

## ON THE $p$ -RANK OF $\text{Ext}$

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ABSTRACT. Assume  $V = L$  and  $\lambda$  is regular smaller than the first weakly compact cardinal. Under those circumstances and with arbitrary requirements on the structure of  $\text{Ext}(G, \mathbb{Z})$  (under well known limitations), we construct an abelian group  $G$  of cardinality  $\lambda$  such that for no  $G' \subseteq G$ ,  $|G'| < \lambda$  is  $G/G'$  free and  $\text{Ext}(G, \mathbb{Z})$  realizes our requirements.

### 1. INTRODUCTION

In section 2 we give a principle true if  $\mathbf{V} = \mathbf{L}$  which is stronger than  $\diamond^+$  (which was enough for building Kurepa tree); of course the proof follows proofs of Jensen for diamonds. It seems we do not use its full strength - we seem to actually need only the cases  $\mathfrak{z}(M_\delta^1) = \text{cf}(\delta)$  (see 2.1). The principle should be helpful for building models on  $\lambda$  with  $\Sigma_2^1$ -properties (on  $\lambda$ ). Also there should be cases where we can prove impossibility (by playing with those cofinalities).

In the third section we apply the principle to construct abelian groups (we drop “abelian”). For a torsion free group  $G$  the group  $\text{Ext}(G, \mathbb{Z})$  is divisible and therefore its structure is determined by ranks  $\nu_0(G)$ ,  $\nu_p(G)$  (the numbers of copies of  $\mathbb{Q}$  and  $Z(p^\infty)$  in the decomposition of the divisible group, where  $p$  ranges over primes) and  $\nu_p(G) \leq 2^{\|G\|}$ . By [HHS<sub>Sh</sub> 91], if  $\mathbf{V} = \mathbf{L}$  and a group  $G$  is not of the form  $G_1 \oplus G_2$ ,  $\|G_1\| < \|G\|$  and  $G_2$  free then  $\nu_0(G) = 2^{\|G\|}$ . If  $\lambda$  is a regular cardinal smaller than the first weakly compact cardinal,  $\lambda_p \leq \lambda^+$  (for  $p$  prime) then assuming  $\mathbf{V} = \mathbf{L}$  we construct an abelian torsion free group  $G$  such that  $\nu_p(G) = \lambda_p$  for each  $p$ ,  $\nu_0(G) = \lambda^+$  and  $\|G\| = \lambda$ . This result can be considered as a generalization of a result of Sageev and Shelah which states the same for  $\lambda = \aleph_1$  but under the assumption of CH only (see [SgSh 138], for an alternative proof see Eklof and Huber [EH] or Theorem XII.2.10 of [EM]).

No advanced knowledge of Group Theory is required, constructions of the third section are purely combinatorial applications of the principle of the second section. On the other hand no advance tools of Set Theory are used — if one accepts 2.1, the rest is elementary.

**Set theoretical notation:** For a cardinal  $\kappa$ ,  $\mathcal{H}(\kappa)$  is the family of sets with transitive closure of cardinality  $< \kappa$ . If  $e$  is a set of ordinals then  $\text{acc}(e)$  denotes the set of accumulation points of  $e$  (i.e. limits of  $e$ ) and  $\text{otp}(e)$  is the order type of  $e$ .

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**Group theoretical notation:**  $\mathbf{P}$  is the set of prime numbers. As all the groups we shall deal with are abelian, we omit this adjective.  $G, H, K$  denote (abelian) groups,  $\oplus$  denotes a direct sum.  $\mathbb{Z}$  is the additive group of integers.  $\text{Hom}(G, H)$  is the group of homomorphisms from  $G$  to  $H$  (with the pointwise addition, i.e.  $(f + g)(x) = f(x) + g(x)$ ). If  $f \in \text{Hom}(G, \mathbb{Z})$  and  $p \in \mathbf{P}$  then  $f/p\mathbb{Z}$  is the following member of  $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ :

$$(f/p\mathbb{Z})(x) = f(x)/p\mathbb{Z} \quad (\text{also called } f(x) + p\mathbb{Z}).$$

For a group  $G$  and its subset  $\{x_n : n \in I\} \subseteq G$ ,  $\langle x_n : n \in I \rangle_G$  denotes the subgroup of  $G$  generated by  $\{x_n : n \in I\}$ .

**History:** The study of the structure of  $\text{Ext}$  has a long history already. For a review of the main results in the area we refer the reader to [EM].

The results of this paper were proved in 1986/87 and a preliminary version of the paper was ready in 1992. The disease and death of the first author<sup>1</sup> stopped his work on the paper. Later the second author joined in finishing the paper.

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## 2. A CONSTRUCTION PRINCIPLE IN $\mathbf{L}$

**Theorem 2.1 (V=L).** *Assume*

- (A)  $\lambda$  a regular uncountable not weakly compact<sup>2</sup> cardinal.
- (B)  $S \subseteq \lambda$  is stationary.
- (C)  $\bar{P}, \bar{Q}, \bar{R}$  are finite pairwise disjoint sequences of predicates and function symbols (so a  $\bar{P}$ -model  $M$  is  $(A, \bar{P}^M) = (A, \dots, P_i^M, \dots)$ ).
- (D) Let  $\varphi = \varphi(\bar{P}, \bar{Q}, \bar{R})$  be a first order sentence.
- (E)  $M^0$  is a  $\bar{P}$ -model with universe  $\lambda$ .
- (F)  $E$  is a club of  $\lambda$  such that  $\delta \in E \Rightarrow M_\delta^0 \stackrel{\text{def}}{=} M^0 \upharpoonright \delta \prec M^0$ .
- (G) For  $\delta \in E \cup \{\lambda\}$  let  $\mathbf{k}_\delta = \{M_\delta^1 : M_\delta^1 \text{ a } (\bar{P} \sim \bar{Q})\text{-model expanding } M_\delta^0\}$  and for  $M_\delta^1 \in \mathbf{k}_\delta$  let  $\mathbf{k}_\delta^+(M_\delta^1) = \{M_\delta^2 : M_\delta^2 \text{ is a } (\bar{P} \sim \bar{Q} \sim \bar{R})\text{-model expanding } M_\delta^1 \text{ and satisfying } \varphi = \varphi(\bar{P}, \bar{Q}, \bar{R})\}$ . Lastly  $\mathbf{k}_\delta^- = \{M_\delta^1 \in \mathbf{k}_\delta : \mathbf{k}_\delta^+(M_\delta^1) \neq \emptyset\}$ .

Then we can find a well ordering  $<^*$  of  $\mathcal{H}(\lambda^+)$  of order type  $\lambda^+$ , a sequence  $\bar{e} = \langle e_\delta : \delta < \lambda \text{ is a singular ordinal} \rangle$  and functions  $\mathfrak{z}, \mathfrak{N}, \mathfrak{B}_\varepsilon$  (for  $\varepsilon < \lambda$ ) such that:

- (a) The domain of the functions  $\mathfrak{z}$  and  $\mathfrak{N}$  is  $\bigcup_{\delta \in E} \mathbf{k}_\delta^-$ ; superscript  $\delta$  means the restriction of the function to  $\mathbf{k}_\delta^-$ .
- (b) For  $\delta \in E$  and  $M_\delta^1 \in \mathbf{k}_\delta^-$  we have:  $\mathfrak{z}(M_\delta^1)$  is zero or a limit ordinal  $< |\delta|^+$ .
- (c) For  $\delta \in E$ ,  $\mathfrak{B}_\varepsilon^\delta$  is a function with domain  $\{M_\delta^1 \in \mathbf{k}_\delta^- : \mathfrak{z}(M_\delta^1) > \varepsilon\}$ , and  $\mathfrak{B}^\delta = \bigcup_{\varepsilon} \mathfrak{B}_\varepsilon^\delta$ .
- (d) For  $\delta \in E$ ,  $M_\delta^1 \in \mathbf{k}_\delta^-$  we have  $\mathfrak{N}(M_\delta^1) \in \mathbf{k}_\delta^+(M_\delta^1)$
- (e) For  $\delta \in E$  and  $M_\delta^1 \in \mathbf{k}_\delta^-$  we have  $\langle \mathfrak{B}_\varepsilon^\delta(M_\delta^1) : \varepsilon < \mathfrak{z}(M_\delta^1) \rangle$  is an increasing sequence of models, isomorphic to some elementary submodels of  $(\mathcal{H}(\lambda^+), \in, <^*)$  but we do not require it to be an elementary chain nor continuous but we do require the following:

<sup>1</sup>Professor Alan H. Mekler died in 1992

<sup>2</sup>can be weakened to the existence of  $Y$  (see the proof)

- (\*)  $x \in \mathfrak{B}_\zeta^\delta(M_\delta^1) \setminus \mathfrak{B}_\varepsilon^\delta(M_\delta^1), y \in \mathfrak{B}_\varepsilon^\delta(M_\delta^1) \Rightarrow \mathfrak{B}_\zeta^\delta(M_\delta^1) \models y <^* x$ .
- (f) For  $M_\delta^1 \in \mathbf{k}_\delta^-$ ,  $\varepsilon < \mathfrak{z}(M_\delta^1)$  the universe of  $\mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  is a transitive set to which  $\delta$  belongs.
- (g) For  $M_\delta^1 \in \mathbf{k}_\delta^-$  for  $\zeta < \mathfrak{z}(M_\delta^1)$  we have  $\langle \mathfrak{B}_\varepsilon^\delta(M_\delta^1) : \varepsilon \leq \zeta \rangle \in \mathfrak{B}_{\zeta+1}^\delta(M_\delta^1)$  (remember:  $\mathfrak{z}(M_\delta^1)$  if not zero, is a limit ordinal) and:

$$\mathfrak{B}_{\zeta+1}^\delta(M_\delta^1) \models \|\mathfrak{B}_\zeta^\delta(M_\delta^1)\| = \|\delta\|.$$

- (h) If  $M_\delta^1 \in \mathbf{k}_\delta^-$  then  $\mathfrak{N}(M_\delta^1) \notin \bigcup \{ \mathfrak{B}_\varepsilon^\delta(M_\delta^1) : \varepsilon < \mathfrak{z}(M_\delta^1) \}$  and  $M_\delta^1 \in \mathfrak{B}_0^\delta(M_\delta^1)$  when  $\mathfrak{z}(M_\delta^1) > 0$ .
- (i) If  $M_\lambda^1 \in \mathbf{k}_\lambda^-$  then for some  $M_\lambda^2 \in \mathbf{k}_\lambda^+$  we have:
- (\*) for some regular  $\sigma \leq \lambda$  (possibly  $\sigma = 0$ ) for every regular  $\theta < \lambda$  (the most interesting case is:  $0 < \sigma < \lambda \Rightarrow \theta = \sigma$ ) such that  $\{ \delta \in S : \text{cf}(\delta) = \theta \}$  is stationary and for stationarily<sup>3</sup> many  $\delta \in S$  we have:
- ( $\alpha$ )  $\text{cf}(\delta) = \theta$   
 ( $\beta$ )  $M_\lambda^2 \upharpoonright \delta \prec M_\lambda^2$   
 ( $\gamma$ )  $\mathfrak{N}(M_\lambda^2 \upharpoonright \delta) = M_\lambda^2 \upharpoonright \delta$ .  
 ( $\delta$ )  $[\sigma < \lambda \Rightarrow \mathfrak{z}(M_\lambda^2 \upharpoonright \delta) = \sigma]$  and  $[\sigma = \lambda \Rightarrow \mathfrak{z}(M_\lambda^2 \upharpoonright \delta) = \text{cf}(\delta)]$ .
- (j) Suppose  $M_\lambda^1 \in \mathbf{k}_\lambda \setminus \mathbf{k}_\lambda^-$  but  $M_\lambda^1 \upharpoonright \delta \in \mathbf{k}_\delta^-$  for  $\delta \in E$ . If  $M_\lambda^3$  is an expansion of  $M_\lambda^1$  with a finite vocabulary then<sup>4</sup> for some club  $E' \subseteq E$  we have

$$\delta \in E' \Rightarrow \{ E' \cap \delta, M_\lambda^3 \upharpoonright \delta \} \in \bigcup \{ \mathfrak{B}_\varepsilon^\delta(M_\lambda^1 \upharpoonright \delta) : \varepsilon < \mathfrak{z}(M_\lambda^1 \upharpoonright \delta) \}.$$

Moreover, if  $M \prec (\mathcal{H}(\lambda^+), \in, <^*)$  is a proper  $<^*$ -initial segment,  $\lambda \subseteq M$ ,  $M = \bigcup_{i < \lambda} M_i$  where  $M_i$  increasing continuous,  $\|M_i\| < \lambda$  for  $i < \lambda$

then for some club  $E' \subseteq E$  for every  $\delta \in E'$ :

if  $\mathbf{j}_\delta$  is the Mostowski collapse of  $M_\delta$  then  
 $\{ E' \cap \delta, \mathbf{j}_\delta[M_\delta] \} \in \bigcup_{\varepsilon < \mathfrak{z}(M_\lambda^1 \upharpoonright \delta)} \mathfrak{B}_\varepsilon^\delta(M_\lambda^1 \upharpoonright \delta)$  and

$\mathbf{j}_\delta[M_\delta]$  is a proper initial segment of  $\bigcup_{\varepsilon < \mathfrak{z}(M_\lambda^1 \upharpoonright \delta)} \mathfrak{B}_\varepsilon^\delta(M_\lambda^1 \upharpoonright \delta)$

(also the order).

(Remark: many  $M$  has a tower  $\langle B_\varepsilon^\lambda : \varepsilon < \varepsilon^* \rangle$  which collapses to an initial segment.)

- (k)  $\bar{e}$  is a square, i.e.  $e_\delta$  a club of  $\delta$  of order type  $< \delta$  and

$$\alpha \in \text{acc}(e_\delta) \Rightarrow e_\alpha = e_\delta \cap \alpha,$$

if  $\mathfrak{z}(M_\delta^1) > 0$  &  $\alpha \in \text{acc}(e_\delta)$  then  $M_\alpha^1 \stackrel{\text{def}}{=} M_\delta^1 \upharpoonright \alpha \prec M_\delta^1$  and  $\mathfrak{z}(M_\alpha^1) > 0$ .

- (l) If  $\delta \in E$ ,  $\alpha < \delta$  and  $\mathfrak{z}(M_\delta^1) > 0$  then  $e_\delta \cap \alpha \in \mathfrak{B}_0^\delta(M_\delta^1)$ .

*Remark 2.2.* The interesting case is when the set  $S$  satisfies:

- ( $\alpha$ ) for every  $\theta = \text{cf}(\theta) < \lambda$ ,  $\{ \delta \in S : \text{cf}(\delta) = \theta \}$  is stationary,  
 ( $\beta$ )  $S$  a set of singular ordinals,  
 ( $\gamma$ )  $\lambda = \mu^+ \Rightarrow S \subseteq [\mu + 1, \lambda)$  and if  $\lambda$  is inaccessible then  $S$  is a set of strong limit singular cardinals and  
 ( $\delta$ )  $S$  does not reflect.

<sup>3</sup>we can get a club  $E_0 \subseteq \lambda$  such that every  $\delta \in S \cap E_0$  is OK.

<sup>4</sup>We can also assign stationary  $S' \subseteq S$  were  $M_\lambda^3$  was guessed but this is not what we need.

*Proof.* Let  $Y \subseteq \lambda$  be such that for every  $\alpha < \lambda$  the set

$$\{\beta > \alpha : (L_\beta[Y \cap \alpha], \in, Y \cap \alpha) \equiv (L_{\lambda^+}, \in, Y)\}$$

is bounded in  $|\alpha|^+$  (e.g.  $Y$  is a non reflecting stationary subset of  $\lambda$ ).

Let  $\bar{e} = \langle \bar{e}_\delta : \delta < \lambda \text{ singular limit ordinal} \rangle$  be as defined by Jensen [Jn].

Let  $<^*$  be the canonical well ordering of  $L$ .

Suppose now that  $\delta \in E \cup \{\lambda\}$  and  $M_\delta^1 \in \mathbf{k}_\delta$ . If  $\delta < \lambda$  and  $M_\delta^1 \in \mathbf{k}_\delta^-$  then we let  $\mathfrak{N}(M_\delta^1)$  be the  $<^*$ -first member of  $\mathbf{k}_\delta^+(M_\delta^1)$ . Let

$$W_\delta^1(M_\delta^1) \stackrel{\text{def}}{=} \{\alpha > \delta : \begin{array}{l} L_\alpha[M_\delta^1, Y \cap \delta] \cap \mathbf{k}_\delta^+(M_\delta^1) = \emptyset \text{ and} \\ (L_\alpha[M_\delta^1, Y \cap \delta], \in, Y \cap \delta) \text{ is elementarily equivalent} \\ \text{to } (L_{\lambda^+}, \in, Y), \text{ moreover it is isomorphic to some} \\ \text{elementary submodel of it (demand } \delta = \sup(Y \cap \delta)\text{)}. \end{array}$$

Let  $W_\delta^2(M_\delta^1) = \text{acc}(W_\delta^1(M_\delta^1))$ . If  $\delta < \lambda$ ,  $W_\delta^2(M_\delta^1) = \emptyset$  then we let  $\mathfrak{z}(M_\delta^1) = 0$ . Otherwise

$$\mathfrak{z}(M_\delta^1) = \text{cf}\left(W_\delta^1(M_\delta^1) \cap \sup W_\delta^2(M_\delta^1)\right),$$

(so we loose at most finitely many members of  $W_\delta^1(M_\delta^1)$ ) and let

$$a_\delta = a_\delta[M_\delta^1] \subseteq W_\delta^1(M_\delta^1) \cap \sup W_\delta^2(M_\delta^1)$$

be unbounded of order type  $\mathfrak{z}(M_\delta^1)$  and  $\gamma \in a_\delta \Rightarrow a_\delta \cap \gamma \in L_\gamma[M_\delta^1, Y \cap \delta]$  (use the definition of the square in Jensen [Jn] for  $L[Y \cap \delta]$  and the ordinal  $\sup(W_\delta^1(M_\delta^1) \cap \sup W_\delta^2(M_\delta^1))$ ). Let  $\mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  be  $(L_\alpha[M_\delta^1, Y \cap \delta], \in, <^*)$  where  $\alpha$  is the  $\varepsilon$ -th member of  $a_\delta$ .

Let us show that it works, i.e. that the clauses **(a)**–**(i)** are satisfied.

**Clause (a):** Directly from the choice.

**Clause (b):** By the use of  $\sup W_\delta^2(M_\delta^1)$ .

**Clause (c):** Directly from the choice of  $\mathfrak{B}_\varepsilon^\delta$  and  $\mathfrak{z}(M_\delta^1)$ .

**Clause (d):** By the choice of  $\mathfrak{N}(M_\delta^1)$ .

**Clause (e):** Since  $\mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  is  $L_\gamma[M_\delta^1, Y \cap \delta]$  for  $\gamma$  the  $\varepsilon$ -th member of  $a_\delta = a_\delta[M_\delta^1]$  and  $a_\delta[M_\delta^1]$  is a subset of  $W_\delta^1(M_\delta^1)$  (see its definition) we are sure that  $\mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  is OK: they are increasing with  $\varepsilon$  as the  $\varepsilon$ -th member of  $a_\delta[M_\delta^1]$  increases with  $\varepsilon$ ; **(\*)** is satisfied as  $<^*$  is the canonical well ordering of  $L$  (or  $L[Y]$ , in our model not a big difference).

**Clause (f):** See the choice of  $W_\delta^1(M_\delta^1)$ .

**Clause (g):** It follows from  $L_\alpha[M_\delta^1, Y \cap \delta] \in L_{\alpha+1}[M_\delta^1, Y \cap \delta]$ , the definition of  $\mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  (and the presence of  $a_\delta[M_\delta^1]$ ), etc.

**Clause (h):** It is a consequence of the first clause in the definition of  $W_\delta^1(M_\delta^1)$ :  $L_\alpha[M_\delta^1, Y \cap \delta] \cap \mathbf{k}_\delta^+(M_\delta^1) = \emptyset$ .

**Clause (i):** Let  $M_\lambda^2 \in \mathbf{k}_\lambda^+(M_\lambda^1)$  be  $<^*$ -minimal. Define  $W_\lambda^1(M_\lambda^1)$  as above. It is bounded in  $\lambda^+$  as  $M_\lambda^2 \in L_{\lambda^+}$ . Then define  $W_\lambda^2(M_\lambda^1)$ ,  $a_\lambda[M_\lambda^1]$  as above and let  $\gamma^* = \sup(W_\lambda^2(M_\lambda^1))$  (so it is a limit ordinal). Let  $\sigma = \text{cf}(\gamma^*)$ . Let  $\gamma < \lambda^+$  be such that

$$(L_\gamma, \in, M_\lambda^1, Y, E, S, M_\lambda^2) \prec (L_{\lambda^+}, \in, M_\lambda^1, Y, E, S, M_\lambda^2).$$

Let  $L_\gamma = \bigcup_{i < \lambda} N_i$ ,  $\|N_i\| < \lambda$ ,  $\langle N_i : i < \lambda \rangle$  increasing continuous and such that

- $\{E, S, M_\lambda^1, M_\lambda^2, Y, W_\lambda^1(M_\lambda^1), W_\lambda^2(M_\lambda^1), \gamma^*, a[M_\lambda^1]\} \in N_0$ ,
- $N_i \prec (L_\gamma, \in)$

- $\langle N_j : j \leq i \rangle \in N_{i+1}$ .

Let  $E' = \{i < \lambda : N_i \cap \lambda = i\}$ ,  $\delta \in S \cap E'$  and let  $\mathbf{j}_\delta$  be the Mostowski collapse of  $N_\delta$ . Note:  $\mathbf{j}_\delta(\langle^* \uparrow N_\delta) = \langle^*_{\mathbf{j}_\delta[N_\delta]}$ , etc. Now clearly  $\mathbf{j}_\delta$  maps  $M_\lambda^2$  to  $\mathfrak{N}(M_\delta^1)$ ,  $W_\lambda^1(M_\lambda^1)$  to  $W_\delta^1(M_\delta^1)$ ,  $W_\lambda^2(M_\lambda^1)$  to  $W_\delta^2(M_\delta^1)$  and  $a[M_\lambda^1]$  to  $a[M_\delta^1]$ . If  $\sigma < \lambda$  then necessarily

$$a[M_\delta^1] = \mathbf{j}_\delta[a[M_\lambda^1] \cap N_\delta] = \{\mathbf{j}(\beta) : \beta \in a[M_\lambda^1] \cap N_\delta\} = \{\mathbf{j}(\beta) : \beta \in a[M_\lambda^1]\}$$

and if  $\sigma = \lambda$  then

$$a[M_\delta^1] = \mathbf{j}_\delta[a[M_\lambda^1] \cap N_\delta] = \{\mathbf{j}(\beta) : \beta \in a[M_\lambda^1], \beta \text{ is } < \delta\text{-th member of } a[M_\lambda^1]\}.$$

Similarly we can check **(j)**, **(k)**, **(l)**. This finishes the proof.  $\square$

*Remark 2.3.* More generally we can phrase parallels of the squared diamond and/or diamond<sup>+</sup>.

*Discussion 2.4.* What is the point of this principle?

You can just read the next section to see how it works. Still let us try to explain it. Diamonds on  $\lambda$  has been very good in helping to build a structure  $M$  with universe  $\lambda$  satisfying some  $\Pi_2^1$  statement (like being Souslin).

We are given  $\bar{P}$ . We build here by induction on  $\delta \in S$  an increasing sequence of models  $M_\delta^1 = (\delta, \bar{P} \upharpoonright \delta, \bar{Q}^\delta)$  carrying some induction hypothesis. We want to have at the end that there is no  $\bar{R}$  such that  $(\lambda, \bar{P}, \bigcup_{\alpha} \bar{Q}^\alpha, \bar{R}) \models \varphi$ , so at stage  $\delta$  we look at  $\mathfrak{N}(M_\delta^1)$  as a candidate for the bad phenomena. For  $\varepsilon < \mathfrak{z}(M_\delta^1)$  (if  $\mathfrak{z}(M_\delta^1) = 0$  then our life is easier) in  $\mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$  we know

$$\mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1) \models \text{“}\mathfrak{B}_\varepsilon^\delta(M_\delta^1) \text{ is a transitive set of cardinality } \delta\text{”}.$$

So we can list all elements of  $\mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  in  $\mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$ , i.e. we have  $f \in \mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$ ,  $f : \delta \rightarrow \mathfrak{B}_\varepsilon^\delta(M_\delta^1)$  which is one-to-one and onto. From the outside point of view  $\delta$  has small cofinality and by **(k)**+**(l)** one can find a sequence  $\langle \alpha_\xi : \xi < \text{cf}(\delta) \rangle$  cofinal in  $\delta$  and such that every proper initial segment is in  $\mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$  (even in  $\mathfrak{B}_0^\delta(M_\delta^1)$ , usually even in  $L_\gamma$ ). So we have a fair chance to diagonalize over those sets to fulfill the obligation in the inductive construction of  $\bar{Q}^\delta$ , while “destroying” the possibility of  $\mathfrak{N}(M_\delta^1)$ .

But doing it for one  $\varepsilon$  does not suffice. However, if  $\text{cf}(\mathfrak{z}(M_\delta^1)) = \text{cf}(\delta)$  then we can do better. We can find  $f : \delta \rightarrow \mathfrak{B}^\delta(M_\delta^1)$ , one-to-one and onto and such that  $(\forall \alpha < \delta)(f \upharpoonright \alpha \in \mathfrak{B}^\delta(M_\delta^1))$  (remember:  $\mathfrak{B}^\delta(M_\delta^1) = \bigcup_{\varepsilon < \mathfrak{z}(M_\delta^1)} \mathfrak{B}_\varepsilon^\delta(M_\delta^1)$ ). This is possible as

$\langle \mathfrak{B}_\varepsilon^\delta(M_\delta^1) : \varepsilon \leq \zeta \rangle \in \mathfrak{B}_{\zeta+1}^\delta(M_\delta^1)$ , so we can choose  $f_\varepsilon \in \mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$  as the first one-to-one mapping from  $\delta$  onto  $\mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$ . So by a demand  $\langle f_\varepsilon : \varepsilon \leq \zeta \rangle \in \mathfrak{B}_{\varepsilon+1}^\delta(M_\delta^1)$ . Now by an easy manipulation we can combine them (using  $\langle \beta_\varepsilon : \varepsilon < \text{cf}(\delta) \rangle$ ).

In the proof of 3.9 and 3.4, to make  $\nu_p(G) = \lambda_p$ , we build together with  $G_\alpha$  also  $f^{p,\zeta}$  (for  $\zeta < \lambda_p$ ). We need that all non trivial combinations  $\sum_{\ell < n} a_\ell f^{p,\zeta^\ell} \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  are not of the form  $f/p\mathbb{Z}$ . This could be a typical application of the diamond. But we also need that for every  $f \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  there will be  $f' = \sum_{\ell < n} a_\ell f^{p,\zeta^\ell}$  and  $f^* \in \text{Hom}(G, \mathbb{Z})$  such that  $f - f' = f^*/p\mathbb{Z}$ . For this the normal thing is to apply  $\diamond_\lambda^+$  and to choose  $\langle a_\ell : \ell < n \rangle$  and  $f^* \upharpoonright G_\delta$  by giving them to approximations of  $f$ . But the two demands seem to be hard to go together without what was said above.

### 3. BUILDING ABELIAN GROUPS

One can think of  $\text{Ext}(G, K)$  as essentially the family of isomorphism types of models  $(K, H, G, g, h, c)_{c \in K \cup G}$  such that (in our case  $K, H, G$  are abelian groups; we will not mention this usually, and)  $h$  is an embedding from  $K$  into  $H$ ,  $g$  a homomorphism from  $H$  onto  $G$  with the range of  $h$  being the kernel of  $g$  (i.e.  $0 \rightarrow K \xrightarrow{h} H \xrightarrow{g} G \rightarrow 0$  being exact) up to isomorphism over  $K \cup G$ . Moreover it has a natural additive structure. So  $\text{Ext}(G, K) = 0$  if and only if for any  $0 \rightarrow K \xrightarrow{h} H \xrightarrow{g} G \rightarrow 0$  as above, the range of  $h$  is a direct summand of  $H$ .

We shall not define  $\text{Ext}(G, \mathbb{Z})$  fully as below we shall quote theorems characterizing it in a convenient way in the relevant cases.

In this section we show how to construct a group  $G$  such that  $\text{Ext}(G, \mathbb{Z})$  satisfies pre-given requirements (within well-known limitations, see below). The main tools in the construction are 2.1 and, as a kind of a single step, 3.4 below.

**Definition 3.1.** The quantifier  $(\forall^* i < \lambda)$  means “for every large enough  $i < \lambda$ ”, so this is an abbreviation for “ $(\exists j < \lambda)(\forall i \in (j, \lambda))$ ”.

**Definition 3.2.** (1) For a sequence  $\bar{\lambda} = \langle \lambda_\ell : \ell < n \rangle$  ( $n < \omega$ ) of pairwise distinct infinite regular cardinals,  $I_{\bar{\lambda}}$  is the ideal on  $\text{Dom}(I_{\bar{\lambda}}) = \prod_{\ell < n} \lambda_\ell$

(called its domain) such that

$$A \in I_{\bar{\lambda}} \text{ iff } (\forall^* i_0 < \lambda_0)(\forall^* i_1 < \lambda_1) \dots (\forall^* i_{n-1} < \lambda_{n-1})[\langle i_0, \dots, i_{n-1} \rangle \notin A].$$

(2) For any  $\lambda \geq \aleph_0$  we define  $\mathfrak{J}_\lambda$  as the set of ideals of the form  $I_{\bar{\lambda}}$  with  $\lambda = \max\{\lambda_\ell : \ell < n\}$ . Let  $\mathfrak{J}_{\leq \lambda} = \bigcup_{\mu \leq \lambda} \mathfrak{J}_\mu$ .

**Lemma 3.3.** *Suppose that  $\bar{\lambda} = \langle \lambda_\ell : \ell < n \rangle$  is a sequence of pairwise distinct infinite regular cardinals and  $\bar{\lambda}^{\text{dec}} = \langle \lambda'_\ell : \ell < n \rangle$  is the re-enumeration of  $\bar{\lambda}$  in the decreasing order. Let  $\pi : \prod_{\ell < n} \lambda_\ell \rightarrow \prod_{\ell < n} \lambda'_\ell$  be the canonical bijection (i.e.  $\pi(\eta)(\ell_0) = \eta(\ell_1)$  provided  $\lambda'_{\ell_0} = \lambda_{\ell_1}$ ). Then*

$$A \in I_{\bar{\lambda}} \quad \Rightarrow \quad \pi[A] \in I_{\bar{\lambda}^{\text{dec}}}.$$

*Proof.* This is an iterated application of the following observation:

**Claim 3.3.1.** *Let  $\lambda_0 < \lambda_1$  be regular infinite cardinals,  $\psi(x, y, \bar{z})$  be a formula. Then*

$$(\forall^* i_0 < \lambda_0)(\forall^* i_1 < \lambda_1) \psi(i_0, i_1, \bar{v}) \quad \Rightarrow \quad (\forall^* i_1 < \lambda_1)(\forall^* i_0 < \lambda_0) \psi(i_0, i_1, \bar{v}).$$

The claim should be clear and so the lemma.  $\square$

**Theorem 3.4** ( $V=L$ ). *Assume  $\lambda$  is a regular cardinal smaller than the first uncountable weakly compact cardinal. Suppose that  $I_k \in \mathfrak{J}_{\leq \lambda}$ , for  $k < k^* < \omega$ ,  $H$  is a free group with the free basis*

$$\{x_t^k : t \in \text{Dom}(I_k) \text{ and } k < k^*\}.$$

*Further, let  $p \in \mathbf{P}$  and let  $f^* \in \text{Hom}(H, \mathbb{Z})$  be a homomorphism such that for some  $\ell_0 < k^*$*

$$\{t \in \text{Dom}(I_{\ell_0}) : f^*(x_t^{\ell_0}) = 0\} \in I_{\ell_0}.$$

*Then there is a free group  $G$ ,  $H \subseteq G$  such that  $\|G\| = \lambda$ ,  $G/H$  is  $\lambda$ -free and:*

( $\alpha$ ) *there is no  $f \in \text{Hom}(G, \mathbb{Z})$  extending  $f^*$ ,*

- ( $\beta$ ) if  $k' < k^*$ ,  $A \in I_{k'}$  then  $G/\langle x_t^k : [k = k' \ \& \ t \in A] \text{ or } k \neq k' \rangle_H$  is free,  
 ( $\gamma$ ) if a homomorphism  $g^* \in \text{Hom}(H, \mathbb{Z})$  is such that for every  $k' < k^*$

$$\{t \in \text{Dom}(I_{k'}) : g^*(x_t^{k'}) \neq 0\} \in I_{k'}$$

and  $g^+ \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  extends  $g^*/p\mathbb{Z}$  then there exists  $g \in \text{Hom}(G, \mathbb{Z})$  such that  $g/p\mathbb{Z} = g^+$  and  $g^* \subseteq g$ ,

- ( $\delta$ ) if  $q \in \mathbf{P}$ ,  $h \in \text{Hom}(H, \mathbb{Z}/q\mathbb{Z})$  is such that for every  $k' < k^*$

$$\{t \in \text{Dom}(I_{k'}) : h(x_t^{k'}) \neq 0\} \in I_{k'}$$

then  $h$  can be extended to an element of  $\text{Hom}(G, \mathbb{Z}/q\mathbb{Z})$ .

*Proof.* Due to lemma 3.3 it is enough to prove the theorem under the assumption that the ideals  $I_k$  are determined by decreasing sequences  $\bar{\lambda}^k$  of regulars. The proof is by induction on  $\lambda$ . To carry out the induction we need the existence of stationary non reflecting sets and  $\diamond_\lambda^+$  only. However we will use this opportunity to show a simpler application of 2.1 and instead of the diamonds we will use our principle. The construction of 3.9, though more complicated, will be similar to the one here.

For  $k < k^*$  let  $I_k = I_{\bar{\lambda}^k}$ ,  $\bar{\lambda}^k = \langle \lambda_\ell^k : \ell < n_k \rangle$ ,  $\lambda_\ell^k \leq \lambda$ . Thus  $\text{Dom}(I_k) = \prod_{\ell < n_k} \lambda_\ell^k$  and according to what we noted earlier we assume that the sequences  $\bar{\lambda}^k$  are decreasing.

If  $\lambda = \aleph_0$  then  $\bigwedge_k n_k = 1$ : this case is easy and can be concluded from [EM], pp 362–363. However for the sake of the completeness we will sketch the construction (skipping only some technical details). The following claim gives us slightly more than needed:

**Claim 3.4.1.** *Suppose that  $H$  is a free group with basis  $\{x_n : n \in \omega\}$ ,  $f^* \in \text{Hom}(H, \mathbb{Z})$  is a homomorphism such that  $(\forall N \in \omega)(\exists n > N)(f^*(x_n) \neq 0)$ . Then there is a free group  $G \supseteq H$  such that  $G/H \cong \mathbb{Q}$  and*

- (1) *there is no  $f \in \text{Hom}(G, \mathbb{Z})$  extending  $f^*$ ,*  
 (2) *if  $A \subseteq \omega$  is infinite and  $h \in \text{Hom}(H, \mathbb{Z}/q\mathbb{Z})$  is such that*

$$\text{Ker}(h) \supseteq \langle x_n : n \in A \rangle_G$$

*then  $G/\langle x_n : n \notin A \rangle_G$  is free and  $h$  can be extended to a homomorphism from  $G$  to  $\mathbb{Z}/p\mathbb{Z}$ ,*

- (3) *if  $g^* \in \text{Hom}(H, \mathbb{Z})$  is such that  $(\exists N \in \omega)(\forall n > N)(g^*(x_n) = 0)$  and  $g^+ \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  extends  $g^*/p\mathbb{Z}$  then there is  $g \in \text{Hom}(G, \mathbb{Z})$  such that  $g^* \subseteq g$  and  $g/p\mathbb{Z} = g^+$ .*

*Proof of the claim.* Let  $A_0 = \{n \in \omega : f^*(x_n) \neq 0\}$  and let  $\{r_n : n \in A_0\}$  enumerate  $\mathbb{Z}$ . Choose inductively positive integers  $s_n$  and integers  $m_n$  such that

- (a)  $m_n \in \{-1, 1\}$   
 (b) if  $n \notin A_0$  then  $m_n = 1$ ,  $s_n = (n + p)!$   
 (c) if  $n \in A_0$  then  
 $r_n + s_0 \cdot \dots \cdot s_{n-1} m_n f^*(x_n) + s_0 \cdot \dots \cdot s_{n-2} m_{n-1} f^*(x_{n-1}) + \dots + s_0 m_1 f^*(x_1) + m_0 f^*(x_0) \neq 0$   
 $s_n = (n + p)! \cdot |r_n + s_0 \cdot \dots \cdot s_{n-1} m_n f^*(x_n) + s_0 \cdot \dots \cdot s_{n-2} m_{n-1} f^*(x_{n-1}) + \dots + s_0 m_1 f^*(x_1) + m_0 f^*(x_0)|$

Now, let  $G$  be the group generated freely by  $\{y_n : n \in \omega\} \cup \{x_n : n \in \omega\}$  except that

$$(*) \quad s_n y_{n+1} = y_n + m_n x_n.$$

Note that the condition  $(*)$  implies that for each  $k > 0$ ,  $n \in \omega$

$$(**)_n^k \quad y_n = s_n s_{n+1} \cdots s_{n+k} y_{n+k+1} - [s_n s_{n+1} \cdots s_{n+k-1} m_{n+k} x_{n+k} + s_n \cdots s_{n+k-2} m_{n+k-1} x_{n+k-1} + \cdots + s_n m_{n+1} x_{n+1} + m_n x_n].$$

0.  $G$  is freely generated by  $\{y_n : n \in \omega\}$  and  $G/H \cong \mathbb{Q}$ .

1. There is no  $f \in \text{Hom}(G, \mathbb{Z})$  extending  $f^*$ .

Why? By  $(**)_0^0$  the value of  $f$  at  $y_0$  determines  $f(y_{n+1})$  in the way that is excluded by the choice of  $s_n$  for  $n \in A_0$  (clause (c)).

2. If  $A \subseteq \omega$  is infinite then  $G/\langle x_n : n \notin A \rangle$  is free.

Why? Let  $\{n_k : k \in \omega\} = A$  be the increasing enumeration. Let  $G_i = \langle y_n : n_{i-1} < n \leq n_i \rangle_G$ ,  $H_i = \langle x_n : n_{i-1} < n < n_i \rangle_G$  (with a convention that  $n_{-1} = -1$ ). Then  $G = \bigoplus_{i \in \omega} G_i$ ,  $H_i = G_i \cap \langle x_n : n \notin A \rangle_G$  and  $\langle x_n : n \notin A \rangle_G = \bigoplus_{i < \omega} H_i$ . The groups  $G_i/H_i$  are (freely) generated by  $y_{n_i}/H_i$ . Hence  $G/H$  is free. Extending suitable homomorphisms into  $\mathbb{Z}/q\mathbb{Z}$  should be clear now.

3. If  $g^* \in \text{Hom}(H, \mathbb{Z})$  is such that  $(\exists N)(\forall n > N)(g^*(x_n) = 0)$  and  $g^+ \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  extends  $g^*/p\mathbb{Z}$  then there is  $g \in \text{Hom}(G, \mathbb{Z})$  such that  $g^* \subseteq g$  and  $g/p\mathbb{Z} = g^+$ .

Why? First note that there is at most one homomorphism  $g^+ \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  such that  $g^+ \supseteq g^*/p\mathbb{Z}$ . This is because  $G/H \cong \mathbb{Q}$ : if  $g_1^+, g_2^+ \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  agree on  $H$  then  $g_1^+ - g_2^+ \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ ,  $\text{Ker}(g_1^+ - g_2^+) \supseteq H$  and hence  $(g_1^+ - g_2^+)/H \in \text{Hom}(G/H, \mathbb{Z}/p\mathbb{Z})$ . But the only homomorphism of  $\mathbb{Q}$  into  $\mathbb{Z}/p\mathbb{Z}$  is the trivial one.

Hence it is enough to show that there is an extension  $g$  of  $g^*$  to a member of  $\text{Hom}(G, \mathbb{Z})$  (as then necessarily  $g/p\mathbb{Z} = g^+$  by the uniqueness).

Now let  $N$  be such that  $(\forall n > N)(g^*(x_n) = 0)$ . Define

$$\begin{aligned} g(y_n) &= 0 \text{ for } n > N, \\ g(y_N) &= -m_N g^*(x_N), \\ g(y_n) &= -[s_n s_{n+1} \cdots s_{N-1} m_N g^*(x_N) + s_n \cdots s_{N-2} m_{N-1} g^*(x_{N-1}) + \cdots + s_n m_{n+1} g^*(x_{n+1}) + m_n g^*(x_n)] \text{ for } n < N \end{aligned}$$

and extend it to a homomorphism from  $\text{Hom}(G, \mathbb{Z})$ . Clearly this  $g$  satisfies  $g^* \subseteq g$ . This finishes the proof of the claim.  $\square$

Assume now that  $\lambda$  is smaller than the first weakly compact uncountable cardinal,  $\lambda > \aleph_0$  and below  $\lambda$  the theorem holds.

We may think that for some  $k_0 \leq k^*$  we have

$$k < k_0 \Rightarrow \lambda_0^k < \lambda \quad \text{and} \quad k_0 \leq k < k^* \Rightarrow \lambda_0^k = \lambda.$$

Of course we may assume that  $k_0 < k^*$  (otherwise the inductive hypothesis applies directly).

Recall that  $\ell_0 < k^*$  is such that

$$\{t \in \text{Dom}(I_{\ell_0}) : f^*(x_t^{\ell_0}) = 0\} \in I_{\ell_0}.$$

Let  $\alpha_0$  be defined as follows:

if  $\ell_0 < k_0$  then  $\alpha_0 = 0$ ,



if  $k_0 \leq \ell_0 < k^*$  and  $n_{\ell_0} > 1$  then  $\alpha_0$  is (the first) such that

$$(\forall \alpha > \alpha_0)(\forall^* i_1 < \lambda_1^{\ell_0}) \dots (\forall^* i_{n_{\ell_0}-1} < \lambda_{n_{\ell_0}-1}^{\ell_0})(f^*(x_{\langle \alpha, i_1, \dots, i_{n_{\ell_0}-1} \rangle}^{\ell_0}) \neq 0),$$

if  $k_0 \leq \ell_0 < k^*$  and  $n_{\ell_0} = 1$  then  $\alpha_0 < \lambda$  is the first ordinal such that  $(\forall i_0 > \alpha_0)(f^*(x_{\langle i_0 \rangle}^{\ell_0}) \neq 0)$ .

We may assume that the group  $H$  has universe  $\{2i : i < \lambda\}$ . Moreover we may have an increasing continuous sequence  $\langle \gamma_\alpha : \alpha < \lambda \rangle$  of limit ordinals such that

$$\boxtimes_1 \{x_t^k : t \in \text{Dom}(I_k) \ \& \ k < k_0\} \subseteq \gamma_0, \ \alpha_0 < \gamma_0 \ \text{and} \ \gamma_1^k < \gamma_0 \ \text{if} \ k_0 \leq k < k^*, \\ 1 < n_k, \ \text{and}$$

$$\boxtimes_2 H_\alpha \stackrel{\text{def}}{=} H \upharpoonright \{2i : i < \gamma_\alpha\} \ \text{is the subgroup of } H \ \text{generated by}$$

$$\{x_t^k : k < k_0 \ \text{or} \ [k_0 \leq k < k^* \ \& \ t(0) < \gamma_\alpha]\}.$$

For  $k < k^*$  we define the reduction  $I_k^{\text{red}}$  of the ideal  $I_k$  by:

if  $k < k_0$  then  $I_k^{\text{red}} = I_k$ ,

if  $k_0 \leq k < k^*$  and  $n_k > 1$  then  $I_k^{\text{red}} = I_{\langle \lambda_1^k, \dots, \lambda_{n_k-1}^k \rangle}$  and

if  $k_0 \leq k < k^*$  and  $n_k = 1$  then  $I_k^{\text{red}} = I_{\langle \aleph_0 \rangle}$ .

Next we define  $y_t^k[\gamma_\alpha]$  (for  $k < k^*$ ,  $t \in \text{Dom}(I_k^{\text{red}})$ ) as

$$\begin{aligned} x_t^k & \quad \text{if } k < k_0, \\ x_{\langle \gamma_\alpha \rangle \smallfrown t}^k & \quad \text{if } k_0 \leq k < k^*, \ n_k > 1, \\ x_{\langle \gamma_\alpha + t(0) \rangle}^k & \quad \text{if } k_0 \leq k < k^*, \ n_k = 1. \end{aligned}$$

It follows from  $\boxtimes_1$ ,  $\boxtimes_2$  and the fact that  $\gamma_\alpha$  are limit ordinals that  $y_t^k[\gamma_\alpha] \in H_{\alpha+1}$  for all  $k < k^*$ ,  $t \in \text{Dom}(I_k^{\text{red}})$ . The subgroup generated by these elements with some side elements will be the one to which we will apply the inductive hypothesis.

Let  $E \subseteq \text{acc}(\{\alpha < \lambda : \alpha = \gamma_\alpha\})$  be a thin enough club of  $\lambda$ . By our assumptions we find a stationary set  $S \subseteq E$  such that

- ( $\alpha$ ) for every  $\theta = \text{cf}(\theta) < \lambda$ ,  $\{\delta \in S : \text{cf}(\delta) = \theta\}$  is stationary,
- ( $\beta$ )  $S$  a set of singular limit ordinals,
- ( $\gamma$ )  $\lambda = \mu^+ \Rightarrow S \subseteq [\mu + 1, \lambda)$  and if  $\lambda$  is inaccessible then  $S$  is a set of strong limit singular cardinals and
- ( $\delta$ )  $S$  does not reflect.

We will use the principle formulated in 2.1 to choose by induction on  $\alpha < \lambda$  a group  $G_\alpha$  with the universe  $\gamma_\alpha$  and extending  $H_\alpha$ . For this we have to define finite vocabularies  $\bar{P}$ ,  $\bar{Q}$ ,  $\bar{R}$  and a formula  $\varphi$ . Thus we declare that  $\bar{P}$  is  $\langle P_0, P_1, \dots \rangle$ ,  $P_0$  a unary predicate and  $P_1$  a unary function symbol,  $\bar{Q} = \langle Q_0, \dots \rangle$  where  $Q_0$  is a binary function symbol, and  $\langle M, \bar{Q}^M, \bar{R}^M \rangle \models \varphi$  means:

- (a)  $\langle M, Q_0^M \rangle$  is a group,  $P_0^M$  is its subgroup (intension:  $H$ ), and  $P_1^M \upharpoonright P_0^M \in \text{Hom}(P_0^M, \mathbb{Z})$  (intension:  $f^*$ )

[we should use some additional predicates to encode  $\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}, \dots$ ],

- (b)  $\bar{R}^M$  encodes a homomorphism  $f \in \text{Hom}(M, \mathbb{Z})$  extending  $f^*$ .

Let functions  $\mathfrak{N}^\delta, \mathfrak{B}_\varepsilon^\delta, \mathfrak{J}$  be given by 2.1 for  $\bar{P}$ ,  $\bar{Q}$ ,  $\bar{R}$  and  $\varphi$  as defined above.

Now, by induction on  $\alpha < \lambda$ , we choose models  $M_\alpha^1$  (i.e. groups  $G_\alpha$ , their subgroups  $P_0^{M_\alpha^1} = H_\alpha$  and homomorphisms  $f_\alpha^+, f_\alpha^*$ ) and  $T_\alpha, R_\alpha, h_{(g^*, g^+, C)}$  (for  $(g^*, g^+, C) \in R_\alpha$ ),  $R_\alpha^q$  and  $h_{(h, C)}^+$  (for  $(h, C) \in R_\alpha^q$ ) such that:

- (1)  $\langle G_\alpha : \alpha < \lambda \rangle$  is an increasing continuous sequence of free groups,
- (2)  $P_0^{M_\alpha^1} = H_\alpha \subseteq G_\alpha$ ,  $G_\alpha$  is a (free) group on  $\gamma_\alpha$ ,
- (3) if  $\beta < \alpha$  then  $G_\alpha/G_\beta$  is free,
- (4)  $G_\alpha/H_\alpha$  is free,
- (5) if  $\alpha \notin S$ ,  $\alpha < \beta$  then  $G_\beta/(G_\alpha + H_\beta)$  is a free group with a basis of size  $\|\gamma_\beta\|$ ,
- (6) if  $\alpha \in S$ ,  $k' < k^*$ ,  $A \in I_{k'}^{\text{red}}$  then  $G_{\alpha+1}/(G_\alpha + H_{\alpha+1}^A)$  is free, where  $H_{\alpha+1}^A$  is the group generated by all elements  $x_t^k$  such that  $k < k^*$ ,  $t \in \text{Dom}(I_k)$ ,  $x_t^k \in H_{\alpha+1}$  but  $(\forall s \in \text{Dom}(I_{k'}^{\text{red}}))(x_t^k = y_s^{k'}[\gamma_\alpha] \Rightarrow s \in A)$ ,
- (7)  $f_\alpha^* = f^* \upharpoonright H_\alpha$ .
- (8) For  $\alpha \in E$  let  $N^\alpha = \mathfrak{B}^\alpha(M_\alpha^1)$ ,  $\mathfrak{z}_\alpha = \mathfrak{z}(M_\alpha^1)$ ,  $N_i^\alpha = \mathfrak{B}_i^\alpha(M_\alpha^1)$  for  $i < \mathfrak{z}_\alpha$ .

*Remark:* Since the group  $G_\alpha/H_\alpha$  is free we have  $M_\alpha^1 \in \mathbf{k}_\alpha^-$ . If  $\mathfrak{z}(M_\alpha^1) = 0$  then  $N^\alpha$  is empty, and below  $T_\alpha = R_\alpha = \emptyset$ .

Assume  $\alpha \in E$  (so  $\gamma_\alpha = \alpha$ ). Then  $T_\alpha$  is the family of all pairs  $(g^*, g^+)$  of homomorphisms  $g^* \in \text{Hom}(H_\alpha, \mathbb{Z}) \cap N^\alpha$ ,  $g^+ \in \text{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z}) \cap N^\alpha$  such that  $g^*/p\mathbb{Z} \subseteq g^+$ .

- (9)  $R_\alpha$  is the family of all triples  $(g^*, g^+, C)$  such that:
  - (a)  $C \in N^\alpha$  is a nonempty closed subset of  $\alpha \cap E$ ,  $(g^*, g^+) \in T_\alpha$ ,
  - (b) for  $\beta \in C$ :  $(g^* \upharpoonright H_\beta, g^+ \upharpoonright G_\beta, C \cap \beta) \in R_\beta$ ,
  - (c) for  $\beta < \gamma$  in  $C$ :  $h_{(g^* \upharpoonright H_\beta, g^+ \upharpoonright G_\beta, C \cap \beta)} \subseteq h_{(g^* \upharpoonright H_\gamma, g^+ \upharpoonright G_\gamma, C \cap \gamma)}$ ,
  - (d) if  $\beta \in [\min C, \alpha) \cap E$  then for all  $k' < k^*$ :

$$\{t \in \text{Dom}(I_{k'}^{\text{red}}) : g^*(y_t^{k'}[\gamma_\beta]) \neq 0\} \in I_{k'}^{\text{red}}.$$

$R_\alpha^q$  (for  $q \in \mathbf{P}$ ) is the family of all pairs  $(h, C)$  such that:

- (a)  $C \in N^\alpha$  is a nonempty closed subset of  $\alpha \cap E$ ,  $h \in \text{Hom}(H_\alpha, \mathbb{Z}/q\mathbb{Z}) \cap N^\alpha$ ,
- (b) for  $\beta \in C$ :  $(h \upharpoonright H_\beta, C \cap \beta) \in R_\beta^q$ ,
- (c) for  $\beta < \gamma$  in  $C$ :  $h_{(h \upharpoonright H_\beta, C \cap \beta)}^+ \subseteq h_{(h \upharpoonright H_\gamma, C \cap \gamma)}^+$ ,
- (d) if  $\beta \in [\min C, \alpha) \cap E$  then for all  $k' < k^*$ :

$$\{t \in \text{Dom}(I_{k'}^{\text{red}}) : h(y_t^{k'}[\gamma_\beta]) \neq 0\} \in I_{k'}^{\text{red}}.$$

- (10) If  $(g^*, g^+, C) \in R_\alpha$  then  $h_{(g^*, g^+, C)} \in \text{Hom}(G_\alpha, \mathbb{Z}) \cap N^\alpha$ ,  $h_{(g^*, g^+, C)}/p\mathbb{Z} = g^+$ ,  $g^* \subseteq h_{(g^*, g^+, C)}$  and

$$\beta \in C \Rightarrow h_{(g^* \upharpoonright H_\beta, g^+ \upharpoonright G_\beta, C \cap \beta)} \subseteq h_{(g^*, g^+, C)},$$

if  $(h, C) \in R_\alpha^q$  then  $h_{(h, C)}^+ \in \text{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z})$  extends  $h \cup \bigcup_{\beta \in C} h_{h \upharpoonright H_\beta, C \cap \beta}^+$ .

- (11) If  $(g^*, g^+, C) \in R_\alpha$ ,  $g_0^* \in \text{Hom}(H_{\alpha+1}, \mathbb{Z})$  is such that  $g^* \subseteq g_0^*$  and for every  $k' < k^*$ :

$$\{t \in \text{Dom}(I_{k'}^{\text{red}}) : g_0^*(y_t^{k'}[\gamma_\alpha]) \neq 0\} \in I_{k'}^{\text{red}}$$

and if  $g_0^+ \in \text{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$  is such that  $g_0^*/p\mathbb{Z} \subseteq g_0^+$ ,  $g^+ \subseteq g_0^+$  then there is  $h' \in \text{Hom}(G_{\alpha+1}, \mathbb{Z})$  extending  $h_{(g^*, g^+, C)}$  and such that  $g_0^* \subseteq h'$ ,  $h'/p\mathbb{Z} = g_0^+$ ;

if  $(h, C) \in R_\alpha^q$ ,  $h_0 \in \text{Hom}(H_{\alpha+1}, \mathbb{Z}/q\mathbb{Z})$  is such that  $h \subseteq h_0$  and for every  $k' < k^*$

$$\{t \in \text{Dom}(I_{k'}^{\text{red}}) : h_0(y_t^{k'}[\gamma_\alpha]) \neq 0\} \in I_{k'}^{\text{red}}$$

then there is  $h' \in \text{Hom}(G_{\alpha+1}, \mathbb{Z}/q\mathbb{Z})$  extending  $h_{(h, C)}^+ \cup h_0$ .

(12) Assume that  $\alpha \in S$  and

$$\mathfrak{N}(M_\alpha^1) = \langle G_\alpha, H_\alpha, f_\alpha^*, \dots, f \rangle,$$

where  $f \in \text{Hom}(G_\alpha, \mathbb{Z})$ .

If there is a free group  $G^*$ ,  $G_\alpha \cup H_{\alpha+1} \subseteq G^*$  such that:  $\|G^*\| = \|\gamma_{\alpha+1}\|$ ,  $G^*$  satisfies (2)—(6), (11) (with  $G^*$  playing the role of  $G_{\alpha+1}$ ) and

(\*) there is no  $g' \in \text{Hom}(G^*, \mathbb{Z})$  extending  $f \cup f_{\alpha+1}^*$

then  $G_{\alpha+1}$  satisfies (\*) too.

The limit stages of the construction are actually determined by the continuity demands of (1), (7). Concerning the requirements (2)—(5) note that (2) is preserved because of (3) at previous stages, (3) is preserved because of (2), (4) is kept due to (5) and the fact that the set  $S$  is not reflecting and finally (5) holds at the limit because of (4) at previous stages and non reflection of  $S$  (see e.g. Proposition IV.1.7 of [EM]). There is some uncertainty in defining  $h_{(g^*, g^+, C)}$  for  $(g^*, g^+, C) \in R_\alpha$  (for  $\alpha \in E$ ). However it is possible to find a suitable  $h_{(g^*, g^+, C)}$  since in the most difficult case when  $\sup C < \alpha$ ,  $\sup C \in S$  we may apply first (11) and then (5). Similarly we handle  $h_{(h, C)}^+$ .

If  $\alpha \notin S$  then we choose a group  $G_{\alpha+1} \supseteq H_{\alpha+1} \cup G_\alpha$  such that  $G_{\alpha+1}/(H_{\alpha+1} + G_\alpha)$  is a free group with a basis of size  $\|\gamma_{\alpha+1}\|$ .

If  $\alpha \in S$  then condition (12) of the construction describes  $G_{\alpha+1}$ . (We will see later that this condition is not empty, i.e. that there is a group  $G^*$  as there.)

Thus we have carried out the definition and we may put  $G = G_\lambda = \bigcup_{\alpha < \lambda} G_\alpha$ .

Let us check that the  $G$  satisfies the desired properties (the main point will be the requirement  $(\alpha)$  of the theorem).

By (2) and (3) the group  $G$  is free of cardinality  $\lambda$  (and the set of elements is  $\lambda$ ) and it extends each  $G_\alpha$ . Due to (4) the quotient  $G/H$  is  $\lambda$ -free.

**Clause  $(\alpha)$  of the assertion**

This is a consequence of the condition (12) of the construction. Suppose that the homomorphism  $f^*$  has an extension to a homomorphism in  $\text{Hom}(G, \mathbb{Z})$ . This means that

$$M^1 = \langle G, H, f^*, \dots \rangle \in \mathbf{k}_\lambda^-.$$

By condition (i) of 2.1 we find a regular cardinal  $\sigma \leq \lambda$  and

$$M^2 = \langle G, H, f^*, \dots, f \rangle \in \mathbf{k}_\lambda^+(M^1) \quad \text{and } \alpha \in S$$

such that  $\mathfrak{N}(M_\alpha^1) = M_\alpha^2 \prec M^2$  and

$$\begin{aligned} \text{cf}(\alpha) = \omega_1 & & \text{if } \lambda = \aleph_2, \\ \text{cf}(\alpha) \neq \text{cf}(\mu) & & \text{if } \lambda = \mu^+, \mu \text{ is an uncountable limit cardinal,} \\ \text{cf}(\alpha) \notin \{\mu, \text{cf}(\theta)\} & & \text{if } \lambda = \mu^+, \mu = \theta^+, \theta > \aleph_0 \end{aligned}$$

(remember (i)( $\alpha$ )). Look now at the stage  $\alpha$  of the construction.

Before we continue with the proof we give a claim which helps us to apply the inductive hypothesis.

**Claim 3.4.2.** (1) If  $R \subseteq \text{Hom}(G_\alpha, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z})$ ,  $2^{\|R\|} < \|\alpha\|$  then

for every  $\beta \in \alpha \setminus S$  large enough there is  $x \in G_\alpha$  such that:

(a)  $(\forall h \in R)(h(x) = 0)$

(b)  $x \in G_{\beta+1}$  is a member of a basis of  $G_{\beta+1}$  over  $H_{\beta+1} + G_\beta$ .

- (2) Suppose that  $\mu < \alpha < \mu^+$ ,  $\mu$  is a limit cardinal (so we are in the case  $\lambda = \mu^+$ ),  $R \subseteq \text{Hom}(G_\alpha, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z})$  and  $\|R\| = \mu$ . Then for each  $\beta \in (\mu, \alpha) \setminus S$  there exist  $x_j \in G_{\beta+1}$  for  $j < \text{cf}(\mu)$  such that:
- (a) if  $h \in R$  then the set  $\{j < \text{cf}(\mu) : h(x_j) \neq 0\}$  is bounded in  $\text{cf}(\mu)$ ,
  - (b)  $\{x_j : j < \text{cf}(\mu)\}$  can be extended to a basis of  $G_{\beta+1}$  over  $H_{\beta+1} + G_\beta$ .
- (3) Suppose that  $\theta^+ = \mu < \alpha < \mu^+$ ,  $R \subseteq \text{Hom}(G_\alpha, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z})$ ,  $\|R\| = \mu$ . Then for every  $\beta \in (\mu, \alpha) \setminus S$  there exists a sequence  $\langle x_{j,k} : j < \mu, k < \text{cf}(\theta) \rangle \subseteq G_{\beta+1}$  such that
- (a) if  $h \in R$  then  $\{(j,k) \in \mu \times \text{cf}(\theta) : h(x_{j,k}) \neq 0\} \in I_{\langle \mu, \text{cf}(\theta) \rangle}$ ,
  - (b)  $\langle x_{j,k} : j < \mu, k < \text{cf}(\theta) \rangle$  can be extended to a basis of  $G_{\beta+1}$  over  $H_{\beta+1} + G_\beta$ .
- (4) Suppose  $\aleph_1 < \alpha < \aleph_2$  (so  $\lambda = \aleph_2$ ),  $R \subseteq \text{Hom}(G_\alpha, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z})$ ,  $\|R\| = \aleph_0$ . Then for each  $\beta \in (\aleph_1, \alpha) \setminus S$  there are  $x_\ell \in G_{\beta+1}$  (for  $\ell < \omega$ ) such that
- (a) if  $h \in R$  then the set  $\{\ell < \omega : h(x_\ell) \neq 0\}$  is finite,
  - (b)  $\{x_\ell : \ell < \omega\}$  can be extended to a basis of  $G_{\beta+1}$  over  $H_{\beta+1} + G_\beta$ .

*Proof of the claim.* 1) Let  $\beta_0 < \alpha$  be such that  $\|\gamma_{\beta_0}\| > 2^{\|R\|}$  (remember  $\alpha \in S \subseteq E$ , see the choice of  $S$ ). Let  $\beta \in \alpha \setminus S$ ,  $\beta > \beta_0$ . Let  $\{y_\xi : \xi < \gamma_{\beta+1}\}$  be a free basis of  $G_{\beta+1}$  over  $H_{\beta+1} + G_\beta$  (exists by condition (5) of the construction). If  $R$  is finite then considering first  $\|R\| + 1$  elements of the basis we find a respective point  $x$  in the group generated by them. If  $R$  is infinite, so  $2^{\|R\|} = \|\mathbb{Z}^R\|$ , then we find  $\xi_0 < \xi_1 < \gamma_{\beta+1}$  such that  $(\forall h \in R)(h(y_{\xi_0}) = h(y_{\xi_1}))$  and we may put  $x = y_{\xi_0} - y_{\xi_1}$ .

2) We follow exactly the lines of (1), but first we have to choose an increasing sequence  $\langle R_j : j < \text{cf}(\mu) \rangle$  such that  $\bigcup_{j < \text{cf}(\mu)} R_j = R$ ,  $\|R_j\| < \|R\|$  (and hence

$2^{\|R_j\|} < \mu$  as  $\mu$  is a limit cardinal). Now if  $\beta \in (\mu, \alpha) \setminus S$  then we find  $\langle x_j : j < \text{cf}(\mu) \rangle \subseteq G_{\beta+1}$  which can be extended to a basis of  $G_{\beta+1}$  over  $H_{\beta+1} + G_\beta$  and such that  $(\forall h \in R_j)(h(x_j) = 0)$ .

3) Similarly: first find  $\langle R_{j,k} : j < \mu, k < \text{cf}(\theta) \rangle$  such that  $\|R_{j,k}\| < \theta$ , the sequence  $\langle \bigcup_{k < \text{cf}(\theta)} R_{j,k} : j < \mu \rangle$  is increasing, for each  $j < \mu$  the sequence  $\langle R_{j,k} : k < \text{cf}(\theta) \rangle$  is

increasing and  $\bigcup_{j < \mu} \bigcup_{k < \text{cf}(\theta)} R_{j,k} = R$ . Next follow as in 2).

4) Represent  $R$  as an increasing (countable) union of finite sets and follow as in 2) above. The claim is proved.  $\square$

Now we are going back to the proof of clause (a). The following claim will finish it.

**Claim 3.4.3.** *Suppose that  $\alpha, f \dots$  are as chosen earlier. Then there exists a free group  $G^* \supseteq H_{\alpha+1} \cup G_\alpha$  such that  $\|G^*\| = \|\gamma_{\alpha+1}\|$ ,  $G^*$  satisfies the conditions (2)–(6), (11) of the construction as  $G_{\alpha+1}$  and there is no  $g' \in \text{Hom}(G^*, \mathbb{Z})$  extending  $f \upharpoonright G_\alpha \cup f_{\alpha+1}^*$ .*

*Proof of the claim.* Let  $R = \{h_{(g^*, g^+, C)} : (g^*, g^+, C) \in R_\alpha\} \cup \{h_{(h, C)}^+ : (h, C) \in R_\alpha^q, q \in \mathbf{P}\}$ . By clauses (g) and (b) of 2.1 we have  $\|R\| \leq \|\alpha\|$  (of course  $R$  may

be empty). Let  $\langle \alpha_i : i < \text{cf}(\alpha) \rangle$  be an increasing continuous sequence cofinal in  $\alpha$  and disjoint from  $S$  (possible by the choice of  $S$ ).

**Case A:**  $\alpha$  is a strongly limit singular cardinal (so we are in the case when  $\lambda$  is inaccessible).

We find an increasing sequence  $\langle R_i^* : i < \text{cf}(\alpha) \rangle$  such that  $\bigcup_{i < \text{cf}(\alpha)} R_i^* = R$  and  $\|R_i^*\| < \|\alpha\|$ . But in this case we have

$$(\forall i < \text{cf}(\alpha))(2^{\|R_i^*\|} < \alpha).$$

So we may apply claim 3.4.2(1) to choose by induction on  $i < \text{cf}(\alpha)$  an increasing sequence  $\langle j_i : i < \text{cf}(\alpha) \rangle \subseteq \text{cf}(\alpha)$  and  $x_i^{k^*} \in G_{\alpha_{j_i+1}}$  such that:

- (a)  $h \in R_i^* \Rightarrow h(x_i^{k^*}) = 0$
- (b)  $x_i^{k^*}$  is a member of a basis of  $G_{\alpha_{j_i+1}}$  over  $H_{\alpha_{j_i+1}} + G_{\alpha_{j_i}}$ .

Since  $\langle \alpha_i : i < \text{cf}(\alpha) \rangle \subseteq \alpha \setminus S$  is increasing continuous (and cofinal in  $\alpha$ ) we get that  $\{x_i^{k^*} : i < \text{cf}(\alpha)\}$  can be extended to a basis of  $G_\alpha$  over  $H_\alpha$ . Now we apply the inductive hypothesis to  $k^* + 1$ ,  $I_k^{\text{red}}$  (for  $k < k^*$ ),  $I_{\langle \text{cf}(\alpha) \rangle}$ , the group  $H^*$  generated by

$$\{y_t^k[\gamma_\alpha] : k < k^*, t \in \text{Dom}(I_k^{\text{red}})\} \cup \{x_i^{k^*} : i < \text{cf}(\alpha)\}$$

and the function  $(f \cup f_{\alpha+1}^*) \upharpoonright H^*$ . This gives us a group  $G_0^* \supseteq H^*$ . Let  $H'$  be such that  $G_\alpha + H_{\alpha+1} = H^* \oplus H'$ . Then put  $G^* = G_0^* \oplus H'$ . It satisfies the requirements of the claim: condition (3) follows from the presence of the  $y_t^k[\gamma_\alpha]$ 's part of  $H^*$  (remember the inductive assumption 3.4( $\beta$ )), condition (4) holds due to the  $x_i^{k^*}$ . It follows from the fact that the  $\alpha_{j_i}$  are cofinal in  $\alpha$  (and from the choice of  $x_i^{k^*} \in G_{\alpha_{j_i+1}}$ ) that (5) is satisfied. Similarly, (11) is a consequence of the choice of  $x_i^{k^*}$  and the inductive hypothesis 3.4( $\gamma, \delta$ ). Finally clause (6) follows from the inductive assumption 3.4( $\beta$ )

**Case B:**  $\aleph_0 < \alpha < \aleph_1$  (so  $\lambda = \aleph_1$ ).

Thus  $R$  is at most countable, so let  $R = \bigcup_{\ell < \omega} R_\ell$ , where  $R_\ell$  are finite increasing with  $\ell$ . Apply 3.4.2(1) to find an increasing sequence  $\langle j_\ell : \ell < \omega \rangle \subseteq \omega$  and  $x_\ell^{k^*} \in G_{\alpha_{j_\ell+1}}$  such that

- (a)  $h \in R_\ell \Rightarrow h(x_\ell^{k^*}) = 0$ ,
- (b)  $x_\ell^{k^*}$  is a member of a basis of  $G_{\alpha_{j_\ell+1}}$  over  $H_{\alpha_{j_\ell+1}} + G_{\alpha_{j_\ell}}$ .

Proceed as in Case A (so apply the inductive hypothesis to  $I_k^{\text{red}}$  (for  $k < k^*$ ) and  $I_{\langle \aleph_0 \rangle}$ ).

**Case C:**  $\alpha \in (\mu, \mu^+)$  for some limit cardinal  $\mu > \aleph_0$  (so  $\lambda = \mu^+$ ).

Then we have  $\|R\| \leq \mu$  and by claim 3.4.2(2) we can choose  $x_{i,j}^{k^*} \in G_{\alpha_i+1}$  (for  $i < \text{cf}(\alpha)$ ,  $j < \text{cf}(\mu)$ ) such that:

- (a) for each  $h \in R$  for every  $i < \text{cf}(\alpha)$  the set  $\{j < \text{cf}(\mu) : h(x_{i,j}^{k^*}) \neq 0\}$  is bounded,
- (b) for each  $i < \text{cf}(\alpha)$  the set  $\{x_{i,j}^{k^*} : j < \text{cf}(\mu)\}$  extends to a basis of  $G_{\alpha_i+1}$  over  $H_{\alpha_i+1} + G_{\alpha_i}$ .

Now apply the inductive hypothesis for  $k^* + 1$ ,  $I_k^{\text{red}}$  (for  $k < k^*$ ) and  $I_{\langle \text{cf}(\alpha), \text{cf}(\mu) \rangle}$  (remember that  $\text{cf}(\alpha) \neq \text{cf}(\mu)$  in this case).

**Case D:**  $\aleph_1 < \alpha < \aleph_2$  (so  $\lambda = \aleph_2$  and  $\text{cf}(\alpha) = \omega_1$ ).

Write  $R$  as an increasing union  $\bigcup_{i < \omega_1} R_i$  of countable sets. Using 3.4.2(4) choose  $x_{i,\ell}^{k^*}$

(for  $i < \omega_1, \ell < \omega$ ) such that

- (a) for each  $h \in R$ , for every sufficiently large  $i < \omega_1$  the set  $\{\ell < \omega : h(x_{i,\ell}^{k^*}) \neq 0\}$  is finite,
- (b) for each  $i < \omega_1$  the set  $\{x_{i,\ell}^{k^*} : \ell < \omega\}$  can be extended to a basis of  $G_{\alpha_i+1}$  over  $H_{\alpha_i+1} + G_{\alpha_i}$ .

Proceed as above (using  $I_k^{\text{red}}$  (for  $k < k^*$ ) and  $I_{(\aleph_1, \aleph_0)}$ ).

**Case E:**  $\alpha \in (\mu, \mu^+)$  for some cardinal  $\mu$  such that  $\mu = \theta^+ > \aleph_1$  (so  $\lambda = \mu^+$ ).

Using claim 3.4.2(3) we choose a sequence  $\langle x_{l,j,i}^{k^*} : l < \text{cf}(\alpha), j < \mu, i < \text{cf}(\theta) \rangle$  such that

- (a) for each  $h \in R$  and for every  $l < \text{cf}(\alpha)$ , for every  $j < \mu$  large enough for every  $i < \text{cf}(\theta)$  large enough,  $h(x_{l,j,i}^{k^*}) = 0$ .
- (b) for every  $l < \text{cf}(\alpha)$  the set  $\{x_{l,j,i}^{k^*} : j < \mu, i < \text{cf}(\theta)\}$  can be extended to a basis of  $G_{\alpha_l+1}$  over  $H_{\alpha_l+1} + G_{\alpha_l}$ .

Now apply the inductive hypothesis to  $k^* + 1$ ,  $I_k^{\text{red}}$  ( $k < k^*$ ) and  $I_{(\text{cf}(\alpha), \mu, \text{cf}(\theta))}$  (remember  $\text{cf}(\alpha) \notin \{\mu, \text{cf}(\theta)\}$  in this case).

This completes the proof of claim 3.4.3. □

It follows from the above claim that at the stage  $\alpha$  of the construction we had a nontrivial application of the condition (12) “killing” the function  $f$ . This gives a contradiction proving the clause  $(\alpha)$ .

**Clause  $(\beta)$  of the assertion**

It follows from conditions (6) and (5) of the construction.

**Clause  $(\gamma)$  of the assertion**

Assume that  $g^*, g^+$  are as there. Then by the clause **(j)** of 2.1 we have a club  $C \subseteq E$  such that for each  $\alpha \in C$

$$(g^* \upharpoonright H_\alpha, g^+ \upharpoonright G_\alpha, C \cap \alpha) \in R_\alpha.$$

Consequently we may use the functions  $h_{(g^* \upharpoonright H_\alpha, g^+ \upharpoonright G_\alpha, C \cap \alpha)}$  for  $\alpha \in C$ .

**Clause  $(\delta)$  of the assertion**

Like clause  $(\gamma)$ . □

Before we state the main result let us recall basic properties of  $\text{Ext}$ . First note that

if  $G$  is an (abelian) group satisfying  $G \models (\forall x)(px = 0)$   
then  $G$  is a vector space over  $\mathbb{Z}/p\mathbb{Z}$ .

**Definition 3.5.** (1) For a group  $G$  and  $p \in \mathbf{P}$  let  $\nu_p(G)$  be the dimension of  $\text{Ext}_p(G, \mathbb{Z})$  as a vector space over  $\mathbb{Z}/p\mathbb{Z}$  where

$$\text{Ext}_p(G, \mathbb{Z}) = \{x \in \text{Ext}(G, \mathbb{Z}) : \text{Ext}(G, \mathbb{Z}) \models px = 0\}.$$

- (2) For a group  $G$  and let  $\nu_0(G)$  be the rank (=maximal cardinality of an independent subset) of the torsion free group  $\text{Ext}(G, \mathbb{Z})/\text{tor}(\text{Ext}(G, \mathbb{Z}))$  where for a group  $G'$ :

$$\text{tor}(G') = \{x \in G' : \text{for some } n, 0 < n \in \mathbb{Z} \text{ we have } G' \models nx = 0\}.$$

**Lemma 3.6** (see Fuchs [Fu] or Eklof Mekler [EM, Ch. XII]).

Let  $G$  be an abelian torsion-free group. Then:

- (1) For  $p \in \mathbf{P}$ ,  $\nu_p(G)$  is the dimension of the vector space

$$\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\text{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$$

over the field  $\mathbb{Z}/p\mathbb{Z}$ , where  $\text{Hom}^-(G, \mathbb{Z}/p\mathbb{Z}) \stackrel{\text{def}}{=} \{f/p : f \in \text{Hom}(G, \mathbb{Z})\}$ .

- (2)  $\text{Ext}(G, \mathbb{Z})$  is a divisible group, hence characterized up to isomorphism by cardinals  $\nu_0(G), \nu_p(G)$  (for  $p \in \mathbf{P}$ ).

**Theorem 3.7** (Hiller, Huber, Shelah [HHS 91]). ( $\mathbf{V}=\mathbf{L}$ )

If a group  $G$  is not free, moreover it is not  $G_1 \oplus G_2$  with  $G_2$  free,  $\|G_1\| < \|G\|$  then  $\nu_0(G) = 2^{\|G\|}$ .

*Remark 3.8.* If  $G = G_1 \oplus G_2$  and  $G_2$  is free then  $\text{Ext}(G, \mathbb{Z}) \cong \text{Ext}(G_1, \mathbb{Z})$  so the demand is reasonable.

**Main Theorem 3.9** ( $\mathbf{V}=\mathbf{L}$ ). Suppose that  $\lambda$  is an uncountable regular cardinal which is smaller than the first weakly compact cardinal. Let  $\lambda_p \leq \lambda^+$  for  $p \in \mathbf{P}$ . Then there exists a (torsion free) strongly  $\lambda$ -free group  $G$  such that  $\|G\| = \lambda$ ,  $\nu_p(G) = \lambda_p$ , and  $\nu_0(G) = \lambda^+$ .

*Proof.* During the proof we will use consequences of the assumption  $\mathbf{V} = \mathbf{L}$  like GCH, the principle proved in 2.1 etc without recalling the main assumption.

The construction is much easier if  $\lambda_p = \lambda^+$  for some  $p \in \mathbf{P}$  and  $\lambda_q = 0$  for all  $q \neq p$  (remember that  $\text{Ext}(\bigoplus_{n \in \omega} G_n, \mathbb{Z}) = \prod_{n \in \omega} \text{Ext}(G_n, \mathbb{Z})$ ). Therefore we assume that we are done with this particular case and we assume that  $\lambda_p \leq \lambda$  for all  $p \in \mathbf{P}$ .

We shall build a  $\lambda$ -free group  $G = G_\lambda = \bigcup_{\alpha < \lambda} G_\alpha$  with universe  $\lambda$  (the sequence  $\langle G_\alpha : \alpha < \lambda \rangle$  increasing continuous,  $G_\alpha$  a group on an ordinal  $\gamma_\alpha < \lambda$  for  $\alpha < \lambda$ ). As witnesses for  $\nu_p(G) \geq \lambda_p$  there will be also homomorphisms  $f_\lambda^{p, \zeta} \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  for  $\zeta < \lambda_p$ ,  $f_\lambda^{p, \zeta} = \bigcup_{\alpha < \lambda} f_\alpha^{p, \zeta}$ . For the witnesses to work we need:

- (\*)<sub>1</sub> If  $p \in \mathbf{P}$ ,  $0 < n < \omega$ ,  $\zeta^0 < \dots < \zeta^{n-1} < \lambda_p$ ,  $a_\ell \in \{1/p\mathbb{Z}, \dots, (p-1)/p\mathbb{Z}\}$  (for  $\ell < n$ ),  
then  $\sum_{\ell < n} a_\ell f_\lambda^{p, \zeta^\ell} \notin \text{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$ ,  
(of course  $\sum_{\ell < n} a_\ell f_\lambda^{p, \zeta^\ell} \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ ).

This is equivalent to

- (\*)<sub>2</sub> there are no  $p \in \mathbf{P}$ ,  $0 < n < \omega$ ,  $\zeta^0 < \dots < \zeta^{n-1} < \lambda_p$ ,  $a_\ell \in \{1/p\mathbb{Z}, \dots, (p-1)/p\mathbb{Z}\}$  (for  $\ell < n$ ) and  $g \in \text{Hom}(G, \mathbb{Z})$  such that  $g/p\mathbb{Z} = \sum_{\ell < n} a_\ell f_\lambda^{p, \zeta^\ell}$ .

We shall also have to take care showing that  $\nu_p(G)$  is not  $> \lambda_p$  (if  $\lambda_p < 2^\lambda$ ) and for this it suffices to show that  $\{f_\lambda^{p, \zeta} : \zeta < \lambda_p\}$  generates  $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  over  $\text{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$ . For this we shall use the  $h_{(g, C)}$  (and  $T_\alpha^p$ ) below.

By induction on  $\alpha < \lambda$  choose an increasing continuous sequence  $\langle \gamma_\alpha : \alpha < \lambda \rangle \subseteq \lambda$  such that  $\lambda_p < \lambda \Rightarrow \lambda_p + \omega < \gamma_0$ ,  $\gamma_{\alpha+1} = \gamma_\alpha + \gamma_\alpha$ .

For our given  $\lambda$ , we want to use 2.1; we use a club  $E \subseteq \text{acc}(\{\alpha < \lambda : \gamma_\alpha = \alpha\})$  thin enough. As  $\mathbf{V} = \mathbf{L}$  we find a stationary set  $S \subseteq E$  such that:

- ( $\alpha$ ) for every  $\theta = \text{cf}(\theta) < \lambda$ ,  $\{\delta \in S : \text{cf}(\delta) = \theta\}$  is stationary,  
( $\beta$ )  $S$  a set of singular limit ordinals,

- ( $\gamma$ )  $\lambda = \mu^+ \Rightarrow S \subseteq [\mu + 1, \lambda)$  and if  $\lambda$  is inaccessible then  $S$  is a set of strong limit singular cardinals and
- ( $\delta$ )  $S$  does not reflect.

Now,  $\bar{P}$  is empty,  $\bar{Q} = \langle Q_0, Q_1, Q_2, Q_3, Q_4 \dots \rangle$  where  $Q_0$  is a binary function symbol,  $Q_1, Q_4$  are 3-place ones and  $Q_2, Q_3$  are binary predicates and  $\langle M, \bar{Q}^M, \bar{R}^M \rangle \models \varphi$  means:

- (a)  $\langle M, Q_0^M \rangle$  is a group,  $Q_1^M(p, \zeta, \cdot)$  is a homomorphism from the group to  $\mathbb{Z}/p\mathbb{Z}$  with  $p, \zeta$  variable (it corresponds to  $f^{p, \zeta}$ ); also  $\mathbb{Z}, \mathbf{P}, \dots$  are coded in some way (see below),
- (b)  $\bar{R}^M$  codes a counterexample to  $(*)_1$ , i.e.  $p, n, \zeta^0, \dots, \zeta^{n-1}, a_0, \dots, a_{n-1}, f, g$  such that  $g \in \text{Hom}(M, \mathbb{Z}), f = \sum_{\ell < n} a_\ell f^{p, \zeta^\ell} = g/p\mathbb{Z} \in \text{Hom}(M, \mathbb{Z}/p\mathbb{Z})$ .

Let  $\aleph^\delta, \mathfrak{B}_\varepsilon^\delta, \mathfrak{z}$  be as gotten in 2.1. Choose a sequence  $\langle A_\alpha : \alpha < \lambda \rangle$  such that  $A_\alpha \subseteq \alpha$  for  $\alpha < \lambda$  and

if  $A \subseteq \beta, \beta < \lambda$

then there is  $\alpha \in (\beta, (\|\beta\| + \aleph_0)^+)$  such that  $A = A_\alpha$

(remember we have GCH).

We now choose by induction on  $\alpha < \lambda$  the following objects:  $M_\alpha^1$  (i.e. a group  $G_\alpha$  and homomorphisms  $f_\alpha^{p, \zeta}$  (for  $p \in \mathbf{P}, \zeta \in \lambda_p \cap \alpha$ ) and  $Q_2^{M_\alpha^1}, Q_3^{M_\alpha^1}, Q_4^{M_\alpha^1}, \dots$ ),  $T_\alpha^p, R_\alpha^p, h_{(g, C)}$  (for  $(g, C) \in R_\alpha^p$ ) such that:

- (1)  $G_\alpha$  is a free group with universe  $\gamma_\alpha$ ,
- (2)  $G_\alpha$  is increasing continuous in  $\alpha$ ,
- (3) if  $\beta < \alpha, \beta \notin S$  then  $G_\alpha/G_\beta$  is a free group of size  $\|\gamma_\alpha\|$ ,
- (4)  $f_\alpha^{p, \zeta} \in \text{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ ,  $M_\alpha^1$  is  $(G_\alpha, f_\alpha^{p, \zeta})$  considering  $f_\alpha^{p, \zeta}(x)$  a function with three places (so  $f_\alpha^{p, \zeta}$  is not defined for  $\zeta \geq \gamma_\alpha$ ),
- (5) if  $\beta < \alpha$  then  $f_\beta^{p, \zeta} \subseteq f_\alpha^{p, \zeta}$  (so that  $f_\alpha^{p, \zeta}$  is increasing continuous in  $\alpha$ ),
- (6) if  $\alpha \notin S$  then there is a basis  $Y_\alpha$  of  $G_{\alpha+1}$  over  $G_\alpha$  such that:  
if  $p \in \mathbf{P}, n < \omega, \zeta^0 < \dots < \zeta^{n-1} < \lambda_p \cap \gamma_\alpha$  and  $a_0, \dots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z}$   
then for  $\|\gamma_\alpha\|$  members  $y \in Y_\alpha, f_{\alpha+1}^{p, \zeta^\ell}(y) = a_\ell$  (for  $\ell < n$ ) and  $f_{\alpha+1}^{q, \zeta}(y) = 0$  if  $q \in \mathbf{P}, \zeta < \lambda_q \cap \gamma_\alpha, (q, \zeta) \notin \{(p, \zeta^0), \dots, (p, \zeta^{n-1})\}$ .
- (7) Let  $\alpha \in E, N^\alpha = \mathfrak{B}^\alpha(M_\alpha^1), \mathfrak{z}_\alpha = \mathfrak{z}(M_\alpha^1), N_i^\alpha = \mathfrak{B}_i^\alpha(M_\alpha^1)$  for  $i < \mathfrak{z}_\alpha$ .

Note:  $M_\alpha^1 \in \mathbf{k}_\alpha^-$  by clause (1), the universe of  $N^\alpha$  is a transitive set (see (f) of 2.1) so  $\text{ord} \cap N^\alpha$  is an ordinal greater than  $\alpha$  (if non zero).

Assume  $\alpha \in S$  (so  $\gamma_\alpha = \alpha$ ) and  $G_\alpha, \langle f_\alpha^{p, \zeta} : \zeta < \lambda_p \cap \alpha \rangle$  belong to  $N^\alpha$ . Then we choose by induction on  $\zeta \in \text{ord} \cap N^\alpha \setminus (\lambda_p \cap \alpha)$  the function  $f_\alpha^{p, \zeta} \in N^\alpha$  as the  ${}_{N^\alpha}^*$ -first member of  $\text{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$  (as a vector space over the field  $\mathbb{Z}/p\mathbb{Z}$ ) which does not depend on  $\{f_\alpha^{p, \xi} : \xi < \zeta\}$ .

Let  $f_\alpha^{p, \zeta}$  be defined if and only if  $\zeta < \zeta(p, \alpha)$ .

Note: if  $\mathfrak{z}(M_\alpha^1) = 0$  then  $\zeta(p, \alpha) = \lambda_p \cap \alpha, N^\alpha$  is empty, and below  $T_\alpha^p = R_\alpha^p = \emptyset$ .

- (8)  $T_\alpha^p = \{f_\alpha^{p, \xi} : \lambda_p \cap \alpha \leq \xi < \zeta(p, \alpha)\}$ .
- (9)  $R_\alpha^p$  is the family of all pairs  $(f_\alpha^{p, \xi}, C)$  such that:
  - (a)  $f_\alpha^{p, \xi} \in T_\alpha^p$  and  $C \in N^\alpha$  is a closed subset of  $\alpha \cap E$ ,
  - (b) for  $\beta \in C: f_\alpha^{p, \xi} \upharpoonright G_\beta \in T_\beta^p$  and  $C \cap \beta \in N^\beta$ ,
  - (c) for  $\beta < \gamma$  in  $C, h_{(f_\alpha^{p, \xi} \upharpoonright G_\beta, C \cap \beta)} \subseteq h_{(f_\alpha^{p, \xi} \upharpoonright G_\gamma, C \cap \gamma)}$ ,



- (10) if  $(g, C) \in R_\alpha^p$  then  $h_{(g,C)} \in \text{Hom}(G_\alpha, \mathbb{Z}) \cap N^\alpha$  and  $h_{(g,C)}/p\mathbb{Z} = g$  and  
 $\bigcup_{\beta \in C} h_{(g \upharpoonright G_\beta, C \cap \beta)} \subseteq h_{(g,C)}$ ,  
(11) if  $(g, C) \in R_\alpha^p$ ,  $g \subseteq g' \in \text{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$   
then there is  $h' \in \text{Hom}(G_{\alpha+1}, \mathbb{Z})$  extending  $h_{(g,C)}$  and  $h'/p\mathbb{Z} = g'$ .  
(12) Assume that  $\alpha \in S$ ,  $0 < n < \omega$  and

$$\mathfrak{N}(M_\alpha^1) = \langle G_\alpha, f_\alpha^{p,\zeta}, \dots, p, n, \zeta^0, \dots, \zeta^{n-1}, a_0, \dots, a_{n-1}, f, g \rangle,$$

where  $f = \sum_{\ell < n} a_\ell f_\alpha^{p,\zeta^\ell} \in \text{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ ,  $a_\ell \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ ,  $f = g/p\mathbb{Z}$ ,  
 $g \in \text{Hom}(G_\alpha, \mathbb{Z})$ .

If there is a free group  $H$ ,  $G_\alpha \subseteq H$  such that: for every  $q \in \mathbf{P}$ ,  $\zeta < \lambda_q \cap \gamma_\alpha$   
the homomorphism  $f_\alpha^{q,\zeta}$  can be extended to a member of  $\text{Hom}(H, \mathbb{Z}/q\mathbb{Z})$ ,  
and the quotient  $H/G_\beta$  is free for  $\beta \in (\alpha \setminus S)$ ,  $\|H\| = \|\gamma_{\alpha+1}\|$ , and  $H$   
satisfies (11) (with  $H$  playing the role of  $G_{\alpha+1}$ ) and there are some  $f_*^{p,\zeta}$   
satisfying (4),(5) (with  $H$  as  $G_{\alpha+1}$  and  $f_*^{p,\zeta}$  as  $f_{\alpha+1}^{p,\zeta}$ ) such that for  $f' =$   
 $\sum_{\ell < n} a_\ell f_*^{p,\zeta^\ell} \in \text{Hom}(H, \mathbb{Z}/p\mathbb{Z})$  we have:

$$(*) \quad \neg(\exists g')[g \subseteq g' \in \text{Hom}(H, \mathbb{Z}) \ \& \ g'/p\mathbb{Z} = f']$$

then  $H = G_{\alpha+1}$ ,  $f' = \sum_{\ell < n} a_\ell f_{\alpha+1}^{p,\zeta^\ell}$  satisfy  $(*)$  too.

- (13) if  $\bigwedge_{p \in \mathbf{P}} \lambda_p = 0$ , (so (12) is an empty demand),  $\alpha \in S$ , and there is a group  
 $H$  such that  $G_\alpha \subseteq H$  and for each  $\beta \in \alpha \setminus S$  the quotient  $H/G_\beta$  free, and  
 $\|H\| = \|\gamma_{\alpha+1}\|$ , and it satisfies (11) (with  $H$  playing the role of  $G_{\alpha+1}$ ) and  
 $H/G_\alpha$  is not free  
then  $G_{\alpha+1}/G_\alpha$  is not free.

- (14)  $Q_2^{M_\alpha^1}, Q_3^{M_\alpha^1} \subseteq \gamma_\alpha \times \gamma_\alpha$  are such that for each  $\beta < \gamma_\alpha$  we have:  
 $A_\beta = \{i < \gamma_\alpha : M_\alpha^1 \models Q_2(\beta, i)\}$  and  
if  $\beta$  is limit then  $\{i < \gamma_\alpha : M_\alpha^1 \models Q_3(\beta, i)\}$  is a cofinal  
subset of  $\beta$  of the order type  $\text{cf}(\beta)$ .

$Q_4^{M_\alpha^1}$  is such that if  $\zeta, \xi < \gamma_\alpha$ ,  $\|\zeta\| = \|\xi\|$  then the function

$$Q_4^{M_\alpha^1}(\zeta, \xi, \cdot) \upharpoonright \zeta : \zeta \longrightarrow \xi$$

is one-to-one and onto. We require that  $Q_2^{M_\alpha^1}, Q_3^{M_\alpha^1}$  and  $Q_4^{M_\alpha^1}$  are increasing  
with  $\alpha$ , of course.

The conditions (6), (12), (13) and (14) fully describe what happens at successor  
stages of the construction. Limit cases are determined by the continuity demands  
(2) and (5). Note that the demands (1), (3) are preserved at the limit stages as  
the set  $S$  is not reflecting (see e.g. [EM, Proposition IV.1.7]). Hence there is no  
problem to carry out the definition and let  $G = G_\lambda = \bigcup_{\alpha < \lambda} G_\alpha$  (though it is not  
so immediate that  $G$  is not free!).

Some of the desired properties are clear:

- ( $\otimes$ )<sub>1</sub>  $G$  is a group of cardinality  $\lambda$  (and the set of elements is  $\lambda$ ) extending each  
 $G_\alpha$   
(by (1)+(2)),  
( $\otimes$ )<sub>2</sub>  $G$  is  $\lambda$ -free and even strongly  $\lambda$ -free

(by (1) each  $G_\alpha$  is free so  $G$  is  $\lambda$ -free; by (3) if  $\beta \in \lambda \setminus S$ ,  $\beta < \alpha < \lambda$  then  $G_\alpha/G_\beta$  is free, so  $G$  is strongly  $\lambda$ -free, see e.g. [EM, pp 87–88]).

Let  $f^{p,\zeta} = f_\lambda^{p,\zeta} = \bigcup_{\alpha < \lambda} f_\alpha^{p,\zeta}$  for  $p \in \mathbf{P}$ ,  $\zeta < \lambda_p$ .

( $\otimes$ )<sub>3</sub>  $f^{p,\zeta} \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  (extending each  $f_\alpha^{p,\zeta}$  for  $\alpha < \lambda$ )

(by (4)+(5)). Before checking the main properties of  $G$  let us note the following two facts (which explain the condition (14) of the construction).

**Claim 3.9.1.** *If  $\delta \in E$ ,  $\mathfrak{z}(M_\delta^1) > 0$  and  $\kappa < \delta$ ,  $\|\kappa\| < \|\delta\|$  then  $\mathcal{P}(\kappa) \subseteq N^\delta$ .*

*Proof of the claim.* By the clauses **(f)**, **(h)** of 2.1 and condition (14) of the construction we have  $A_i \in N_0^\delta$  for all  $i \in (\kappa, \|\kappa\|^+)$ . By the choice of the sequence  $\langle A_i : i < \lambda \rangle$  we are done.  $\square$

**Claim 3.9.2.** *If  $\delta \in E$ ,  $\mathfrak{z}(M_\delta^1) > 0$  then there is an increasing cofinal in  $\delta$  sequence  $\langle \beta_i^\delta : i < \text{cf}(\delta) \rangle$  such that for every  $i^* < \text{cf}(\delta)$  we have*

$$\langle \beta_i^\delta : i < i^* \rangle \in N^\delta.$$

*Proof of the claim.* By the clauses **(k)**, **(l)** of 2.1 we have a club  $e_\delta \subseteq \delta$  such that  $\text{otp}(e_\delta) < \delta$  and for each  $\alpha < \delta$  the intersection  $e_\delta \cap \alpha$  is in  $N^\delta$ . The set  $b^* \stackrel{\text{def}}{=} \{i < \gamma_\delta : M_\delta^1 \models Q_3(\text{otp}(e_\delta), i)\}$  is an increasing cofinal subset of  $\text{otp}(e_\delta)$  of the order type  $\text{cf}(\delta) = \text{cf}(\text{otp}(e_\delta))$ . It follows from the condition **(h)** of 2.1 that  $b^* \in N^\delta$ . But with  $e_\delta$  and  $b^*$  in hands we may easily build  $\langle \beta_i^\delta : i < \text{cf}(\delta) \rangle$  as required.  $\square$

Now comes the main point:

( $\otimes$ )<sub>4</sub> if  $p \in \mathbf{P}$ ,  $0 < n < \omega$ ,  $\zeta^0 < \dots < \zeta^{n-1} < \lambda_p$ ,  $a_0, \dots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ ,  
 $f = \sum_{\ell < n} a_\ell f^{p,\zeta^\ell} \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$   
 then  $f \notin \text{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$ .

Why ( $\otimes$ )<sub>4</sub>?

Assume that ( $\otimes$ )<sub>4</sub> fails, so there are  $p \in \mathbf{P}$ ,  $\zeta^0 < \dots < \zeta^{n-1}$ ,  $a_0, \dots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$  and  $g \in \text{Hom}(G, \mathbb{Z})$  with  $f = \sum_{\ell < n} a_\ell f^{p,\zeta^\ell} = g/p\mathbb{Z}$ . Let

$$M^2 = \langle G, f^{p,\zeta}, \dots, p, n, \zeta^0, \dots, \zeta^{n-1}, a_0, \dots, a_{n-1}, f, g \rangle, \quad M_\delta^2 = M^2 \upharpoonright \delta.$$

By 2.1 condition **(i)** without loss of generality (i.e. possibly replacing

$$p, n, \zeta^0, \dots, \zeta^{n-1}, a_0, \dots, a_{n-1}, f, g$$

by some other

$$p^*, n^*, \zeta_*^0, \dots, \zeta_*^{n^*-1}, a_0^*, \dots, a_{n^*-1}^*, f^*, g^*$$

with the same properties) we have: the set

$$S^* \stackrel{\text{def}}{=} \{ \delta \in S : \mathfrak{N}(M_\delta^1) = M_\delta^2 \text{ and } M_\delta^2 \prec M^2 \text{ and } \mathfrak{z}(M_\delta^1) = \text{cf}(\delta) \text{ or } \mathfrak{z}(M_\delta^1) = 0 \}$$

is stationary. (Just applying **(i)** choose  $\theta = \sigma$  when  $0 < \sigma < \lambda$  and that arbitrary regular  $\theta < \lambda$  in other cases.) Choose  $\delta \in S^*$ . Remember that  $S^* \subseteq S$ , so e.g.  $\delta = \gamma_\delta$ ,  $\text{cf}(\delta) < \delta$  and

$\lambda = \mu^+ \Rightarrow \delta \in [\mu + 1, \lambda)$  and

if  $\lambda$  is inaccessible then  $\delta$  is a strongly limit singular cardinal.

Let us first consider the case  $\mathfrak{z}(M_\delta^1) \neq 0$  (so  $\text{cf}(\delta) = \mathfrak{z}(M_\delta^1)$ ). To show  $(\otimes_4)$  we will need the following technical but useful claims.

**Claim 3.9.3.** If  $R \in N^\delta$ ,  $R \subseteq \text{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})$ ,  $2^{\|R\|} < \|\delta\|$  (so  $2^{2^{\|R\|}} \leq \delta$ ) then for every  $\beta \in \delta \setminus S$  large enough there is  $x \in G_\beta$  such that:

- (a)  $h \in R \Rightarrow h(x) = 0$ ,
- (b)  $x \in G_{\beta+1}$ , moreover  $G_\beta \oplus \langle \mathbb{Z}x \rangle$  is a direct summand of  $G_{\beta+1}$ ,
- (c)  $g(x) \neq 0$ .

*Proof of the claim.* First assume that  $R$  is infinite, so  $2^{\|R\|} = \|\mathbb{R}\mathbb{Z}\|$ .

As  $2^{\|R\|} < \|\delta\|$  clearly there are  $x \in G_\delta \setminus \{0\}$  satisfying (a). If  $x$  satisfies (a)+(c),  $x \in G_\beta$  and  $\beta \in \delta \setminus S$  large enough then we can find  $y \in G_{\beta+1}$  which is a member of a basis of  $G_{\beta+1}$  over  $G_\beta$  and which satisfies (a) and  $g(y) = 0$ . Then the element  $x + y$  satisfies (b) (and (a), (c)). So it suffices to find  $x \in G_\delta$  satisfying (a)+(c). If this fails then for every  $x_1, x_2 \in G_\delta$  we have

$$\left[ \bigwedge_{h \in R} h(x_1) = h(x_2) \right] \Rightarrow g(x_1) = g(x_2).$$

So there is a function  $F : {}^R\mathbb{Z} \rightarrow \mathbb{Z}$  such that  $g(x) = F(\dots, h(x), \dots)_{h \in R}$ . Take  $i < \mathfrak{z}(M_\delta^1)$  such that  $R \in N_i^\delta$ . Then  $N_{i+1}^\delta \models \|R\| \leq \|\delta\|$  (remember 2.1(**f**), **g**)) and necessarily  $N_{i+1}^\delta \models \|R\| < \|\delta\|$  (as  $\|R\| < \|\delta\|$  in **V**). Applying 2.1(**h**) we get that  $Q_4^{M_\delta^1} \in N_{i+1}^\delta$  and therefore  $N_{i+1}^\delta \models \|R\| = \kappa_0$ , where  $\kappa_0 = \|R\|$  (in **V**). Let  $\kappa_1 = 2^{\kappa_0}$  (so  $\kappa_1 < \|\delta\|$ ). Then, by 3.9.1, we get  $\mathcal{P}(\kappa_0) \subseteq N_{i+1}^\delta$  and  $N_{i+1}^\delta \models \|\kappa_0\mathbb{Z} \times \mathbb{Z}\| = \kappa_1$ . Again by 3.9.1, we get  $\mathcal{P}(\kappa_1) \subseteq N_{i+1}^\delta$ . But this implies that  $\mathcal{P}(\kappa_0\mathbb{Z} \times \mathbb{Z}) \subseteq N_{i+1}^\delta$  and  $\mathcal{P}({}^R\mathbb{Z} \times \mathbb{Z}) \subseteq N_{i+1}^\delta$ . In particular  $F \in N^\delta$ . Since  $\{G_\delta, R\} \in N^\delta$  ( $G_\delta$  by clause (h) of 2.1,  $R$  by the assumption) we conclude that  $g \in N^\delta$  — a contradiction to condition (h) of 2.1. The case when  $R$  is finite is much easier. We start as above, but getting  $x \in G_\delta$  with (a)+(c) we give purely algebraical arguments. The claim is proved.  $\square$

**Claim 3.9.4.** (1) Suppose that  $\text{cf}(\delta) \neq \text{cf}(\mu) < \mu < \delta < \mu^+$  (so we are in the case  $\lambda = \mu^+$ ),  $R \in N_i^\delta$ ,  $R \subseteq \text{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})$ ,  $i < \mathfrak{z}(M_\delta^1)$

and  $N_i^\delta \models \|R\| = \mu$ . Then for each sufficiently large  $\beta \in (\mu, \delta) \setminus S$  there exist  $x_j \in G_{\beta+1}$  for  $j < \text{cf}(\mu)$  such that:

- (a) if  $h \in R$  then the set  $\{j < \text{cf}(\mu) : h(x_j) \neq 0\}$  is bounded in  $\text{cf}(\mu)$ ,
- (b)  $G_\beta \oplus \langle x_j : j < \text{cf}(\mu) \rangle_{G_{\beta+1}}$  is a direct summand of  $G_{\beta+1}$  (and so of  $G_\delta$ ),
- (c)  $g(x_j) \neq 0$  for all  $j < \text{cf}(\mu)$ .

(2) In 1), if we change the assumptions to:

$$\text{cf}(\delta) = \theta^+ = \mu < \delta < \mu^+, \quad \|R\| = \theta$$

then the assertion holds true after replacing  $\text{cf}(\mu)$  by  $\text{cf}(\theta)$  (so  $x_j$  are being chosen for  $j < \text{cf}(\theta)$ ).

(3) Suppose that  $\text{cf}(\delta) \neq \text{cf}(\theta)$ ,  $\text{cf}(\delta) < \theta^+ = \mu < \delta < \mu^+$ ,  $R \in N_i^\delta$ ,  $R \subseteq \text{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})$ ,  $i < \mathfrak{z}(M_\delta^1)$  and  $N_i^\delta \models \|R\| = \mu$ . Then

for sufficiently large  $\beta \in (\mu, \delta) \setminus S$  there exists a sequence  $\langle x_{j,k} : j < \mu, k < \text{cf}(\theta) \rangle \subseteq G_{\beta+1}$  such that

- (a) if  $h \in R$  then  $\{(j, k) \in \mu \times \text{cf}(\theta) : h(x_{j,k}) \neq 0\} \in I_{\langle \mu, \text{cf}(\theta) \rangle}$ ,
- (b)  $G_\beta \oplus \langle x_{j,k} : j < \mu, k < \text{cf}(\theta) \rangle_{G_{\beta+1}}$  is a direct summand of  $G_{\beta+1}$ ,
- (c)  $g(x_{j,k}) \neq 0$  for all  $j < \mu, k < \text{cf}(\theta)$ .

*Proof of the claim.* We follow exactly the lines of the proof of 3.9.3, but first we have to choose an increasing sequence  $\langle R_j : j < \text{cf}(\mu) \rangle \in N_i^\delta$  such that  $\bigcup_{j < \text{cf}(\mu)} R_j = R$ ,

$\|R_j\| < \|R\|$  (and hence  $2^{\|R_j\|} < \mu$  as  $\mu$  is a limit cardinal). To find the  $R_j$  use condition (14) of the construction (and  $Q_3, Q_4$ ). Then use 3.9.3 to find  $\beta_0 \in (\mu, \delta)$  such that there are  $x_j^* \in G_{\beta_0}$  (for  $j < \text{cf}(\mu)$ ) with

$$(\forall h \in R_j)(h(x_j^*) = 0) \text{ and } g(x_j^*) \neq 0$$

(remember that  $\text{cf}(\delta) \neq \text{cf}(\mu)$ ). Now if  $\beta \in (\beta_0, \delta) \setminus S$  then we find a sequence  $\langle y_j : j < \text{cf}(\mu) \rangle \subseteq G_{\beta+1}$  which can be extended to a basis of  $G_{\beta+1}$  over  $G_\beta$  and such that for all  $j < \text{cf}(\mu)$

$$(\forall h \in R_j)(h(y_j) = 0) \text{ and } g(y_j) = 0.$$

Put  $x_j = y_j + x_j^*$ .

2) Similarly (note that if  $R \in N_i^\delta$ ,  $\|R\| = \theta$  then  $N_i^\delta \models \|R\| = \theta$ ).

3) Similarly: first find  $\langle R_{j,k} : j < \mu, k < \text{cf}(\theta) \rangle \in N_i^\delta$  such that  $\|R_{j,k}\| < \theta$ , the sequence  $\langle \bigcup_{k < \text{cf}(\theta)} R_{j,k} : j < \mu \rangle$  is increasing, for each  $j < \mu$  the sequence  $\langle R_{j,k} : k < \text{cf}(\theta) \rangle$  is increasing and  $\bigcup_{j < \mu} \bigcup_{k < \text{cf}(\theta)} R_{j,k} = R$ . Next follow as in 1).  $\square$

Now we are going to finish the proof of  $(\otimes)_4$  (in the case  $\mathfrak{z}(M_\delta^1) \neq 0$ ). Note that by  $(\otimes)_3$ ,  $f^{p, \zeta^\ell} \upharpoonright G_\delta = f_\delta^{p, \zeta^\ell}$ , so we can try to apply condition (12). But condition (12) says “help only those who can help themselves”. More specifically we have to prove that there are  $H, f', f_{\delta+1}^{p, \zeta}$  as required there (in particular  $(*)$ ) and then by (12),  $g \upharpoonright G_{\delta+1}$  gives the desired contradiction. But this is done by the following claim.

**Claim 3.9.5.** *Suppose that  $\delta, g \dots$  are as chosen earlier. Then there exists a free group  $H$  such that  $G_\delta \subseteq H$ ,  $H/G_\beta$  is free for  $\beta \in \delta \setminus S$ ,  $\|H\| = \|\delta\|$ , the homomorphism  $g$  cannot be extended to a homomorphism  $g' \in \text{Hom}(H, \mathbb{Z})$  and*

- ( $\alpha$ ) for every  $h \in \text{Hom}(G_\delta, \mathbb{Z}) \cap N^\delta$  and  $h^+ \in \text{Hom}(H, \mathbb{Z}/p\mathbb{Z})$  such that  $h/p\mathbb{Z} \subseteq h^+$  there is  $h^* \in \text{Hom}(H, \mathbb{Z})$  with  $h \subseteq h^*$  and  $h^*/p\mathbb{Z} = h^+$ ;
- ( $\beta$ ) if  $q \in \mathbf{P}$ ,  $\zeta < \lambda_q \cap \delta$  then  $f_\delta^{q, \zeta}$  can be extended to an element of  $\text{Hom}(H, \mathbb{Z}/q\mathbb{Z})$ .

*Proof of the claim.* Let us recall that  $\mathfrak{z}(M_\delta^1) = \text{cf}(\delta)$ . For  $\varepsilon < \mathfrak{z}(M_\delta^1)$  let  $R_\varepsilon = \mathfrak{B}_\varepsilon^\delta(M_\delta^1) \cap (\text{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \text{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z}))$  and let  $\langle f_\varepsilon : \varepsilon < \mathfrak{z}(M_\delta^1) \rangle$  be a sequence of functions such that for each  $\varepsilon < \mathfrak{z}(M_\delta^1)$  we have:

$$f_\varepsilon : \delta \xrightarrow{\text{onto}} R_\varepsilon \text{ and } \langle f_\zeta : \zeta \leq \varepsilon \rangle \in N_{\varepsilon+1}^\delta$$

(see clauses **(e)** and **(g)** of 2.1). Let  $\langle \beta_i^\delta : i < \text{cf}(\delta) \rangle$  be an increasing sequence with limit  $\delta$  such that  $\beta_0^\delta > \text{cf}(\delta)$  and  $\langle \beta_i^\delta : i < i^* \rangle \in N^\delta$  for all  $i^* < \text{cf}(\delta)$  (see claim 3.9.2). Finally for  $i < \text{cf}(\delta)$  let

$$R_i^* \stackrel{\text{def}}{=} \{f_\varepsilon(\zeta) : \zeta < \beta_i^\delta \ \& \ \varepsilon \leq i\}.$$

Then  $R_i^*$  are increasing with  $i$  and

$$\bigcup_{i < \text{cf}(\delta)} R_i^* = \bigcup_{\varepsilon < \mathfrak{J}(M_\delta^1)} R_\varepsilon$$

and for each  $i^* < \text{cf}(\delta)$  the sequence  $\langle R_i^* : i < i^* \rangle$  belongs to  $N_{\zeta_{i^*}}^\delta$  (for some  $\zeta_{i^*} < \mathfrak{J}(M_\delta^1)$ ). Moreover  $N_{\zeta_{i^*}}^\delta \models \|R_i^*\| = \|\beta_i^\delta\|$  for each  $i < i^*$ .

Let  $\langle \alpha_i^\delta : i < \text{cf}(\delta) \rangle$  be an increasing continuous sequence cofinal in  $\delta$  and disjoint from  $S$  (possible by the choice of  $S$ ).

**Case A:**  $\delta$  is a strongly limit singular cardinal.

In this case we have

$$(\forall i < \text{cf}(\delta))(2^{\|R_i^*\|} < \delta).$$

Thus we may apply claim 3.9.3 and choose by induction on  $i < \text{cf}(\delta)$  an increasing sequence  $\langle j_i : i < \text{cf}(\delta) \rangle \subseteq \text{cf}(\delta)$  and  $x_i^\delta \in G_{\alpha_{j_i}^\delta + 1}$  such that:

- (a)  $h \in R_i^* \Rightarrow h(x_i^\delta) = 0$ ;
- (b)  $G_{\alpha_{j_i}^\delta} \oplus \langle \mathbb{Z}x_i^\delta \rangle$  is a direct summand of  $G_\delta$ ;
- (c)  $g(x_i^\delta) \neq 0$ .

Since  $\langle \alpha_i^\delta : i < \text{cf}(\delta) \rangle \subseteq \delta \setminus S$  is increasing continuous (and cofinal in  $\delta$ ) we get that the subgroup  $H_\delta = \langle x_i^\delta : i < \text{cf}(\delta) \rangle_{G_\delta}$  is a direct summand of  $G_\delta$ , say  $G_\delta = H_\delta \oplus H_\delta^*$  (and  $\{x_i^\delta : i < \text{cf}(\delta)\}$  is a free basis of  $H_\delta$ ). Let  $I = \{A \subseteq \text{cf}(\delta) : A \text{ is bounded}\}$  and apply 3.4 for  $\text{cf}(\delta)$ ,  $I$  and  $H_\delta$  and get the respective free group  $H' \supseteq H_\delta$  ( $H' \cap G_\delta = H_\delta$ ). We claim that the group  $H = H' \oplus H_\delta^*$  is as required. For this first note that if  $\beta \in \delta \setminus S$ ,  $\alpha_{j_{i_0}}^\delta > \beta$ ,  $A = [0, j_{i_0})$  then  $H'/\langle x_i^\delta : i \in A \rangle$  is free and hence  $H/G_{\alpha_{j_{i_0}}^\delta}$  is free. But  $G_{\alpha_{j_{i_0}}^\delta}/G_\beta$  is free so we conclude that  $H/G_\beta$  is free.

As  $g$  cannot be extended to a member of  $\text{Hom}(H', \mathbb{Z})$  it has no extension in  $\text{Hom}(H, \mathbb{Z})$ . Suppose now that  $h \in \text{Hom}(G_\delta, \mathbb{Z}) \cap N^\delta$ , so  $h \in R_{i_0}^*$  for some  $i_0 < \text{cf}(\delta)$ . Let  $h^+ \in \text{Hom}(H, \mathbb{Z}/p\mathbb{Z})$  extend  $h/p\mathbb{Z}$ . Since for all  $i \geq i_0$  we have  $h(x_i^\delta) = 0$ , we may apply clause  $(\gamma)$  of 3.4 to get a suitable lifting  $h^* \in \text{Hom}(H, \mathbb{Z})$  of  $h^+$ . Similarly, we use 3.4 $(\delta)$  to show that  $f_\delta^{q, \xi}$  can be extended onto  $H$  (remember  $f_\delta^{q, \xi} \in N^\delta$ ).

**Case B:**  $\delta \in (\mu, \mu^+)$  for some cardinal  $\mu$  such that  $\text{cf}(\mu) = \text{cf}(\delta) < \mu$ .

By condition (14) of the construction and the use of  $Q_3, Q_4$  we have that, letting  $\alpha = \text{cf}(\delta)$ , for each  $i < \text{cf}(\delta)$

$$N_{\zeta_{i+1}}^\delta \models " \|\beta_i^\delta\| = \|\mu\| \ \& \ \text{cf}(\mu) = \alpha "$$

This allows us to build  $R_i^{**}$  such that

$$i < j < \text{cf}(\delta) \Rightarrow R_i^{**} \subseteq R_j^{**} \in N^\delta,$$

$$\bigcup_{i < \text{cf}(\delta)} R_i^{**} = \bigcup_{i < \text{cf}(\delta)} R_i^* \quad \text{and} \quad \|R_i^{**}\| < \mu.$$

Now we can continue as in the previous case.

**Case C:**  $\delta \in (\mu, \mu^+)$  for some cardinal number  $\mu$  such that  $\text{cf}(\delta) \neq \text{cf}(\mu) < \mu$

By claim 3.9.4(1) we can choose  $x_{i,j}^\delta$  (for  $i < \text{cf}(\delta)$ ,  $j < \text{cf}(\mu)$ ) such that:

- (a) for each  $h \in R_i^*$ , for every  $j < \text{cf}(\mu)$  large enough  $h(x_{i,j}^\delta) = 0$ ;

- (b)  $\{x_{i,j}^\delta : i < \text{cf}(\delta), j < \text{cf}(\mu)\}$  is a free basis of a direct summand of  $G_\delta$ , moreover for some increasing sequence  $\langle j_i : i < \text{cf}(\delta) \rangle \subseteq \text{cf}(\delta)$ , for each  $i^* < \text{cf}(\delta)$ , the family  $\{x_{i,j}^\delta : i^* \leq i < \text{cf}(\delta), j < \text{cf}(\mu)\}$  is a free basis of a subgroup  $H \subseteq G_\delta$  such that  $G_{\alpha_{j_{i^*}}^\delta} \oplus H$  is a direct summand of  $G_\delta$ ;
- (c)  $g(x_{i,j}^\delta) \neq 0$ .

Let

$$I = I_{\langle \text{cf}(\delta), \text{cf}(\mu) \rangle} = \{A \subseteq \text{cf}(\delta) \times \text{cf}(\mu) : (\forall^* i < \text{cf}(\delta))(\forall^* j < \text{cf}(\mu))((i, j) \notin A)\}.$$

Again apply 3.4 (with  $\lambda$  there standing for  $\text{cf}(\delta) + \text{cf}(\mu)$ ).

**Case D:**  $\delta \in (\mu, \mu^+)$  for some cardinal number  $\mu$  such that  $\text{cf}(\delta) = \mu = \text{cf}(\mu)$  is inaccessible.

Similar to Case B.

**Case E:**  $\delta \in (\mu, \mu^+)$  for some cardinal number  $\mu$  such that  $\mu = \theta^+ = \text{cf}(\delta)$ .

First find an increasing sequence  $\langle R_i^{**} : i < \theta^+ \rangle$  such that  $R_i^{**} \in N^\delta$ ,  $\bigcup_{i < \theta^+} R_i^{**} =$

$\bigcup_{i < \text{cf}(\delta)} R_i^*$  and  $\|R_i^{**}\| \leq \theta$ . Then apply claim 3.9.4(2) to choose a sequence  $\langle x_{i,j}^\delta : i < \text{cf}(\delta), j < \text{cf}(\theta) \rangle$  similarly as in case C.

**Case F:**  $\delta \in (\mu, \mu^+)$  for some cardinal  $\mu$  such that  $\text{cf}(\delta) < \mu = \theta^+$ ,  $\text{cf}(\delta) \neq \text{cf}(\theta)$ .

Using claim 3.9.4(3) we choose an increasing sequence  $\langle j_i : i < \text{cf}(\delta) \rangle \subseteq \text{cf}(\delta)$  and a sequence  $\langle x_{i,j,k}^\delta : i < \text{cf}(\delta), j < \mu, k < \text{cf}(\theta) \rangle$  such that

- (a) for each  $h \in R_i^*$ , for every  $j < \mu$  large enough for every  $\kappa < \text{cf}(\theta)$  large enough,  $h(x_{i,j,k}^\delta) = 0$ ;
- (b)  $\{x_{i,j,k}^\delta : i < \text{cf}(\delta), j < \mu, \kappa < \text{cf}(\theta)\}$  is a free basis of a direct summand of  $G_\delta$ ; moreover for each  $i^* < \text{cf}(\delta)$  the set

$$\{x_{i,j,k}^\delta : i^* \leq i < \text{cf}(\delta), j < \mu, k < \text{cf}(\theta)\}$$

is a free basis of a subgroup  $H \subseteq G_\delta$  such that  $G_{\alpha_{j_{i^*}}^\delta} \oplus H$  is a direct summand of  $G_\delta$ ;

- (c)  $g(x_{i,j,k}^\delta) \neq 0$ .

Let  $I = I_{\langle \text{cf}(\delta), \mu, \text{cf}(\theta) \rangle}$  and apply 3.4.

**Case G:**  $\delta \in (\mu, \mu^+)$  for some cardinal  $\mu$  such that  $\mu = \theta^+$ ,  $\text{cf}(\delta) = \text{cf}(\theta)$ .

This is similar to the case F though we have to modify the application of 3.4. First we choose increasing sequences  $\langle R_{i,j}^{**} : j < \mu \rangle \in N^\delta$  (for  $i < \text{cf}(\delta)$ ) such that

$$\|R_{i,j}^{**}\| < \mu, \quad \bigcup_{j < \mu} R_{i,j}^{**} = R_i^* \quad \text{and}$$

$$\langle \langle R_{i,j}^{**} : j < \mu \rangle : i < i^* \rangle \in N^\delta \quad \text{for each } i^* < \text{cf}(\delta).$$

Then we choose increasing sequences  $\langle R_{i,j,k}^{***} : k < \text{cf}(\theta) \rangle$  (for  $i < \text{cf}(\delta) = \text{cf}(\theta)$ ,  $j < \mu$ ) such that

$$\|R_{i,j,k}^{***}\| < \theta, \quad \bigcup_{k < \text{cf}(\theta)} R_{i,j,k}^{***} = R_{i,j}^{**} \quad \text{and}$$

$$\langle \langle R_{i,j,k}^{***} : j < \mu, k < \text{cf}(\theta) \rangle : i < i^* \rangle \in N^\delta.$$

Now, for  $\ell < \text{cf}(\theta) = \text{cf}(\delta)$ ,  $j < \mu$  put  $R_{j,\ell}^+ = \bigcup_{i,k < \ell} R_{i,j,k}^{***}$ . Note that  $R_{j,\ell}^+ \in N^\delta$  and  $\|R_{j,\ell}^+\| < \theta$ . Moreover if  $h \in \bigcup_{i < \text{cf}(\delta)} R_i^*$  then  $(\forall^* j < \mu)(\forall^* \ell < \text{cf}(\delta))(h \in R_{j,\ell}^+)$ . Next, as in the proof of 3.9.4 we choose  $x_{j,\ell}^*, y_{j,\ell}$  such that

$$(\forall h \in R_{j,\ell}^+)(h(x_{j,\ell}^*) = h(y_{j,\ell}) = 0), \quad g(x_{j,\ell}^*) \neq 0, g(y_{j,\ell}) = 0,$$

if  $\rho_{j,\ell} \stackrel{\text{def}}{=} \min\{\alpha_{\ell_0}^\delta : x_{j,\ell}^* \in G_{\alpha_{\ell_0}^\delta}\}$  then  $\ell < \ell_0$  and

$$\{y_{j,\ell} : \rho_{j,\ell} = \beta\} \subseteq G_{\beta+1} \text{ can be extended to a basis of } G_{\beta+1} \text{ over } G_\beta.$$

Then we put  $x_{j,\ell} = x_{j,\ell}^* + y_{j,\ell}$  (for  $j < \mu$ ,  $\ell < \text{cf}(\delta)$ ) and we apply 3.4 as earlier.

**Case H:**  $\delta \in (\mu, \mu^+)$  for some inaccessible cardinal  $\mu$  such that  $\text{cf}(\delta) < \text{cf}(\mu) = \mu$ .

This is similar to case C.

This completes the proof of claim 3.9.5.  $\square$

The case  $\mathfrak{z}(M_\delta^1) = 0$  is much easier and can be done similarly. We do not have  $N^\delta$  and we have to take care of extending homomorphisms  $f_\delta^{q,\zeta}$  only. We basically follow the lines of the previous case, but proving the suitable variants of 3.9.3, 3.9.4 instead of the fact that  $g \notin N^\delta$  we use clause (6) of the inductive construction.

This completes the proof of  $(\otimes)_4$ .

To finish the proof of the theorem we have to show

$(\otimes)_5$  if  $p \in \mathbf{P}$ ,  $f \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  then there are  $n < \omega$ ,  $\zeta^0, \dots, \zeta^{n-1} < \lambda_p$ ,  $a_0, \dots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z}$  such that  $f - \sum_{\ell < n} a_\ell f^{p,\zeta^\ell} \in \text{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$  (i.e. the difference can be lifted to a homomorphism to  $\mathbb{Z}$ ).

For this we inductively define a sequence  $\langle f_\xi : \xi < \xi(*) \rangle \subseteq \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  by:

$f_\xi$  is the  $<^*$ -first member of the vector space  $\text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$  over the field  $\mathbb{Z}/p\mathbb{Z}$  which does not depend on

$$\{f^{p,\zeta} : \zeta < \lambda_p\} \cup \{f_\zeta : \zeta < \xi\}.$$

(So  $\xi(*)$  is the maximal length of a sequence with the property stated above.) It is enough to show that all the homomorphisms  $f_\xi$  (for  $\xi < \xi(*)$ ) can be lifted to homomorphisms  $f_\xi^* \in \text{Hom}(G, \mathbb{Z})$ . But by  $(\otimes)_4$  we know that  $M_\lambda^1 \in \mathbf{k}_\lambda \setminus \mathbf{k}_\lambda^-$ , so we may apply **(j)**. Thus we have a club  $C \subseteq E$  such that for each  $\delta \in C$ :

$$\{C \cap \delta, \langle f_\zeta \upharpoonright \delta : \zeta < \delta \rangle\} \in N^\delta.$$

Applying the “moreover” part of **(j)** of 2.1 we may make use of conditions (8), (9) of the construction. (Remember that in (7) the sequence  $\langle f_\alpha^{p,\xi} : \lambda_p \cap \alpha < \xi < \zeta(p, \alpha) \rangle$  has the same definition as our sequence  $\langle f_\xi : \xi < \xi(*) \rangle$ .)  $\square$

*Remark 3.10.* The main theorem 3.9 can be proved for all regular cardinals which are not weakly compact. This requires some changes in the construction (and 3.4).

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