alpha_\ell : \ell < n\rangle) = s^*_n$.

Also clearly $\sup(N_\eta) = \delta$ and $\{k_n : n < \omega\} = \{|C_\delta \cap \beta| : \beta \in N_\eta\} = \{\alpha_{\delta,k_n} : n < \omega\}\}$.

Letting a be the universe of N_η it follows that $a \in \mathscr{S}_\zeta$ so N_a is well defined and isomorphic to $\mathfrak{A}^\zeta$ hence to N_η using $<^M$ we get $N_a = N_\eta$. But $N_\eta \prec M$. So $\langle N_a : a \in \mathscr{S}\rangle$ is really a diamond sequence, well for τ_1-models rather then τ-models, but this does no harm and will help for $\kappa > \aleph_2$.

Second, we consider the case $\kappa > \aleph_2$. For each $c \in [\kappa]^{\aleph_0}$, if $\mathrm{otp}(c) = \mathrm{otp}(c \cap \omega_2, <^{N_{c\cap\omega_2}})$, let g_c be the unique isomorphism from $(c \cap \omega_2, <^{N_{c\cap\omega_2}})$ onto $(c, <), <$ the usual order, and let M_c be the τ–model with universe c such that g is an isomorphism from $N_{c\cap\omega_2}{\restriction}\tau$ onto M_c. Clearly it is an isomorphism and the M_c's form a diamond sequence.

[Why? For notational simplicity τ has predicates only. Let $M_0 = M$ be a τ-model with universe κ, let M_1 be an elementary submodel of M of cardinality $\aleph_2$ such that $\omega_2 \subseteq M_1$, let h be a one-to-one function from M_1 onto ω_2 let M_2 be a τ-model with universe ω_2 such that h is an isomorphism from M_1 onto M_2, and let M_3 be the τ_1-model expanding M_2 such that $<^{M_3} = \{(h(\alpha), h(\beta)) : \alpha < \beta$ are from $M_1\}$.

So for some $a \in \mathscr{S} \subseteq [\kappa]^{\aleph_0}$ we have $N_a \prec M_3$ and $h(\alpha) = \beta \in N_a \wedge \alpha < \omega_2 \Rightarrow \alpha \in a$ (the set of a-s satisfying this contains a club of $[\aleph_2]^{\aleph_0}$). Let $c = \{\alpha : h(\alpha) \in a\}$, so clearly $c \cap \omega_2 = a$ and $M_c \prec M_1$ hence $M_c \prec M$, so we are done.] $\square_{4.2}$

Discussion 4.3. Some concluding remark are:
1) We can use other cardinals, but it is natural if we deal with $D_{\kappa,<\theta,\aleph_0}$ (see below).
2) The context is very near to §3, but the stress is different.

Definition 4.4. Let $\kappa \geq \theta \geq \sigma$, θ uncountable regular. If $\theta = \mu^+$ we may write μ instead of $< \theta$.
1) Let $D = D_1 = D^1_{\kappa,<\theta,\aleph_0}$ be the filter $[\kappa]^{<\theta}$ generated by $\{A^1_x : x \in \mathscr{H}(\chi)\}$ where

$$\begin{aligned} A^1_x = \{N \cap \kappa : \ & N \text{ is an elementary submodel of } (\mathscr{H}(\chi), \in) \text{ and} \\ & N \text{ is } \bigcup_{n<\omega} N_n,\ N_n \text{ increasing and } N_n \in N_{n+1} \\ & \text{and } \|N_n\| < \theta \text{ and } N_n \cap \theta \in \theta\}. \end{aligned}$$

2) Let $D = D_2 = D^2_{\kappa,<\theta,\sigma}$ be the filter on $[\kappa]^{<\theta}$ generated by $\{A^2_x : x \in \mathscr{H}(\chi)\}$ where

$$\begin{aligned} A^2_x = \{N \cap \kappa : \ & N textisanelementarysubmodelof (\mathscr{H}(\chi), \in) \text{ and} \\ & N \text{ is } \bigcup_{\zeta<\sigma} N_\zeta,\ N_\zeta \text{ increasing and} \\ & \langle N_\varepsilon : \varepsilon \leq \zeta \rangle \in N_{\zeta+1} \text{ and } N_\varepsilon \cap \theta \in \theta\}. \end{aligned}$$

3) For a filter D on $[\kappa]^{<\theta}$ let $\diamondsuit_D$ mean: fixing any countable vocabulary τ there are $S \in D$ and $N = \langle N_a : a \in S \rangle$, each N_a a τ–model with universe a, such that for every τ–model M with universe λ we have $\{a \in S : N_a \subseteq M\} \neq \emptyset \mod D$.
4) Instead $< \theta$ we may write θ.

Claim 4.5. *Assume $\theta \leq \sigma$ and $\kappa > \sigma^+$ and let $D = D_{\kappa,\theta,\aleph_0}$.*
1) $[\kappa]^\theta$ can be partitioned to $\sigma^{\aleph_0}$ (pairwise disjoint) D–positive sets.
2) Assume in addition that $\sigma^{\aleph_0} \geq 2^\theta$. Then

- (α) *we can find $A_\alpha \subseteq [\kappa]^\theta$ for $\alpha < \lambda := 2^{\kappa^\theta}$ such that each is D–positive but they are pairwise disjoint* $\mod D$,
- (β) *if $\lambda = \kappa^\theta$ and τ is a countable vocabulary <u>then</u> $\diamondsuit_{\lambda,\theta,\aleph_0}$; moreover there are $S^* \subseteq [\lambda]^\theta$ and function N^* with domain S^* such that*
  - (a) *for distinct a, b from S^* we have $a \cap \kappa \neq b \cap \kappa$,*
  - (b) *for $a \in S^*$ we have $N^*(a) = N^*_a$ is a τ–model with universe a,*

(c) *for a τ–model M with universe λ, the set $\{a : N^*_a = M \restriction a\}$ is stationary.*

Proof. Similar to earlier ones : part (1) like Claim 4.1 case (a), part (2) like the proof of Claim 4.2. $\square_{4.5}$

Claim 4.6. *1) If $\theta \leq \kappa_0 \leq \kappa_1$ and $\diamondsuit_{S_0}$ i.e. $\diamondsuit_{D_{\kappa_0,\theta,\sigma} S_0}$, where S_0 is a subset of $[\kappa_0]^\theta$ which is $D_{\kappa_0,\theta,\sigma}$–positive and $S_1 := \{a \in [\kappa_1]^\theta : a \cap \kappa_0 \in S_0\}$, then $\diamondsuit_{S_1}$, i.e. $\diamondsuit_{D_{\kappa_1,\theta,\sigma} S_1}$.*
2) In part (1), if in addition $\kappa_0 = (\kappa_0)^\theta$ and $\kappa_2 = (\kappa_1)^\theta$ then we can find $S_2 \subseteq [\kappa_2]^\theta$ such that:

(a) *$a \in S_2$ implies $a \cap \kappa_0 \in S_0$,*
(b) *if b, c are distinct members of S_2 then $b \cap \kappa_1$, $c \cap \kappa_1$ are distinct, and*
(c) *$\diamondsuit_{S_2}$.*

3) If $\kappa = \kappa^\theta$ then $\diamondsuit_{D_{\kappa,\theta,\sigma}}$.

Remark 4.7. This works for other uniform definition of normal filters.

Above, $\kappa^{\theta^\sigma} = \kappa$ can be replaced by: every tree with $\leq \theta$ nodes has at most θ^*–branches and $\kappa^{\theta^*} = \kappa$.

Proof. 1) Easy.
2) Implicit in earlier proof, 4.2.
3) See [She86d], [She86a] $\square_{4.6}$

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