ZERO ONE LAWS FOR GRAPHS WITH EDGE PROBABILITIES DECAYING WITH DISTANCE, PART II
SH517

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Abstract. Fixing an irrational $\alpha \in (0,1)$ we prove the 0-1 law for the random graph on $[n]$ with the probability of $\{i,j\}$ being an edge being essentially $\frac{1}{|i-j|^\alpha}$.

§ 0. Introduction

This continues [She02] which is Part I, background and a description of the results are given in [I,§0]; as this is the second part, our sections are named §4 - §6 and not §1 - §3.

In §5 we just state the necessary probabilistic inequalities quoting [**?**]. In [**?**] we prove the case with a successor function (this was originally §7).

Recall that we fix an irrational $\alpha \in (0,1)_{\mathbb{R}}$ and the random graph $\mathscr{M}_n = \mathscr{M}_n^0$ is drawn as follows:

(a) its set of elements is $[n] = \{1,\dots,n\}$

(b) for $i < j$ in $[n]$ the probability of $\{i,j\}$ being an edge is $p_{|i-j|}$ where $p_\ell = 1/\ell^\alpha$ if $\ell > 1$ as is $1/2^\alpha$ if $\ell = 1$ or just [1] $p_\ell = 1/\ell^\alpha$ for $\ell > 1$

(c) the drawing for the edges are independent

(d) $\mathscr{K}_n$ is the set of possible values of $\mathscr{M}_n, \mathscr{K}$ is the class of graphs.

Our main interest is to prove the 0-1 laws (for first order logic) for this 0-1 context, but also to analyze the limit theory.

We can now explain our intentions.

Zero Step: We define relations $<^*_x$ on the class of graphs with no apparent relation to the probability side.

First Step: We can prove that these $<^*_x$ have the formal properties of $<_x$ (which we defined in [I,§1] via probabilistic expected value), for example "$<^*_i$ is a partial order", this is done in §4, e.g., in 1.16 below.

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[1] originally we assume the former but actually also the latter works, and also just using $\frac{1}{\ell^\alpha}$ works (i.e. gives the same limit theory)

Remember from [I,§1] that $A <_a B \Leftrightarrow$ for random enough $\mathscr{M}_n$ and $f : A \hookrightarrow \mathscr{M}_n$, the maximal number of pairwise disjoint $g \supseteq f$ satisfying $g : B \to \mathscr{M}_n$ is $< n^\varepsilon$ (for every fixed ε).

Second Step: We shall start dealing with the two version of $<_a$: the $<_a$ from [She02, §1] and $<^*_a$ defined in 1.10(5) below. We intend to prove:

$(*)$ $A <^*_a B \quad \Rightarrow \quad A <_a B.$

For this it suffices to show that for every $f : A \hookrightarrow \mathscr{M}_n$ and positive real ε, the expected value of the following is $\leq 1/n^\varepsilon$: the number of extensions $g : B \to \mathscr{M}_n$ of f satisfying "the sets $\mathrm{Rang}(g \restriction (B \setminus A)), f(A)$ are with distance $\geq n^\varepsilon$". Then, the expected value of the number of k-tuples of such (pairwise) disjoint g is $\leq \frac{1}{n^{k\varepsilon}}$. So if $k\varepsilon > |A|$, the expected value of the number of functions f with k pairwise disjoint such extensions g is $< \frac{1}{n^{k\varepsilon - |A|}}$. Hence for random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow \mathscr{M}_n$ there are no such k-tuples of pairwise disjoint g's. This will help to prove that $<^*_i = <_i$. We do this and more probability arguments in §5.
But the full proofs of those probabilistic inequalities are delayed to [**?**].

Third step: We deduce from §5 that $<^*_x = <_x$ for all relevant x and prove that the context is weakly nice. We then work somwhat more to prove the existential part of nice (the simple goodness (see Definition [I,2.12,(1)]) of appropriate candidate). I.e. we first prove "weakly niceness" by proving that $A <^*_i B$ implies (A, B) satisfy the demand for $<_i$ of [I,§1], and in a strong way the parallel thing for $\leq_s$. Those involve probability estimation, i.e., quoting §5. But we need more: sufficient conditions for appropriate tuples to be simply good and this is the first part of §6.

Fourth step: This is the universal part from niceness. This does not involve any probability, just weight computations (and previous stages), in other words, purely model theoretic investigation of the "limit" theory. By the "universal part of nice" we mean (A) of [I,2.13,(1)] which includes:

> if $\bar a \in {}^k(\mathscr{M}_n), b \in \mathscr{M}_n$ then there are $m_1 < m$, $B \subseteq c\ell^{k,m_1}(\bar a)$ such that $\bar a \subseteq B$ and
>
> $$c\ell^k(B) \mathop{\bigcup^{\mathscr{M}_n}}_{B} (c\ell^k(\bar a b, \mathscr{M}_n) \backslash c\ell^k(B, \mathscr{M})) \cup B.$$

This is done in the latter part of §6.

Because of the request of the referee and editors to shorten the paper, the computational part (in §5) in full was moved to [**?**], and the generalization to the case we have a successor function (which was §7) was moved to [**?**].

We thank Ilya Tsindlekht for pointing out a gap in the proof of **??**.

Notation 0.1. As in [She02].

§ 1. Applications

§ 1(A). The Basics for the case of $\frac{1}{\ell^\alpha}$.

We intend to apply the general theorems (Lemmas [I,2.17,2.19]), to our problem. That is, we try to answer: does the main context $\mathscr{M}_n^0$ with $p_i = 1/i^\alpha$ for $i > 1$ satisfies the 0-1 law? So here our irrational number $\alpha \in (0,1)_{\mathbb{R}}$ is fixed. We work in Main Context (see 1.1 below, the other one, $\mathscr{M}_n^1$, would work out as well, see §7).

Context 1.1. A particular case of [I,1.1]: $p_i = 1/i^\alpha$ for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_{\mathbb{R}}$ is a fix irrational) and the n-th random structure is $\mathscr{M}_n = \mathscr{M}_n^0 = ([n], R)$ (i.e. only the graph with the probability of $\{i,j\}$ being $p_{|i-j|}$).

Fact 1.2. 1) For any (finite) graph A, we have $A \in \mathscr{K}_\infty$, that is

$$1 = \lim_n \text{ Prob}(A \text{ is embeddable into } \mathscr{M}_n).$$

2) Moreover [2] for every $\varepsilon > 0$

$$1 = \lim_n \text{ Prob}(A \text{ has } \geq n^{1-\varepsilon} \text{ disjoint copies in } \mathscr{M}_n).$$

This is easy, still, before proving it, note that since by our definition of the closure $A \subseteq c\ell^{m,k}(\emptyset, \mathscr{M}_n)$ implies that A has $< n^\varepsilon$ embeddings into $\mathscr{M}_n$ we get:

Conclusion 1.3. *$\langle c\ell^{m,k}_{\mathscr{M}_n}(\emptyset) : n < \omega\rangle$ satisfies the 0–1 law (being a sequence of empty models).*

Hence (see [I,Def.1.4,Conclusion 2.19])

Conclusion 1.4. *$\mathscr{K}_\infty = \mathscr{K}$, the class of finite graph, and for our main theorem it suffices to prove <u>simple almost niceness</u> of $\mathfrak{K}$ (see Def.[I,2.13]).*

(Now 1.3 explicate one part of what in fact we always meant by "random enough" in previous discussions.)

Proof. Let the nodes of A be $\{a_0, \dots, a_{k-1}\}$. Let the event $\mathscr{E}_r^n$ be:

$$a_\ell \mapsto 2rk + 2\ell \text{ is an embedding of } A \text{ into } \mathscr{M}_n.$$

The point of this is that for various values of r these tries are going to speak on pairwise disjoint sets of nodes, so we get independent events.

Now <u>subfact</u> $\text{Prob}(\mathscr{E}_r^n) = q > 0$ (i.e. > 0 but it does <u>not</u> depend on n, r).
(Note: this is not true,e.g. in the close context where the probability of $\{i,j\}$ being an edge when $i \neq j$ is $1/n^\alpha + 1/2^{|i-j|}$, as in that case the probability depends on n. But still, we can have $\geq q > 0$ which suffices); where:

$$q = \prod_{\ell<m<k, \{\ell,m\} \text{ edge}} 1/(2(m-\ell))^\alpha \times \prod_{\ell<m<k,\{\ell,m\} \text{not an edge}} \left(1 - \frac{1}{(2(m-\ell))^\alpha}\right).$$

(What we need is that all the relevant edges have probability $> 0, < 1$. Note: if we have retained $p = 1/i^\alpha$ this is false for the pairs $(i, i+1)$, so we have changed

[2] Actually also "$\geq cn$" works for $c \in \mathbb{R}^{>0}$ depending on A only.

p_1. Anyway, in our case we multiplied by 2 to avoid this (in the definition of the event)). For the second case, (the probability of edge being $1/n^\alpha + 1/2^{(i-j)}$)

$$q \geq \prod_{\ell<m<k, \{\ell,m\} \text{ edge}} \frac{1}{2^{|m-\ell|}} \times \prod_{\ell<m<k, \{\ell,m\} \text{ not an edge}} (1 - \frac{1}{(3/2)^{|m-\ell|}}).$$

So these $\mathrm{Prob}(\mathscr{E}^n_r)$ have a positive lower bound which does not depend on r.

Also the events $\mathscr{E}^n_0, \ldots, \mathscr{E}^n_{[\frac{n}{2k}]-1}$ are independent. So the probability that they all fail is

$$\prod_{i<\lfloor \frac{n}{2k} \rfloor} (1 - \mathrm{Prob}(\mathscr{E}^n_i)) \leq \prod_i (1-q) \leq (1-q)^{\frac{n}{2k}}$$

which goes to 0 quite fast. The "moreover" is left to the reader. $\square_{1.2}$

Definition 1.5. 1) Let

$$\begin{aligned} \mathscr{T} = \{(A,B,\lambda) : \quad & A \subseteq B \text{ graphs (generally: models from } \mathscr{K}) \text{ and} \\ & \lambda \text{ an equivalence relation on } B \setminus A\} \end{aligned}$$

We may write (A, B, λ) instead of $(A, B, \lambda \restriction (B \setminus A))$.
2) We say that $X \subseteq B$ is λ-closed if:

$$x \in X \text{ and } x \in B \cap \mathrm{Dom}(\lambda) \text{ implies } x/\lambda \subseteq X.$$

3) $A \leq B$ means A is a submodel of B, and $A \leq^* B$ if [3] $A \leq B \in \mathscr{K}_\infty$ (clearly $\leq^*$ is a partial order).

Story:
We would like to ask for any given copy of A in $\mathscr{M}_n$, is there a copy of B above it, and how many, we hope for a dichotomy: i.e. usually none, always few or always many. The point of λ is to take distance into account, because for our present distribution being near is important, $b_1 \lambda b_2$ will indicate that b_1 and b_2 are near. Note that being near is not transitive, but "luck" helps us, we will succeed to "pretend" it is. We will look at many candidates for a copy of $B \setminus A$ and compute the expected value. We would like to show that saying "variance small" says that the true value is near the expected value.

Definition 1.6. 1) For $(A, B, \lambda) \in \mathscr{T}$ let

$$\mathbf{v}(A, B, \lambda) = \mathbf{v}_\lambda(A, B) = |(B \setminus A)/\lambda|$$

that is, the number of λ–equivalence classes in $B \setminus A$ ($\mathbf{v}$ stands for vertices).

(This measures degrees of freedom in choosing candidates for B over a given copy of A.)
2) Let

[3] Note: here $A \leq^* B$ iff $A \leq B$; anyhow the choice of $\leq^*$ is for our present specific context, so this definition does not apply to §1, §2, §3,§7; in fact, in §7, i.e. in [?] we give a different definition for a different context.

$$\mathbf{e}(A,B,\lambda) = \mathbf{e}_\lambda(A,B) = |e_\lambda(A,B)| \text{ where}$$

$$e_\lambda(A,B) = \{e : e \text{ an edge of } B, e \nsubseteq A, \text{ and } e \nsubseteq x/\lambda \text{ for } x \in B \setminus A\}.$$

[This measures the number of "expensive", "long" edges ($\mathbf{e}$ stands for edges).]

Story:
$\mathbf{v}$ larger means that there are more candidates for copies of B,
$\mathbf{e}$ larger means that the probability per candidate is smaller.

Definition 1.7. 1) For $(A,B,\lambda) \in \mathscr{T}$ and our given irrational $\alpha \in (0,1)_\mathbb{R}$ we define ($\mathbf{w}$ stands for weight)

$$\mathbf{w}(A,B,\lambda) = \mathbf{w}_\lambda(A,B) = \mathbf{v}_\lambda(A,B) - \alpha\mathbf{e}_\lambda(A,B).$$

2) Let

$$\Xi(A,B) =: \{\lambda : \ (A,B,\lambda) \in \mathscr{T}, \text{ and if } C \subseteq B \setminus A \text{ is a non-empty } \lambda\text{-closed set then } \mathbf{w}_\lambda(A, C \cup A) > 0\}.$$

3) If $A \leq^* B$ then we let $\xi(A,B) = \mathrm{Max}\{\mathbf{w}_\lambda(A,B) : \lambda \in \Xi(A,B)\}$.

Observation 1.8. *1) $(A,B,\lambda) \in \mathscr{T} and A \neq B \Rightarrow \mathbf{w}_\lambda(A,B) \neq 0$.*
2) If $A \leq^* B \leq^* C$ (hence $A \leq^* C$) and $(A,C,\lambda) \in \mathscr{T}$ and B is λ-closed then

(a) $(A,B,\lambda \restriction (B \setminus A)) \in \mathscr{T}$
(b) $(B,C,\lambda \restriction (C \setminus B)) \in \mathscr{T}$
(c) $\mathbf{w}_\lambda(A,C) = \mathbf{w}_{\lambda \restriction (B \setminus A)}(A,B) + \mathbf{w}_{\lambda \restriction (C \setminus B)}(B,C)$
(d) *similarly for* $\mathbf{v}$ *and* $\mathbf{e}$.

3) Note that 1.8(2) legitimizes our writing λ instead of $\lambda \restriction (C \setminus A)$ or $\lambda \restriction (B \setminus (C \cup A))$ when $(A,B,\lambda) \in \mathscr{T}$ and C is a λ-closed subset of B. Thus we may write, e.g., $\mathbf{w}_\lambda(A \cup C, B)$ for $\mathbf{w}(A \cup C, B, \lambda \restriction (B \setminus A \setminus C))$.
4) If $(A,B,\lambda) \in \mathscr{T}$ and $D \subseteq B \setminus A$ and $D^+ = \bigcup\{x/\lambda : x \in D\}$ then $\mathbf{w}_{\lambda \restriction D^+}(A, A \cup D^+) \leq \mathbf{w}_{\lambda \restriction D}(A, A \cup D)$ and D^+ is λ-closed.

Proof. 1) As α is irrational and $\mathbf{v}_\lambda(A,B)$ is not zero.
2) Clauses (a),(b) are totally immediate, and clauses (c),(d) are easy or see the proof of 1.15 below.
3) Left to the reader.
4) Clearly by the choice of D^+ we have $\mathbf{v}_{\lambda \restriction D^+}(A, A \cup D^+) = |D^+/\lambda \restriction D^+| = |D/(\lambda \restriction D)| = \mathbf{v}_{\lambda \restriction D}(A, A \cup D)$ and $\mathbf{e}_{\lambda \restriction D^+}(A, A \cup D^+) \geq \mathbf{e}_{\lambda \restriction D}(A, A \cup D)$ hence $\mathbf{w}_{\lambda \restriction D^+}(A, A \cup D^+) \leq \mathbf{w}_{\lambda \restriction D}(A, A \cup D)$. $\square_{1.8}$

Discussion 1.9. Note: $\mathbf{w}_\lambda(A,B)$ measures in a sense the expected value of the number of copies of B over a given copy of A with λ saying when one node is "near to" another. Of course, when λ is the identity this degenerates to the definition in [SS88].
We would like to characterize $\leq_i$ and $\leq_s$ (from Definition [I,1.4,(3)] and Definition [I,1.4,(4)]), using $\mathbf{w}$ and to prove that they are O.K. (meaning that they form a

nice context). Looking at the expected behaviour, we attempt to give an "effective" definition (depending on α only).
All of this, of course, just says what the intention of these relations and functions is (i.e. $<^*_i$, $<^*_s$, $<^*_{pr}$ and $\mathbf{v}$, $\mathbf{e}$, $\mathbf{w}$ below); we still will not prove anything on the connections to $\le_i, \le_s, \le_{\rm pr}$. We may view it differently: We are, for our fix α, defining $\mathbf{w}_\lambda(A,B)$ and investigating the $\le^*_i, \le^*_s, \le^*_{pr}$ defined below per se ignoring the probability side.

Definition 1.10. 1) Recall $A \le^* B$ if $A \le B \in \mathfrak{K}_\infty$.
2) $A <^*_c B$ <u>if</u> $A <^* B$ (i.e. $A \le^* B$ and $A \ne B$) and for every λ, we have

$$(A,B,\lambda) \in \mathscr{T} \Rightarrow \mathbf{w}_\lambda(A,B) < 0,$$

3) $A \le^*_i B$ <u>if</u> $A \le^* B$ and for every A' we have

$$A \le^* A' <^* B \Rightarrow A' <^*_c B.$$

Of course, $A <^*_i B$ means $A \le^*_i B and A \ne B$.
4) $A \le^*_s B$ <u>if</u> $A \le^* B$ and for no A' do we have

$$A <^*_i A' \le^* B,$$

Of course, $A <^*_s B$ means $A \le^*_s B and A \ne B$.
5) $A <^*_a B$ <u>if</u> $A \le^* B$, $\neg(A <^*_s B)$ (i.e. $A \le^* B$ and there is $A' \subseteq B \setminus A$ such that $A <^*_i A \cup A' \le^* B$).
6) $A <^*_{pr} B$ <u>if</u> $A \le^* B$ and $A <^*_s B$ but for no C do we have $A <^*_s C <^*_s B$.

Remark 1.11. We <u>intend</u> to prove that usually $\le^*_x = \le_x$ but it will take time.

Lemma 1.12. *Suppose $A' <^* B, (A',B,\lambda) \in \mathscr{T}$ and $\mathbf{w}_\lambda(A',B) > 0$. <u>Then</u> there is A'' satisfying $A' \le^* A'' <^* B$ such that A'' is λ-closed and*

$(*)_1 = (*)_1$ $[A'',B,\lambda]$ *we have* $\mathbf{w}_\lambda(A'',B) > 0$ *and <u>if</u>* $C \subseteq B \setminus A''$, $C \notin \{\emptyset, B \setminus A''\}$ *and* C *is λ-closed <u>then</u>* $\mathbf{w}_\lambda(A'', A'' \cup C) > 0$ *and* $\mathbf{w}_\lambda(A'' \cup C, B) < 0$.

Proof. Let C' be a maximal λ-closed subset of $B \setminus A'$ such that $\mathbf{w}_\lambda(A' \cup C', B) > 0$. Such a C' exists since $C' = \emptyset$ is as required and B is finite. Let $A'' = A' \cup C'$. Since C' is λ-closed, $B \setminus A''$ is λ-closed and $(A'',B,\lambda \restriction (B \setminus A'')) \in \mathscr{T}$ and clearly $\mathbf{w}_\lambda(A'',B) > 0$. Now suppose $D \subseteq B \setminus A''$ is λ-closed, $D \notin \{\emptyset, B \setminus A''\}$. By the maximality of C', $\mathbf{w}_\lambda(A'' \cup D, B) < 0$. Now (by 1.8(2)(c))

$$\mathbf{w}_\lambda(A'',B) = \mathbf{w}_\lambda(A'', A'' \cup D) + \mathbf{w}_\lambda(A'' \cup D, B)$$

and the term in the left side is positive by the choice of C' and A'', but in the right side, the term $\mathbf{w}_\lambda(A'', A'' \cup D)$ is negative by a previous sentence the maximality of C' so together we conclude $\mathbf{w}_\lambda(A'', A'' \cup D) > 0$ contradicting the maximality of C'. $\square_{1.12}$

§ 1(B). **Investigating the $\le^*_x$'s.**

Claim 1.13. *Assume $A <^* B$. The following statements are equivalent:*

(*i*) $A <^*_i B$,

(*ii*) *for no A' and λ do we have:*
$(*)_2 = (*)_2[A, A', B, \lambda]$ *we have* $A \le^* A' <^* B, (A', B, \lambda) \in \mathscr{T}$ *and* $\mathbf{w}_\lambda(A', B) > 0$,

(*iii*) *for no A', λ do we have:*
$(*)_3 = (*)_3[A, A', B, \lambda]$ *we have* $A \le^* A' <^* B, (A', B, \lambda) \in \mathscr{T}, \mathbf{w}_\lambda(A', B) > 0$ *and moreover* $(*)_1[A', B, \lambda]$ *of 1.12.*

Proof. For the equivalence of the first and the second clauses read Definition 1.10(2), 1.10(3) (remembering 1.8(1)). Trivially $(*)_3 \Rightarrow (*)_2$ and hence the second clause implies the third one. Now we will see that $(iii) \Rightarrow (ii)$. So suppose $\neg(ii)$, so let this be exemplified by A',λ i.e. they satisfy $(*a)_2$ so by 1.12 there is A'' such that $A' \le^* A'' <^* B$ and $(*)_1[A'', B, \lambda]$ of 1.12 holds. So A'', λ exemplified that $\neg(iii)$ holds. $\square_{1.13}$

Observation 1.14. *1) If $(*)_3[A, A', B, \lambda]$ from 1.13(iii) holds, <u>then</u> we have: if $C \subseteq B \setminus A'$ is λ-closed non-empty then $\mathbf{w}(A', A' \cup C, \lambda \restriction C) > 0$.*
[Why? If $C \ne B \setminus A'$ this is stated explicitly, otherwise this means $\mathbf{w}(A', B, \lambda) > 0$ which holds.]
2) In $(*)_3$ of 1.13(iii), i.e., 1.12$(*)_1[A', B, \lambda]$, we can allow any λ–closed $C \subseteq B \setminus A'$ if we make the inequalities non-strict.
[Why? If $C = \emptyset$ then $\mathbf{w}_\lambda(A', A' \cup C) = \mathbf{w}_\lambda(A', A') = 0$, $\mathbf{w}_\lambda(A' \cup C, B) = \mathbf{w}_\lambda(A', B) > 0$. If $C = B \setminus A'$ then $\mathbf{w}_\lambda(A', A' \cup C) = \mathbf{w}_\lambda(A', B) > 0$ and $\mathbf{w}_\lambda(A' \cup C, B) = \mathbf{w}_\lambda(B, B) = 0$. Lastly if $C \notin \{\emptyset, B \setminus A'\}$ we use 1.12$(*)_1[A', B, \lambda]$ itself.]
3) If $(A, B, \lambda) \in \mathscr{T}, A' \le^* A, B' \le^* B, A' \le^* B'$ and $B \setminus A = B' \setminus A'$ <u>then</u> $(A', B', \lambda) \in \mathscr{T}, \mathbf{w}(A', B', \lambda) \ge \mathbf{w}(A, B, \lambda)$ also $\mathbf{e}(A', B', \lambda) \le \mathbf{e}(A, B, \lambda), \mathbf{v}(A', B', \lambda) = \mathbf{v}(A, B, \lambda)$.
4) In (3) if in addition $A \underset{A'}{\overset{M}{\biguplus}} B'$, i.e., no edge $\{x, y\}$ with satisfying $x \in A \setminus A'$ and $y \in B' \setminus A'$ <u>then</u> the equalities hold.

Claim 1.15. *$A \le^*_s B$ <u>if and only if</u> either $A = B$ or for some λ we have:*
$(A, B, \lambda) \in \mathscr{T}$ and for every non-empty λ-closed $C \subseteq B \setminus A$, we have $\mathbf{w}(A, A \cup C, \lambda \restriction C) > 0$, that is $\Xi(A, B) \ne \emptyset$.

Proof. So we have $A \le^*_s B$. If $A = B$ we are done: the left side holds as its first possibility is $A = B$. So assume $A <^*_s B$. Let C be such that

$(*)_1$ C is minimal such that $A \le^* C \le^* B$ and for some λ_0 the triple $(C, B, \lambda_0) \in \mathscr{T}$ satisfies: for every non-empty λ_0-closed $C' \subseteq B \setminus C$ we have $\mathbf{w}(C, C \cup C', \lambda_0 \restriction C') > 0$

(exists as $C = B$ is O.K. because then there is no such C').
By 1.8(4):

$(*)_2$ for every non-empty $C' \subseteq B \backslash C$ so not necessarily λ_0-closed we have $\mathbf{w}(C, C \cup C', \lambda_0 \restriction C') > 0$ hence $\neg(C <^*_i C \cup C')$ by $(i) \Leftrightarrow (ii)$ of 1.13.

If $C = A$ we have finished by the choice of C. Otherwise, the hypothesis $A \leq^*_s B$ implies that $\neg(A <^*_i C)$, hence by 1.13 the third clause (iii) fails which means that (recalling 1.10(1)), for some C', λ_1 we have

$(*)_3$ $A \leq^* C' <^* C$, $(C', C, \lambda_1) \in \mathscr{T}, \mathbf{w}_{\lambda_1}(C', C) > 0$ and for every λ_1-closed $D \subseteq C \setminus C'$ satisfying $D \notin \{\emptyset, C \setminus C'\}$ we have
- $\mathbf{w}(C', C' \cup D, \lambda_1 \restriction D) > 0$
  and
- $\mathbf{w}\big(C' \cup D, C, \lambda_1 \restriction (C \setminus C' \setminus D)\big) < 0.$

Define an equivalence relation λ on $B \backslash C'$: an equivalence class of λ is an equivalence class of λ_0 or an equivalence class of λ_1.

We shall show that (C', B, λ) satisfies the requirements in $(*)_1$ above on C, thus contradicting the minimality of C. Clearly $A \leq^* C' \leq^* B$. So let $D \subseteq B \setminus C'$ be λ-closed and we define $D_0 = D \cap (B \setminus C)$, $D_1 = D \cap (C \setminus C')$. Clearly D_0 is λ_0-closed so $\mathbf{w}(C, C \cup D_0, \lambda \restriction D_0) \geq 0$ (see **??**(2)), and D_1 is λ_1-closed so $\mathbf{w}(C', C' \cup D_1, \lambda \restriction D_1) \geq 0$ (this follows from: for every λ_1-closed $D \subseteq C \setminus C'$ satisfying $D \notin \{\emptyset, C \setminus C'\}$ we have $\mathbf{w}_\lambda(C', C' \cup D, \lambda_1 \restriction D) > 0$ and by **??**(2)). Now (in the last line we change C' to C twice), by **??**(3) we will get

$$\begin{aligned}
\mathbf{v}(C', C' \cup D, \lambda) = \quad & |D/\lambda| = |D_1/\lambda_1| + |D_0/\lambda_0| \\
& = \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda \restriction D_0) \\
& = \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda \restriction D_0),
\end{aligned}$$

and (using **??**(3)):

$$\begin{aligned}
\mathbf{e}(C', C' \cup D, \lambda) \quad & = \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) \\
& +\mathbf{e}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda \restriction D_0) \\
& \leq \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{e}(C, C \cup D_0, \lambda \restriction D_0),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{w}(C', C' \cup D, \lambda) \quad & = \mathbf{v}(C', C' \cup D, \lambda) - \alpha \mathbf{e}(C', C' \cup D, \lambda) \\
& = \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda \restriction D_0) \\
& -\alpha \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) \\
& -\alpha \mathbf{e}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda \restriction D_0) \\
& \geq \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda \restriction D_0) \\
& -\alpha \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) - \alpha \mathbf{e}(C, C \cup D_0, \lambda \restriction D_0) \\
& = \mathbf{w}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{w}(C, C \cup D_0, \lambda \restriction D_0) \geq 0,
\end{aligned}$$

and the (strict) inequality holds by the irrationality of α, i.e. by 1.8(1). So actually (C', B, λ) satisfies the requirements on C, λ_0 thus giving contradiction to the minimality of C.

<u>The if direction</u>:

As the case $A = B$ is obvious, we can assume that the second half of 1.15 holds. So let λ be as required in the second half of 1.15.

Suppose $A <^* C \leq^* B$, and we shall prove that $\neg(A <^*_i C)$ thus finishing by Definition 1.10. We shall show that $\lambda' = \lambda \restriction (C \backslash A)$ satisfies $(*)_2[A, A, C, \lambda']$ from 1.13, thus (ii) of 1.13 fail hence (i) of 1.13 fails, i.e., $\neg(A <^*_i C)$ as required. Let $D = \bigcup\{x/\lambda : x \in C \setminus A\}$, so D is a non-empty λ-closed subset of $B \setminus A$. Hence by the present assumption on A, B, λ we have $\mathbf{w}(A, A \cup D, \lambda \restriction D) > 0$. Now

$$\mathbf{v}(A, C, \lambda \restriction C) = |C/\lambda| = |D/\lambda| = \mathbf{v}(A, D, \lambda \restriction D)$$

and

$$\mathbf{e}(A, C, \lambda \restriction C) \leq \mathbf{e}(A, D, \lambda \restriction D)$$

so $\mathbf{w}(A, C, \lambda \restriction C) \geq \mathbf{w}(A, D, \lambda \restriction D) > 0$ as requested. $\square_{1.15}$

Claim 1.16. *1) $\leq^*_i$ is transitive.*
2) $\leq^*_s$ is transitive.
3) For any $A \leq^* C$ for some B we have $A \leq^*_i B \leq^*_s C$.
4) If $A <^* B$ and $\neg(A \leq^*_s B)$ <u>then</u> $A <^*_c B$ or there is C such that $A <^* C <^* B$, $\neg(A <^*_s C)$.
5) Smoothness holds (with $<^*_i$ instead of $<_i$ see [I,2.5(4)]), that is

(a) if $A \leq^* C \leq^* M \in \mathscr{K}, A \leq^* B \leq^* M$, $B \cap C = A$ <u>then</u> $A <^*_c B \Rightarrow C <^*_c B \cup C$ and $A \leq^*_i C \Rightarrow B \leq^*_i B \cup C$

(b) if in addition $C \underset{A}{\overset{M}{\bigcup}} B$ then $A <^*_c B \Leftrightarrow C \leq^*_c B \cup C$ and $A \leq^*_i C \Leftrightarrow B \leq^*_i B \cup C$ and $A \leq^*_s B \Leftrightarrow C \leq^*_s C \cup B$.

6) For $A <^* B$ we have $\neg(A \leq^*_s B)$ <u>iff</u> $(\exists C)(A <^*_c C \leq^* B)$.
7) If $A \leq^* B \leq^* C$ and $A \leq^*_s C$ <u>then</u> $A \leq^*_s B$.
8) If $A_\ell \leq^*_s B_\ell$ for $\ell = 1, 2, A_1 \leq^* A_2, B_1 \leq^* B_2$ and $B_2 \backslash A_2 = B_1 \backslash A_1$ <u>then</u>

$$\xi(A_1, B_1) \geq \xi(A_2, B_2).$$

9) In (8), equality holds iff A_2, B_1 are freely amalgamated over A_1 inside B_2.
10) If $A <^*_s B_\ell$ for $\ell = 1, 2$ and $B_1 <^*_i B_2$ and for some edge $\{x, y\}$ of B_2 we have $x \in A, y \in B_2 \backslash B_1$ <u>then</u> $\xi(A, B_1) > \xi(A, B_2)$.
11) If $B_1 <^* B_2$ and for no $x \in B_1, y \in B_2 \backslash B_1$ is $\{x, y\}$ an edge of B_2 <u>then</u> $B_1 <^*_s B_2$.
12) If $A \leq^* B \leq^* C$ and $A \leq^*_i C$ <u>then</u> $B \leq^*_i C$.
13) If $A <^*_{\mathrm{pr}} B$ and $a \in B \backslash A$ <u>then</u> $A \cup \{a\} \leq^*_i B$.
14) If $A_1 <^*_{\mathrm{pr}} B_1, A_1 \leq^* A_2 \leq^* B_2$ and $B_1 \leq^* B_2$ and $B_2 = A_2 \cup B_1$ <u>then</u> $A_2 \leq^*_s B_2$ or $A_2 <^*_{\mathrm{pr}} B_2$.

Proof. 1) So assume $A \leq^*_i B \leq^*_i C$ and we shall prove $A \leq^*_i C$. By 1.13 it suffice to prove that clause (ii) there holds with A, C here standing for A, B there. So assume $A \leq^* A' <^* C$, $(A', C, \lambda) \in \mathscr{T}$ and we shall prove that $\mathbf{w}_\lambda(A', C) \leq 0$, this suffice. Let $A'_1 = A' \cap B$, $A'_0 =: B \cup A' \cup \cup\{x/\lambda : x \in B\}$, now as $A \leq^*_i B$ by 1.13 + **??**(2) we have $\mathbf{w}_\lambda(A'_1, B) \leq 0$, and by **??**(3) we have $\mathbf{w}_\lambda(A', B \cup A') \leq \mathbf{w}_\lambda(A'_1, B)$ and by 1.8(4) we have $\mathbf{w}_\lambda(A', A'_0) \leq \mathbf{w}_\lambda(A', A' \cup B)$. Those three inequalities together gives $\mathbf{w}_\lambda(A', A'_0) \leq 0$ and as $B \leq^*_i C \wedge B \subseteq A'_0$ by 1.13 we have $\mathbf{w}_\lambda(A'_0, C) \leq 0$.

By 1.8(2)(c) we have $\mathbf{w}_\lambda(A', C) = \mathbf{w}_\lambda(A', A'_0) + \mathbf{w}_\lambda(A'_0, C)$ and by the previous sentence the latter is $\leq 0 + 0 = 0$, so $\mathbf{w}_\lambda(A, A') \leq 0$ as required.
2) We use the condition from 1.15. So assume $A_0 \leq^*_s A_1 \leq^*_s A_2$ and λ_ℓ witness $A_\ell \leq^*_s A_{\ell+1}$ (i.e. $(A_\ell, A_{\ell+1}, \lambda_\ell)$ is as in 1.15). Let λ be the equivalence relation on $A_2 \setminus A_0$ such that for $x \in A_{\ell+1} \setminus A_\ell$ we have $x/\lambda = x/\lambda_\ell$. Easily $(A_0, A_2, \lambda) \in \mathscr{T}$. Now, by 1.8(2)(c), **??**(3) and 1.15 the triple (A_0, A_2, λ) satisfies the second condition in 1.15 so $A_0 \leq^*_s A_2$.
3) Let B be maximal such that $A \leq^*_i B \leq^* C$, such B exists as C is finite [4] and for $B = A$ we get $A \leq^*_i B \leq^* C$. Now if $B \leq^*_s C$ we are done, otherwise by the definition of $\leq^*_s$ in 1.10(4) there is B' such that $B <^*_i B' \leq^* C$, now by part (1) we have $A \leq^*_i B' \leq^* C$, contradicting the maximality of B, so really $B \leq^*_s C$ and we are done.
4) We assume $A <^* B$ and now if $A <^*_c B$ we are done hence we can assume $\neg(A <^*_c B)$, by Definition 1.10(2) there is λ such that $(A, B, \lambda) \in \mathscr{T}$ and $\mathbf{w}_\lambda(A, B) \geq 0$. So by the irrationality of α the inequality is strict and by 1.12 there is C such that $A \leq^* C <^* B$, C is λ-closed, $\mathbf{w}_\lambda(C, B) > 0$ and if $C' \subseteq B \backslash C$ is non-empty λ-closed and $\neq B \backslash C$ then $\mathbf{w}_\lambda(C, C \cup C') > 0 and \mathbf{w}_\lambda(C \cup C', B) < 0$. So by 1.15 + inspection, $C <^*_s B$, so by 1.16(2), $A \leq^*_s C \Rightarrow A^* \leq^*_s B$, but we know that $\neg(A <^*_s B)$ hence by part (2) we have $\neg(A \leq^*_s C)$, so the second possibility in the conclusion holds.
5) <u>Clause (a)</u>: Assume $A \leq^* C \leq^* M \in \mathscr{K}$ and $A \leq^* B \leq^* M$ and $B \cap C$. <u>Then</u>

- $A \leq^*_c B \Rightarrow C <^*_c B \cup C$ and
- $A \leq^*_i C \Rightarrow B \leq^*_i B \cup C$.

[Why? Note by our assumption $C <^* B \cup C$ and $B <^* B \cup C$. The first desired conclusion is easier, so we prove the second so assume $A \leq^*_i C$. If $B \leq^* D <^* B \cup C$, and $(D, B \cup C, \lambda) \in \mathscr{T}$ then $A \leq^* D \cap C <^* C$ so as $A \leq^*_i C$, by the definition of $\leq_i$ we have $\mathbf{w}_\lambda(D \cap C, C) < 0$ hence (noting $C \setminus D \cap C = B \cup C \setminus D$) by Observation **??**(3) we have $\mathbf{w}_\lambda(D, B \cup C) \leq \mathbf{w}_\lambda(D \cap C, C) < 0$ hence (by the Def. of $\leq^*_c$) $D \leq^*_c B \cup C$. As this holds for any such D by Definition 1.10(3) we have $B \leq^*_i B \cup C$ as required.]

<u>Clause (b)</u>: If in addition $C \bigcup_{A}^{M} B$ then

- $A <^*_c C \Leftrightarrow B \leq^*_c B \cup C$ and
- $A \leq_i C \Leftrightarrow B \leq_i B \cup C$ and
- $A \leq^*_s B \Leftrightarrow B \leq^*_s B \cup C$.

[Why? Immediate by **??**(4), Definition 1.10 and clause (a) which was proved.]
6) The "only if" direction can be prove by induction on $|B|$, using 1.16(4). For the if direction assume that for some C, $A <^*_c C \leq^* B$ and choose a minimal C like that. Now if $A \leq^* A^* <^* C$, and λ_1 is an equivalence relation on $C \setminus A^*$ then let λ_0 be an equivalence relation on $A^* \setminus A$ such that $\mathbf{w}_{\lambda_0}(A, A^*) \geq 0$ (exists by the minimality of C) and let $\lambda = \lambda_0 \cup \lambda_1$ so $(A, C, \lambda) \in \mathscr{T}$ and by 1.8(2)(i) we have $\mathbf{w}_\lambda(A^*, C) = \mathbf{w}_\lambda(A, C) - \mathbf{w}_\lambda(A, A^*)$; but as $A <^*_c C$ we have $\mathbf{w}_\lambda(A, C) < 0$, and by the choice of λ_0 we have $\mathbf{w}_\lambda(A, A^*) \geq 0$ hence $\mathbf{w}_\lambda(A^*, C) < 0$ hence $\mathbf{w}_{\lambda_1}(A^*, C) =$

[4] Actually the finiteness is not needed if for possibly infinite A, B we define $A \leq^*_i B$ iff for every finite $B' \leq^* B$ there is a finite B'' such that $B' \leq^* B'' \leq^* B$, and $B'' \cap A <^*_i B''$.

$\mathbf{w}_\lambda(A^*, C) < 0$. As λ_1 was any equivalence relation on $C \setminus A^*$ by Definition 1.10(2) we have shown that $A^* <^*_c C$. By the definition of $\leq^*_i$ (1.10(3)), as A^* was arbitrary such that $A \leq^* A^* <^* C$ by Definition 1.10(3) we get that $A <^*_i C$, hence by the definition of $\leq_s$ (1.10(4)), we can deduce $\neg(A \leq^*_s B)$ as required.
7) Immediate by Definition 1.10(4).
8) It is enough to prove that

$\circledast$ if $\lambda \in \Xi(A_2, B_2)$ then $\lambda \in \Xi(A_1, B_1)$ and $\mathbf{w}_\lambda(A_1, B_1) \geq \mathbf{w}_\lambda(A_2, B_2)$.

So assume λ is an equivalence relation over $B_2 \backslash A_2$ which is equal to $B_1 \backslash A_1$, now for every non-empty λ-closed $C \subseteq B_1 \backslash A_1$ we have

(i) $\mathbf{v}_\lambda(A_1, A_1 \cup C) = |C/\lambda| = \mathbf{v}_\lambda(A_2, A_2 \cup C)$
(ii) $\mathbf{e}_\lambda(A_1, A_1 \cup C) \leq \mathbf{e}_\lambda(A_2, A_2 \cup C)$
[as any edge in $e_\lambda(A_1, A_1 \cup C)$ belongs to $e_\lambda(A_2, A_2 \cup C)$]
hence
(iii) $\mathbf{w}_\lambda(A_1, A_1 \cup C) \geq \mathbf{w}_\lambda(A_2, A_2 \cup C)$.

So by the definition of $\Xi(A_1, B_1)$ we have $\lambda \in \Xi(A_2, B_2) \Rightarrow \lambda \in \Xi(A_1, B_1)$ and, moreover, the desired inequality in $\circledast$ holds.
9) First assume $A_2 \displaystyle\biguplus_{A_1}^{B_2} B_1$, in the proof of (8) we get $e_\lambda(A_1, A_1 \cup C) = e_\lambda(A_2, A_2 \cup C)$ hence $\mathbf{w}_\lambda(A_1, A_1 \cup C) = \mathbf{w}_\lambda(A_2, A_2 \cup C)$, in particular $\mathbf{w}_\lambda(A_1, B_1) = \mathbf{w}_\lambda(A_2, B_2)$. Also now the proof of (8) gives $\lambda \in \Xi(A_1, B_1) \Rightarrow \lambda \in \Xi(A_2, B_2)$ so trivially $\xi(A_1, B_1) = \xi(A_2, B_2)$.

Second, assume $\neg(A_2 \displaystyle\biguplus_{A_1}^{B_2} B_1)$ then for every equivalence relation λ on $B_1 \backslash A_1 = B_2 \backslash A_2$ we have

(ii)$^+$ $\mathbf{e}_\lambda(A_1, B_1) < \mathbf{e}_\lambda(A_2, B_2)$.

[Why? As $e_\lambda(A_1, B_1)$ is a proper subset of $e_\lambda(A_2, B_2)$ by our present assumption.]
hence

(iii)$^+$ $\mathbf{w}_\lambda(A_1, B_1) > \mathbf{w}_\lambda(A_2, B_2)$.

As the number of such λ is finite and as we have shown $\Xi(A_2, B_2) \subseteq \Xi(A_1, B_1)$ we get $\xi(A_1, B_1) > \xi(A_2, B_2)$.
10) This follows from $\circledast_0 + \circledast_1 + \circledast_2$ below and the finiteness of $\Xi(A, B_2)$ recalling Definition 1.7(3).

$\circledast_0$ $\Xi(A, B_2) \neq \emptyset$.

[Why? As we are assuming $A <_s B_2$ and so recall claim 1.15.]

$\circledast_1$ $\lambda \in \Xi(A, B_2) \Rightarrow \lambda \restriction (B_1 \backslash A) \in \Xi(A, B_1)$.

[Why? If λ is an equivalence relation on $B_2 \backslash A$ and let $\lambda_1 = \lambda \restriction (B_1 \backslash A)$ so λ_1 is an equivalence relation on $B_1 \backslash A$ and for any non-empty λ_1-closed $C_1 \subseteq B_1 \backslash A$, letting $C_2 = \cup\{x/\lambda : x \in C_1\}$ we have $\mathbf{w}_\lambda(A, A \cup C_1) \geq \mathbf{w}_\lambda(A, A \cup C_2)$ by 1.8(4) and the latter is positive because $\lambda \in \Xi(A, B_2)$.]
And

$\circledast_2$ $\lambda \in \Xi(A, B_2) \Rightarrow \mathbf{w}_\lambda(A, B_2) < \mathbf{w}_\lambda(A, B_1)$.

[Why? Otherwise let λ be from $\Xi(A, B_2)$ and let $C_\lambda = \cup\{x/\lambda : x \in B_1 \backslash A\} \cup A$ so $B_1 \leq^* C_\lambda \leq^* B_2$.]

Case 1: $C_\lambda = B_2$.

So $\mathbf{v}_\lambda(A, B_1) = \mathbf{v}(A, B_1, \lambda{\restriction}(B_1\backslash A)) = |(B_1\backslash A)/\lambda| = |(B_2\backslash A)/\lambda| = \mathbf{v}_\lambda(A, B_2, \lambda)$. By an assumption of part (10), for some $x \in A, y \in B_2\backslash B_1$ the pair $\{x, y\}$ is an edge so $e(A, B_1, \lambda \restriction (B_1\backslash A))$ is a proper subset of $e(A, B_2, \lambda)$ hence

$$\mathbf{e}_\lambda(A, B_1) < \mathbf{e}_\lambda(A, B_2)$$

hence

$$\mathbf{w}_\lambda(A, B_1) > \mathbf{w}_\lambda(A, B_2)$$

is as required.

Case 2: $C_\lambda \neq B_2$.

As in case 1, $\mathbf{w}_\lambda(A, B_1) \geq \mathbf{w}_\lambda(A, C_\lambda)$. Now $B_1 \leq^* C_\lambda \leq^* B_2$, (using the case assumption) and $B_1 <^*_i B_2$ by an assumption so by part (12) below we have $C_\lambda \leq^*_i B_2$ but we are assuming $C_\lambda \neq B_2$ hence $C_\lambda <^*_i B_2$ and by 1.13 this implies $\mathbf{w}_\lambda(C_\lambda, B_2) < 0$. So $\mathbf{w}_\lambda(A, B_2) = \mathbf{w}_\lambda(A, C_\lambda) + \mathbf{w}_\lambda(C_\lambda, B_2) \leq \mathbf{w}_\lambda(A, B_1) + \mathbf{w}_\lambda(C_\lambda, B_2) < \mathbf{w}_\lambda(A, B_1)$ as required.

11) Define λ, it is the equivalence relation with exactly one class on $B_2\backslash B_1$ so $(B_1, B_2, \lambda) \in \mathscr{T}, \mathbf{v}_\lambda(B_1, B_2) = 1, \mathbf{e}_\lambda(B_1, B_2) = 0$ so $\mathbf{w}_\lambda(B_1, B_2) \geq 0$ hence $\lambda \in \Xi(B_2, B_2)$ hence $B_1 <_s B_2$.

12) By the Definition 1.10(3).

13) Clearly $A \cup \{a\} \leq^* B$ hence by part (3) for some C we have $A \cup \{a\} \leq^*_i C \leq^*_s B$. If $C = B$ we are done, otherwise $A <^*_s C$ by part (7) so we have $A <^*_s C <^*_s B$, contradiction.

14) Easy by Definition 1.10(6). $\square_{1.16}$

§ 2. The probabilistic inequalities

In this section we deal with probabilistic inequalities about the number of extensions for the context $\mathscr{M}_n^0$. Mostly the (computational) proofs are delayed to [?].

Note: the proof of almost simply niceness of $\mathfrak{K}$ is in the next section.

Context 2.1. As in §4, so $p_i = 1/i^\alpha$, for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_{\mathbb{R}}$ irrational) and $\mathscr{M}_n = \mathscr{M}_n^0$ (i.e. only the graph).

Definition 2.2. Let $\varepsilon > 0, k \in \mathbb{N}, \mathscr{M}_n \in \mathscr{K}$ and $A <^* B$ be in $\mathscr{K}_\infty$. Assume $f : A \hookrightarrow \mathscr{M}_n$ is an embedding or just $f : A \hookrightarrow [n]$ which means it is one to one. Define

$$\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n) := \{\bar g : \begin{array}{ll} (1) & \bar g = \langle g_\ell : \ell < k\rangle, \\ (2) & \bullet \quad f \subseteq g_\ell \\ & \bullet \quad g_\ell \text{ a one-to-one function from } B \text{ into } |\mathscr{M}_n| \\ (3) & g_\ell : B \hookrightarrow_f \mathscr{M}_n, \text{ for } \ell \le k \text{ or we may write } g_\ell : B \hookrightarrow_A \mathscr{M}_n \\ & \text{which means :} \\ & \bullet \quad g : A \hookrightarrow [n] \text{ so } g \text{ is one-to-one} \\ & \bullet \quad \{a,b\} \in \mathrm{Edge}(B)\backslash \mathrm{Edge}(A) \Rightarrow \{g(b), g(b)\} \in \mathrm{Edge}(\mathscr{M}_n) \\ & \bullet \quad g \text{ extends } f \\ (4) & \ell_1 \ne \ell_2 \Rightarrow \mathrm{Rang}(g_{\ell_1}) \cap \mathrm{Rang}(g_{\ell_2}) = \mathrm{Rang}(f) \\ (5) & [\ell < k \text{ and } x \in B\backslash A \text{ and } y \in A] \Rightarrow |g_\ell(x) - g_\ell(y)| \ge n^\varepsilon\}. \end{array}$$

The size of this set has natural connection with the number of pairwise disjoint extensions $g : B \hookrightarrow \mathscr{M}_n$ of f, hence with the holding of $A <_s B$, see 2.3 below.

Fact 2.3. For every ε and k and $A \le^* B$ we have:

($*$) for every n and $M \in \mathscr{K}_n$ and one to one $f : A \hookrightarrow_A M_n$ we have: if $\mathscr{G}^{\varepsilon,k}_{A,B}(f, M_n) = \emptyset$ <u>then</u>

$$2(|A|)n^\varepsilon + (k-1) \ge \max\{\ell : \begin{array}{l} \text{there are } g_m : B \hookrightarrow_A M \text{ for } m < \ell \text{ such that } f \subseteq g_m \text{ and} \\ [m_1 < m_2 \Rightarrow \mathrm{Rang}(g_{m_1}) \cap \mathrm{Rang}(g_{m_2}) \subseteq \mathrm{Rang}(f)]\}. \end{array}$$

Proof. Assume that there are g_m for $m < \ell^*$ where $\ell^* > 2|A|n^\varepsilon + k - 1$ as above.

By renaming without loss of generality for some $\ell^{**} \le \ell^*$ we have $\mathrm{Rang}(g_m) \backslash \mathrm{Rang}(f)$ when $m < \ell^{**}$ is with distance $\ge n^\varepsilon$ from $\mathrm{Rang}(f)$ but if $\ell \in [\ell^{**}, \ell^*]$ then $\mathrm{Rang}(g_\ell) \backslash \mathrm{Rang}(f)$ has distance $< n^\varepsilon$ to $\mathrm{Rang}(f)$. Recall that by one of our assumptions $\ell^{**} \le k - 1$. Now for each $x \in \mathrm{Rang}(f)$, there are $\le 2n^\varepsilon$ numbers $m \in [\ell^{**}, \ell^*)$ such that $\min\{|x - g_m(y)| : y \in B \setminus A\} \le n^\varepsilon$. So by the demand on ℓ^{**} we have $\ell^* - \ell^{**} \le (|A|) \times (2n^\varepsilon) = 2|A|n^\varepsilon$ and as $\ell^{**} < k$ we are done. $\square_{2.3}$

The following is central, it does not yet prove almost niceness, but the parallels (to 2.4) from [SS88], [BS97] were immediate, and here we see the main additional difficulty we have which is that we are looking for copies of B over A but we have to take into account the distance, the closeness of images of points in B under embeddings into $\mathscr{M}_n$. Now, in order to prove 2.4 we will have to look for different types of g's which satisfy Condition (5) from the Def. of $\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n)$ and restricting ourselves to one kind we will calculate the expected value of "relevant part" of $\mathscr{G}^{\varepsilon,1}_{A,B}(f, \mathscr{M}_n)$ and we will show that it is small enough.

Theorem 2.4. *Assume $A <^* B$ (so both in $\mathscr{K}_\infty$). Then $\otimes_2 \Rightarrow \otimes_1$ where*

- $\otimes_1$ *for every $\varepsilon > 0$, for some $k \in \mathbb{N}$, for every random enough $\mathscr{M}_n$ we have:*
  - $(*)$ *if $f : A \hookrightarrow [n]$ then $\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n) = \emptyset$*
- $\otimes_2$ *$A <^*_a B$ (which by Definition 1.10(5) means $A <^* B \wedge \neg(A <_s B)$).*

Remark 2.5. From $\otimes_1$ we can conclude: for every $\varepsilon \in \mathbb{R}^+$ we have: for every random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow_A \mathscr{M}_n$, there cannot be $\geq n^\varepsilon$ extensions $g : B \hookrightarrow_A \mathscr{M}_n$ of f pairwise disjoint over f.

Explanation 2.6. For this, first choose $\varepsilon_1 < \varepsilon$. Note that for any k we have $\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n) \subseteq \mathscr{G}^{\varepsilon_1,k}_{A,B}(f, \mathscr{M}_n)$. Choose k_1 for ε_1 by $\otimes_1$. Then by 2.3 the number of pairwise disjoint extensions $g : B \hookrightarrow_A \mathscr{M}_n$ of f is $\leq 2(|A|)n^{\varepsilon_1} + (k_1 - 1)$. For sufficiently large n this is $< n^\varepsilon$.

Remark 2.7. We think of g extending f such that $g : B \hookrightarrow \mathscr{M}_n$ that satisfies, for some constants c_1 and c_2 with $c_2 > 2c_1$ and equivalence relation λ:

$$x\lambda y \Rightarrow |g(x) - g(y)| < c_1$$

and

$$[\{x, y\} \subseteq B \wedge \{x, y\} \nsubseteq A \wedge \neg x\lambda y] \Rightarrow |g(x) - g(y)| \geq c_2.$$

Explanation 2.8. So the number of such $g \supset f$ is $\sim n^{|(B\setminus A)/\lambda|} = n^{\mathbf{v}(A,B,\lambda)}$, the probability of each being an embedding, assuming f is one to one, is $\sim n^{-\alpha \mathbf{e}(A,B,\lambda)}$, hence the expected value is $\sim n^{\mathbf{w}_\lambda(A,B)}$ ($\sim$ means "up to a constant"). So $A <^*_i B$ implies that usually there are few such copies of B over any copy of A, i.e. the expected value is < 1. In [SS88], λ is equality, things here are more complicated.

By 2.4 we have sufficient conditions for (given $A \leq^* B$) "every $f : A \hookrightarrow \mathscr{M}_n$ has few pairwise disjoint extension to $g : B \hookrightarrow \mathscr{M}_n$". Now we try to get a dual, a sufficient condition for: (given $A \leq^* B$) for every random enough $\mathscr{M}_n$, every $f : A \hookrightarrow \mathscr{M}_n$ has "many" pairwise disjoint extensions to $g : A \hookrightarrow \mathscr{M}_n$.

Now 2.9 is used in 3.2.

Lemma 2.9. *If (A),(B),(C) then $\otimes$ where:*

- *(A)* *$\lambda \in \Xi_\gamma(A, B)$ that is*
  - *(a)* $(A, B, \lambda) \in \mathscr{T}$
  - *(b)* *$(\forall B')[A <^* B' \leq^* B and B'$ is λ-closed $\Rightarrow \mathbf{w}_\lambda(A, B') > 0]$, (recall that "$B'$ is λ-closed" means $x\lambda y \wedge x \in B' \Rightarrow y \in B'$).*
- *(B)* *let $\zeta \in \mathbb{R}^+$ be defined by*

$$\zeta =: \min\{\mathbf{w}_{\lambda\restriction B'}(A, B') : A \subseteq B' \subseteq B \text{ and } B' \text{ is } \lambda\text{-closed}\},$$

- *(C)* *k_* large enough than $|B|$ or just $|A| + |(B\setminus A)/\lambda|$ and $m \in \mathbb{N}$*

$\otimes$ *for every small enough $\varepsilon > 0$, for every random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow [n]$ there are $\geq n^{(1-\varepsilon)\cdot\zeta}$ pairwise disjoint extensions g of f satisfying*

(i) $g : B \hookrightarrow_A \mathscr{M}_n$,

(ii) *if $x\lambda y$ (so $x, y \in B\backslash A$) then $|x-y| \le 2|B|$*

(iii) *if $x \in B\backslash A, y \in B$ and $\neg(x\lambda y)$ then $|x-y| \ge n/k_*$*

(iv) *there are no $b \in B\backslash A$ and extension $g' : B^+ \hookrightarrow \mathscr{M}_n$ of $g\restriction(b/\lambda)$ when (recall b/λ denote $B\restriction(b/\lambda)$):*

(a) $|B^+\backslash B| \le m$

(b) $(b/\lambda) <_i ((b/\lambda) \cup (B^+\backslash B)$

(c) *if $b_1 \in B^+\backslash B$ then $|g^+(b) - g^+(b_1)| \le n^\varepsilon$.*

Now 2.4, 2.9 are enough for proving $<^*_i = <_i$, $<^*_s = <_s$, weakly nice and similar things, see §(6A) and proof of 3.2, 3.3, 3.4.

Claim 2.10. *1) Assume $A <^*_{pr} B$ and we let $\xi = \xi(A, B)$ that is*

$$\xi = \max\{\mathbf{w}_\lambda(A,B) : \begin{array}{l} (A,B,\lambda) \in \mathscr{T} \text{ and } \lambda \in \Xi(A,B), \text{ i.e.} \\ \text{for every } \lambda\text{-closed non-empty } C \subseteq B \setminus A \\ \text{we have } \mathbf{w}(A, A \cup C, \lambda \restriction C) > 0\}. \end{array}$$

Then for every $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathscr{M}_n$ for every $f : A \hookrightarrow \mathscr{M}_n$ we have

(∗) *the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is at most $n^{\xi+\varepsilon}$.*

2) Assume that $A <^*_s B, \lambda \in \Xi(A, B)$ and $\xi = \mathbf{w}_\lambda(A, B)$. Then for every reals $\zeta, \varepsilon > 0$ for every random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow \mathscr{M}_n$ the set $\mathscr{G}^{\varepsilon,\zeta}_{f,A,B,\lambda}(\mathscr{M}_n)$ defined below has $\le n^{\xi+\varepsilon}$ members, where

$$\mathscr{G}^{\varepsilon,\zeta}_{f,A,B,\lambda}(\mathscr{M}_n) = \{g : g : \begin{array}{l} B \hookrightarrow \mathscr{M}_n \text{ extend } f \text{ and} \\ x \in B\backslash A \wedge y \in B \wedge \neg(x\lambda y) \Rightarrow n^\zeta \le |x-y|\}. \end{array}$$

Remark 2.11. Part (1) is used in the proof of 3.5 before $(*)_4$, pg. 27 and part (2) before $(*)_6$, pg.28.

Claim 2.12. *For every $c \in \mathbb{R}^+$ for $A <_s B$ and $\lambda \in \Xi(A, B)$ for every $\zeta \in \mathbb{R}^+$ for every $\varepsilon \in \mathbb{R}^+$ small enough we have:*

⊞ *for $\mathscr{M}_n$ random enough, if $f : A \hookrightarrow [n]$ then $|\mathscr{F}^{\varepsilon,c}_{A,B}(f, \mathscr{M}_n)|$ is $\le n^{\mathbf{w}_\lambda(A,B)+\zeta}$ where $\mathscr{F}^{\varepsilon,c}_{A,B}(f, \mathscr{M}_n)$ is the set of functions g such that*

(a) $g : B \hookrightarrow_A \mathscr{M}_n$

(b) *if $b_1 \lambda b_2$ then $|g(b_1) - g(b_2)| \le c_1 n^\varepsilon$*

(c) *if $b_2 \in B\backslash A$ and $b_2 \in B\backslash(b_1/\lambda)$ then $|g(b_1) - g(b_2)| \ge 3c_1 n^\varepsilon$.*

Proof. Like the proof of 2.4. □

Claim 2.13. *Like 2.12 but $|\mathscr{F}^{\varepsilon,c}_{A,B}(f, \mathscr{M}_n)|$ is $\ge n^{\mathbf{w}_\lambda(A,B)-\zeta}$.*

Proof. Like the proof of 2.4. □

§ 3. The conclusion

§ 3(A). An Outline.

Comment: In this section it is shown that $<^*_i$ and $<^*_s$ (introduced in §4) agree with the $<_i$ and $<_s$ of [I,§1] by using the probabilistic information from §5. Then it is proven that the main context $\mathscr{M}_n$ is simply nice (hence simply almost nice) and it satisfies the $0-1$ law.

Context 3.1. As in §4 and §5, so $p_i = 1/i^\alpha$, for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_{\mathbb{R}}$ irrational) and $\mathscr{M}_n = \mathscr{M}^0_n$ (only the graph) and $\leq_i$, $\leq_s$, cl are as defined §1. (So $\mathscr{K}_\infty = \mathscr{K}$ by 1.4).

Note that actually the section has two parts of distinct flavours: in 3.2 - 3.5 we use the probabilistic information from §5. First we show that the definition of $<_i$ from [I,§1] and of $<^*_i$ from §4 give the same relation. This is done in 3.2, we then deduce that $<_x=<^*_x$ for the relevant x's (in 3.3, 3.4). We then (in 3.5) prove the existence of $g : B \hookrightarrow \mathscr{M}_n$ extending a given f (for $\mathscr{M}_n$ random enough) for which no "unintended algebraic elements" occur. In 3.7 we rephrase it by sufficient conditions for simply good tuple, ([She02, 2.12(1)]); all this is §(6B). But to actually prove almost niceness, we need more work on the relations $\leq^*_x$ defined in §4; this is done in 3.8, 3.10, 3.11, i.e. §(6C).

Lastly, (in §(6D)) we put everything together.

The argument in 3.2 - 3.5 parallels that in [BS97] which is more hidden in [SS88]. The most delicate step is to establish clauses (A)(δ) and (ε) of Definition [I,2.13,(1)], (almost simply nice). For this, we consider $f : A \hookrightarrow \mathscr{M}_n$ and try to extend f to $g : B \hookrightarrow \mathscr{M}_n$ where $A \leq_s B$ such that Rang(g) and $c\ell^k(f(A), \mathscr{M}_n)$ are "freely amalgamated" over Rang(f). The key facts have been established in Section 5. If $\zeta = \mathbf{w}(A, B, \lambda)$ we have shown (Claim 2.9) that for every $\varepsilon > 0$ for every random enough $\mathscr{M}_n$, there are $\geq n^{\zeta-\varepsilon}$ embeddings of B into $\mathscr{M}_n$ extending f. But we also show (using 2.10) that for each obstruction to free amalgamation there is a $\zeta' < \zeta$ such that for every $\varepsilon_1 > 0$ the number of embeddings satisfying this obstruction is $< n^{\zeta'+\varepsilon_1}$, where $\zeta' = \mathbf{w}(A, B', \lambda)$ (for some B' exemplifying the obstruction) with $\zeta' + \alpha \leq \zeta$. So if $\alpha > \varepsilon + \varepsilon_1$ we overcome the obstruction. The details of this computation for various kinds of obstructions are carried out in proving Claim 3.5.

§ 3(B). Using the inequalities.

Claim 3.2. *Assume $A <^* B$. Then the following are equivalent:*

(i) *$A <^*_i B$ (i.e. from Definition 1.10(3)),*

(ii) *it is not true that: for some ε, for every random enough $\mathscr{M}_n$ for every $f : A \hookrightarrow \mathscr{M}_n$, the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is $\geq n^\varepsilon$,*

(iii) *for every $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathscr{M}_n$ for every $f : A \hookrightarrow \mathscr{M}_n$ the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is $< n^\varepsilon$ (this is the definition of $A <_i B$ in [I,§1]).*

Proof. We shall use the well known finite Δ-system lemma: if $f_i : B \to [n]$ is one to one and $f_i \restriction A = f$ for $i < k$ then for some $w \subseteq \{0, \dots, k-1\}$, $|w| \geq \frac{1}{|B\setminus A|^2} k^{1/2^{|B\setminus A|}}$, and $A' \subseteq B$ and f^* we have: $\bigwedge\limits_{i \in w} f_i \restriction A' = f^*$ and $\langle$ Rang$(f_i \restriction (B \setminus A') : i \in w\rangle$ are pairwise disjoint (so if the f_i's are pairwise distinct then $B \setminus A' \neq \emptyset$).

We use freely Fact 1.2. First, clearly $(iii) \Rightarrow (ii)$.

Second, if $\neg(i)$, i.e., $\neg(A <^*_i B)$ then by 1.13 (equivalence of first and last possibilities + 1.12(1)) there are A', λ as there, that is such that:

$A \leq^* A' <^* B$ and $(A', B, \lambda) \in \mathscr{T}$ and if $C \subseteq B \setminus A'$ is non-empty λ-closed then $\mathbf{w}(A', A' \cup C, \lambda \restriction C) > 0$ (see 1.13).

So (A', B, λ) satisfies the assumptions of 2.9 which gives $\neg(ii)$, i.e., we have proved $\neg(i) \Rightarrow \neg(ii)$ hence $(ii) \Rightarrow (i)$.

Lastly, to prove $(i) \Rightarrow (iii)$ assume $\neg(iii)$ (and we shall prove $\neg(i)$). So for some $\varepsilon \in \mathbb{R}^+$:

$(*)_1$ $0 < \limsup_{n \to \infty}$ Prob (for some $f : A \hookrightarrow \mathscr{M}_n$, the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is $\geq n^\varepsilon$).

But by the first paragraph of this proof it follows that from $(*)_1$ we can deduce that for some $\zeta \in \mathbb{R}^+$

$(*)_2$ $0 < \limsup_{n \to \infty}$ Prob (for some $A', A \leq^* A' <^* B$, and $f' : A' \hookrightarrow \mathscr{M}_n$ there are $\geq n^\zeta$ functions $g : B \hookrightarrow \mathscr{M}_n$ which are pairwise disjoint extensions of f').

So for some A', $A \leq^* A' <^* B$ and

$(*)_3$ $0 < \limsup_{n \to \infty}$ Prob (for some $f' : A' \hookrightarrow \mathscr{M}_n$ there are $\geq n^\zeta$ functions $g : B \hookrightarrow \mathscr{M}_n$ which are pairwise disjoint extensions of f^*).

By 2.4 (and 2.3, 2.2) we have $\neg(A' <^*_a B)$, which by Definition 1.10(5) means that $A' <^*_s B$ which (by Definition 1.10(4)) implies $\neg(A <^*_i B)$ so $\neg$(i) holds as required. $\square_{3.2}$

Claim 3.3. *For $A <^* B \in \mathscr{K}_\infty$, the following conditions are equivalent:*

(i) $A <^*_s B$,

(ii) it is not true that: "for every $\varepsilon \in \mathbb{R}^+$, for every random enough $\mathscr{M}_n$ for every $f : A \hookrightarrow \mathscr{M}_n$, there are no n^ε pairwise disjoint extensions $g : B \hookrightarrow \mathscr{M}_n$ of f",

(iii) for some $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathscr{M}_n$ for every $f : A \hookrightarrow \mathscr{M}_n$ there are $\geq n^\varepsilon$ pairwise disjoint extensions $g : B \hookrightarrow \mathscr{M}_n$ of f.

Proof. Reflection shows that $(iii) \Rightarrow (ii)$.

If $\neg(i)$, i.e., $\neg(A <^*_s B)$ then by Definition 1.10(4) for some $B', A <^*_i B' \leq^* B$, hence by 3.2 easily $\neg(ii)$, so $(ii) \Rightarrow (i)$.

Lastly, it suffices to prove $(i) \Rightarrow (iii)$. Now by (i) and 1.15 for some λ the assumptions of 2.9 holds hence the conclusion which give clause (iii). $\square_{3.3}$

Conclusion 3.4. *1) $<^*_s = <_s$ and $<^*_i = <_i$, and $\mathfrak{K}$ is weakly nice where $<_s, <_i$ are defined in [I,1.4(4),1.4(5)] hence $<^*_{pr} = \leq_{pr}$.*
2) $(\mathscr{K}, c\ell)$ is as required in [I,§2], and the $\leq_i$, $\leq_s$ defined in [I,§2] are the same as those defined in [I,§1] for our context, of course when for $A \leq B \in \mathscr{K}_\infty$ we let $c\ell(A, B)$ be minimal A' such that $A \leq A' \leq_s B$.
3) Also $\mathfrak{K}$ (that is $(\mathscr{K}, c\ell)$) is transitive local transparent and smooth, (see [I,2.2(3),2.3(2),2.5(5),2.5(4)]).
4) $<^*_x = <_x$ for the other x's (not used).

Proof. 1) $<^*_s = <_s$ and $<^*_i = <_i$ by 3.2, 3.3 and see Definition in [I,§1].
Lastly, $\mathfrak{K}$ being weakly nice follows from 3.3, see Definition in [I,§1].
2) By [I,2.6].
3) By [I,2.6] the transitive local and transparent follows (see clauses $(\delta),(\varepsilon),(\zeta)$ there). As for smoothness, use 1.16(5).
4) Check. $\square_{3.4}$

Note that we are in a "nice" case, in particular no successor function. Toward proving it we characterize "simply good".

Claim 3.5. *If $A \leq^*_s B$ and $k,t \in \mathbb{N}$ satisfies $k + |B| \leq t$, then for every random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow \mathscr{M}_n$, we can find $g : B \hookrightarrow \mathscr{M}_n$ extending f such that:*

(*i*) $\mathrm{Rang}(g) \cap c\ell^t(\mathrm{Rang}(f), \mathscr{M}_n) = \mathrm{Rang}(f)$,

(*ii*) $\mathrm{Rang}(g) \underset{\mathrm{Rang}(f)}{\overset{\mathscr{M}_n}{\bigcup\!\!\!\!\!\bigcup}} c\ell^t(\mathrm{Rang}(f), \mathscr{M}_n)$,

(*iii*) $c\ell^k(\mathrm{Rang}(g), \mathscr{M}_n) \subseteq \mathrm{Rang}(g) \cup c\ell^k(\mathrm{Rang}(f), \mathscr{M}_n)$.

Remark 3.6. Note that we can in clauses (i), (ii) of 3.5 replace t by k - this just demands less. We shall use this freely. Have we put t in the second appearance of k in clause (iii) of 3.5 the loss would not be great: just in [I], we should systematically use [I2.12(2)] instead of [I2.12(1)]. It even suffices to demand "given $A \leq^*_s B$ and k for some $t \ldots$".

Proof. We prove this by induction on $|B \setminus A|$, but by the character of the desired conclusion, if $A <^*_s B <^*_s C$, to prove it for the pair (A, C) it suffices to prove it for the pairs (A, B) and (B, C). Also, if $B = A$ the statement is trivial (because we can take $f = g$). So, without loss of generality, $A <^*_{pr} B$ (see Definition 1.10(6)).

Let $\lambda \in \Xi(A, B)$, that is λ is such that $(A, B, \lambda) \in \mathscr{T}$ and for every λ-closed $C \subseteq B \setminus A$ we have $\mathbf{w}_\lambda(A, A \cup C) > 0$ and recall

$$\xi := \mathbf{w}_\lambda(A, B) = \max\{\mathbf{w}_{\lambda_1}(A, B) : \lambda_1 \in \Xi(A, B)\}.$$

Choose $k_* = k(*)$ large enough and $\varepsilon \in \mathbb{R}^+$ small enough. The requirements on ε, $k(*)$ will be clear by the end of the argument.

Let $\mathscr{M}_n$ be random enough, and $f : A \hookrightarrow \mathscr{M}_n$. Now by 3.2 and the definition of $c\ell^t$ we have $(*)$ and by 2.9 for $\zeta = \xi$ we have $(*)_1$ where

$(*)_0$ $|c\ell^t(f(A), \mathscr{M}_n)| \leq n^{\varepsilon/k(*)}$,
$(*)_1$ $|\mathscr{G}| \geq n^{\xi - \varepsilon/2}$,

where $\mathscr{G}$ is the family from $\otimes$ of 2.9, in particular each $g \in \mathscr{G}$ extends f to an embedding of B into $\mathscr{M}_n$ and also saisfies (ii)-(iv) from there.

Recall that

$\circledast_1$ if $A' \subseteq M \in \mathscr{K}$ and $a \in c\ell^k(A', M)$ then for some C we have $C \subseteq c\ell^k(A', M), |C| \leq k, a \in C$ and $c\ell^k(C \cap A', C) = C$

(by the Definition of $c\ell^k$, see [I,§1]).

We intend to find $g \in \mathscr{G}$ satisfying the requirements in the claim. Now g being an embedding of B into $\mathscr{M}_n$ extending f follows from $g \in \mathscr{G}$. So it is enough to prove that $< n^{\xi-\varepsilon}$ members g of $\mathscr{G}$ fail clause (i) and similarly for clauses (ii) and (iii) of 3.5.

More specifically, let

$\boxplus_1$ $\mathscr{G}^1 = \{g \in \mathscr{G} : g(B) \cap c\ell^t(f(A), \mathscr{M}_n) \neq f(A)$ or just for some $a \in B \backslash A, b \in c\ell^+(f(A), \mathscr{M}_n) \backslash f(A)$ we have $|g(b) - a| \leq n^\varepsilon\}$,

$\boxplus_2$ $\mathscr{G}^2 = \{g \in \mathscr{G} : g \notin \mathscr{G}^1$ but clause (ii) fails for $g\}$

$\boxplus_3$ $\mathscr{G}^3 = \{g \in \mathscr{G}$: clause (iii) fails for g but $g \notin \mathscr{G}^1 \cup \mathscr{G}^2\}$.

So clearly it is enough to prove $\mathscr{G} \nsubseteq \mathscr{G}^1 \cup \mathscr{G}^2 \cup \mathscr{G}^3$ (because (i) fails for $g \Rightarrow g \in \mathscr{G}^1$, (ii) fails for $g \Rightarrow g \in \mathscr{G}^2 \vee g \in \mathscr{G}^1$ and (iii) fails for $g \Rightarrow g \in \mathscr{G}^3 \vee g \in \mathscr{G}^2 \vee g \in \mathscr{G}^1$.

So it is enough to prove that $|\mathscr{G}^\iota| < |\mathscr{G}|/3$ for $\iota = 1, 2, 3$; naturally much more is proved.

<u>On the number of $g \in \mathscr{G}^1$</u>: For $a \in B \backslash A$ and $x \in c\ell^t(f(A), \mathscr{M}_n)$ let $\mathscr{G}^1_{a,x} = \{g \in \mathscr{G}^2 : g(a) = x$ or just $|g(a) - x| \leq n^\varepsilon\}$, so $\mathscr{G}^1 = \cup\{\mathscr{G}^1_{a,x} : a \in B \backslash A$ and $x \in c\ell^t(f(A), \mathscr{M}_n)\}$, and by 1.16(13) clearly $A \cup \{a\} \leq_i B$ (as $A <_{pr} B$). The rest is as in the proof for $\mathscr{G}^2$ below, only easier.

<u>On the number of $g \in \mathscr{G}^2$</u>:

If $g \in \mathscr{G}^2$ then for some

$$x_g \in c\ell^t(\mathrm{Rang}(f), \mathscr{M}_n) \setminus \mathrm{Rang}(f) \text{ and } y \in B \setminus A$$

we have: $\{x_g, g(y)\}$ is an edge of $\mathscr{M}_n$. Note $x_g \notin g(B)$ as $g \in \mathscr{G}_2$.

We now form a new structure B^2 with a set of elements $B \cup \{x^*\}$, $(x^* \notin B)$, such that $g \cup \{\langle x^*, x_g \rangle\} : B^2 \hookrightarrow \mathscr{M}_n$ and let $A^2 = B^2 \restriction (A \cup \{x^*\})$. Now up to isomorphism over B there is a finite number (i.e., with a bound not depending on n) of such B^2, say $\langle B^2_j : j < j^*_2 \rangle$.

For $x \in c\ell^t(\mathrm{Rang}(f), \mathscr{M}_n)$ and $j < j^*$ let

$$\mathscr{G}^2_{j,x} =: \{g : g \text{ is an embedding of } B^2_j \text{ into } \mathscr{M}_n \text{ extending } f \text{ and satisfying } g(x^*) = x\}$$

$$\mathscr{G}^2_j =: \bigcup_{x \in c\ell^t(f(A), \mathscr{M}_n)} \mathscr{G}^2_{j,x}.$$

So:

$(*)_2$ if $g \in \mathscr{G}^2$ then

$$g \in \bigcup \{\{g' \restriction B : g' \in \mathscr{G}^2_{j,x}\} : j < j^*_2 \text{ and } x \in c\ell^t(f(A), \mathscr{M}_n)\}.$$

Now, if $\neg(A^2_j <_s B^2_j)$ then as $A <^*_{pr} B$ easily $A^2_j <_i B^2_j$ so by 3.2 using $(*)$ (with $\varepsilon/2 - \varepsilon/k(*)$ here standing for ε in (iii) there) we have

$(*)_3$ if $\neg(A^2_j <_s B^2_j)$ then $|\mathscr{G}^2_j| \leq n^{\varepsilon/2}$.

[Why? As $A^2_j <_i B^2_j$ on the one hand for each $x \in c\ell^t(\mathrm{Rang}(f), \mathscr{M}_n)$ by 3.2 the number of $g : B^2_j \hookrightarrow \mathscr{M}_n$ extending $f \cup \{\langle x^*, x\rangle\} : A^2_j \hookrightarrow \mathscr{M}_n$ is $< n^{\varepsilon/k(*)}$ and on the other hand the number of candidates for x is $\le |c\ell^t(\mathrm{Rang}(f), \mathscr{M}_n)| \le n^{\frac{\varepsilon}{k(*)}}$. So $|\mathscr{G}^2_j| \le n^{\varepsilon/k(*)} \times n^{\varepsilon/k(*)} \le n^{2\varepsilon/k(*)} \le n^{\frac{\varepsilon}{2}}$.]

If $A^2_j <_s B^2_j$, then by 1.16(14) still $A^2_j <_{pr} B^2_j$ and letting

$$\xi^2_j =: \max\{\mathbf{w}_\lambda(A^2_j, B^2_j) : \begin{array}{l} (A^2_j, B^2_j, \lambda) \in \mathscr{T} \text{ and for every} \\ \lambda\text{-closed non-empty } C \subseteq B^2_j \setminus A^2_j \\ \text{we have } \mathbf{w}(A^2_j, A^2_j \cup C, \lambda \restriction C) > 0\} \end{array}$$

clearly $\xi^2_j < \xi - 2\varepsilon$ (as we retain the "old" edges, and by at least one we actually enlarge the number of edges but we keep the number of "vertexes", i.e., equivalence classes see 1.16(9)).

So, by 2.10,

$(*)_4$ if $A^2_j <^*_{pr} B^2_j$ then $|\mathscr{G}^2_j| \le n^{\xi - 2\varepsilon}$.

As $\xi - 2\varepsilon > \varepsilon$ by $(*)_3 + (*)_4$, multiplying by j^*, as n is large enough

$(*)_5$ $|\mathscr{G}^2|$, the number of $g \in \mathscr{G} \backslash \mathscr{G}^1$ failing clause (ii) of 3.5 is $\le n^{\xi - \varepsilon}$.

<u>On the number of $g \in \mathscr{G}^3$</u>:

First if $g \in \mathscr{G}^3$ then necessarily there are $A^+ = A^+_g, B^+ = B^+_g, C = C_g, g^+$ such that

$\otimes_1$ (a) $B \le^* B^+$

(b) $c\ell^k(B, B^+) = B^+$; moreover $B^+ = B \cup C, |C| \le k$ and $C \cap A \le_i C$

(c) $A \le_i A^+ \le_s B^+$ hence $B \cap A^+ = A, |B^+| \le |B| + k$

(d) $C \nsubseteq B \cup A^+$ and $B \underset{A}{\overset{B^+}{\biguplus}} A^+$

(e) $g \subseteq g^+$ and $g^+ : B^+ \hookrightarrow \mathscr{M}_n$ and $g^+(A^+) \subseteq c\ell^t(f(A), \mathscr{M}_n)$

(f) without loss of generality $|B^+|$ is minimal under (a)-(d).

[Why? As $g \in \mathscr{G}^3$ there is $y_g \in c\ell^k(g(B), \mathscr{M}_n)$ such that $y_g \notin g(B)$ and moreover $y_g \notin c\ell^k(f(A), \mathscr{M}_n)$. By the first statement (and $\circledast_1$ above) there is $C^* \subseteq c\ell^k(g(B), \mathscr{M}_n)$ with $\le k$ elements such that $y_g \in C^*$ and $C^* \cap g(B) \le_i C^*$. Let $B^* = g(B) \cup C^* \le \mathscr{M}_n$. Let B^+, g^+ be such that $B \le^* B^+ \in \mathscr{K}, g \subseteq g^+, g^+$ an isomorphism from B^+ onto B^*, and let $C = g^{-1}(C^*)$.

Lastly, choose A^+ such that $A' \le_i A^+ \le_s B^+$, clearly it exists by 1.16(2). Now $|A^+| \le |B| + |C| \le |B| + k \le t$ by the assumptions on A, B, k, t hence $g^+(A^+) \subseteq c\ell^t(f(A), \mathscr{M}_n)$ but as $g \in \mathscr{G}^3$ we have $g \notin \mathscr{G}^1$ hence $A = g(B) \cap c\ell^t(f(A), \mathscr{M}_n)$ so we have $A^+ \cap B = A$. Also $C \nsubseteq B \cup A^+$, otherwise, as $g \notin \mathscr{G}^2, g \notin \mathscr{G}^1$ we have $B \underset{A}{\overset{B^+}{\biguplus}} A^+$ hence $C \cap B \underset{C \cap A}{\biguplus} C \cap A^+$ but as $C \cap B <^*_i C$ so by smoothness (e.g. 1.16(5)) we get $C \cap A <^*_i C \cap A^+$ hence $C \cap A^+ \subseteq c\ell^k(A, B^+)$ hence $C^* \backslash g(B) = g^+(C \backslash B) \subseteq g^+(C \cap A^+) \subseteq c\ell^k(f(A), \mathscr{M}_n)$ hence $y_g \in c\ell^k(f(A), \mathscr{M}_n)$, contradiction. So clauses (a)-(e) of $\otimes_1$ hold and so without loss of generality also clause (f).]

$\otimes_2$ if $g \in \mathscr{G}^3$ <u>then</u> there are (unique) $k_g, \lambda_g, h_g, A^*_g$ such that

(a) $0 < k_g < k_*/3$

(b) λ_g is an equivalence relation on $B^+ \backslash A$

(c) $\lambda_g = \{(b_1, b_2) : b_1, b_2 \in B^+_g \backslash A$ and $|g^+(b_1) - g^+(b_2)| \le k_g n^\varepsilon / k_*\}$

(d) also $\lambda_g = \{(b_1, b_2) : b_1, b_2 \in B^+_g \backslash A$ and $|g^+(b_1) - g^+(b_2)| \le 3kn^\varepsilon / k_*\}$

(e) $A^*_g = \cup\{b/\lambda_g : b \in A\}$

[Why? Obvious, e.g. $3^{|B^+|\times|B^+|} \le k_*$ suffice; actually can use 2 instead of 3.]

$\otimes_3$ (a) if $a \in B\backslash A$ and $b \in A^+_g$ then $|g^*(a) - (b)| > n^\varepsilon > k_g n^\varepsilon / k_*$

(b) $\mathbf{w}_{\lambda_g}(A^+_g, B^+_g) < \mathbf{w}_\lambda(A, B)$

(c) moreover the difference is $> \varepsilon$.

Why? Clause (a) as $g \notin \mathscr{G}^1$ see the "or just..." in its definition, clause (c) follows from clause (b), so we concentrate on clause (b).

Let $B^*_g = \cup\{b/\lambda_g : b \in B\backslash A\} \cup A^+$ so $B^*_g \supseteq B$. rm Without loss of generality

$\otimes_{3.1}$ $B^+_g = B^*_g$.

[Why? Toward contradiction assume that $B^+_g \ne B^*_g$. Easily $\mathbf{w}_\lambda(A, B) \ge \mathbf{w}_{\lambda_g}(A^+, B^*_g)$.

Also $B^*_g <^*_i B^+_g$ as $B <^*_i B^+_g, B \le^* B^*_g \le^* B^+_g$ and $B^*_g \ne B^+_g$ by the case assumption. As B^*_g is λ_g-closed we have $w_{\lambda_g}(A^+, B^+_g) = w_{\lambda_g}(A^+, B^*_g) + w_{\lambda_g}(B^*_g, B^+_g) \le w_\lambda(A, B) + w_{\lambda_g}(B^*_g, B^+_g) < w_g(A, B)$.

Why? By §4, as $w_\lambda(A, B) \ge w_{\lambda_g}(A, B)$ and as $w_{\lambda_g}(B^*_g, B^+_g)$ is negative as $B^*_g \prec^*_i B^+_g$. So we have proved $\otimes_3(b)$ indeed; so without loss of generality $B^+_g = B^*_g$ as promised in $\otimes_{3.1}$.]

As $B^+_g = B^*_g$, necessarily $B <_i B^*_g$.

Let $\langle b_\ell : \ell < \ell(*)\rangle$ be representatives of $(B\backslash A)/\lambda$ hence of $B^*_g \backslash A^+_g$, so necessarily $\ell(*) > 0$.

rm Without loss of generality

$\otimes_{3.2}$ $E := e_\lambda(A^+_g, B^+_g)$ is empty.

[Why? If $E \ne \emptyset$ then clearly $\mathbf{w}_\lambda(A, B) > \mathbf{w}(A^+_g, B^*_g)$ as required in $\otimes_3(b)$.]

Another formulation

$\otimes'_{3.2}$ without loss of generality there are no $\ell < \ell(*)$ and edge $\{a, b\}$ of B^+ <u>when</u> one of the following occurs:

(a) $a \in (b_\ell/\lambda_g)\backslash B$ and $b \in \cup\{b_{\ell_1}/\lambda_g : \ell_1 \ne \ell$ and $\ell_1 < \ell(*)\}$

(b) $b \in B^+_g \backslash A^+_g$ and $a \in A^+_g \backslash A$

(c) $b \in (b_\ell/\lambda_g)\backslash(b_\ell/\lambda)$ and $a \in A$

(d) $a \in (b_\ell/\lambda_g)\backslash B$ and $b \in A^+$.

[Why? In clause (a) otherwise $\mathbf{v}_\lambda(A, B) = \mathbf{v}_{\lambda_g}(A, B^*_g) = \mathbf{v}_{\lambda_g}(A, B^+_g)$ and $e_\lambda(A, B) \subsetneqq e_{\lambda_g}(A, B^*_g)$ we get the inequality of $\otimes_3(b)$. Clauses (b),(c) are similar.]

$\otimes_{3.3}$ $\ell(*) = 1 \wedge A^+_g = A$.

[Why? Note $\langle (b_\ell/\lambda_g)\backslash B : \ell < \ell(*)\rangle$ are pairwise disjoint with no edges between the (by $\otimes_{3.3}(a)$, so by smoothness clearly for some $\ell < \ell(*)$ we have $(A^+_g \cup B) <_i (A^+_g \cup B \cup (b_\ell/\lambda_g))$, so necessarily $(b_\ell/\lambda_g) \nsubseteq B$. By $\otimes_1(f)$ we have $\ell(*) \le 1$. But $\ell(*) = 0$ is impossible as $B^+_g \ne B \cup A^+_g$ by $\otimes_1(a)$ so $\ell(*) = 1$. By $\otimes'_{3.2}(a)$ also we have $A \cup (b/\lambda_g) \underset{A \cup (b_0/\lambda)}{\overset{B^+}{\biguplus}} A^+ \cup (b_0/\lambda_g)$ hence $A^+_g \cup (b_0/\lambda) <_i A^+_g \cup (b_0/\lambda_g)$. By $\otimes'_{3.2}(d)$ we have $A \cup (b_0/\lambda_g) \underset{A}{\overset{B^+}{\biguplus}} A^+$ so by smoothness we have $((b_0/\lambda_g) \cup A) <_i (b/\lambda_g) \cup A$, so by the minimality of B^+, see $\otimes_1(f)$, we have $A^+_g = A$.]

$\otimes_{3.4}$ $b_0/\lambda_g \underset{b_0/\lambda}{\overset{B^+}{\biguplus}} B$ hence by smoothness $(b_0/\lambda_g) <^*_i (b_0/\lambda)$.

[Why? By $\otimes_{3.4}$.]

Together we contradict the demand in $2.9\otimes(iv)$ $(((b_\ell/\lambda) \cup A^+) <_s B'$, so $\otimes_3(b)$ holds hence $\otimes_3$ holds.]

- $\otimes_4$ for $g \in \mathscr{G}$
  - (a) let $\mathbf{x}_g$, i.e. $\mathbf{x}_g[\mathscr{M}_n]$ be $(B^+_g, A^+_g, C_g, k_g, \lambda_g)$
  - (b) let $h_g = h_g[\mathscr{M}_n] := g{\restriction}A^+_g$
- $\otimes_5$ without loss of generality $\{\mathbf{x}_g : g \in \mathscr{G}$ for some $\mathscr{M}_n\}$ is a finite set, i.e. of size not depending on $\mathscr{G}$.

[Why? Just think.]

Hence

- $\otimes_6$ let $\langle \mathbf{x}_j = (B_j, A^+_j, C_j, k_j, \lambda_j) : j < j_3\rangle$ list $\mathbf{X}$
- $\otimes_7$ let $\mathscr{H}_j[\mathscr{M}_n] = \{h_g : g \in \mathscr{G}[\mathscr{M}_n]\}$
- $\otimes_8$ $\mathscr{H}_j[\mathscr{M}_n]$ has $\le |c\ell^k(g(A), \mathscr{M}_n)| \le (n^{\varepsilon/k(*)})^{|A^+_j|} \le n^{m\varepsilon/k(*)}$.

[Why? By $(*)_0$ and the definition above.]

- $\otimes_9$ for $h \in \mathscr{H}_j = \cup\{\mathscr{H}_j[\mathscr{G}] : j < j_3\}$ let $\mathscr{G}^3_{j,h}[\mathscr{M}_n] = \{g \in \mathscr{G}[\mathscr{M}_n] : \mathbf{x}_g = \mathbf{x}_j$ and $h_g = h\}$.

Now

$\otimes_{10}$ $\mathscr{G}_{j,h}$ has at most $n^{(\xi-\varepsilon)+m\varepsilon/k(*)}$ members.

[Why? Like 2.4, fully by 2.12.]

$\otimes_{11}$ $|\mathscr{G}^3| \le n^{\xi-\varepsilon+(m+2)/k(*)}$.

[Why? Put together.]

So we are done. $\square_{3.5}$

Conclusion 3.7. *If $A <^*_s B$ and $B_0 \subseteq B$ and $k \in \mathbb{N}$ <u>then</u> the tuple (B, A, B_0, k) is simply good (see Definition* **??***(I,(1)).*

Proof. Read 3.5 and Definition [She02, 2.12]. $\square_{3.7}$

§ 3(C). **Analyzing the Model Theory.**

Toward simple niceness the "only" thing left is the universal part, i.e., Definition [She02, 2.13].

The following claims 3.8, 3.10 do not use §5 and have nothing to do with probability; they are the crucial step for proving the satisfaction of Definition **??**[I,(1)(A)] in our case; claim 3.8 is a sufficient condition for goodness (by 3.7). Our preceding the actual proof (of 3.11) by the two claims (3.8,3.10) and separating them is for clarity, though it has a bad effect on the bound; see also - 3.8 using "$c\ell^{k,m}(\bar{a}b, M)$" instead of $c\ell^k(\bar{a}b, M)$ when $k' < k$ may improve the bound.

Claim 3.8. *For every k and ℓ (from $\mathbb{N}$) there are natural numbers $t = t(k,\ell)$ and $k^*(k,\ell) \geq t, k$ such that for any $k^* \geq k^*(k,\ell)$ we have:*

(∗) *if $m^\otimes \in \mathbb{N}$ and $M \in \mathscr{K}$, $\bar{a} \in {}^{\ell\geq}M$, $b \in M$ then*

$\otimes$ *the set*

$$\mathscr{R} =: \{(c,d) : \begin{array}{l} d \in c\ell^k(\bar{a}b, M) \setminus c\ell^{k^*,m^\otimes+k}(\bar{a}, M) \text{ and} \\ c \in c\ell^{k^*,m^\otimes}(\bar{a}, M) \text{ and } \{c,d\} \text{ is an edge of } M\} \end{array}$$

has less than t members.

Proof. If $k = 0$ this is trivial so without loss of generality $k > 0$. Choose $\varepsilon \in \mathbb{R}^+$ small enough such that

$(*)_1$ $C_0 <^* C_1 and (C_0, C_1, \lambda) \in \mathscr{T} and |C_1| \leq k \Rightarrow \mathbf{w}_\lambda(C_0, C_1) \notin [-\varepsilon, \varepsilon]$

(in fact we can restrict ourselves to the case $C_0 <_i^* C_1$)).

Choose $\mathbf{c} \in \mathbb{R}^+$ large enough such that

$(*)_2$ $(C_0, C_1, \lambda) \in \mathscr{T}, |(C_1 \setminus C_0)/\lambda| \leq \Rightarrow \mathbf{w}_\lambda(C_0, C_1) \leq \mathbf{c}$.

(so actually $\mathbf{c} = k$ is enough). Choose $t_1 > 0$ such that $t_1 > \mathbf{c}/\varepsilon$ and $t_1 > 2$.

Choose $t_2 \geq 2^{2^{t_1+k+\ell}}$ (overkill, we mainly need to apply twice the Δ-system lemma; but note that in the proof of 3.10 below we will use Ramsey Theorem). Lastly, choose $t > k^2 t_2$ and let $k^* \in \mathbb{N}$ be large enough which actually means that $k^* > k and k^* \geq (k+1) \times t_2$ so $k^*(k,\ell) =: (k+1) \times t_2$ is O.K.

Suppose we have $m^\otimes$, M, $\bar{a}$, b as in (∗) but such that the set $\mathscr{R}$ has at least t members. Let $(c_i, d_i) \in \mathscr{R}$ for $i < t$ be pairwise distinct [5].

As $d_i \in c\ell^k(\bar{a}b, \mathscr{M})$, we can choose for each $i < t$ a set $C_i \leq M$ such that:

(*i*) $C_i \leq M$,
(*ii*) $|C_i| \leq k$,
(*iii*) $d_i \in C_i$,
(*iv*) $C_i \restriction (C_i \cap (\bar{a}b)) <_i C_i$.

For each $i < t$, as $C_i \cap c\ell^{k^*,m^\otimes+k}(\bar{a}, M)$ is a proper subset of C_i (this is witnessed by d_i, i.e., as $d_i \in C_i \setminus (C_i \cap c\ell^{k^*,m^\otimes+k}(\bar{a}, M_n)))$ clearly this set has $< k$ elements and hence for some $k[i] < k$ we have

[5]Note: we do not require the d_i's to be distinct; though if $w = \{i : d_i = d^*\}$ has $\geq k' > \frac{1}{\alpha}$ elements then $d^* \in c\ell^{k'}(c\ell^{k^*,m^\otimes+k}(\bar{a}, M))$

(v) $C_i \cap c\ell^{k^*,m^\otimes+k[i]+1}(\bar a, M) \subseteq c\ell^{k^*,m^\otimes+k[i]}(\bar a, M)$.

So without loss of generality

(vi) $i < t/k^2 \Rightarrow k[i] = k[0] and |C_i| = |C_0| = k' \le k$,

remember $t_2 < t/k^2$; also

(vii) $b \in C_i$.

[Why? If not then by clause (iv) we have $(C_i \cap \bar a) <_i C_i$, hence $d_i \in C_i \subseteq c\ell^k(\bar a, M) \subseteq c\ell^{k^*,m^\otimes+k}(\bar a, M)$, contradiction.]

As $k^* \ge k^*(k,\ell) \ge t_2 \times (k+1)$ (by the assumption on k^*), clearly $|\bigcup\limits_{i<t_2} C_i \cup \{c_i : i < t_2\}| \le \sum\limits_{i<t_2} |C_i| + t_2 \le \sum\limits_{i<t_2} k + t_2 \le t_2 \times (k+1) \le k^*$ and we define

$$D = \bigcup_{i<t_2} C_i \cup \{c_i : i < t_2\}$$

$$D' = D \cap c\ell^{k^*,m^\otimes+k[0]}(\bar a, M).$$

So by the previous sentence we have $|D'| \le |D| \le k^*$.

Now

$\otimes_0$ $D' <_s D$.

[Why? As otherwise "there is D'' such that $D' <_i D'' \le_s D$ so as $|D''| \le |D| \le k^*$ clearly $D'' \subseteq c\ell^{k^*,m^\otimes+k[0]+1}(\bar a b, M)$; contradiction.]

So we can choose $\lambda \in \Xi(D', D)$ (see Definition 1.7(2)). Let $C_i = \{d_{i,s} : s < k'\}$, with $d_{i,0} = d_i$ and recalling (vii) also $b \ne d_i \Rightarrow b = d_{i,1}$ and with no repetitions.

Clearly $d_{i,0} = d_i \notin D'$. By the finite Δ-system lemma for some $S_0, S_1, S_2 \subseteq \{0, \dots, k'-1\}$ and $u \subseteq \{0, \dots, t_2 - 1\}$ with $\ge t_1$ elements we have:

$\oplus_1$ (a) $\lambda' =: \{(s_1, s_2) : d_{i,s_1} \lambda d_{i,s_2}\}$ is the same for all $i \in u$ and $S_0 = \{0, \dots, k'-1\} \backslash \mathrm{Dom}(\lambda')$ so $d_{i,s} \in D' \Rightarrow i \in S_0$

(b) for each $j < \ell g(\bar a) + 1$, and $s < k'$, the truth value of $d_{i,s} = (\bar a b)_j$ is the same for all $i \in u$ and each $s \in S_0 = \{0, \dots, k'-1\} \backslash \mathrm{Dom}(\lambda')$

(c) $d_{i_1,s_1} = d_{i_2,s_2} \Rightarrow s_1 = s_2$ for $i_1, i_2 \in u$

(d) $d_{i_1,s} = d_{i_2,s} \Leftrightarrow s \in S_1$ for $i_1 \ne i_2 \in u$

(e) $d_{i_1,s_1} \lambda d_{i_2,s_2} \Rightarrow d_{i_1,s_1} \lambda d_{i_1,s_2}$ and $d_{i_1,s_2} \lambda d_{i_2,s_2}$ for $i_1 \ne i_2 \in u$

(f) $d_{i_1,s} \lambda d_{i_2,s} \Leftrightarrow s \in S_2$ for $i_1 \ne i_2 \in u$: so $i \in u$ and $s \in S_2 \Rightarrow d_{i,s} \notin D'$

(g) the statement $b = d_{i,0}$ has the same truth value for all $i \in u$

(h) in §7 we also demand $(d_{i_1,s_1} S d_{i_1,s_2}) = (d_{i_2,s_1} S d_{i_1,s_2})$ where S is the successor relation.

Now we necessarily have:

$\oplus_2$ $0 \notin S_2$ (i.e., $\lambda \restriction \{d_i : i \in u\}$ is equality).

[Why? Otherwise, letting $X = d_i/\lambda$ for any $i \in u$, the triple $(D', D' \cup X, \lambda \restriction X) \in \mathscr{T}$ has weight

$$\begin{aligned}\mathbf{w}(D', D' \cup X, \lambda \restriction X) &= \mathbf{v}(D', D' \cup X, \lambda \restriction X)\\ &-\alpha\mathbf{e}(D', D' \cup X, \lambda \restriction X)\\ &= 1 - \alpha \times |\{e : e \text{ an edge of } M \text{ with one end in}\\ &\quad D' \text{ and the other in } X\}|.\end{aligned}$$

Now as $c_i \in c\ell^{k^*,m^\otimes}(\bar{a}, M)$ clearly $c_i \in D'$ and the pairs $\{c_i, d_i\} \in$ edge(M) are distinct for different i clearly the number above is $\leq 1 - \alpha \times |\{(c_i, d_i) : i \in u\}| = 1 - \alpha \times |u| = 1 - \alpha \times t_1 < 0$; contradiction to $\lambda \in \Xi(D', D)$.]

Let $D_0 = \bar{a} \cup \bigcup\{d_{i,s}/\lambda : s \in S_2 \text{ and } i \in u\}$, clearly D_0 is λ-closed subset of D though not necessarily $\subseteq \mathrm{Dom}(\lambda) = D \backslash D'$ because of $\bar{a}$.

$\oplus_3$ $b = d_{i,1}$ and $1 \in S_1 \backslash S_0$ and $0 \notin S_0 \cup S_1 \cup S_2$ and $S_1 \backslash S_0 \subseteq S_2$ (hence $b \in D_0$).

[Why? The first two clauses hold as $b \in C_i$, $b \in \{d_{i,0}, d_{i,1}\}$ and $\oplus_2$ and (g) of $\oplus_1$. The last clause holds by $\oplus_1$(d),(f) and the "hence $b \in D_0$" by the definition of $D_0, S_1 \backslash \mathrm{Dom}(\lambda') \subseteq S_2$ and the first clause. Also $0 \notin S_0 \cup S_1 \cup S_2$ should be clear.]

$\oplus_4$ For each $i \in u$ we have $\mathbf{w}_\lambda(C_i \cap D_0, C_i) < 0$.

[Why? As $C_i \cap (\bar{a}b) \subseteq C_i \cap D_0$ by clauses (b) + (f) of $\oplus_1$ and by monotonicity of $<_i$ we have $C_i \restriction (C_i \cap \bar{a}b) <_i C_i \Rightarrow C_i \cap D_0 \leq_i C_i$ but $d_{i,0} = d_i \in C_i \backslash C_i \cap D_0$.] Hence

$\oplus_5$ $\mathbf{w}_\lambda(C_i \cap D_0, C_i) \leq -\varepsilon$ for $i \in u$.

[Why? See the choice of ε.]

Let

$$D_1 =: D' \cup \bigcup\{d_{i,s}/\lambda : i \in u, s < k'\} = D' \cup D_0 \cup \{d_{i,s}/\lambda : i \in u, s < k' ands \notin S_2\}$$

so clearly D_1 is λ-closed subset of D including D' but $D_1 \neq D'$ as $i \in u \Rightarrow d_i \in D_1$ by $\oplus_2$. Also clearly

$\oplus_6$ $D' \subseteq D' \cup D_0 \subseteq D_1 \subseteq D$ and D_0, D_1 are λ-closed.

So, as we know $\lambda \in \Xi(D', D)$, we have

$\oplus_7$ $\mathbf{w}_\lambda(D', D_1) > 0$.

Now:

$$\begin{aligned}
\mathbf{w}_\lambda(D',D_1) &= \mathbf{w}_\lambda(D',\bigcup\{x/\lambda : x\in \bigcup_{i\in u} C_i\setminus D'\}\cup D') \\
&= \mathbf{w}_\lambda(D',D'\cup D_0)+\mathbf{w}_\lambda(D'\cup D_0, D'\cup D_0 \\
&\qquad\qquad \cup\bigcup\{d_{i,s}/\lambda : i\in u, s<k', s\notin S_0\cup S_2\}) \\
&\quad [\text{now by 3.9 below with } B_i=\{d_{i,s}: s<k', s\notin S_2\cup S_0\} \text{ and} \\
&\quad\ B_i^+=\cup\{d_{i,s}/\lambda : s<k', s\notin S_2\}] \\
&\le \mathbf{w}_\lambda(D',D'\cup D_0)+\sum_{i\in u}\mathbf{w}_\lambda(D'\cup D_0, D'\cup D_0\cup\{d_{i,s}: s<k', s\notin S_2\cup S_0\}) \\
&\quad [\text{now as } C_i=\{d_{i,s}: s<k'\} \text{ and } d_{i,s}\in D'\cup D_0 \text{ if } s<k', s\in S_0\cup S_2, i\in u] \\
&\le \mathbf{w}_\lambda(D',D'\cup D_0)+\sum_{i\in u}\mathbf{w}_\lambda(D'\cup D_0, D'\cup D_0\cup C_i) \\
&\quad [\text{now as } \mathbf{w}_\lambda(A_1,B_1)\le \mathbf{w}_\lambda(A,B) \\
&\quad \text{when } A\le A_1\le B_1, A\le B\le B_1, B_1\setminus A_1=B\setminus A) \text{ by } \mathbf{??}(3)] \\
&\le \mathbf{w}_\lambda(D'\cap D_0, D_0)+\sum_{i\in u}\mathbf{w}_\lambda(C_i\cap D_0, C_i) \\
&\quad [\text{now by the choice of } \mathbf{c}, D_0 \text{ i.e., } (*)_2 \text{ and the choice of } \varepsilon, u+\oplus_5 \text{ respectively}] \\
&\le \mathbf{c}+|u|\times(-\varepsilon)=\mathbf{c}-t_1\varepsilon<0,
\end{aligned}$$

contradicting the choice of λ, i.e., $\oplus_7$. $\square_{3.8}$

Observation 3.9. *Assume*

- (a) $A\le^* A\cup B_i\le^* A\cup B_i^+\le^* B$ *for* $i\in u$,
- (b) $B\backslash A$ *is the disjoint union of* $\langle B_i^+ : i\in u\rangle$,
- (c) λ *is an equivalence relation on* $B\backslash A$,
- (d) *each* B_i^+ *is* λ*-closed in fact,*
- (e) $B_i^+=\cup\{x/\lambda : x\in B_i\backslash A\}$.

Then $\mathbf{w}_\lambda(A,B)\ge\Sigma\{\mathbf{w}_\lambda(A,B_i) : i\in u\}$.

Proof. By clause (b) + (d)

$$\mathbf{v}_\lambda(A,B)=\Sigma\{\mathbf{v}_\lambda(A,A\cup B_i^+) : i\in u\}=\Sigma\{\mathbf{v}_\lambda(A,A\cup B_i) : i\in u\}$$

and by clause (b) the set $e_\lambda(A,B)$ contains the disjoint union of $\langle e_\lambda(A,B_i) : i\in u\rangle$. Together the result follows. $\square_{3.9}$

Claim 3.10. *For every k, m and ℓ from $\mathbb{N}$ for some $m^*=m^*(k,\ell,m)$, for any $k^*\ge k^*(k,\ell)$ (the function $k^*(k,l)$ is the one from claim 3.8) satisfying $k^*\ge k\times m^*$ we have*

- $(*)$ *if $M\in\mathscr{K}, \bar a\in {}^{\ell\ge}M$ and $b\in M\backslash c\ell^{k^*,m^*}(\bar a,M)$ then for some $m^\otimes\le m^*-m$ we have*

$$c\ell^k(\bar a b,M)\cap c\ell^{k^*,m^\otimes+m}(\bar a,M)\subseteq c\ell^{k^*,m^\otimes}(\bar a,M).$$

Proof. For $k=0$ this is trivial so without loss of generality $k>0$. Let $t=t(k,\ell)$ be as in the previous claim 3.8. Choose m^* such that, e.g., $\lfloor m^*/(km)\rfloor\to(t+5)^2_{2^{k!+\ell}}$ in the usual notation in Ramsey theory. We could get more reasonable bounds but no need as for now. Remember that $k^*(k,\ell)$ is from 3.8 and k^* is any natural number $\ge k^*(k,\ell)$ such that $k^*\ge km^*$.

If the conclusion fails, then the set

$$Z =: \{j \leq m^* - k : c\ell^k(\bar{a}b, M) \cap c\ell^{k^*,j+1}(\bar{a}, M) \nsubseteq c\ell^{k^*,j}(\bar{a}, M)\}$$

satisfies:

$$j \leq m^* - m - k \Rightarrow Z \cap [j, j+m) \neq \emptyset.$$

Hence $|Z| \geq (m^* - m - k)/m$.

For $j \in Z$ there are $C_j \leq M$ and d_j such that

$|C_j| \leq k$ and $(C_j \cap (\bar{a}b)) <^*_i C_j$, and $d_j \in C_j \cap c\ell^{k^*,j+1}(\bar{a}, M) \setminus c\ell^{k^*,j}(\bar{a}, M)$. Now we use the same argument as in the proof of 3.8.

As $d_j \in C_j \cap c\ell^{k^*,j+1}(\bar{a}, M) \setminus c\ell^{k^*,j}(\bar{a}, M)$ we will get that $C_j \cap c\ell^{k^*,j}(\bar{a}, M)$ is a proper subset of $C_j \cap c\ell^{k^*,j+1}(\bar{a}, M)$ (witnessed by d_j) so $|C_j \cap c\ell^{k^*,j}(\bar{a}, M)| < |C_j \cap c\ell^{k^*,j+1}(\bar{a}, M)| \leq k$ so $|C_j \cap c\ell^{k^*,j}(\bar{a}, M)| < k$. Hence for some $k_j \in \{1, \ldots, k\}$ we have $C_j \cap c\ell^{k^*,m^*-k_j+1}(\bar{a}, M) \subseteq c\ell^{k^*,m^*-k_j}(\bar{a}, M)$ hence for some $k' \in \{1, \ldots, k\}$ we have $|Z'| \geq (m^* - m - k)/(mk)$ where $Z' = \{j \in Z : k_j = k'\}$.

Let $C_j = \{d_{j,s} : s < s_j \leq k\}$ with $d_{j,0} = d_j$ and no repetitions. We can find $s^* \leq k$ and $S_1, S_0 \subseteq \{0, \ldots, s^* - 1\}$ and $u \subseteq Z'$ satisfying $|u| = t + 5$ such that (because of the partition relation):

- (a) $i \in u \Rightarrow s_j = s^*$,
- (b) for each $j < \ell g(\bar{a}) + 1$ and $s < s^*$ the truth value of $d_{i,s} = (\bar{a}b)_j$ is the same for all $i \in u$,
- (c) if $i \neq j$ are from u then $|i - j| > k + 1$, i.e., the C_i's for $i \in u$ are quite far from each other
- (d) the truth value of "$\{d_{i,s_1}, d_{i,s_2}\}$ is an edge" is the same for all $i \in u$,
- (e) for all $i_0 < i_1$ from u:
$$[d_{i_0,s} \in c\ell^{k^*,i_1}(\bar{a}, M)] \Leftrightarrow s \in S_0,$$
- (f) for all $i_0 < i_1$ from u:
$$d_{i_1,s} \in c\ell^{k^*,i_0}(\bar{a}, M) \Leftrightarrow s \in S_1,$$
- (g) for each $s < s^*$, the sequence $\langle d_{i,s} : i \in u \rangle$ is constant or with no repetition,
- (h) if $d_{i_1,s_1} = d_{i_2,s_2}$ then $d_{i_1,s_1} = d_{i_1,s_2} = d_{i_2,s_2}$, moreover, $s_1 = s_2$ (recalling that $\langle d_{i,s} : s < s_j \rangle$ is with no repetitions).

Now let $i(*)$ be, e.g., the third element of the set u and

$$B_1 =: C_{i(*)} \cap c\ell^{k^*,\min(u)}(\bar{a}, M), \text{ and } B_2 =: C_{i(*)} \cap c\ell^{k^*,\max(u)}(\bar{a}, M).$$

So

- $\circledast_1$ $B_1 <^* B_2 \leq^* C_{i(*)}$ (note: that $B_1 \neq B_2$ because $d_{i(*)} \in B_2 \setminus B_1$) and
- $\circledast_2$ $(\bar{a}b) \cap B_2 \subseteq B_1$ by clause (b) and
- $\circledast_3$ there is no edge in $(C_{i(*)} \setminus B_2) \times (B_2 \setminus B_1)$.

Why? Toward contradiction assume that this fails. Let the edge be $\{d_{i(*),s_1}, d_{i(*),s_2}\}$ with

$$d_{i(*),s_1} \in C_{i(*)} \setminus B_2 \text{ and } d_{i(*),s_2} \in B_2 \setminus B_1;$$

hence

$(*)_1$ $d_{i(*),s_1} \in C_{i(*)} \backslash c\ell^{k^*,\max(u)}(\bar{a}, M)$ and $d_{i(*),s_2} \in c\ell^{k^*,\max(u)}(\bar{a}, M) \backslash c\ell^{k^*,\min(u)}(\bar{a}, M)$ and $\{d_{i(*),s_1}, d_{i(*),s_2}\}$ is an edge.

Hence by clause (d)

$(*)_2$ $\{d_{i,s_1}, d_{i,s_2}\}$ is an edge for every $i \in u$

and by clauses $(e), (f)$ we have

$(*)_3$ if $i_0 < i_1 < i_2$ are in u then $d_{i_1,s_2} \notin c\ell^{k^*,i_0}(\bar{a}, M)$ and $d_{i_1,s_2} \in c\ell^{k^*,i_2}(\bar{a}, M)$ and $d_{i_1,s_1} \notin c\ell^{k^*,i_2}(\bar{a}, M)$

and so necessarily

$(*)_4$ $\langle d_{i,s_2} : i \in u \rangle$ is with no repetitions.

[Why? By clause (g) and $(*)_3$.]

So the set of edges $\{\{d_{i,s_1}, d_{i,s_2}\} : i \in u$ but $|u \cap i| \geq 2$ and $|u \setminus i| \geq 2\}$ contradicts 3.8 using $m^{\otimes} = \max(u) - k$ there (and our choice of parameters and $C_i \subseteq c\ell^k(\bar{a}b, M)$). So $\circledast_3$ holds.

As $C_{i(*)} \restriction (\bar{a}b) <_i C_{i(*)}$ and $B_2 \cap (\bar{a}b) \subseteq B_1$ (by $\circledast_2$), clearly $C_{i(*)} \cap \bar{a}b \subseteq C_{i(*)} \setminus (B_2 \backslash B_1) \subset C_{i(*)}$, the strict $\subset$ as $d_{i(*)} \in C_{i(*)} \cap (c\ell^{k^*,i(*)+1}(\bar{a}b, M) \backslash c\ell^{k^*,i(*)}(\bar{a}b, M)) \subseteq B_2 \backslash B_1$. But, as stated above, $C_{i(*)} \backslash (B_2 \backslash B_1) \bigcup\limits_{B_1} B_2$, hence by the previous sentence (and smoothness, see 1.16(5)) we get $B_1 <^*_i B_2$; also $|B_2| \leq |C_{i(*)}| \leq k \leq k^*$. By their definitions, $B_1 \subseteq c\ell^{k^*,\min(u)}(\bar{a}, M)$, but $B_1 \leq^*_i B_2$, $|B_2| \leq k \leq k^*$ and hence $B_2 \subseteq c\ell^{k^*,2^{\text{nd member of } u}}(\bar{a}, M)$. Contradiction to the choice of $d_{i(*)}$. $\square_{3.10}$

Claim 3.11. *For every k, m and ℓ (from $\mathbb{N}$), for some m^*, k^* and t^* we have:*

$(*)$ *if $M \in \mathscr{K}, \bar{a} \in {}^{\ell \geq}M$ and $b \in M \backslash c\ell^{k^*,m^*}(\bar{a}, M)$ <u>then</u> for some $m^{\otimes} \leq m^* - m$ and B we have*

(i) $|B| \leq t^*$,

(ii) $\bar{a} \subseteq B \subseteq c\ell^k(B, M) \subseteq c\ell^{k^*,m^{\otimes}}(\bar{a}, M)$,

(iii) $c\ell^{k^*,m^{\otimes}+m}(\bar{a}, M), (c\ell^k(\bar{a}b, M) \setminus c\ell^{k^*,m^{\otimes}+m}(\bar{a}, M)) \cup B$ *are free over B inside M,*

(iv) $B \leq^*_s B^* =: M \restriction ((c\ell^{k^*}(\bar{a}b, M) \setminus c\ell^{k^*,m^{\otimes}+m}(B, M)) \cup B)$.

Remark 3.12. Clearly this will finish the proof of simply nice.

Comments 3.13. Let us describe the proof below.
1) In the proof we apply the last two claims. By them we arrive to the following situation: inside $c\ell^k(\bar{a}b, M)$ we have $B \leq B^*, |B| \leq t^*$ and there is no "small" D such that $B <^*_i D \leq B^*$ and we have to show that $B <^*_s B^*$, a kind of compactness lemma.

2) Note that for each $d \in c\ell^k(\bar{a}b, M)$ there is $C_d \subseteq c\ell^k(\bar{a}b, M)$ witnessing it, i.e., $C_d \cap (\bar{a}b) \leq_i C_d, d \in C_d, |C_d| \leq k$. To prove the statement above we choose an increasing sequence $\langle D_i : i \leq i(*)\rangle$ of subsets of $B^*, D_0 = B \cup \{b\}, |D_i|$ has an a priori bound, D_{i+1} "large" enough. So by our assumption toward contradiction $B <_s^* D_{i(*)}$ hence there is $\lambda \in \Xi(B, D_{i(*)})$, without loss of generality, $B^* = B \cup \bigcup\{C_d : d \in D_{i(*)}\}$. For each $i < i(*)$ we try to "lift" $\lambda \restriction (D_i \backslash B)$ to $\lambda^+ \in \Xi(B, B^*)$, a failure will show that we could have put elements satisfying some conditions in D_{i+1} so we had done so. As this occurs for every $i < i(*)$, by weight computations we get a contradiction.

Proof. Without loss of generality $k > 0$. Let $t = t(k, \ell), k^*(k, \ell)$ be as required in 3.8 (for our given k, ℓ).

Choose $m(1) = t \times (m + 1) + k + 2$ and let $t^* = t + \ell + k$.

Choose m^* as in 3.10 for k (given in 3.11), $m(1)$ (chosen above) and ℓ (given in 3.11), i.e., $m^* = m^*(k, m(1), \ell)$. Let $\varepsilon^* \in \mathbb{R}^{>0}$ be such that

$$(A', B', \lambda) \in \mathscr{T} and |B'| \leq k and A' \neq B' \Rightarrow \mathbf{w}_\lambda(A', B') \notin (-\varepsilon^*, \varepsilon^*).$$

Let $i(*) > \frac{1}{\varepsilon^*}$. Define inductively k_i^* for $i \leq i(*)$ as follows

$$k_0^* = \max\{k^*(k, \ell), m \times k, m^* \times t^* + 1\}$$

$$k_{i+1}^* = 2^{2^{k_i^*}}$$

and lastly let

$$k^* = k \times k_{i(*)}^*.$$

We shall prove that m^*, k^*, t^* are as required in 3.11. So let M, $\bar{a}$, b be as in the assumption of $(*)$ of 3.11. So $M \in \mathscr{K}$, $\bar{a} \in {}^{\ell \geq}M$ and $b \in M \setminus c\ell^{k^*, m^*}(\bar{a}, M)$, but this means that the assumption of $(*)$ in 3.10 holds for k, $m(1)$, ℓ, so we can apply it (i.e., as $m^* = m^*(k, m(1), \ell), k^* \geq k^*(k, \ell)$ where $k^*(k, \ell)$ is from 3.8 and $k^* \geq k \times m^*$ as $k^* \geq k_{i(*)}^* > k_0^* \geq m^* \times k$).

So for some $r \leq m^* - m(1)$ we have

$$\oplus_1 \quad c\ell^k(\bar{a}b, M) \cap c\ell^{k^*, r+m(1)}(\bar{a}, M) \subseteq c\ell^{k^*, r}(\bar{a}, M).$$

Let us define

$$\begin{aligned} \mathscr{R} = \{(c, d) : \quad & d \in c\ell^k(\bar{a}b, M) \setminus c\ell^{k^*, r+m(1)}(\bar{a}, M) \text{ and} \\ & c \in c\ell^{k^*, r+m(1)-k}(\bar{a}, M) \text{ and} \\ & \{c, d\} \text{ is an edge of } M\}. \end{aligned}$$

How many members does $\mathscr{R}$ have? By 3.8 (with $r + m(1) - k$ here standing for $m^{\otimes}$ there are (as $k^* \geq k^*(k, \ell)$) at most t members. But by $\oplus_1$ above

$$\begin{aligned} \mathscr{R} = \{(c, d) : \quad & d \in c\ell^k(\bar{a}b, M) \setminus c\ell^{k^*, r}(\bar{a}, M) \text{ and} \\ & c \in c\ell^{k^*, r+m(1)-k}(\bar{a}, M) \text{ and} \\ & \{c, d\} \text{ is an edge of } M\}. \end{aligned}$$

But $t \times (m + 1) + 1 < m(1) - k$ by the choice of $m(1)$ (and, of course, $c\ell^{k^*, i}(\bar{a}, M)$ increase with i) hence for some $m^{\otimes} \in \{r + 1, \ldots, r + m(1) - k - m\}$ we have

$\oplus_2$ $(c,d) \in \mathscr{R} \Rightarrow c \notin c\ell^{k^*,m^\otimes+m}(\bar a,M) \setminus c\ell^{k^*,m^\otimes-1}(\bar a,M)$.

So

$\oplus_3$ $r \le m^\otimes - 1 < m^\otimes + m \le r + m(1) - k$.

Let

$$B =: \{c \in c\ell^{k^*,m^\otimes-1}(\bar a,M) : \text{ for some } d \text{ we have } (c,d) \in \mathscr{R}\} \cup \bar a.$$

So by the above $B = \{c \in c\ell^{k^*,m^\otimes+m}(\bar a,M) : (\exists d)((c,d) \in \mathscr{R})\} \cup \bar a$.

Let us check the demands $(i)-(iv)$ of $(*)$ of 3.11, remember that we are defining $B^* = (c\ell^k(\bar ab,M) \setminus c\ell^{k^*,m^\otimes+m}(\bar a,M)) \cup B$, that is the submodel of M with this set of elements. $\square$

Clause (i): $|B| \le t^*$.

As said above, $|\mathscr{R}| \le t$, hence clearly $|B| \le t + \ell g(\bar a) \le t + \ell \le t^*$.

Clause (ii): $\bar a \subseteq B \subseteq c\ell^k(B,M) \subseteq c\ell^{k^*,m^\otimes}(\bar a,M)$.

As by its definition $B \subseteq c\ell^{k^*,m^\otimes-1}(\bar a,M)$, and $k \le k^*$ clearly $c\ell^k(B,M) \subseteq c\ell^{k^*,m^\otimes}(\bar a,M)$ and $B \subseteq c\ell^k(B,M)$ always and $\bar a \subseteq B$ by its definition.

Clause (iii):

Clearly

$$\begin{aligned} B &= c\ell^{k^*,m^\otimes+m}(\bar a,M) \cap ((c\ell^k(\bar ab,M) \setminus c\ell^{k^*,m^\otimes+m}(\bar a,M)) \cup B) \\ &= (c\ell^{k^*,m^\otimes+m}(\bar a,M)) \cap B^*. \end{aligned}$$

Now the "no edges" holds by the definitions of B and $\mathscr{R}$.

Clause (iv): $B \le^*_s B^*$.

Clearly $B \subseteq B^*$ by the definition of B^* before the proof of clause (i). Toward contradiction assume $\neg(B \le^*_s B^*)$ then 1.16(2) for some D, $B <_i D \le B^*$, and choose such D with minimal number of elements. Note that as $B \subseteq c\ell^{k^*,m^\otimes-1}(\bar a,M)$ and $B^* \cap c\ell^{k^*,m^\otimes+m}(\bar a,M) = B$, necessarily $|D| > k^*$ (and $B <^* D \le B^*$). For every $d \in D \setminus B$, as $d \in B^*$ clearly $d \in c\ell^k(\bar ab,M)$ hence there is a set $C_d \le M$, $|C_d| \le k$ such that $C_d \restriction (\bar ab) \le_i C_d$, $d \in C_d$; note that $C_d \subseteq c\ell^k(\bar ab,M)$ by the definition of $c\ell^k$, hence by the choice of B^* and $m^\otimes$ and $\oplus_1$ we have $C_d \subseteq B^* \cup c\ell^{k^*,m^\otimes-1}(\bar a,M)$. Let $C'_d = C_d \cap (B \cup \{b\})$, $C''_d = C_d \cap B^*$. Clearly $C_d \cap (\bar ab) \le C'_d \le C''_d \le C_d$ hence $C'_d \le_i C_d$. Now by clause (iii), $C''_d \underset{C'_d}{\overset{M}{\biguplus}} C'_d \cup (C_d \setminus C''_d)$ hence (by smoothness) we have $C'_{d_i} \le_i C''_{d_i}$. Of course, $|C''_d| \le |C_d| \le k$. For $d \in B$ let $C_d = C'_d = C''_d = \{d\}$.

We now choose a set D_i by induction on $i \le i(*)$, such that (letting $C^{**}_i = \bigcup\limits_{d \in D_i} C''_d$):

(a) $D_0 = B \cup \{b\}$

(b) $j < i \Rightarrow D_j \subseteq D_i \subseteq D$

(c) $|D_i| \le k^*_i$

(d) if λ is an equivalence relation on $C^{**}_i \setminus B$ and for some $d \in D \setminus D_i$ one of the clauses below holds then there is such $d \in D_{i+1}$ where

$\otimes^1_{\lambda,d}$ for some $x \in C''_d \setminus C^{**}_i$, there are no $y \in C''_d \cap C^{**}_i, j^* \in \mathbb{N}$ and $\langle y_j : j \le j^* \rangle$ such that $y_j \in C''_d, y_{j^*} = x, y_0 = y, \{y_j, y_{j+1}\}$ an edge of M, (actually an empty case, i.e., never occurs see $(*)_{14}$ below)

$\otimes^2_{\lambda,d}$ there are $x \in C''_d \setminus C^{**}_i, y \in (C^{**}_i \setminus C''_d) \cup B$ and $y' \in C''_d \cap C^{**}_i$ such that $\{x, y\}$ is an edge of M and y is connected by a path $\langle y_0, \dots, y_1 \rangle$ inside C''_d to x so $x = y_j, y = y_0$ and $[y_i \in C^{**}_i \equiv i = 0]$ and $\neg(y' \lambda y)$

$\otimes^3_{\lambda,d}$ there is an edge $\{x_1, x_2\}$ of M such that we have:

(A) $\{x_1, x_2\} \subseteq C''_d$

(B) $\{x_1, x_2\}$ is disjoint to C^{**}_i

(C) for $s \in \{1, 2\}$ there is a path $\langle y_{s,0}, \dots, y_{s,j_s} \rangle$ in $C''_d, y_{s,j_s} = x_s, [y_{s,j} \in C^{**}_i \equiv j = 0]$ and $\neg(y_{1,0} \lambda y_{2,0})$

(e) if λ is an equivalence relation on $C^{**}_i \setminus B$ to which clause (d) does not apply but there are $d_1,\ d_2 \in D$ satisfying one of the following then we can find such $d_1,\ d_2 \in D_{i+1}$

$\otimes^4_{\lambda,d_1,d_2}$ for some $x_1 \in C''_{d_1} \setminus C^{**}_i, x_2 \in C''_{d_2} \setminus C^{**}_i$ and $y_1 \in C''_{d_1} \cap C^{**}_i, y_2 \in C''_{d_2} \cap C^{**}_i$ we have: for $s = 1, 2$ there is a path $\langle y_{s,0}, \dots, y_{s,j_s} \rangle$ in $C''_{d_s}, y_{s,j_s} = x, y_{s,0} = y_s, [y_{s,j} \in C^{**}_i \Leftrightarrow j = 0]$ and: $x_1 = x_2 and \neg(y_1 \lambda y_2)$

$\otimes^5_{\lambda,d_1,d_2}$ for some x_1, x_2, y_1, y_2 as in $\otimes^4_{\lambda,d_1,d_2}$ we have: $\neg(y_1 \lambda y_2)$ and $\{x_1, x_2\}$ an edge.

So $|D_{i(*)}| \le k^*/k$ (by the choice of k^*, $i(*)$ and clause (c)), hence $C^{**}_{i(*)} =: \bigcup\limits_{d \in D_{i(*)}} C''_d$ has $\le k^*$ members, $\bar{a}b \subseteq B \cup \{b\} \subseteq D_0 \subseteq C^{**}_{i(*)} \subseteq c\ell^k(\bar{a}b, M)$ and $C^{**}_{i(*)} \cap c\ell^{k^*, m^{\otimes}+m}(\bar{a}, M) = B \subseteq c\ell^{k^*, m^{\otimes}-1}(\bar{a}, M)$ hence necessarily $B \le_s C^{**}_{i(*)}$ hence there is $\lambda \in \Xi(B, C^{**}_{i(*)})$. Let $\lambda_i = \lambda \restriction (C^{**}_i \setminus B)$.

Now

$$\boxdot\ (B, C^{**}_i, \lambda_i) \in \Xi(B, C^{**}_i).$$

[Why? Easy.]

Case 1: For some i and an equivalence relation λ_i on $D_i \setminus B$, clauses (d) and (e) are vacuous for λ_i.

Let λ^*_i be the set of pairs (x, y) from $C^{**} \setminus B$ where $C^{**} = \bigcup\limits_{d \in D} C''_d$ which satisfies (α) or (β) where

(α) $x, y \in C^{**}_i \setminus B$ and $x \lambda_i y$

(β) for some $d \in D$ we have $x \in C^{**} \setminus C^{**}_i, x \in C''_d, y \in C^{**}_i \cap C''_d$ and there is a sequence $\langle y_j : j \le j^* \rangle, j^* \ge 1$ such that $y_{j^*} = x, y_j \in C''_d, y_0 = y, \{y_j, y_{j+1}\}$ an edge of M and $[j > 0 \Rightarrow y_j \notin C^{**}_i]$.

This in general is not an equivalence relation.

Let $C^{\otimes} = \{x :$ for some $(x_1, x_2) \in \lambda^*_i$ we have $x \in \{x_1, x_2\}\}$

$$\lambda_i^+ = \{(x_1, x_2) : \begin{array}{l} \text{for some } y_1, y_2 \in D_i \text{ we have} \\ y_1 \lambda y_2, (x_1, y_1) \in \lambda_i^*, (x_2, y_2) \in \lambda_i^*\}. \end{array}$$

Now

$(*)_1$ λ_i^+ is a set of pairs from $C^{\otimes}$ with $\lambda_i^+ \restriction D_i = \lambda_i$

$(*)_2$ $x \in C^{\otimes} \Rightarrow (x, x) \in \lambda_i^+$.

[Why? Read clauses $(\alpha), (\beta)$ and the choice of λ_i^+.]

$(*)_3$ for every $x \in C^{\otimes}$ for some $y \in C_i^{**}$ we have $x \lambda_i^* y$.

[Why? Read the choice of λ_i^+, λ_i^*.]

$(*)_4$ λ_i^+ is a symmetric relation on $C^{\otimes}$.

[Why? Read the definition of λ_i^+ recalling λ is symmetric.]

$(*)_5$ λ_i^+ is transitive.

[Why? Looking at the choice of λ_i^* this is reduced to the case excluded in $(*)_6$ below]

$(*)_6$ if $(x, y_1), (x, y_2) \in \lambda_i^*, \{y_1, y_2\} \subseteq D_i, x \notin D_i$, then $y_1 \lambda y_2$.

[Why? Because clause (e) in the choice of D_{i+1} is vacuous. More fully, otherwise possibility $\otimes^4_{\lambda, d_1, d_2}$ holds for λ_i.]

$(*)_7$ for every $x \in C^{**} \setminus C_i^{**}$, clause (β) apply to $x \in C^{\otimes}$, i.e., $C^{\otimes} = C^{**}$.

[Why? As $x \in C^{**}$ there is $d \in D$ such that $x \in C''_d$, hence by $\otimes^1_{\lambda, d}$ of clause (d) of the choice of D_{i+1} holds for x hence is not vacuous contradicting the assumption on i in the present case.]

$(*)_8$ λ_i^+ is an equivalence relation on $C^{**} \backslash B$.

[Why? Its domain is $C^{**} \backslash B$ by $(*)_7$, it is an equivalence relation on its domain by $(*)_1 + (*)_2 + (*)_4 + (*)_5$.]

Also

$(*)_9$ $\lambda_i^+ \restriction C_i^{**} = \lambda_i$.

[Why? By the choice of λ_i^+ that is by $(*)_1$.]

$(*)_{10}$ every λ_i^+-equivalence class is represented in C_i^{**}.

[Why? By the choice of λ_i^+ and λ_i^*.]

$(*)_{11}$ if $x_1, x_2 \in C^{**} \setminus B$ and $\neg(x_1 \lambda_i^+ x_2)$ but $\{x_1, x_2\}$ is an edge <u>then</u> $\{x_1, x_2\} \subseteq C_i^{**}$.

[Why $(*)_{11}$ holds?

Assume $\{x_1, x_2\}$ is a counterexample, so $\{x_1, x_2\} \nsubseteq C^{**}_i$, so without loss of generality $x_1 \notin C^{**}_i$. Now for $\ell = 1, 2$ if $x_\ell \notin C^{**}_i$ then we can choose $d_\ell \in D_i$ and $y_\ell \in C''_{d_\ell} \cap C^{**}_i$ such that d witnesses that $(x_\ell, y_\ell) \in \lambda^*_i$ that is, as in clause (β) there is a path $\langle y_{\ell,0}, \dots, y_{\ell,j_\ell}\rangle$ such that $y_{\ell,0} = y_\ell, y_{\ell,j_\ell} = x_\ell$ and $(j > 0 \Rightarrow y_{\ell,j} \notin C^{**}_i)$.

We separate to cases:

- (A) $x_1, x_2 \notin C^{**}_i, d_1 = d_2$. This case can't happen as $\otimes^3_{\lambda,d_1}$ of clause (d) is vacuous.
- (B) $x_1, x_2 \notin C^{**}_i, d_1 \neq d_2$. In this case by the vacuousness of $\otimes^5_{\lambda_i,d_1,d_2}$ of clause (e) we get contradiction.
- (C) $x_1 \in C''_{d_1}$ and $x_2 \in C^{**}_i$. By the vacuousness of $\otimes^2_{\lambda_i,d_1}$ of clause (d) we get a contradiction.

Together we have proved $(*)_{11}$.]

As $\lambda_i \in \Xi(B, C^{**}_i)$ by $(*)_8 + (*)_9 + (*)_{10} + (*)_{11}$ and $\boxdot$, easily $\lambda^+_i \in \Xi(B, C^{**})$, hence (see 1.15) $B <^*_s C^{**}$, so as $B \subseteq D \subseteq C^{**}$ we have $B <^*_s D$, the desired contradiction.

Case 2: For every $i < i(*)$, at least one of the clauses (d), (e) is non vacuous for λ_i.

Let $\mathbf{w}_i = \mathbf{w}_{\lambda_i}(B, C^{**}_i)$. For each i let $\langle d_{i,j} : j < j_i\rangle$ list $D_{i+1} \setminus D_i$, such that: if clause (d) applies to λ_i then $d_{i,0}$ form a witness and if clause (e) applies to λ_i then $d_{i,0}$, $d_{i,1}$ form a witness. For $j \le j_i$ let $C^{**}_{i,j} = C^{**}_i \cup \bigcup\limits_{s<j} C''_{d_{i,s}}$, so $C^{**}_{i,0} = C^{**}_i$, $C^{**}_{i,j_i} = C^{**}_{i+1}$. Let $\mathbf{w}_{i,j} = \mathbf{w}_{\lambda_i}(B, C^{**}_{i,j})$.

So it suffice to prove:

- (A) $\mathbf{w}_{i,j} \ge \mathbf{w}_{i,j+1}$
- (B) $\mathbf{w}_{i,0} - \varepsilon^* \ge \mathbf{w}_{i,1}$ or $\mathbf{w}_{i,1} - \varepsilon^* \ge \mathbf{w}_{i,2}$.

Let $i < i(*)$, $j < j_i$.

Clearly $C^{**}_{i,j+1} \setminus C^{**}_{i,j} \subseteq C''_{d_{i,j}} \subseteq C^{**}_{i,j+1}$, let

$$A_{i,j} = \{x \in C''_{d_{i,j}} : x \in B \text{ or } x/\lambda \text{ is not disjoint to } C^{**}_{i,j}\}.$$

Clearly $A_{i,j}\backslash B$ is $(\lambda \restriction C''_{d_{i,j}})$-closed hence $A_{i,j} \le^* C''_{d_{i,j}}, C''_{d_{i,j}} \setminus A_{i,j}$ is disjoint to $C^{**}_{i,j}$ and $C'_{d_{i,j}} = C_{d_{i,j}} \cap (B \cup \{b\}) \subseteq C^{**}_{i,j}$, and $C'_{d_{i,j}} \subseteq C''_{d_{i,j}}$ hence $C'_{d_{i,j}} \subseteq A_{i,j}, A_{i,j} \le^* C''_{d_{i,j}}$, but $C'_{d_{i,j}} \le_i C''_{d_{i,j}}$ so $A_{i,j} \le^*_i C''_{d_{i,j}}$ (the $\le^*$ in this sentence serves §7 where we say, "repeat the proof of 3.16").

Clearly

$(*)_{12}$ $\mathbf{w}_{i,j+1} = \mathbf{w}_{i,j} + \mathbf{w}_\lambda(A_{i,j}, C''_{d_{i,j}}) - \alpha \mathbf{e}^1_{i,j} - \alpha \mathbf{e}^2_{i,j}$ where

$$\mathbf{e}^1_{i,j} = |\{\{x,y\} : \begin{array}{l}\{x,y\} \text{ an edge of } M, \{x,y\} \subseteq A_{i,j},\\ \neg(x\lambda y) \text{ but } \{x,y\} \nsubseteq C^{**}_{i,j}\}|\end{array}$$

$$\mathbf{e}^2_{i,j} = |\{\{x,y\} : \begin{array}{l}\{x,y\} \text{ an edge of } M, x \in C''_{d_{i,j}} \setminus C^{**}_{i,j},\\ y \in C^{**}_{i,j} \setminus C''_{d_{i,j}} \text{ but } \neg(x\lambda y)\}|.\end{array}$$

Note

$(*)_{13}$ $\mathbf{w}_\lambda(A_{i,j}, C''_{d_{i,j}})$ can be zero if $A_{i,j} = C''_{d_{i,j}}$ and is $\le -\varepsilon^*$ otherwise.

[Why? As $A_{i,j} \le^*_i C''_{d_{i,j}}$.]

$(*)_{14}$ in clause (d), $\otimes^1_{\lambda,d}$ never occurs.

[Why? If $x \in C''_d$ is as there, let $Y = \{y \in C''_d : y, x$ are connected in $M \restriction C''_d\}$. So $x \in Y \subseteq C''_d, Y \cap C^{**}_i = \emptyset$ and $C'_d = C''_d \cap (B \cup \{b\}) = C''_d \cap C^{**}_0 \subseteq C^{**}_i$. Hence $(C''_d \backslash Y) <^*_i C''_d$, but the equivalence relation $\{(y', y'') : y', y'' \in Y\}$ exemplify that this fail.]

Proof. <u>Proof of (A)</u>

Easy by $(*)_{12}$, because $\mathbf{w}_\lambda(A_{i,j}, C''_{d_{i,j}}) \le 0$ holds by $(*)_{13}, -\alpha\mathbf{e}^1_{i,j} \le 0$, and $-\alpha\mathbf{e}^2_{i,j} \le 0$ as $\mathbf{e}^1_{i,j}$, $\mathbf{e}^2_{i,j}$ are natural numbers. $\square$

Proof. <u>Proof of (B)</u>

It suffice to prove that $\mathbf{w}_{i,0} \ne \mathbf{w}_{i,1}$ or $\mathbf{w}_{i,1} \ne \mathbf{w}_{i,2}$ (as inequality implies the right order (by clause (A)) and the difference is $\ge \varepsilon^*$ by definition of ε^* (if $\mathbf{w}_\lambda(A_{i,1}, C''_{d_{i,j}}) \ne 0$) and $\ge \alpha$ (if $\mathbf{e}^1_{i,j} \ne 0$ or $\mathbf{e}^2_{i,j} \ne 0$)). But if $\mathbf{w}_{i,0} = \mathbf{w}_{i,1}$ recalling $(*)_{14}$ easily clause (d) does not apply to λ_i, and if $\mathbf{w}_{i,0} = \mathbf{w}_{i,1} = \mathbf{w}_{i,2}$ also clause (e) does not apply.

So (A),(B) holds so we are done proving case 2 hence the claim. $\square_{3.11}$

Remark 3.14.

(*a*) We could use smaller k^* by building a tree $\langle (D_t, D^+_t, C_t, \lambda_t) : t \in T\rangle$, T a finite tree with a root Λ, $D_\Lambda = \emptyset$, $D^+_\Lambda = B \cup \{b\}$, for each t we have λ_t an equivalence relation on $C_t \backslash B$ and $C_t = \cup\{C''_d : d \in D_t\} \cup B, s \in \mathrm{suc}_T(t) \Rightarrow D^+_t = D_s$ and $D^+_t \backslash D_t$ is $\{d\}$ or $\{d_1, d_2\}$ which witness clause (d) or clause (e) for (D_t, λ_t) when $t \ne \Lambda$ and

$$\begin{aligned}\{(D_s, \lambda_s) :\quad & s \in \mathrm{suc}_T(t)\} = \{(D^+_t, \lambda) : \lambda \restriction D_t = \lambda_t, \lambda \\ & \text{an equivalence relation on } D^+_t \setminus B\}.\end{aligned}$$

(*b*) We can make the argument separated that is prove as a separate claim that is for any k and ℓ there is k^* such that: if A, $B \subseteq M \in \mathscr{K}$, $|B|$, $|A| \le \ell$, $B \subseteq B^*$, $c\ell^k(A, M) \setminus c\ell^k(B, M) \subseteq B^* \setminus B \subseteq c\ell^k(A, M)$ and $(\forall C)(B \subseteq C \subseteq B^* \wedge |C| \le k^* \Rightarrow B <_s C)$ <u>then</u> $B <_s B^*$.

This is a kind of compactness.

§ 3(D). **The Conclusions.**

Conclusion 3.15. *Requirements (A) of* [She02, 2.13(1)] *and even (B) + (C) of* [She02, 2.13(3)] *hold.*

Proof. Requirement (B) of [She02, 2.13(3)] holds by 3.7. Requirement (A) of [She02, 2.13(2)] holds by 3.11 (and the previous sentence). $\square_{3.15}$

Conclusion 3.16.

(1) $\mathfrak{K}$ *is smooth and transitive and local and transparent*

(2) $\mathfrak{K}$ *is simply nice (hence simply almost nice)*

(3) $\mathfrak{K}$ *satisfies the 0-1 law.*

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