

**NON-EXISTENCE OF UNIVERSALS FOR CLASSES LIKE  
REDUCED TORSION FREE ABELIAN GROUPS UNDER  
EMBEDDINGS WHICH ARE NOT NECESSARILY PURE**

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ABSTRACT. We consider a class  $K$  of structures, e.g. trees with  $\omega + 1$  levels, metric spaces and mainly, classes of Abelian groups like the one mentioned in the title and the class of reduced separable (Abelian)  $p$ -groups. We say  $M \in K$  is universal for  $K$  if any member  $N$  of  $K$  of cardinality not bigger than the cardinality of  $M$  can be embedded into  $M$ . This is a natural, often raised, problem. We try to draw consequences of cardinal arithmetic to non-existence of universal members for such natural classes.

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## 0. INTRODUCTION

**Context.** In this paper, model theoretic notions (like superstable, elementary classes) appear in the introduction but not in the paper itself (so the reader does not need to know them). Only naive set theory and basic facts on Abelian groups (all in [Fuc73]) are necessary for understanding the paper. The basic definitions are reviewed at the end of the introduction. On the history of the problem of the existence of universal members, see Kojman, Shelah [KS92a]; for more direct predecessors see Kojman, Shelah [KS92b], [KS95] and [She96c], but we do not rely on them. For other advances see [She93c], [She96b] and Džamonja, Shelah [DS04]. Lately [She01] continues this paper.

A class  $\mathfrak{K}$  is a class of structures with an embeddability notion. If not said otherwise, an embedding, is a one to one function preserving atomic relations and their negations. If  $\mathfrak{K}$  is a class and  $\lambda$  is a cardinal, then  $\mathfrak{K}_\lambda$  stands for the collection of all members of  $\mathfrak{K}$  of cardinality  $\lambda$ .

We similarly define  $\mathfrak{K}_{\leq\lambda}$ .

A member  $M$  of  $\mathfrak{K}_\lambda$  is universal, if every  $N \in \mathfrak{K}_{\leq\lambda}$ , embeds into  $M$ . An example is  $M =: \bigoplus_\lambda \mathbb{Q}$ , which is universal in  $\mathfrak{K}_\lambda$  if  $\mathfrak{K}$  is the class of all torsion-free Abelian groups, under usual embeddings.

We give some motivation to the present paper by a short review of the above references. The general thesis in these papers, as well as the present one is:

**Thesis 0.1.** General Abelian groups and trees with  $\omega + 1$  levels behave in universality theorems like stable non-superstable theories.

The simplest example of such a class is the class  $\mathfrak{K}^{tr} =:$  trees  $T$  with  $(\omega + 1)$ -levels, i.e.  $T \subseteq \omega^{\geq\alpha}$  for some  $\alpha$ , with the relations  $\eta E_n^0 \nu =: \eta \upharpoonright n = \nu \upharpoonright n$  &  $\text{lg}(\eta) \geq n$ . For  $\mathfrak{K}^{tr}$  we know that  $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$  implies there is no universal for  $\mathfrak{K}_\lambda^{tr}$  (by [KS92b]). Classes as  $\mathfrak{K}^{rtf}$  (defined in the title), or  $\mathfrak{K}^{rs(p)}$  (reduced separable Abelian  $p$ -groups) are similar (though they are not elementary classes) when we consider pure embeddings (by [KS95]). But it is not less natural to consider usual embeddings (remembering they, the (Abelian) groups under consideration, are reduced). The problem is that the invariant has been defined using divisibility, and so under non-pure embedding those seemed to be erased.

Then in [She96c] the non-existence of universals is proved restricting ourselves to  $\lambda > 2^{\aleph_0}$  and  $(< \lambda)$ -stable groups (see there). These restrictions hurt the generality of the theorem; because of the first requirement we lose some cardinals. The second requirement changes the class to one which is not established among Abelian group theorists (though to me it looks natural).

Our aim was to eliminate those requirements, or show that they are necessary. Note that the present paper is mainly concerned essentially with

results in ZFC, but they have roots in “difficulties” in extending independence results thus providing a case for the

**Thesis 0.2.** Even if you do not like independence results you better look at them, as you will not even consider your desirable ZFC results when they are camouflaged by a litany of many independence results you can prove things about.

Of course, independence has interest *per se*; still for a given problem in general a solution in ZFC is for me preferable to an independence result. But if it gives a method of forcing (so relevant to a series of problems) the independence result is preferable (of course, I assume there are no other major differences; the depth of the proof would be of first importance to me).

As occurs often in my papers lately, quotations of **pcf** theory appear.

This paper is also a case of, mainly

**Thesis 0.3.** Assumption of cases of not GCH, mainly at singular (more generally  $\text{pp}(\lambda) > \lambda^+$ ) are “good”, “helpful” assumptions; i.e. traditionally uses of GCH proliferate mainly not from conviction but as you can prove many theorems assuming  $2^{\aleph_0} = \aleph_1$  but very few from  $2^{\aleph_0} > \aleph_1$ , but assuming  $2^{\aleph_\omega} > \aleph_\omega^+$  is helpful in proving.

Unfortunately, most results are only almost in ZFC as they use extremely weak assumptions from **pcf**, assumptions whose independence is not known. So practically it is not tempting to try to remove them as they may be true, and it is unreasonable to try to prove independence results before independence results on **pcf** will advance.

In §1 we give an explanation of the earlier difficulties: the problem (of the existence of universals for  $\mathfrak{R}^{rs(p)}$ ) is not like looking for  $\mathfrak{R}^{tr}$  (trees with  $\omega + 1$  levels) but for  $\mathfrak{R}_{\langle \lambda_\alpha : \alpha < \omega \rangle}^{tr}$  where

$$(\oplus): \lambda_n^{\aleph_0} < \lambda_{n+1} < \mu, \lambda_n \text{ are regular and } \mu^+ < \lambda = \lambda_\omega = \text{cf}(\lambda) < \mu^{\aleph_0} \\ \text{and } \mathfrak{R}_{\langle \lambda_n : n < \omega \rangle}^{tr} \text{ is}$$

$$\{T : T \text{ a tree with } \omega + 1 \text{ levels, in level } n < \omega \text{ there are } \lambda_n \text{ elements}\}.$$

We also consider  $\mathfrak{R}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^{tr}$ , which is defined similarly but the level  $\omega$  of  $T$  is required to have  $\lambda_\omega$  elements.

For  $\mathfrak{R}^{rs(p)}$  this is proved fully, for  $\mathfrak{R}^{rtf}$  this is proved for the natural examples.

In §2 we define two such basic examples: one is  $\mathfrak{R}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^{tr}$ , ?? and the second is  $\mathfrak{R}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^{fc}$ . The first is a tree with  $\omega + 1$  levels; in the second we have slightly less restrictions. We have  $\omega$  kinds of elements and a function from the  $\omega$ -th-kind to the  $n$ th kind. We can interpret a tree  $T$  as a member of the second example:  $P_\alpha^T = \{x : x \text{ is of level } \alpha\}$  and

$$F_n(x) = y \iff x \in P_\omega^T \ \& \ y \in P_n^T \ \& \ y <_T x.$$

For the second we recapture the non-existence theorems.

But this is not one of the classes we considered originally.

In §3 we return to  $\mathfrak{R}^{rtf}$  (reduced torsion free Abelian groups) and prove the non-existence of universal ones in  $\lambda$  if  $2^{\aleph_0} < \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$  and an additional very weak set theoretic assumption (the consistency of its failure is not known).

Note that (it will be proved in [She01]):

( $\otimes$ ): if  $\lambda < 2^{\aleph_0}$  then  $\mathfrak{R}_\lambda^{rtf}$  has no universal members.

Note: if  $\lambda = \aleph_0$  then  $\mathfrak{R}_\lambda^{tr}$  has universal member also  $\mathfrak{R}_\lambda^{rs(p)}$  (see [Fuc73]) but not  $\mathfrak{R}_\lambda^{rtf}$  (see [Shear, Ch IV, VI]).

We have noted above that for  $\mathfrak{R}_\lambda^{rtf}$  requiring  $\lambda \geq 2^{\aleph_0}$  is reasonable: we can prove (i.e. in ZFC) that there is no universal member. What about  $\mathfrak{R}_\lambda^{rs(p)}$ ? By §1 we should look at  $\mathfrak{R}_{\langle \lambda_i : i \leq \omega \rangle}^{tr}$ ,  $\lambda_\omega = \lambda < 2^{\aleph_0}$ ,  $\lambda_n < \aleph_0$ .

In §4 we prove the consistency of the existence of universals for  $\mathfrak{R}_{\langle \lambda_i : i \leq \omega \rangle}^{tr}$  when  $\lambda_n \leq \omega$ ,  $\lambda_\omega = \lambda < 2^{\aleph_0}$  but of cardinality  $\lambda^+$ ; this is not the original problem but it seems to be a reasonable variant, and more seriously, it shoots down the hope to use the present methods of proving non-existence of universals. Anyhow this is  $\mathfrak{R}_{\langle \lambda_i : i \leq \omega \rangle}^{tr}$  not  $\mathfrak{R}_{\lambda_\omega}^{rs(p)}$ , so we proceed to reduce this problem to the previous one under a mild variant of MA. The intentions are to deal with “there is universal of cardinality  $\lambda$ ” in Džamonja Shelah [DS04].

The reader should remember that the consistency of e.g.

$2^{\aleph_0} > \lambda > \aleph_0$  and there is no  $M$  such that  $M \in \mathfrak{R}^{rs(p)}$  is of cardinality  $< 2^{\aleph_0}$  and universal for  $\mathfrak{R}_\lambda^{rs(p)}$

is much easier to obtain, even in a wider context (just add many Cohen reals).

As in §4 the problem for  $\mathfrak{R}_\lambda^{rs(p)}$  was reasonably resolved for  $\lambda < 2^{\aleph_0}$  (and for  $\lambda = \aleph_0$ , see [KS95]), we now, in §5 turn to  $\lambda > 2^{\aleph_0}$  (and  $\mu, \lambda_n$ ) as in ( $\oplus$ ) above. As in an earlier proof we use  $\langle C_\delta : \delta \in S \rangle$  guessing clubs for  $\lambda$  (see references or later here), so  $C_\delta$  is a subset of  $\delta$  (so the invariant depends on the representation of  $G$  but this disappears when we divide by suitable ideal on  $\lambda$ ). What we do is: rather than trying to code a subset of  $C_\delta$  (for  $\bar{G} = \langle G_i : i < \lambda \rangle$  a representation or filtration of the structure  $G$  as the union of an increasing continuous sequence of structures of smaller cardinality) by an element of  $G$ , we do it, say, by some set  $\bar{x} = \langle x_t : t \in \text{Dom}(I) \rangle$ ,  $I$  an ideal on  $\text{Dom}(I)$  (really by  $\bar{x}/I$ ). At first glance if  $\text{Dom}(I)$  is infinite we cannot list *a priori* all possible such sequences for a candidate  $H$  for being a universal member, as their number is  $\geq \lambda^{\aleph_0} = \mu^{\aleph_0}$ . But we can find a family

$$\mathcal{F} \subseteq \{ \langle x_t : t \in A \rangle : A \subseteq \text{Dom}(I), A \notin I, x_t \in \lambda \}$$

of cardinality  $< \mu^{\aleph_0}$  such that for any  $\bar{x} = \langle x_t : t \in \text{Dom}(I) \rangle$ , for some  $\bar{y} \in \mathcal{F}$  we have  $\bar{y} = \bar{x} \upharpoonright \text{Dom}(\bar{y})$ .

As in §3 there is such  $\mathcal{F}$  except when some set theoretic statement related to **pcf** holds. This statement is extremely strong, also in the sense that we do not know how to prove its consistency at present. But again, it seems unreasonable to try to prove its consistency before the **pcf** problem was dealt with. Of course, we may try to improve the combinatorics to avoid the use of this statement, but are naturally discouraged by the possibility that the **pcf** statement can be proved in ZFC; thus we would retroactively get the non-existence of universals in ZFC.

In §6, under weak **pcf** assumptions, we prove: if there is a universal member in  $\mathfrak{K}_\lambda^{fc}$  then there is one in  $\mathfrak{K}_\lambda^{rs(p)}$ ; so making the connection between the combinatorial structures and the algebraic ones closer.

In §7 we give other weak **pcf** assumptions which suffice to prove non-existence of universals in  $\mathfrak{K}_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^x$  (with  $x$  one of the “legal” values):  $\max \text{pcf}\{\lambda_n : n < \omega\} = \lambda$  and  $\mathcal{P}(\{\lambda_n : n < \omega\})/J_{< \lambda}\{\lambda_n : n < \omega\}$  is an infinite Boolean Algebra (and  $\oplus$  holds, of course).

In [KS92a], for singular  $\lambda$  results on non-existence of universals (there on orders) can be gotten from these weak **pcf** assumptions.

In §8 we get parallel results from, in general, more complicated assumptions.

In §9 we turn to a closely related class: the class of metric spaces with (one to one) continuous embeddings, similar results hold for it. We also phrase a natural criterion for deducing the non-existence of universals from one class to another.

In §10 we deal with modules and in §11 we discuss the open problems of various degrees of seriousness.

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The sections are written in the order the research was done.

*Notation* 0.4. Note that we deal with trees with  $\omega + 1$ , levels rather than, say, with  $\kappa + 1$ , and related situations, as those cases are quite popular. But inherently the proofs of §1-§3, §5-§9 work for  $\kappa + 1$  as well (in fact, **pcf** theory is stronger).

For a structure  $M$ ,  $\|M\|$  is its cardinality.

For a model, i.e. a structure,  $M$  of cardinality  $\lambda$ , where  $\lambda$  is regular uncountable, we say that  $\bar{M}$  is a representation (or filtration) of  $M$  if  $\bar{M} = \langle M_i : i < \lambda \rangle$  is an increasing continuous sequence of submodels of cardinality  $< \lambda$  with union  $M$ .

For a set  $A$ , we let  $[A]^\kappa = \{B : B \subseteq A \text{ and } |B| = \kappa\}$ .

For a set  $C$  of ordinals,

$$\text{acc}(C) = \{\alpha \in C : \alpha = \sup(\alpha \cap C)\}, \text{ (set of accumulation points)}$$

$\text{nacc}(C) = C \setminus \text{acc}(C)$  (= the set of non-accumulation points).

We usually use  $\eta, \nu, \rho$  for sequences of ordinals; let  $\eta \triangleleft \nu$  means  $\eta$  is an initial segment of  $\nu$ .

Let  $\text{cov}(\lambda, \mu, \theta, \sigma) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{<\mu}, \text{ and for every } A \in [\lambda]^{<\theta} \text{ for some } \alpha < \sigma \text{ and } B_i \in \mathcal{P} \text{ for } i < \alpha \text{ we have } A \subseteq \bigcup_{i < \alpha} B_i\}$ . Remember that for an ordinal  $\alpha$ , e.g. a natural number,  $\alpha = \{\beta : \beta < \alpha\}$ .

*Notation 0.5.*  $\mathfrak{K}^{rs(p)}$  is the class of (Abelian) groups which are  $p$ -groups (i.e.  $(\forall x \in G)(\exists n)[p^n x = 0]$ ) reduced (i.e. have no divisible non-zero subgroups) and separable (i.e. every cyclic pure subgroup is a direct summand). See [Fuc73].

For  $G \in \mathfrak{K}^{rs(p)}$  define a norm  $\|x\| = \inf\{2^{-n} : p^n \text{ divides } x\}$ . Now every  $G \in \mathfrak{K}^{rs(p)}$  has a basic subgroup  $B = \bigoplus_{\substack{n < \omega \\ i < \lambda_n}} \mathbb{Z}x_i^n$ , where  $x_i^n$  has order  $p^{n+1}$ ,

and every  $x \in G$  can be represented as  $\sum_{\substack{n < \omega \\ i < \lambda_n}} a_i^n x_i^n$ , where for each  $n$ ,  $w_n(x) =$

$\{i < \lambda_n : a_i^n x_i^n \neq 0\}$  is finite.

$\mathfrak{K}^{rtf}$  is the class of Abelian groups which are reduced and torsion free (i.e.  $G \models nx = 0, n > 0 \Rightarrow x = 0$ ).

For a group  $G$  and  $A \subseteq G$  let  $\langle A \rangle_G$  be the subgroup of  $G$  generated by  $A$ , we may omit the subscript  $G$  if clear from the context.

Group will mean an Abelian group, even if not stated explicitly.

Let  $H \subseteq_{pr} G$  means  $H$  is a pure subgroup of  $G$ .

Let  $nG = \{nx : x \in G\}$  and let  $G[n] = \{x \in G : nx = 0\}$ .

*Notation 0.6.*  $\mathfrak{K}$  will denote a class of structures with the same vocabulary, with a notion of embeddability, equivalently a notion  $\leq_{\mathfrak{K}}$  of submodel.

### 1. THEIR PROTOTYPE IS $\mathfrak{K}_{(\lambda_n: n < \omega)}^{tr}$ NOT $\mathfrak{K}^{tr}$ !

If we look for universal member in  $\mathfrak{K}_{\lambda}^{rs(p)}$ , thesis 0.1 suggests to us to think it is basically  $\mathfrak{K}_{\lambda}^{tr}$  (trees with  $\omega + 1$  levels, i.e.  $\mathfrak{K}_{\lambda}^{tr}$  is our prototype), a way followed in [KS95], [She96c]. But, as explained in the introduction, this does not give an answer for the case of usual embedding for the family of all such groups. Here we show that for this case the thesis should be corrected. More concretely, the choice of the prototype means the choice of what we expect is the division of the possible classes. That is for a family of classes a choice of a prototype assert that we believe that they all behave in the same way.

We show that looking for a universal member  $G$  in  $\mathfrak{K}_{\lambda}^{rs(p)}$  is like looking for it among the  $G$ 's with density  $\leq \mu$  ( $\lambda, \mu$ , as usual, as in  $(\oplus)$  from §0). For  $\mathfrak{K}_{\lambda}^{rtf}$  we get weaker results which still cover the examples usually constructed, so showing that the restrictions in [KS95] (to pure embeddings) and [She96c] (to  $(< \lambda)$ -stable groups) were natural.

**Proposition 1.1.** *Assume that  $\mu = \sum_{n < \omega} \lambda_n = \limsup_n \lambda_n$ ,  $\mu \leq \lambda \leq \mu^{\aleph_0}$ , and  $G$  is a reduced separable  $p$ -group such that*

$$|G| = \lambda \quad \text{and} \quad \lambda_n(G) =: \dim((p^n G)[p]/(p^{n+1} G)[p]) \leq \mu$$

(this is a vector space over  $\mathbb{Z}/p\mathbb{Z}$ , hence the dimension is well defined).

Then there is a reduced separable  $p$ -group  $H$  such that  $|H| = \lambda$ ,  $H$  extends  $G$  and  $(p^n H)[p]/(p^{n+1} H)[p]$  is a group of dimension  $\lambda_n$  (so if  $\lambda_n \geq \aleph_0$ , this means cardinality  $\lambda_n$ ).

*Remark 1.2.* So for  $H$  the invariants from [KS95] are trivial.

*Proof.* (See Fuchs [Fuc73]). We can find  $z_i^n$  (for  $n < \omega$ ,  $i < \lambda_n(G) \leq \mu$ ) such that:

- (a):  $z_i^n$  has order  $p^n$ ,
- (b):  $B = \sum_{n,i} \langle z_i^n \rangle_G$  is a direct sum,
- (c):  $B$  is dense in  $G$  in the topology induced by the norm

$$\|x\| = \min\{2^{-n} : p^n \text{ divides } x \text{ in } G\}.$$

For each  $n < \omega$  and  $i < \lambda_n(G) (\leq \mu)$  choose  $\eta_i^n \in \prod_{m < \omega} \lambda_m$ , pairwise distinct such that for  $(n^1, i^1) \neq (n^2, i^2)$  for some  $n(*)$  we have:

$$\lambda_n \geq \lambda_{n(*)} \quad \Rightarrow \quad \eta_{i^1}^{n^1}(n) \neq \eta_{i^2}^{n^2}(n).$$

Let  $H$  be generated by  $G$ ,  $x_i^m$  ( $i < \lambda_m$ ,  $m < \omega$ ),  $y_i^{n,k}$  ( $i < \lambda_n$ ,  $n < \omega$ ,  $n \leq k < \omega$ ) freely except for:

- ( $\alpha$ ): the equations of  $G$ ,
- ( $\beta$ ):  $y_i^{n,n} = z_i^n$ ,
- ( $\gamma$ ):  $py_i^{n,k+1} - y_i^{n,k} = x_{\eta_i^n(k)}^k$ ,
- ( $\delta$ ):  $p^{n+1}x_i^n = 0$ ,
- ( $\varepsilon$ ):  $p^{k+1}y_i^{n,k} = 0$ .

Now check, (also see the proof of 1.5). □<sub>1.1</sub>

**Definition 1.3.** (1)  $\mathbf{t}$  denotes a sequence  $\langle t_i : i < \omega \rangle$ ,  $t_i$  a natural number  $> 1$ .

(2) For a group  $G$  we define

$$G^{[\mathbf{t}]} = \{x \in G : \bigwedge_{j < \omega} [x \in (\prod_{i < j} t_i)G]\}.$$

(3) We can define a semi-norm  $\| - \|_{\mathbf{t}}$  on  $G$

$$\|x\|_{\mathbf{t}} = \min\{2^{-i} : x \in (\prod_{j < i} t_j)G\}$$

and so the semi-metric

$$d_{\mathbf{t}}(x, y) = \|x - y\|_{\mathbf{t}}.$$

*Remark 1.4.* So, if  $\| - \|_{\mathbf{t}}$  is a norm,  $G$  has a completion under  $\| - \|_{\mathbf{t}}$ , which we call  $\| - \|_{\mathbf{t}}$ -completion; if  $\mathbf{t} = \langle i! : i < \omega \rangle$  we refer to  $\| - \|_{\mathbf{t}}$  as  $\mathbb{Z}$ -adic norm, and this induces  $\mathbb{Z}$ -adic topology, so we can speak of  $\mathbb{Z}$ -adic completion.

**Proposition 1.5.** *Suppose that*

- ( $\otimes_0$ ):  $\mu = \sum_n \lambda_n$  and  $\mu \leq \lambda \leq \mu^{\aleph_0}$  for simplicity,  $2 < 2 \cdot \lambda_n \leq \lambda_{n+1}$   
(maybe  $\lambda_n$  is finite!),
- ( $\otimes_1$ ):  $G$  is a torsion free group,  $|G| = \lambda$ , and  $G^{[\mathbf{t}]} = \{0\}$ ,
- ( $\otimes_2$ ):  $G_0 \subseteq G$ ,  $G_0$  is free and  $G_0$  is  $\mathbf{t}$ -dense in  $G$  (i.e. in the topology induced by the metric  $d_{\mathbf{t}}$ ), where  $\mathbf{t}$  is a sequence of primes.

Then there is a torsion free group  $H$ ,  $G \subseteq H$ ,  $H^{[\mathbf{t}]} = \{0\}$ ,  $|H| = \lambda$  and, under  $d_{\mathbf{t}}$ ,  $H$  has density  $\mu$ .

*Proof.* Let  $\{x_i : i < \lambda\}$  be a basis of  $G_0$ . Let  $\eta_i \in \prod_{n < \omega} \lambda_n$  for  $i < \mu$  be distinct such that  $\eta_i(n+1) \geq \lambda_n$  and

$$i \neq j \quad \Rightarrow \quad (\exists m)(\forall n)[m \leq n \quad \Rightarrow \quad \eta_i(n) \neq \eta_j(n)].$$

Let  $H$  be generated by

$$G, \quad x_i^m \text{ (for } i < \lambda_m, m < \omega), \quad y_i^n \text{ (for } i < \mu, n < \omega)$$

freely except for

- (a): the equations of  $G$ ,
- (b):  $y_i^0 = x_i$ ,
- (c):  $t_n y_i^{n+1} + y_i^n = x_{\eta_i(n)}^n$ .

**Fact A**  $H$  extends  $G$  and is torsion free.

[Why? As  $H$  can be embedded into the divisible hull of  $G$ .]

**Fact B**  $H^{[\mathbf{t}]} = \{0\}$ .

*Proof.* Let  $K$  be a countable pure subgroup of  $H$  such that  $K^{[\mathbf{t}]} \neq \{0\}$ . Now without loss of generality  $K$  is generated by

- (i):  $K_1 \subseteq G \cap [\text{the } d_{\mathbf{t}}\text{-closure of } \langle x_i : i \in I \rangle_G]$ , where  $I$  is a countable infinite subset of  $\lambda$  and  $K_1 \supseteq \langle x_i : i \in I \rangle_G$ ,
- (ii):  $y_i^m, x_j^n$  for  $i \in I, m < \omega$  and  $(n, j) \in J$ , where  $J \subseteq \omega \times \lambda$  is countable and

$$i \in I, n < \omega \quad \Rightarrow \quad (n, \eta_i(n)) \in J.$$

Moreover, the equations holding among those elements are deducible from the equations of the form

- (a)<sup>-</sup>: equations of  $K_1$ ,
- (b)<sup>-</sup>:  $y_i^0 = x_i$  for  $i \in I$ ,
- (c)<sup>-</sup>:  $t_n y_i^{n+1} + y_i^n = x_{\eta_i(n)}^n$  for  $i \in I, n < \omega$ .

We can find  $\langle k_i : i < \omega \rangle$  such that

$$[n \geq k_i \ \& \ n \geq k_j \ \& \ i \neq j \quad \Rightarrow \quad \eta_i(n) \neq \eta_j(n)].$$

Let  $y \in K[t] \setminus \{0\}$ . Then for some  $j$ ,  $y \notin (\prod_{i < j} t_i)G$ , so for some finite  $I_0 \subseteq I$  and finite  $J_0 \subseteq J$  and

$$y^* \in \langle \{x_i : i \in I_0\} \cup \{x_\alpha^n : (n, \alpha) \in J_0\} \rangle_K$$

we have  $y - y^* \in (\prod_{i < j} t_i)G$ . Without loss of generality  $J_0 \cap \{(n, \eta_i(n)) : i \in I, n \geq k_i\} = \emptyset$ . Now there is a homomorphism  $\varphi$  from  $K$  into the divisible hull  $K^{**}$  of

$$K^* = \langle \{x_i : i \in I_0\} \cup \{x_j^n : (n, j) \in J_0\} \rangle_G$$

such that  $\text{Rang}(\varphi)/K^*$  is finite. This is enough.

**Fact C**  $H_0 =: \langle x_i^n : n < \omega, i < \lambda_n \rangle_H$  is dense in  $H$  by  $d_t$ .

*Proof.* Straight as each  $x_i$  is in the  $d_t$ -closure of  $H_0$  inside  $H$ .

Noting then that we can increase the dimension easily, we are done.  $\square_{1.5}$

## 2. ON STRUCTURES LIKE $(\prod_n \lambda_n, E_m)_{m < \omega}$ , $\eta E_m \nu =: \eta(m) = \nu(m)$

*Discussion 2.1.* We discuss the existence of universal members in cardinality  $\lambda$ ,  $\mu^+ < \lambda < \mu^{\aleph_0}$ , for certain classes of groups. The claims in §1 indicate that the problem is similar not to the problem of the existence of a universal member in  $\mathfrak{R}_\lambda^{tr}$  (the class of trees with  $\lambda$  nodes,  $\omega + 1$  levels) but to the one where the first  $\omega$  levels, are each with  $< \mu$  elements. We look more carefully and see that some variants are quite different.

The major concepts and Lemma (2.4) are similar to those of §3, but easier. Since detailed proofs are given in §3, here we give somewhat shorter proofs.

**Definition 2.2.** For a sequence  $\bar{\lambda} = \langle \lambda_i : i \leq \delta \rangle$  of cardinals we define:

(A):  $\mathfrak{R}_\lambda^{tr} = \{T : T \text{ is a tree with } \delta + 1 \text{ levels (i.e. a partial order such that}$

for  $x \in T$ ,  $\text{lev}_T(x) =: \text{otp}(\{y : y < x\})$  is an ordinal  $\leq \delta$ )

such

that:  $\text{lev}_i(T) =: \{x \in T : \text{lev}_T(x) = i\}$  has cardinality  $\leq \lambda_i\}$ ,

(B):  $\mathfrak{R}_\lambda^{fc} = \{M : M = (\|M\|, P_i, F_i)_{i \leq \delta}$ ,  $\|M\|$  is the disjoint union of  $\langle P_i : i \leq \delta \rangle$ ,  $F_i$  is a function from  $P_\delta$  to  $P_i$ ,  $\|P_i\| \leq \lambda_i$ ,  $F_\delta$  is the identity (so can be omitted)\},

(C): If  $[i \leq \delta \Rightarrow \lambda_i = \lambda]$  then we write  $\lambda, \delta + 1$  instead of  $\langle \lambda_i : i \leq \delta \rangle$ .

**Definition 2.3.** Embeddings for  $\mathfrak{R}_\lambda^{tr}$ ,  $\mathfrak{R}_\lambda^{fc}$  are defined naturally: for  $\mathfrak{R}_\lambda^{tr}$  embeddings preserve  $x < y$ ,  $\neg x < y$ ,  $\text{lev}_T(x) = \alpha$ ; for  $\mathfrak{R}_\lambda^{fc}$  embeddings are defined just as for models.

If  $\delta^1 = \delta^2 = \delta$  and  $[i < \delta \Rightarrow \lambda_i^1 \leq \lambda_i^2]$  and  $M^\ell \in \mathfrak{R}_{\lambda^\ell}^{fc}$ , (or  $T^\ell \in \mathfrak{R}_{\lambda^\ell}^{tr}$ ) for  $\ell = 1, 2$ , then an embedding of  $M^1$  into  $M^2$  ( $T^1$  into  $T^2$ ) is defined naturally.

**Lemma 2.4.** *Assume  $\bar{\lambda} = \langle \lambda_i : i \leq \delta \rangle$  and  $\theta, \chi$  satisfy (for some  $\bar{C}$ ):*

(a):  $\lambda_\delta, \theta$  are regular,  $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ ,  $\emptyset \neq S \subseteq \lambda =: \lambda_\delta$ ,  $C_\alpha \subseteq \alpha$ , for every club  $E$  of  $\lambda$  for some  $\alpha$  we have  $C_\alpha \subseteq E$ ,  $\lambda_\delta < \chi \leq |C_\alpha|^\theta$  and  $\text{otp}(C_\alpha) \geq \theta$ ,

(b):  $\lambda_i \leq \lambda_\delta$ ,

(c): there are  $\theta$  pairwise disjoint sets  $A \subseteq \delta$  such that  $\prod_{i \in A} \lambda_i \geq \lambda_\delta$ .

Then

( $\alpha$ ): there is no universal member in  $\mathfrak{R}_\lambda^{fc}$ ; moreover

( $\beta$ ): if  $M_\alpha^* \in \mathfrak{R}_\lambda^{fc}$  or even  $M_\alpha^* \in \mathfrak{R}_{\lambda_\delta}^{fc}$  for  $\alpha < \alpha^* < \chi$  then some  $M \in \mathfrak{R}_\lambda^{fc}$  cannot be embedded into any  $M_\alpha^*$ .

*Remark 2.5.* Note that clause ( $\beta$ ) is relevant to our discussion in §1: the non-universality is preserved even if we increase the density and, also, it is witnessed even by non-embeddability in many models.

*Proof.* Let  $\langle A_\varepsilon : \varepsilon < \theta \rangle$  be as in clause (c) and let  $\eta_\alpha^\varepsilon \in \prod_{i \in A_\varepsilon} \lambda_i$  for  $\alpha < \lambda_\delta$  be

pairwise distinct. We fix  $M_\alpha^* \in \mathfrak{R}_{\lambda_\delta}^{fc}$  for  $\alpha < \alpha^* < \chi$ .

For  $M \in \mathfrak{R}_\lambda^{fc}$ , let  $\bar{M} = (|M|, P_i^M, F_i^M)_{i \leq \delta}$  and let  $\langle M_\alpha : \alpha < \lambda_\delta \rangle$  be a representation (=filtration) of  $M$ ; for  $\alpha \in S$ ,  $x \in P_\delta^M$ , let

$$\text{inv}(x, C_\alpha; \bar{M}) = \{ \beta \in C_\alpha : \text{for some } \varepsilon < \theta \text{ and } y \in M_{\min(C_\alpha \setminus (\beta+1))} \}$$

$$\text{we have } \bigwedge_{i \in A_\varepsilon} F_i^M(x) = F_i^M(y)$$

but there is no such  $y \in M_\beta$ .

$$\text{Inv}(C_\alpha, \bar{M}) = \{ \text{inv}(x, C_\alpha, \bar{M}) : x \in P_\delta^M \}.$$

$$\text{INv}(\bar{M}, \bar{C}) = \langle \text{Inv}(C_\alpha, \bar{M}) : \alpha \in S \rangle.$$

$$\text{INV}(\bar{M}, \bar{C}) = \text{INv}(\bar{M}, \bar{C}) / \text{id}^a(\bar{C}).$$

Recall that

$$\text{id}^a(\bar{C}) = \{ T \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ for no } \alpha \in T \text{ is } C_\alpha \subseteq E \}.$$

The rest should be clear (for more details see proofs in §3), noticing

**Fact 2.6.** (1)  $\text{INV}(\bar{M}, \bar{C})$  is well defined, i.e. if  $\bar{M}^1, \bar{M}^2$  are representations (=filtrations) of  $M$  then  $\text{INV}(\bar{M}^1, \bar{C}) = \text{INV}(\bar{M}^2, \bar{C})$ .

(2)  $\text{Inv}(C_\alpha, \bar{M})$  has cardinality  $\leq \lambda$ .

(3)  $\text{inv}(x, C_\alpha; \bar{M})$  is a subset of  $C_\alpha$  of cardinality  $\leq \theta$ .

□<sub>2.4</sub>

*Conclusion 2.7.* If  $\mu = \sum_{n < \omega} \lambda_n$  and  $\lambda_n^{\aleph_0} < \lambda_{n+1}$  and  $\mu^+ < \lambda_\omega = \text{cf}(\lambda_\omega) < \mu^{\aleph_0}$ , then in  $\mathfrak{K}_{(\lambda_\alpha: \alpha \leq \omega)}^{fc}$  there is no universal member and even in  $\mathfrak{K}_{(\lambda_\omega: \alpha \leq \omega)}^{fc}$  we cannot find a member universal for it.

*Proof.* Should be clear or see the proof in §3. □<sub>2.7</sub>

**Claim 2.8.** *Suppose  $T$  is a first order complete theory such that*

- ⊠: *there are formulas  $\varphi_i(\bar{x}, \bar{y}_i)$  for  $i < \delta$ , such that*
  - (a)  $(\forall \bar{y}', \bar{y}'')(\bar{y}' \neq \bar{y}'' \rightarrow \neg(\exists \bar{x})(\varphi_i(\bar{x}, \bar{y}') \wedge \varphi_i(\bar{x}, \bar{y}'')))$  and
  - (b)  $\Gamma = \{\varphi_i(x_\eta, \bar{y}_{i,\alpha})^{if(\eta(i)=\alpha)} : \eta \in {}^\delta \omega, i < \delta, \alpha < \omega\}$  *is consistent with  $T$ .*

*If  $T$  has a universal model in  $\lambda$  and  $\lambda_i = \lambda$  for  $i < \delta$  then  $\mathfrak{K}_\lambda^k$  has a universal member.*

*Proof.* Should be clear. □

### 3. REDUCED TORSION FREE GROUPS: NON-EXISTENCE OF UNIVERSALS

We try to choose torsion free reduced groups and define invariants so that in an extension to another such group  $H$  something survives. To this end it is natural to stretch “reduced” near to its limit.

- Definition 3.1.**
- (1)  $\mathfrak{K}^{tf}$  is the class of torsion free (abelian) groups.
  - (2)  $\mathfrak{K}^{rtf} = \{G \in \mathfrak{K}^{tf} : \mathbb{Q} \text{ is not embeddable into } G \text{ (i.e. } G \text{ is reduced)}\}$ .
  - (3)  $\mathbf{P}^*$  denotes the set of primes.
  - (4) For  $x \in G$ ,  $\mathbf{P}(x, G) =: \{p \in \mathbf{P}^* : \bigwedge_n x \in p^n G\}$ .
  - (5)  $\mathfrak{K}_\lambda^x = \{G \in \mathfrak{K}^x : \|G\| = \lambda\}$ .
  - (6) If  $H \in \mathfrak{K}_\lambda^{rtf}$ , we say  $\bar{H}$  is a representation or filtration of  $H$  if  $\bar{H} = \langle H_\alpha : \alpha < \lambda \rangle$  is increasing continuous and  $H = \bigcup_{\alpha < \lambda} H_\alpha$ ,  $H \in \mathfrak{K}^{rtf}$  and each  $H_\alpha$  has cardinality  $< \lambda$ .

**Proposition 3.2.** (1) *If  $G \in \mathfrak{K}^{rtf}$ ,  $x \in G \setminus \{0\}$ ,  $Q \cup \mathbf{P}(x, G) \subsetneq \mathbf{P}^*$ ,  $G^+$  is the group generated by  $G, y, y_{p,\ell}$  ( $\ell < \omega$ ,  $p \in Q$ ) freely, except for the equations of  $G$  and*

$$y_{p,0} = y, \quad py_{p,\ell+1} = y_{p,\ell} \quad \text{and} \quad y_{p,\ell} = z \text{ when } z \in G, p^\ell z = x$$

*then  $G^+ \in \mathfrak{K}^{rtf}$ ,  $G \subseteq_{pr} G^+$  (pure extension).*

- (2) *If  $G_i \in \mathfrak{K}^{rtf}$  ( $i < \alpha$ ) is  $\subseteq_{pr}$ -increasing then  $G_i \subseteq_{pr} \bigcup_{j < \alpha} G_j \in \mathfrak{K}^{rtf}$  for every  $i < \alpha$ .*

The proof of the following lemma introduces a method quite central to this paper.

**Lemma 3.3.** *Assume that*

- (\*)<sub>λ</sub><sup>1</sup>:  $2^{\aleph_0} + \mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$ ,
- (\*)<sub>λ</sub><sup>2</sup>: *for every  $\chi < \lambda$ , there is  $S \subseteq [\chi]^{\leq \aleph_0}$ , such that:*

- (i):  $|S| < \lambda$ ,
- (ii): if  $D$  is a non-principal ultrafilter on  $\omega$  and  $f : D \rightarrow \chi$  then for some  $a \in S$  we have

$$\bigcap \{X \in D : f(X) \in a\} \notin D.$$

Then

- ( $\alpha$ ): in  $\mathfrak{K}_\lambda^{rtf}$  there is no universal member (under usual embeddings (i.e. not necessarily pure)),
- ( $\beta$ ): moreover, for any  $G_i \in \mathfrak{K}_\lambda^{rtf}$ , for  $i < i^* < \mu^{\aleph_0}$  there is  $G \in \mathfrak{K}_\lambda^{rtf}$  not embeddable into any one of  $G_i$ .

Before we prove 3.3 we consider the assumptions of 3.3 in 3.4, 3.5.

- Claim 3.4.** (1) In 3.3 we can replace  $(*)_\lambda^1$  by
- $(**)_\lambda^1$  (i):  $2^{\aleph_0} < \mu < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$ ,
  - (ii): there is  $\bar{C} = \langle C_\delta : \delta \in S^* \rangle$  such that  $S^*$  is a stationary subset of  $\lambda$ , each  $C_\delta$  is a subset of  $\delta$  with  $\text{otp}(C_\delta)$  divisible by  $\mu$ ,  $C_\delta$  closed in  $\text{sup}(C_\delta)$  (which is normally  $\delta$ , but not necessarily so) and

$$(\forall \alpha)[\alpha \in \text{nacc}(C_\delta) \Rightarrow \text{cf}(\alpha) > 2^{\aleph_0}]$$

(where  $\text{nacc}$  stands for “non-accumulation points”), and such that  $\bar{C}$  guesses clubs of  $\lambda$  (i.e. for every club  $E$  of  $\lambda$ , for some  $\delta \in S^*$  we have  $C_\delta \subseteq E$ ) and  $[\delta \in S^* \Rightarrow \text{cf}(\delta) = \aleph_0]$ .

- (2) In  $(*)_\lambda^1$  and in  $(*)_\lambda^2$ , without loss of generality  $(\forall \theta < \mu)[\theta^{\aleph_0} < \mu]$  and  $\text{cf}(\mu) = \aleph_0$ .

*Proof.* 1) This is what we actually use in the proof (see below).

2) Replace  $\mu$  by  $\mu' = \min\{\mu_1 : \mu_1^{\aleph_0} \geq \mu \text{ (equivalently } \mu_1^{\aleph_0} = \mu^{\aleph_0})\}$ .  $\square_{3.4}$

Compare to, say, [KS92b], [KS95]; the new assumption is  $(*)_\lambda^2$ , note that it is a very weak assumption, in fact it might be that it is always true.

**Claim 3.5.** Assume that  $2^{\aleph_0} < \mu < \lambda < \mu^{\aleph_0}$  and  $(\forall \theta < \mu)[\theta^{\aleph_0} < \mu]$  (see 3.4(2)). Then each of the following is a sufficient condition to  $(*)_\lambda^2$ :

- ( $\alpha$ ):  $\lambda < \mu^{+\omega_1}$ ,
- ( $\beta$ ): if  $\mathfrak{a} \subseteq \text{Reg} \cap \lambda \setminus \mu$  and  $|\mathfrak{a}| \leq 2^{\aleph_0}$  then we can find  $h : \mathfrak{a} \rightarrow \omega$  such that:

$$\lambda > \sup\{\max \text{pcf}(\mathfrak{b}) : \mathfrak{b} \subseteq \mathfrak{a} \text{ countable, and } h \upharpoonright \mathfrak{b} \text{ constant}\}.$$

*Proof.* Clause ( $\alpha$ ) implies Clause ( $\beta$ ): just use any one-to-one function  $h : \text{Reg} \cap \lambda \setminus \mu \rightarrow \omega$ .

Clause ( $\beta$ ) implies (by [She93b, §6] + [She96a, §2]) that for  $\chi < \lambda$  there is  $S \subseteq [\chi]^{\aleph_0}$ ,  $|S| < \lambda$  such that for every  $Y \subseteq \chi$ ,  $|Y| = 2^{\aleph_0}$ , we can find  $Y_n$  such that  $Y = \bigcup_{n < \omega} Y_n$  and  $[Y_n]^{\aleph_0} \subseteq S$ . (Remember:  $\mu > 2^{\aleph_0}$ .) Without loss

of generality (as  $2^{\aleph_0} < \mu < \lambda$ ):

- ( $*$ ):  $S$  is downward closed.

So if  $D$  is a non-principal ultrafilter on  $\omega$  and  $f : D \rightarrow \chi$  then letting  $Y = \text{Rang}(f)$  we can find  $\langle Y_n : n < \omega \rangle$  as above. Let  $h : D \rightarrow \omega$  be defined by  $h(A) = \min\{n : f(A) \in Y_n\}$ . So

$$X \subseteq D \ \& \ |X| \leq \aleph_0 \ \& \ h \upharpoonright X \text{ constant} \Rightarrow f''(X) \in S \quad (\text{remember } (*)).$$

Now for each  $n$ , for some countable  $X_n \subseteq D$  (possibly finite or even empty) we have:

$$h \upharpoonright X_n \text{ is constantly } n,$$

$$\ell < \omega \ \& \ (\exists A \in D)(h(A) = n \ \& \ \ell \notin A) \Rightarrow (\exists B \in X_n)(\ell \notin B).$$

Let  $A_n =: \bigcap \{A : A \in X_n\} = \bigcap \{A : A \in D \text{ and } h(X) = n\}$ . If the desired conclusion fails, then  $\bigwedge_{n < \omega} A_n \in D$ . So

$$(\forall A)[A \in D \Leftrightarrow \bigvee_{n < \omega} A \supseteq A_n].$$

So  $D$  is generated by  $\{A_n : n < \omega\}$  but then  $D$  cannot be a non-principal ultrafilter.  $\square_{3.5}$

*Proof. of Lemma 3.3*

Let  $\bar{C} = \langle C_\delta : \delta \in S^* \rangle$  be as in  $(**)_\lambda^1$  (ii) from 3.4 (for 3.4(1) its existence is obvious, for 3.3 - use [Shear, VI,old III 7.8]). Let us suppose that  $\bar{A} = \langle A_\delta : \delta \in S^* \rangle$ ,  $A_\delta \subseteq \text{nacc}(C_\delta)$  has order type  $\omega$  ( $A_\delta$  like this will be chosen later) and let  $\eta_\delta$  enumerate  $A_\delta$  increasingly. Let  $G_0$  be generated by  $\{x_i : i < \lambda\} \cup \{x_{i,p,\ell} : i = (i-p) + p \text{ that is } (\exists j)(i = j+p), p \in \mathbf{P}^* \text{ and } i < \omega\}$  freely except

$$x_{i,p,0} = x_i, px_{i,p,\ell+1} = x_{i,p,\ell}.$$

Let  $R$  be

$\{\bar{a} : \bar{a} = \langle a_n : n < \omega \rangle$  is a sequence of pairwise disjoint subsets of  $\mathbf{P}^*$ ,  
with  $a_n \cap \{0, \dots, n-1\} = \emptyset$  such that  
for infinitely many  $n$ ,  $a_n \neq \emptyset\}$ .

Let  $G$  be a group generated by

$$G_0 \cup \{y_{\bar{a}}^{\alpha,n}, z_{\bar{a},p}^{\alpha,n} : \alpha < \lambda, \bar{a} \in R, n < \omega, p \text{ prime}\}$$

freely except for:

- (a): the equations of  $G_0$ ,
- (b):  $pz_{\bar{a},p}^{\alpha,n+1} = z_{\bar{a},p}^{\alpha,n}$  when  $p \in a_n$ ,  $\alpha < \lambda$ ,
- (c):  $z_{\bar{a},p}^{\delta,0} = y_{\bar{a}}^{\delta,n} - x_{\eta_\delta(n)+n}$  when  $p \in a_n$  and  $\delta \in S^*$ .

Now  $G \in \mathfrak{K}_\lambda^{rtf}$  by inspection.

For  $\alpha < \lambda$  let  $G_\alpha$  be the subgroup of  $G$  generated by  $\{x_i, x_{i,p,\ell} : i < \alpha, \ell < \omega \text{ and } i = (i-p) + p\} \cup \{y_{\bar{a}}^{\delta,n} : \delta \in S^* \cap \alpha, \bar{a} \in R, n < \omega\} \cup \{z_{\bar{a},p}^{\delta,n} : \delta \in S^* \cap \alpha, \bar{a} \in R, n < \omega, p \in a_n\}$   $\square$

Before continuing the proof of 3.3 we present a definition and some facts.

**Definition 3.6.** For a representation  $\bar{H}$  of  $H \in \mathfrak{R}_\lambda^{rtf}$ , and  $x \in H$ ,  $\delta \in S^*$  let

- (1)  $\text{inv}(x, C_\delta; \bar{H}) =: \{\alpha \in C_\delta : \text{for some } Q \subseteq \mathbf{P}^*, \text{ there is } y \in H_{\min[C_\delta \setminus (\alpha+1)]}$   
such that  $Q \subseteq \mathbf{P}(x - y, H)$  but for no  $y \in H_\alpha$  is  $Q \subseteq \mathbf{P}(x - y, H)\}$   
(so  $\text{inv}(x, C_\delta; \bar{H})$  is a subset of  $C_\delta$  of cardinality  $\leq 2^{\aleph_0}$ ).
- (2)  $\text{Inv}^0(C_\delta, \bar{H}) =: \{\text{inv}(x, C_\delta; \bar{H}) : x \in \bigcup_i H_i\}$ .
- (3)  $\text{Inv}^1(C_\delta, \bar{H}) =: \{a : a \subseteq C_\delta \text{ countable and for some } x \in H, a \subseteq \text{inv}(x, C_\delta; \bar{H})\}$ .
- (4)  $\text{INv}^\ell(\bar{H}, \bar{C}) =: \text{Inv}^\ell(H, \bar{H}, \bar{C}) =: \langle \text{Inv}^\ell(C_\delta; \bar{H}) : \delta \in S^* \rangle$  for  $\ell \in \{0, 1\}$ .
- (5)  $\text{INV}^\ell(H, \bar{C}) =: \text{INv}^\ell(H, \bar{H}, \bar{C})/\text{id}^a(\bar{C})$ , where  
 $\text{id}^a(\bar{C}) =: \{T \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ for no } \delta \in T \text{ is } C_\delta \subseteq E\}$ .
- (6) If  $\ell$  is omitted,  $\ell = 0$  is understood.

**Fact 3.7.** (1)  $\text{INV}^\ell(H, \bar{C})$  is well defined.

- (2) The  $\delta$ -th component of  $\text{INV}^\ell(\bar{H}, \bar{C})$  is a family of  $\leq \lambda$  subsets of  $C_\delta$  each of cardinality  $\leq 2^{\aleph_0}$  and if  $\ell = 1$  each member is countable and the family is closed under subsets.
- (3) If  $G_i^* \in \mathfrak{R}_\lambda^{rtf}$  for  $i < i^*$ ,  $i^* < \mu^{\aleph_0}$ ,  $\bar{G}^i = \langle \bar{G}_{i,\alpha} : \alpha < \lambda \rangle$  is a representation of  $G_i^*$ ,  
then we can find  $A_\delta \subseteq \text{nacc}(C_\delta)$  of order type  $\omega$  such that:  $i < i^*$ ,  
 $\delta \in S^* \Rightarrow$  for no  $a$  in the  $\delta$ -th component of  $\text{INV}^\ell(G_i^*, \bar{G}^i, \bar{C})$   
do we have  $|a \cap A_\delta| \geq \aleph_0$ .

*Proof.* Straightforward. (For (3) note  $\text{otp}(C_\delta) \geq \mu$ , so there are  $\mu^{\aleph_0} > \lambda$  pairwise almost disjoint subsets of  $C_\delta$  each of cardinality  $\aleph_0$  and every  $A \in \text{Inv}(C_\delta, \bar{G}^i)$  disqualifies at most  $2^{\aleph_0}$  of them.)  $\square_{3.7}$

**Fact 3.8.** Let  $G$  be as constructed above for  $\langle A_\delta : \delta \in S^* \rangle$ ,  $A_\delta \subseteq \text{nacc}(C_\delta)$ ,  $\text{otp}(A_\delta) = \omega$  (where  $\langle A_\delta : \delta \in S^* \rangle$  are chosen as in 3.7(3) for the sequence  $\langle G_i^* : i < i^* \rangle$  given for proving 3.3, see  $(\beta)$  there).

Assume  $G \subseteq H \in \mathfrak{R}_\lambda^{rtf}$  and  $\bar{H}$  is a filtration of  $H$ . Then

$$B =: \left\{ \delta : A_\delta \text{ has infinite intersection with some } a \in \text{Inv}(C_\delta, \bar{H}) \right\} = \lambda \pmod{\text{id}^a(\bar{C})}.$$

*Proof.* We assume otherwise and derive a contradiction. Let for  $\alpha < \lambda$ ,  $S_\alpha \subseteq [\alpha]^{\leq \aleph_0}$ ,  $|S_\alpha| < \lambda$  be as guaranteed by  $(*)_\lambda^2$ .

Let  $\chi > 2^\lambda$ ,  $\mathfrak{A}_\alpha \prec (\mathcal{H}(\chi), \in, <_\chi^*)$  for  $\alpha < \lambda$  increasing continuous,  $\|\mathfrak{A}_\alpha\| < \lambda$ ,  $\langle \mathfrak{A}_\beta : \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$ ,  $\mathfrak{A}_\alpha \cap \lambda$  an ordinal and:

$$\langle S_\alpha : \alpha < \lambda \rangle, G, H, \bar{C}, \langle A_\delta : \delta \in S^* \rangle, \bar{H}, \langle x_i, x_{i,p,\ell}, y_{\bar{a}}^{\delta,n}, z_{\bar{a},p}^{\delta,n} : i, \delta, \bar{a}, n, p \rangle$$

all belong to  $\mathfrak{A}_0$  and  $2^{\aleph_0} + 1 \subseteq \mathfrak{A}_0$ . Then  $E = \{\delta < \lambda : \mathfrak{A}_\delta \cap \lambda = \delta\}$  is a club of  $\lambda$ , note that  $\delta \in E \Rightarrow H_\delta \cap G = G_\delta$ . Choose  $\delta \in S^* \cap E \setminus B$  such that  $C_\delta \subseteq E$ . (Why can we? As  $\text{id}^a(\bar{C})$  contains all non stationary subsets of  $\lambda$ , in particular  $\lambda \setminus E$ , and  $\lambda \setminus S^*$  and  $B$ , but  $\lambda \notin \text{id}^a(\bar{C})$ .) Remember

that  $\eta_\delta$  enumerates  $A_\delta$  (in the increasing order). For each  $\alpha = \eta_\delta(n) \in A_\delta$  (so  $\alpha \in E$  hence  $\mathfrak{A}_\alpha \cap \lambda = \alpha$  but  $\bar{H} \in \mathfrak{A}_\alpha$  hence  $H \cap \mathfrak{A}_\alpha = H_\alpha$ ) and  $Q \subseteq \mathbf{P}^*$  choose, if possible,  $y_{\alpha,Q} \in H_\alpha$  such that:

$$Q \subseteq \mathbf{P}(x_{\alpha+n} - y_{\alpha,Q}, H).$$

Let  $I_\alpha =: \{Q \subseteq \mathbf{P}^* : y_{\alpha,Q} \text{ well defined}\}$ . Note (see 3.4 (\*\*) $^1_\lambda$  and remember  $\eta_\delta(n) \in A_\delta \subseteq \text{nacc}(C_\delta)$ ) that  $\text{cf}(\alpha) > 2^{\aleph_0}$  (by (ii) of 3.4 (\*\*) $^1_\lambda$ ) and hence for some  $\beta_\alpha < \alpha$ ,

$$\{y_{\alpha,Q} : Q \in I_\alpha\} \subseteq H_{\beta_\alpha}.$$

Now:

$\otimes_1$ :  $I_\alpha$  is a downward closed family of subsets of  $\mathbf{P}^*$ ,  $\mathbf{P}^* \notin I_\alpha$  for  $\alpha \in A_\delta$ .

[Why? See the definition for the first phrase and note also that  $H$  is reduced for the second phrase.]

$\otimes_2$ :  $I_\alpha$  is closed under unions of two members (hence is an ideal on  $\mathbf{P}^*$ ).

[Why? If  $Q_1, Q_2 \in I_\alpha$  then (as  $x_{\alpha+n} \in G \subseteq H$  witnesses this):

$$(\mathcal{H}(\chi), \in, <^*_\chi) \models (\exists x)(x \in H \ \& \ Q_1 \subseteq \mathbf{P}(x - y_{\alpha,Q_1}, H) \ \& \ Q_2 \subseteq \mathbf{P}(x - y_{\alpha,Q_2}, H)).$$

All the parameters are in  $\mathfrak{A}_\alpha$  so there is  $y \in \mathfrak{A}_\alpha \cap H$  such that

$$Q_1 \subseteq \mathbf{P}(y - y_{\alpha,Q_1}, H) \quad \text{and} \quad Q_2 \subseteq \mathbf{P}(y - y_{\alpha,Q_2}, H).$$

By algebraic manipulations,

$$Q_1 \subseteq \mathbf{P}(x_{\alpha+n} - y_{\alpha,Q_1}, H), \quad Q_1 \subseteq \mathbf{P}(y - y_{\alpha,Q_1}, H) \quad \Rightarrow \quad Q_1 \subseteq \mathbf{P}(x_{\alpha+n} - y, H);$$

similarly for  $Q_2$ . So  $Q_1 \cup Q_2 \subseteq \mathbf{P}(x_{\alpha+n} - y, H)$  and hence  $Q_1 \cup Q_2 \in I_\alpha$ .]

$\otimes_3$ : If  $\bar{Q} = \langle Q_n : n \in \Gamma \rangle$  are pairwise disjoint subsets of  $\mathbf{P}^*$  and  $Q_n$  disjoint to  $\{0, \dots, n\}$ , for some infinite  $\Gamma \subseteq \omega$ , then for some  $n \in \Gamma$  we have  $Q_n \in I_{\eta_\delta(n)}$ .

[Why? Otherwise let  $a_n$  be  $Q_n$  if  $n \in \Gamma$ , and  $\emptyset$  if  $n \in \omega \setminus \Gamma$ , and let  $\bar{a} = \langle a_n : n < \omega \rangle$ . Now  $n \in \Gamma \Rightarrow \eta_\delta(n) \in \text{inv}(y_{\bar{a}}^{\delta,0}, C_\delta; \bar{H})$  and hence

$$A_\delta \cap \text{inv}(y_{\bar{a}}^{\delta,0}, C_\delta; \bar{H}) \supseteq \{\eta_\delta(n) : n \in \Gamma\},$$

which is infinite, contradicting the choice of  $A_\delta$ .]

$\otimes_4$ : for all but finitely many  $n$  the Boolean algebra  $\mathcal{P}(\mathbf{P}^*)/I_{\eta_\delta(n)}$  is finite.

[Why? If not, then by  $\otimes_1$  second phrase, for each  $n$  there are infinitely many non-principal ultrafilters  $D$  on  $\mathbf{P}^*$  disjoint to  $I_{\eta_\delta(n)}$ , so for  $n < \omega$  we can find an ultrafilter  $D_n$  on  $\mathbf{P}^*$  disjoint to  $I_{\eta_\delta(n)}$ , distinct from  $D_m$  for  $m < n$ . Thus we can find  $\Gamma \in [\omega]^{\aleph_0}$  and  $Q_n \in D_n$  for  $n \in \Gamma$  such that  $\langle Q_n : n \in \Gamma \rangle$

are pairwise disjoint (as  $Q_n \in D_n$  clearly  $|Q_n| = \aleph_0$ ) and without loss of generality  $Q_n \cap \{0, \dots, n\} = \emptyset$ . Why? Look: if  $B_n \in D_0 \setminus D_1$  for  $n \in \omega$  then

$$(\exists^\infty n)(B_n \in D_n) \quad \text{or} \quad (\exists^\infty n)(\mathbf{P}^* \setminus B_n \in D_n),$$

etc. Let  $Q_n = \emptyset$  for  $n \in \omega \setminus \Gamma$ , now  $\bar{Q} = \langle Q_n : n < \omega \rangle$  contradicts  $\otimes_3$ .]

$\otimes_5$ : If the conclusion (of 3.8) fails, then for no  $\alpha \in A_\delta$  do we have  $I_\alpha \cap [\omega]^{\aleph_0} = \emptyset$  &  $\mathcal{P}(\mathbf{P}^*)/I_\alpha$  finite.

[Why? If not, choose such an  $\alpha$  and  $Q^* \subseteq \mathbf{P}^*$ ,  $Q^* \notin I_\alpha$  such that  $I = I_\alpha \upharpoonright Q^*$  is a maximal ideal on  $Q^*$ . So  $D =: \mathcal{P}(Q^*) \setminus I$  is a non-principal ultrafilter. Remember  $\beta = \beta_\alpha < \alpha$  is such that  $\{y_{\alpha, Q} : Q \in I_\alpha\} \subseteq H_\beta$ . Now,  $H_\beta \in \mathfrak{A}_{\beta+1}$ ,  $|H_\beta| < \lambda$ . Hence  $(*)_\lambda^2$  from 3.3 (note that it does not matter whether we consider an ordinal  $\chi < \lambda$  or a cardinal  $\chi < \lambda$ , or any other set of cardinality  $< \lambda$ ) implies that there is  $S_{H_\beta} \in \mathfrak{A}_{\beta+1}$ ,  $S_{H_\beta} \subseteq [H_\beta]^{\leq \aleph_0}$ ,  $|S_{H_\beta}| < \lambda$  as there. Now it does not matter if we deal with functions from an ultrafilter on  $\omega$  or an ultrafilter on  $Q^*$ . We define  $f : D \rightarrow H_\beta$  as follows: for  $U \in D$  we let  $f(U) = y_{\alpha, Q^* \setminus U}$ . (Note:  $Q^* \setminus U \in I_\alpha$ , hence  $y_{\alpha, Q^* \setminus U}$  is well defined.) So, by the choice of  $S_{H_\beta}$  (see (ii) of  $(*)_\lambda^2$ ), for some countable  $f' \subseteq f$ ,  $f' \in \mathfrak{A}_{\beta+1}$  and  $\bigcap \{U : U \in \text{Dom}(f')\} \notin D$  (reflect for a minute). Let  $\text{Dom}(f') = \{U_0, U_1, \dots\}$ . Then  $\bigcup_{n < \omega} (Q^* \setminus U_n) \notin I_\alpha$ . But as in the proof of

$\otimes_2$ , as

$$\langle y_\alpha, (Q^* \setminus U_n) : n < \omega \rangle \in \mathfrak{A}_{\beta+1} \subseteq \mathfrak{A}_\alpha,$$

we have  $\bigcup_{n < \omega} (Q^* \setminus U_n) \in I_\alpha$ , an easy contradiction.]

Now recalling  $\{0, \dots, n\} \in I_{\eta_\delta(n)}$  by the choice of  $G_0$  and  $I_{\eta_\delta(n)}$  clearly  $\otimes_4, \otimes_5$  give a contradiction.  $\square_{3.3}$

*Remark 3.9.* We can deal similarly with  $R$ -modules,  $|R| < \mu$  if  $R$  has infinitely many prime ideals  $I$ . Also the treatment of  $\mathfrak{K}_\lambda^{rs(p)}$  is similar to the one for modules over rings with one prime.

Note: if we replace “reduced” by

$$x \in G \setminus \{0\} \quad \Rightarrow \quad (\exists p \in \mathbf{P}^*)(x \notin pG)$$

then here we could have defined

$$\mathbf{P}(x, H) =: \{p \in \mathbf{P}^* : x \in pH\}$$

and the proof would go through with no difference (e.g. choose a fixed partition  $\langle \mathbf{P}_n^* : n < \omega \rangle$  of  $\mathbf{P}^*$  to infinite sets, and let  $\mathbf{P}'(x, H) = \{n : x \in pH \text{ for every } p \in \mathbf{P}_n^*\}$ ). Now the groups are less divisible.

*Remark 3.10.* We can get that the groups are slender, in fact, the construction gives it.

#### 4. BELOW THE CONTINUUM THERE MAY BE UNIVERSAL STRUCTURES

Both in [She96c] (where we deal with universality for  $(< \lambda)$ -stable (Abelian) groups, like  $\mathfrak{K}_\lambda^{rs(p)}$ ) and in §3, we restrict ourselves to  $\lambda > 2^{\aleph_0}$ , a restriction

which does not appear in [KS92b], [KS95]. Is this restriction necessary? In this section we shall show that at least to some extent, it is.

We first show under MA that for  $\lambda < 2^{\aleph_0}$ , any  $G \in \mathfrak{K}_\lambda^{rs(p)}$  can be embedded into a “nice” one; our aim is to reduce the consistency of “there is a universal in  $\mathfrak{K}_\lambda^{rs(p)}$ ” to “there is a universal in  $\mathfrak{K}_{\langle \aleph_0: n < \omega \rangle}^{tr \wedge \langle \lambda \rangle}$ ”. Then we proceed to prove the consistency of the latter. Actually a weak form of MA suffices.

**Definition 4.1.** (1)  $G \in \mathfrak{K}_\lambda^{rs(p)}$  is *tree-like* if:

(a): we can find a basic subgroup  $B = \bigoplus_{\substack{i < \lambda_n \\ n < \omega}} \mathbb{Z}x_i^n$ , where

$$\lambda_n = \lambda_n(G) =: \dim((p^n G)[p]/p^{n+1}(G)[p])$$

(see Fuchs [Fuc73]) such that:  $\mathbb{Z}x_i^n \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$  and

$\otimes_0$ : every  $x \in G$  has the form

$$\sum_{n,i} \{a_i^n p^{n-k} x_i^n : n \in [k, \omega) \text{ and } i < \lambda\}$$

where  $a_i^n \in \mathbb{Z}$  and

$$n < \omega \Rightarrow w_n[x] =: \{i : a_i^n p^{n-k} x_i^n \neq 0\} \text{ is finite}$$

(this applies to any  $G \in \mathfrak{K}_\lambda^{rs(p)}$  we considered so far; we write  $w_n[x] = w_n[x, \bar{Y}]$  when  $\bar{Y} = \langle x_i^n : n, i \rangle$ ). Moreover

(b):  $\bar{Y} = \langle x_i^n : n, i \rangle$  is tree-like inside  $G$ , which means that we can find  $F_n : \lambda_{n+1} \rightarrow \lambda_n$  such that letting  $\bar{F} = \langle F_n : n < \omega \rangle$ ,  $G$  is generated by some subset of  $\Gamma(G, \bar{Y}, \bar{F})$  where:

$$\Gamma(G, \bar{Y}, \bar{F}) = \{x : \text{for some } \eta \in \prod_{n < \omega} \lambda_n, \text{ for each } n < \omega \text{ we have}$$

$$F_n(\eta(n+1)) = \eta(n) \text{ and } x = \sum_{n \geq k} p^{n-k} x_{\eta(n)}^n\}.$$

(2)  $G \in \mathfrak{K}_\lambda^{rs(p)}$  is *semi-tree-like* if above we replace (b) by

(b)': we can find a set  $\Gamma \subseteq \{\eta : \eta \text{ is a partial function from } \omega \text{ to } \sup_{n < \omega} \lambda_n \text{ with } \eta(n) < \lambda_n\}$  such that:

$$(\alpha): \eta_1 \in \Gamma, \eta_2 \in \Gamma, \eta_1(n) = \eta_2(n) \Rightarrow \eta_1 \upharpoonright n = \eta_2 \upharpoonright n,$$

(\beta): for  $\eta \in \Gamma$  and  $n \in \text{Dom}(\eta)$ , there is

$$y_{\eta,n} = \sum \{p^{m-n} x_{\eta(m)}^m : m \in \text{Dom}(\eta) \text{ and } m \geq n\} \in G,$$

(\gamma):  $G$  is generated by

$$\{x_i^n : n < \omega, i < \lambda_n\} \cup \{y_{\eta,n} : \eta \in \Gamma, n \in \text{Dom}(\eta)\}.$$

(3)  $G \in \mathfrak{K}_\lambda^{rs(p)}$  is *almost tree-like* if in (b)' we add

(\delta): for some  $A \subseteq \omega$  for every  $\eta \in \Gamma$ ,  $\text{Dom}(\eta) = A$ .

**Proposition 4.2.** (1) *Suppose  $G \in \mathfrak{K}_\lambda^{rs(p)}$  is almost tree-like, as witnessed by  $A \subseteq \omega$ ,  $\lambda_n$  (for  $n < \omega$ ),  $x_i^n$  (for  $n \in A$ ,  $i < \lambda_n$ ), and if  $n_0 < n_2$  are successive members of  $A$ ,  $n_0 < n < n_2$  then  $\lambda_n \geq \lambda_{n_0}$  or just*

$$\lambda_n \geq |\{\eta(n_0) : \eta \in \Gamma\}|.$$

Then  $G$  is tree-like (possibly with other witnesses).

(2) *If in 4.1(3) we just demand  $\eta \in \Gamma \Rightarrow \bigvee_{n < \omega} \text{Dom}(\eta) \setminus n = A \setminus n$ ; then changing the  $\eta$ 's and the  $y_{\eta,n}$ 's we can regain the “almost tree-like”.*

*Proof.* 1) For every successive members  $n_0 < n_2$  of  $A$  for

$$\alpha \in S_{n_0} =: \{\alpha : (\exists \eta)[\eta \in \Gamma \ \& \ \eta(n_0) = \alpha]\},$$

choose ordinals  $\gamma(n_0, \alpha, \ell)$  for  $\ell \in (n_0, n_2)$  such that

$$\gamma(n_0, \alpha_1, \ell) = \gamma(n_0, \alpha_2, \ell) \Rightarrow \alpha_1 = \alpha_2.$$

We change the basis by replacing for  $\alpha \in S_{n_0}$ ,  $\{x_\alpha^n\} \cup \{x_{\gamma(n_0, \alpha, \ell)}^\ell : \ell \in (n_0, n_2)\}$  (note:  $n_0 < n_2$  but possibly  $n_0 + 1 = n_2$ ), by:

$$\left\{ \begin{aligned} &x_\alpha^{n_0} + px_{\gamma(n_0, \alpha, n_0+1)}^{n_0+1}, x_{\gamma(n_0, \alpha, n_0+1)}^{n_0+1} + px_{\gamma(n_0, \alpha, n_0+2)}^{n_0+2}, \dots, \\ &x_{\gamma(n_0, \alpha, n_2-2)}^{n_2-2} + px_{\gamma(n_0, \alpha, n_2-1)}^{n_2-1}, x_{\gamma(n_0, \alpha, n_2-1)}^{n_2-1} \end{aligned} \right\}.$$

2) For  $\eta \in \Gamma$  let  $n(\eta) = \min\{n : n \in A \cap \text{Dom}(\eta) \text{ and } \text{Dom}(\eta) \setminus n = A \setminus n\}$ , and let  $\Gamma_n = \{\eta \in \Gamma : n(\eta) = n\}$  for  $n \in A$ . We choose by induction on  $n < \omega$  the objects  $\nu_\eta$  for  $\eta \in \Gamma_n$  and  $\rho_\alpha^n$  for  $\alpha < \lambda_n$  such that:  $\nu_\eta$  is a function with domain  $A$ ,  $\nu_\eta \upharpoonright (A \setminus n(\eta)) = \eta \upharpoonright (A \setminus n(\eta))$  and  $\nu_\eta \upharpoonright (A \cap n(\eta)) = \rho_{\eta(n)}^n$ ,  $\nu_\eta(n) < \lambda_n$  and  $\rho_\alpha^n$  is a function with domain  $A \cap n$ ,  $\rho_\alpha^n(\ell) < \lambda_\ell$  and  $\rho_\alpha^n \upharpoonright (A \cap \ell) = \rho_{\rho_\alpha^n(\ell)}^\ell$  for  $\ell \in A \cap n$ . There are no problems and  $\{\nu_\eta : \eta \in \Gamma_n\}$  is as required.  $\square_{4.2}$

**Theorem 4.3** (MA). *Let  $\lambda < 2^{\aleph_0}$ . Any  $G \in \mathfrak{K}_\lambda^{rs(p)}$  can be embedded into some  $G' \in \mathfrak{K}_\lambda^{rs(p)}$  with countable density which is tree-like.*

*Proof.* By 4.2 it suffices to get  $G'$  “almost tree-like” and  $A \subseteq \omega$  which satisfies 4.2(1). The ability to make  $A$  thin helps in proving Fact E below. By 1.1 without loss of generality  $G$  has a base (i.e. a dense subgroup of the form)  $B = \bigoplus_{\substack{n < \omega \\ i < \lambda_n}} \mathbb{Z}x_i^n$ , where  $\mathbb{Z}x_i^n \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$  and  $\lambda_n = \aleph_0$  (in fact  $\lambda_n$  can

be  $g(n)$  if  $g \in {}^\omega\omega$  is not bounded (by algebraic manipulations), this will be useful if we consider the forcing from [She92, §2]).

Let  $B^+$  be the extension of  $B$  by  $y_i^{n,k}$  ( $k < \omega$ ,  $n < \omega$ ,  $i < \lambda_n$ ) generated freely except for  $py_i^{n,k+1} = y_i^{n,k}$  (for  $k < \omega$ ),  $y_i^{n,\ell} = p^{n-\ell}x_i^n$  for  $\ell \leq n$ ,  $n < \omega$ ,  $i < \lambda_n$ . So  $B^+$  is a divisible  $p$ -group, let  $G^+ =: B^+ \bigoplus_B G$ . Let

$\{z_\alpha^0 : \alpha < \lambda\} \subseteq G[p]$  be a basis of  $G[p]$  over  $\{p^n x_i^n : n, i < \omega\}$  (as a vector space over  $\mathbb{Z}/p\mathbb{Z}$  i.e. the two sets are disjoint, their union is a basis); remember  $G[p] = \{x \in G : px = 0\}$ . So we can find  $z_\alpha^k \in G$  (for  $\alpha < \lambda$ ,  $k < \omega$  and  $k \neq 0$ ) such that

$$pz_\alpha^{k+1} - z_\alpha^k = \sum_{i \in w(\alpha, k)} a_i^{k, \alpha} x_i^k,$$

where  $w(\alpha, k) \subseteq \omega$  is finite (reflect on the Abelian group theory).

We define a forcing notion  $P$  as follows: a condition  $p \in P$  consists of (in brackets are explanations of intentions):

**(a):**  $m < \omega$ ,  $M \subseteq m$ ,

[ $M$  is intended as  $A \cap \{0, \dots, m-1\}$ ]

**(b):** a finite  $u \subseteq m \times \omega$  and  $h : u \rightarrow \omega$  such that  $h(n, i) \geq n$ ,

[our extensions will not be pure, but still we want that the group produced will be reduced, now we add some  $y_i^{n, k}$ 's and  $h$  tells us how many]

**(c):** a subgroup  $K$  of  $B^+$ :

$$K = \langle y_i^{n, k} : (n, i) \in u, k < h(n, i) \rangle_{B^+},$$

**(d):** a finite  $w \subseteq \lambda$ ,

[ $w$  is the set of  $\alpha < \lambda$  on which we give information]

**(e):**  $g : w \rightarrow m+1$ ,

[ $g(\alpha)$  is in what level  $m' \leq m$  we “start to think” about  $\alpha$ ]

**(f):**  $\bar{\eta} = \langle \eta_\alpha : \alpha \in w \rangle$  (see (i)),

[of course,  $\eta_\alpha$  is the intended  $\eta_\alpha$  restricted to  $m$  and the set of all  $\eta_\alpha$  forms the intended  $\Gamma$ ]

**(g):** a finite  $v \subseteq m \times \omega$ ,

[this approximates the set of indices of the new basis]

**(h):**  $\bar{t} = \{t_{n, i} : (n, i) \in v\}$  (see (j)),

[approximates the new basis]

**(i):**  $\eta_\alpha \in {}^M \omega$ ,  $\bigwedge_{\alpha \in w} \bigwedge_{n \in M} (n, \eta_\alpha(n)) \in v$ ,

[toward guaranteeing clause  $(\delta)$  of 4.1(3) (see 4.2(2))]

**(j):**  $t_{n, i} \in K$  and  $\mathbb{Z}t_{n, i} \cong \mathbb{Z}/p^n \mathbb{Z}$ ,

**(k):**  $K = \bigoplus_{(n, i) \in v} (\mathbb{Z}t_{n, i})$ ,

[so  $K$  is an approximation to the new basic subgroup]

**(l):** if  $\alpha \in w$ ,  $g(\alpha) \leq \ell \leq m$  and  $\ell \in M$  then

$$z_\alpha^\ell - \sum \{t_{n, \eta_\alpha(n)}^{n-\ell} : \ell \leq n \in \text{Dom}(\eta_\alpha)\} \in p^{m-\ell}(K + G),$$

[this is a step toward guaranteeing that the full difference (when  $\text{Dom}(\eta_\alpha)$  is possibly infinite) will be in the closure of  $\bigoplus_{\substack{n \in [i, \omega) \\ i < \omega}} \mathbb{Z}x_i^n$ ].

We define the order by:

$p \leq q$  if and only if

- ( $\alpha$ ):  $m^p \leq m^q$ ,  $M^q \cap m^p = M^p$ ,
- ( $\beta$ ):  $u^p \subseteq u^q$ ,  $h^p \subseteq h^q$ ,
- ( $\gamma$ ):  $K^p \subseteq_{pr} K^q$ ,
- ( $\delta$ ):  $w^p \subseteq w^q$ ,
- ( $\varepsilon$ ):  $g^p \subseteq g^q$ ,
- ( $\zeta$ ):  $\eta_\alpha^p \trianglelefteq \eta_\alpha^q$ , (i.e.  $\eta_\alpha^p$  is an initial segment of  $\eta_\alpha^q$ )
- ( $\eta$ ):  $v^p \subseteq v^q$ ,
- ( $\theta$ ):  $t_{n,i}^p = t_{n,i}^q$  for  $(n, i) \in v^p$ .

**A Fact**  $(P, \leq)$  is a partial order.

*Proof of the Fact:* Trivial.

**B Fact**  $P$  satisfies the c.c.c. (even is  $\sigma$ -centered).

*Proof of the Fact:* It suffices to observe the following.

Suppose that

- (\*) (**i**):  $p, q \in P$ ,
- (**ii**):  $M^p = M^q$ ,  $m^p = m^q$ ,  $h^p = h^q$ ,  $u^p = u^q$ ,  $K^p = K^q$ ,  $v^p = v^q$ ,  
 $t_{n,i}^p = t_{n,i}^q$ ,
- (**iii**):  $\langle \eta_\alpha^p : \alpha \in w^p \cap w^q \rangle = \langle \eta_\alpha^q : \alpha \in w^p \cap w^q \rangle$ ,
- (**iv**):  $g^p \upharpoonright (w^p \cap w^q) = g^q \upharpoonright (w^p \cap w^q)$ .

Then the conditions  $p, q$  are compatible (in fact have an upper bound with the same common parts): take the common values (in (ii)) or the union (for (iii)).

**C Fact** For each  $\alpha < \lambda$  the set  $\mathcal{I}_\alpha =: \{p \in P : \alpha \in w^p\}$  is dense (and open).

*Proof of the Fact:* For  $p \in P$  let  $q$  be like  $p$  except that:

$$w^q = w^p \cup \{\alpha\} \quad \text{and} \quad g^q(\beta) = \begin{cases} g^p(\beta) & \text{if } \beta \in w^p \\ m^p & \text{if } \beta = \alpha, \beta \notin w^p. \end{cases}$$

**D Fact** For  $n < \omega$ ,  $i < \omega$  the following set is a dense subset of  $P$ :

$$\mathcal{J}_{(n,i)}^* = \{p \in P : x_i^n \in K^p \text{ \& } (\forall n < m^p)(\{n\} \times \omega) \cap u^p \text{ has } > m^p \text{ elements}\}.$$

*Proof of the Fact:* Should be clear.

**E Fact** For each  $m < \omega$  the set  $\mathcal{J}_m =: \{p \in P : m^p \geq m\}$  is dense in  $P$ .

*Proof of the Fact:* Let  $p \in P$  be given such that  $m^p < m$ . Let  $w^p = \{\alpha_0, \dots, \alpha_{r-1}\}$  be without repetitions; we know that in  $G$ ,  $pz_{\alpha_\ell}^0 = 0$  and

$\{z_{\alpha_\ell}^0 : \ell < r\}$  is independent mod  $B$ , hence also in  $K + G$  the set  $\{z_{\alpha_\ell}^0 : \ell < r\}$  is independent mod  $K$ . Clearly

$$\text{(A): } pz_{\alpha_\ell}^{k+1} = z_{\alpha_\ell}^k \text{ mod } K \text{ for } k \in [g(\alpha_\ell), m^p], \text{ hence}$$

$$\text{(B): } p^{m^p} z_{\alpha_\ell}^{m^p} = z_{\alpha_\ell}^{g(\alpha_\ell)} \text{ mod } K.$$

Remember

$$\text{(C): } z_{\alpha_\ell}^{m^p} = \sum \{a_i^{k, \alpha_\ell} p^{k-m^p} x_i^k : k \geq m^p, i \in w(\alpha_\ell, k)\},$$

and so, in particular, (from the choice of  $z_{\alpha_\ell}^0$ )

$$p^{m^p+1} z_{\alpha_\ell}^{m^p} = 0 \quad \text{and} \quad p^{m^p} z_{\alpha_\ell}^{m^p} \neq 0.$$

For  $\ell < r$  and  $n \in [m^p, \omega)$  let

$$s_\ell^n =: \sum \{a_i^{k, \alpha_\ell} p^{k-m^p} x_i^k : k \geq m^p \text{ but } k < n \text{ and } i \in w(\alpha_\ell, k)\}.$$

But  $p^{k-m^p} x_i^k = y_i^{k, m^p}$ , so

$$s_\ell^n = \sum \{a_i^{k, \alpha_\ell} y_i^{k, m^p} : k \in [m^p, n) \text{ and } i \in w(\alpha_\ell, k)\}.$$

Hence, for some  $m^* > m, m^p$  we have:  $\{p^{m^*} s_\ell^{m^*} : \ell < r\}$  is independent in  $G[p]$  over  $K[p]$  and therefore in  $\langle x_i^k : k \in [m^p, m^*], i < \omega \rangle$ . Let

$$s_\ell^* = \sum \{a_i^{k, \alpha_\ell} : k \in [m^p, m^*) \text{ and } i \in w(\alpha_\ell, k)\}.$$

Then  $\{s_\ell^* : \ell < r\}$  is independent in

$$B_{[m, m^*)}^+ = \langle y_i^{l, m^*-1} : k \in [m^p, m^*) \text{ and } i < \omega \rangle.$$

Let  $i^* < \omega$  be such that:  $w(\alpha_\ell, k) \subseteq \{0, \dots, i^* - 1\}$  for  $k \in [m^p, m^*)$ ,  $\ell = 1, \dots, r$ . Let us start to define  $q$ :

$$m^q = m^*, \quad M^q = M^p \cup \{m^* - 1\}, \quad w^q = w^p, \quad g^q = g^p,$$

$$u^q = u^p \cup ([m^p, m^*) \times \{0, \dots, i^* - 1\}),$$

$$h^q \text{ is } h^p \text{ on } u^p \text{ and } h^q(k, i) = m^* - 1 \text{ otherwise,}$$

$$K^q \text{ is defined appropriately, let } K^q = \langle x_i^n : n \in [m^p, m^*), i < i^* \rangle.$$

Complete  $\{s_\ell^* : \ell < r\}$  to  $\{s_\ell^* : \ell < r^*\}$ , a basis of  $K^q[p]$ , and choose  $\{t_{n,i} : (n, i) \in v^*\}$  such that:  $[p^{m^*} t_{n,i} = 0 \Leftrightarrow m > n]$ , and for  $\ell < r$

$$p^{m^*-1-\ell} t_{m^*-1, \ell} = s_\ell^*.$$

The rest should be clear.

The generic gives a variant of the desired result: almost tree-like basis; the restriction to  $M$  and  $g$  but by 4.2 we can finish.  $\square_{4.1}$

*Conclusion 4.4* ( $\text{MA}_\lambda(\sigma\text{-centered})$ ). For  $(*)_0$  to hold it suffices that  $(*)_1$  holds where

$(*)_0$ : in  $\mathfrak{K}_\lambda^{rs(p)}$ , there is a universal member,

$(*)_1$ : in  $\mathfrak{K}_\lambda^{tr}$  there is a universal member, where:

**(a)**:  $\lambda_n = \aleph_0$ ,  $\lambda_\omega = \lambda$ ,  $\ell g(\bar{\lambda}) = \omega + 1$  or

**(b)**:  $\lambda_\omega = \lambda$ ,  $\lambda_n \in [n, \omega)$ ,  $\ell g(\bar{\lambda}) = \omega + 1$ .

*Remark 4.5.* Any  $\langle \lambda_n : n < \omega \rangle$ ,  $\lambda_n < \omega$  which is not bounded suffices.

*Proof.* For case (a) - by 4.3.

For case (b) - the same proof. □<sub>4.4</sub>

**Theorem 4.6.** *Assume  $\lambda < 2^{\aleph_0}$  and*

(a): *there are  $A_i \subseteq \lambda$ ,  $|A_i| = \lambda$  for  $i < 2^\lambda$  such that  $i \neq j \Rightarrow |A_i \cap A_j| \leq \aleph_0$ .*

Let  $\bar{\lambda} = \langle \lambda_\alpha : \alpha \leq \omega \rangle$ ,  $\lambda_n = \aleph_0$ ,  $\lambda_\omega = \lambda$ .

Then there is  $P$  such that:

- ( $\alpha$ ):  $P$  is a c.c.c. forcing notion,
- ( $\beta$ ):  $|P| = 2^\lambda$ ,
- ( $\gamma$ ): in  $V^P$ , there is  $T \in \mathfrak{R}_\lambda^{tr}$  into which every  $T' \in (\mathfrak{R}_\lambda^{tr})^V$  can be embedded.

*Proof.* Let  $\bar{T} = \langle T_i : i < 2^\lambda \rangle$  list the trees  $T$  of cardinality  $\leq \lambda$  satisfying

$$\omega^{>\omega} \subseteq T \subseteq \omega^{\geq\omega} \quad \text{and} \quad T \cap \omega^\omega \text{ has cardinality } \lambda, \text{ for simplicity.}$$

Let  $T_i \cap \omega^\omega = \{\eta_\alpha^i : \alpha \in A_i\}$ .

We shall force  $\rho_{\alpha,\ell} \in \omega^\omega$  for  $\alpha < \lambda$ ,  $\ell < \omega$ , and for each  $i < 2^\lambda$  a function  $g_i : A_i \rightarrow \omega$  such that: there is an automorphism  $f_i$  of  $(\omega^{>\omega}, \triangleleft)$  which induces an embedding of  $T_i$  into  $((\omega^{>\omega}) \cup \{\rho_{\alpha,g_i(\alpha)} : \alpha < \lambda\}, \triangleleft)$ . We shall define  $p \in P$  as an approximation.

A condition  $p \in P$  consists of:

- (a):  $m < \omega$  and a finite subset  $u$  of  $\omega^{\geq\omega}$ , closed under initial segments such that  $\langle \rangle \in u$ ,
- (b): a finite  $w \subseteq 2^\lambda$ ,
- (c): for each  $i \in w$ , a finite function  $g_i$  from  $A_i$  to  $\omega$ ,
- (d): for each  $i \in w$ , an automorphism  $f_i$  of  $(u, \triangleleft)$ ,
- (e): a finite  $v \subseteq \lambda \times \omega$ ,
- (f): for  $(\alpha, n) \in v$ ,  $\rho_{\alpha,n} \in u \cap (\omega^m)$ ,

such that

- (g): if  $i \in w$  and  $\alpha \in \text{Dom}(g_i)$  then:
  - ( $\alpha$ ):  $(\alpha, g_i(\alpha)) \in v$ ,
  - ( $\beta$ ):  $\eta_\alpha^i \upharpoonright m \in u$ ,
  - ( $\gamma$ ):  $f_i(\eta_\alpha^i \upharpoonright m) = \rho_{\alpha,g_i(\alpha)}$ ,
- (h):  $\langle \rho_{\alpha,n} : (\alpha, n) \in v \rangle$  is with no repetition (all of length  $m$ ),
- (i): for  $i \in w$ ,  $\langle \eta_\alpha^i \upharpoonright m : \alpha \in \text{Dom}(g_i) \rangle$  is with no repetition.

The order on  $P$  is:  $p \leq q$  if and only if:

- ( $\alpha$ ):  $u^p \subseteq u^q$ ,  $m^p \leq m^q$ ,
- ( $\beta$ ):  $w^p \subseteq w^q$ ,
- ( $\gamma$ ):  $f_i^p \subseteq f_i^q$  for  $i \in w^p$ ,
- ( $\delta$ ):  $g_i^p \subseteq g_i^q$  for  $i \in w^p$ ,
- ( $\varepsilon$ ):  $v^p \subseteq v^q$ ,
- ( $\zeta$ ):  $\rho_{\alpha,n}^p \leq \rho_{\alpha,n}^q$ , when  $(\alpha, n) \in v^p$ ,

( $\eta$ ): if  $i \neq j \in w^p$  then for every  $\alpha \in A_i \cap A_j \setminus (\text{Dom}(g_i^p) \cap \text{Dom}(g_j^p))$  we have  $g_i^q(\alpha) \neq g_j^q(\alpha)$ .

**A Fact**  $(P, \leq)$  is a partial order.

*Proof of the Fact:* Trivial.

**B Fact** For  $i < 2^\lambda$  the set  $\{p : i \in w^p\}$  is dense in  $P$ .

*Proof of the Fact:* If  $p \in P$ ,  $i \in 2^\lambda \setminus w^p$ , define  $q$  like  $p$  except  $w^q = w^p \cup \{i\}$ ,  $\text{Dom}(g_i^q) = \emptyset$ .

**C Fact** If  $p \in P$ ,  $m_1 \in (m^p, \omega)$ ,  $\eta^* \in u^p$ ,  $m^* < \omega$ ,  $i \in w^p$ ,  $\alpha \in \lambda \setminus \text{Dom}(g_i^p)$  then we can find  $q$  such that  $p \leq q \in P$ ,  $m^q > m_1$ ,  $\eta^* \hat{\ } \langle m^* \rangle \in u^q$  and  $\alpha \in \text{Dom}(g_i)$  and  $\langle \eta_\beta^j \upharpoonright m^q : j \in w^q \text{ and } \beta \in \text{Dom}(g_j^q) \rangle$  is with no repetition, more exactly  $\eta_{\beta_1}^{j(1)} \upharpoonright m^q = \eta_{\beta_2}^{j(2)} \upharpoonright m^q \Rightarrow \eta_{\beta_1}^{j(1)} = \eta_{\beta_2}^{j(2)}$ .

*Proof of the Fact:* Let  $n_0 \leq m^p$  be maximal such that  $\eta_\alpha^i \upharpoonright n_0 \in u^p$ . Let  $n_1 < \omega$  be minimal such that  $\eta_\alpha^i \upharpoonright n_1 \notin \{\eta_\beta^i \upharpoonright n_1 : \beta \in \text{Dom}(g_i^p)\}$  and moreover the sequence

$$\langle \eta_\beta^j \upharpoonright n_1 : j \in w^p \ \& \ \beta \in \text{Dom}(g_j^p) \ \text{ or } \ j = i \ \& \ \beta = \alpha \rangle$$

is with no repetition. Choose a natural number  $m^q > m^p + 1, n_0 + 1, n_1 + 2$  and let  $k^* =: 3 + \sum_{i \in w^p} |\text{Dom}(g_i^p)|$ . Choose  $u^q \subseteq m^q \geq \omega$  such that:

- (i):  $u^p \subseteq u^q \subseteq m^q \geq \omega$ ,  $u^q$  is downward closed,
- (ii): for every  $\eta \in u^q$  such that  $\ell g(\eta) < m^q$ , for exactly  $k^*$  numbers  $k$ ,  $\eta \hat{\ } \langle k \rangle \in u^q \setminus u^p$ ,
- (iii):  $\eta_\beta^j \upharpoonright \ell \in u^q$  when  $\ell \leq m^q$  and  $j \in w^p$ ,  $\beta \in \text{Dom}(g_j^p)$ ,
- (iv):  $\eta_\alpha^i \upharpoonright \ell \in u^q$  for  $\ell \leq m^q$ ,
- (v):  $\eta^* \hat{\ } \langle m^* \rangle \in u^q$ .

Next choose  $\rho_{\beta,n}^q$  (for pairs  $(\beta, n) \in v^p$ ) such that:

$$\rho_{\beta,n}^p \trianglelefteq \rho_{\beta,n}^q \in u^q \cap m^q \omega.$$

For each  $j \in w^p$  separately extend  $f_j^p$  to an automorphism  $f_j^q$  of  $(u^q, \triangleleft)$  such that for each  $\beta \in \text{Dom}(g_j^p)$  we have:

$$f_j^q(\eta_\beta^j \upharpoonright m^q) = \rho_{\beta, g_j^p}^q(\beta).$$

This is possible, as for each  $\nu \in u^p$ , and  $j \in w^p$ , we can separately define

$$f_j^q \upharpoonright \{\nu' : \nu \triangleleft \nu' \in u^q \text{ and } \nu' \upharpoonright (\ell g(\nu) + 1) \notin u^p\}$$

–its range is

$$\{\nu' : f_j^p(\nu) \triangleleft \nu' \in u^q \text{ and } \nu' \upharpoonright (\ell g(\nu) + 1) \notin u^p\}.$$

The point is: by Clause (ii) above those two sets are isomorphic and for each  $\nu$  at most one  $\rho_{\beta,n}^p$  is involved (see Clause (h) in the definition of  $p \in P$ ). Next let  $w^q = w^p$ ,  $g_j^q = g_j^p$  for  $j \in w \setminus \{i\}$ ,  $g_i^q \upharpoonright \text{Dom}(g_i^p) = g_i^p$ ,

$g_i^q(\alpha) = \min(\{n : (\alpha, n) \notin v^p\})$ ,  $\text{Dom}(g_i^q) = \text{Dom}(g_i^p) \cup \{\alpha\}$ , and  $\rho_{\alpha, g_i^q(\alpha)}^q = f_i^q(\eta_\alpha^i \upharpoonright m^q)$  and  $v^q = v^p \cup \{(\alpha, g_i^q(\alpha))\}$ .

**D Fact**  $P$  satisfies the c.c.c.

*Proof of the Fact:* Assume  $p_\varepsilon \in P$  for  $\varepsilon < \omega_1$ . By Fact C, without loss of generality each

$$\langle \eta_\beta^j \upharpoonright m^{p_\varepsilon} : j \in w^{p_\varepsilon} \text{ and } \beta \in \text{Dom}(g_j^{p_\varepsilon}) \rangle$$

is with no repetition. Without loss of generality, for all  $\varepsilon < \omega_1$

$$U_\varepsilon =: \{\alpha < 2^\lambda : \alpha \in w^{p_\varepsilon} \text{ or } \bigvee_{i \in w^p} [\alpha \in \text{Dom}(g_i)] \text{ or } \bigvee_k (k, \alpha) \in v^{p_\varepsilon}\}$$

has the same number of elements and for  $\varepsilon \neq \zeta < \omega_1$ , there is a unique one-to-one order preserving function from  $U_\varepsilon$  onto  $U_\zeta$  which we call  $\text{OP}_{\zeta, \varepsilon}$ , which also maps  $p_\varepsilon$  to  $p_\zeta$  (so  $m^{p_\zeta} = m^{p_\varepsilon}$ ;  $u^{p_\zeta} = u^{p_\varepsilon}$ ;  $\text{OP}_{\zeta, \varepsilon}(w^{p_\varepsilon}) = w^{p_\zeta}$ ; if  $i \in w^{p_\varepsilon}$ ,  $j = \text{OP}_{\zeta, \varepsilon}(i)$ , then  $f_i \circ \text{OP}_{\varepsilon, \zeta} \equiv f_j$ ; and if  $\beta = \text{OP}_{\zeta, \varepsilon}(\alpha)$  and  $\ell < \omega$  then

$$(\alpha, \ell) \in v^{p_\varepsilon} \Leftrightarrow (\beta, \ell) \in v^{p_\zeta} \Rightarrow \rho_{\alpha, \ell}^{p_\varepsilon} = \rho_{\beta, \ell}^{p_\zeta}.$$

Also this mapping is the identity on  $U_\zeta \cap U_\varepsilon$  and  $\langle U_\zeta : \zeta < \omega_1 \rangle$  is a  $\Delta$ -system.

Let  $w =: w^{p_0} \cap w^{p_1}$ . As  $i \neq j \Rightarrow |A_i \cap A_j| \leq \aleph_0$ , without loss of generality

(\*): if  $i \neq j \in w$  then

$$U_\varepsilon \cap (A_i \cap A_j) \subseteq w.$$

We now start to define  $q \geq p_0, p_1$ . Choose  $m^q$  such that  $m^q \in (m^{p_\varepsilon}, \omega)$  and

$$m^q > \max \{ \ell g(\eta_{\alpha_0}^{i_0} \cap \eta_{\alpha_1}^{i_1}) + 1 : i_0 \in w^{p_0}, i_1 \in w^{p_1}, \text{OP}_{1,0}(i_0) = i_1, \\ \alpha_0 \in \text{Dom}(g_{i_0}^{p_0}), \alpha_1 \in \text{Dom}(g_{i_1}^{p_1}), \\ \text{OP}_{1,0}(\alpha_0) = \alpha_1 \}.$$

Let  $u^q \subseteq m^q \geq \omega$  be such that:

(A):  $u^q \cap (m^{p_0} \geq \omega) = u^q \cap (m^{p_1} \geq \omega) = u^{p_0} = u^{p_1}$ ,

(B): for each  $\nu \in u^q$ ,  $m^{p_0} \leq \ell g(\nu) < m^q$ , for exactly two numbers  $k < \omega$ ,  $\nu \hat{\ } (k) \in u^q$ ,

(C):  $\eta_\alpha^i \upharpoonright \ell \in u^q$  for  $\ell \leq m^q$  when:  $i \in w^{p_0}$ ,  $\alpha \in \text{Dom}(g_i^{p_0})$  or  $i \in w^{p_1}$ ,  $\alpha \in \text{Dom}(g_i^{p_1})$ .

[Possible as  $\{\eta_\alpha^i \upharpoonright m^{p_\varepsilon} : i \in w^{p_\varepsilon}, \alpha \in \text{Dom}(g_i^{p_\varepsilon})\}$  is with no repetitions (the first line of the proof).]

Let  $w^q =: w^{p_0} \cup w^{p_1}$  and  $v^q =: v^{p_0} \cup v^{p_1}$  and for  $i \in w^q$

$$g_i^q = \begin{cases} g_i^{p_0} & \text{if } i \in w^{p_0} \setminus w^{p_1}, \\ g_i^{p_1} & \text{if } i \in w^{p_1} \setminus w^{p_0}, \\ g_i^{p_0} \cup g_i^{p_1} & \text{if } i \in w^{p_0} \cap w^{p_1}. \end{cases}$$

Next choose  $\rho_{\alpha, \ell}^q$  for  $(\alpha, \ell) \in v^q$  as follows. Let  $\nu_{\alpha, \ell}$  be  $\rho_{\alpha, \ell}^{p_0}$  if defined,  $\rho_{\alpha, \ell}^{p_1}$  if defined (no contradiction). If  $(\alpha, \ell) \in v^q$  choose  $\rho_{\alpha, \ell}^q$  as any  $\rho$  such that:

$$\otimes_0: \nu_{\alpha, \ell} \triangleleft \rho \in u^q \cap (m^q) \omega.$$

But not all choices are O.K., as we need to be able to define  $f_i^q$  for  $i \in w^q$ . A possible problem will arise only when  $i \in w^{p_0} \cap w^{p_1}$ . Specifically we need just (remember that  $\langle \rho_{\alpha, \ell}^{p_\varepsilon} : (\alpha, \ell) \in v^{p_\varepsilon} \rangle$  are pairwise distinct by clause (b) of the Definition of  $p \in P$ ):

$$\begin{aligned} \otimes_1: & \text{ if } i_0 \in w^{p_0}, (\alpha_0, \ell) = (\alpha_0, g_{i_0}(\alpha_0)), \alpha_0 \in \text{Dom}(g_{i_0}^{p_0}), i_1 = \text{OP}_{1,0}(i_0) \\ & \text{ and } \alpha_1 = \text{OP}_{1,0}(\alpha_0) \text{ and } i_0 = i_1 \\ & \text{ then } \ell g(\eta_{\alpha_0}^{i_0} \cap \eta_{\alpha_1}^{i_1}) = \ell g(\rho_{\alpha_0, \ell}^q \cap \rho_{\alpha_1, \ell}^q). \end{aligned}$$

We can, of course, demand  $\alpha_0 \neq \alpha_1$  (otherwise the conclusion of  $\otimes_1$  is trivial). Our problem is expressible for each pair  $(\alpha_0, \ell), (\alpha_1, \ell)$  separately as: first the problem is in defining the  $\rho_{(\alpha, \ell)}^q$ 's and second, if  $(\alpha'_1, \ell'), (\alpha'_2, \ell)$  is another such pair then  $\{(\alpha_1, \ell), (\alpha_2, \ell)\}, \{(\alpha'_1, \ell'), (\alpha'_2, \ell)\}$  are either disjoint or equal. Now for a given pair  $(\alpha_0, \ell), (\alpha_1, \ell)$  how many  $i_0 = i_1$  do we have? Necessarily  $i_0 \in w^{p_0} \cap w^{p_1} = w$ . But if  $i'_0 \neq i''_0$  are like that then  $\alpha_0 \in A_{i'_0} \cap A_{i''_0}$ , contradicting (\*) above because  $\alpha_0 \neq \alpha_1 = \text{OP}_{1,0}(\alpha_0)$ . So there is at most one candidate  $i_0 = i_1$ , so there is no problem to satisfy  $\otimes_1$ . Now we can define  $f_i^q$  ( $i \in w^q$ ) as in the proof of Fact C.

The rest should be clear. □<sub>4.4</sub>

*Conclusion 4.7.* Suppose  $V \models GCH$ ,  $\aleph_0 < \lambda < \chi$  and  $\chi^\lambda = \chi$ . Then for some c.c.c. forcing notion  $P$  of cardinality  $\chi$ , not collapsing cardinals nor changing cofinalities, in  $V^P$ :

- (i):  $2^{\aleph_0} = 2^\lambda = \chi$ ,
- (ii):  $\mathfrak{K}_\lambda^{tr}$  has a universal family of cardinality  $\lambda^+$ ,
- (iii):  $\mathfrak{K}_\lambda^{rs(p)}$  has a universal family of cardinality  $\lambda^+$ .

*Proof.* First use a preliminary forcing  $Q^0$  of Baumgartner [Bau70], adding  $\langle A_\alpha : \alpha < \chi \rangle$ ,  $A_\alpha \in [\lambda]^\lambda$ ,  $\alpha \neq \beta \Rightarrow |A_\alpha \cap A_\beta| \leq \aleph_0$  (we can have  $2^{\aleph_0} = \aleph_1$  here, or  $[\alpha \neq \beta \Rightarrow A_\alpha \cap A_\beta \text{ finite}]$ , but not both). Next use an FS iteration  $\langle P_i, \dot{Q}_i : i < \chi \times \lambda^+ \rangle$  such that each forcing from 4.4 appears and each forcing as in 4.6 appears. □<sub>4.7</sub>

*Remark 4.8.* We would like to have that there is a universal member in  $\mathfrak{K}_\lambda^{rs(p)}$ ; this sounds very reasonable but we did not try.

In our framework, the present result shows limitations to ZFC results which the methods applied in the previous sections can give.

## 5. BACK TO $\mathfrak{K}_\lambda^{rs(p)}$ , REAL NON-EXISTENCE RESULTS

By §1 we know that if  $G$  is an Abelian group with set of elements  $\lambda$ ,  $C \subseteq \lambda$ , then for an element  $x \in G$  the distance from  $\{y : y < \alpha\}$  for  $\alpha \in C$  does not code an appropriate invariant. If we have infinitely many such distance functions, e.g. have infinitely many primes, we can use more complicated invariants related to  $x$  as in §3. But if we have one prime, this approach does not help.

If one element fails, can we use infinitely many? A countable subset  $X$  of  $G$  can code a countable subset of  $C$ :

$$\{\alpha \in C : \text{closure}(\langle X \rangle_G) \cap \alpha \not\subseteq \text{sup}(C \cap \alpha)\},$$

but this seems silly - we use heavily the fact that  $C$  has many countable subsets (in particular  $> \lambda$ ) and  $\lambda$  has at least as many. However, what if  $C$  has a small family (say of cardinality  $\leq \lambda$  or  $< \mu^{\aleph_0}$ ) of countable subsets such that every subset of cardinality, say continuum, contains one? Well, we need more: we catch a countable subset for which the invariant defined above is infinite (necessarily it is at most of cardinality  $2^{\aleph_0}$ , and because of §4 we are not trying any more to deal with  $\lambda \leq 2^{\aleph_0}$ ). The set theory needed is expressed by  $T_J$  below, and various ideals also defined below, and the result itself is 5.9.

Of course, we can deal with other classes like torsion free reduced groups, as they have the characteristic non-structure property of unsuperstable first order theories; but the relevant ideals will vary: the parallel to  $I_\mu^0$  for  $\bigwedge_n \mu_n = \mu$ ,  $J_{\bar{\mu}}^2$  seems to be always O.K.

**Definition 5.1.** (1) For  $\bar{\mu} = \langle \mu_n : n < \omega \rangle$  let  $B_{\bar{\mu}}$  be

$$\bigoplus \{K_\alpha^n : n < \omega, \alpha < \mu_n\}, \quad K_\alpha^n = \langle {}^*t_\alpha^n \rangle_{K_\alpha^n} \cong \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

Let  $B_{\bar{\mu}|n} = \bigoplus \{K_\alpha^m : \alpha < \mu_m, m < n\} \subseteq B_{\bar{\mu}}$  (they are in  $\mathfrak{K}_{\leq \sum_n \mu_n}^{rs(p)}$ ).

Let  $\hat{B}$  be the  $p$ -torsion completion of  $B$  (i.e. completion under the norm  $\|x\| = \min\{2^{-n} : p^n \text{ divides } x\}$ ).

(2) Let  $I_{\bar{\mu}}^1$  be the ideal on  $\hat{B}_{\bar{\mu}}$  generated by  $I_{\bar{\mu}}^0$ , where

$$I_{\bar{\mu}}^0 = \left\{ A \subseteq \hat{B}_{\bar{\mu}} : \begin{array}{l} \text{for every large enough } n, \\ \text{for no } y \in \bigoplus \{K_\alpha^m : m \leq n \text{ and } \alpha < \mu_m\} \\ \text{but } y \notin \bigoplus \{K_\alpha^m : m < n \text{ and } \alpha < \mu_m\} \text{ we have :} \\ \text{for every } m \text{ for some } z \in \langle A \rangle \text{ we have:} \\ p^m \text{ divides } z - y \}. \end{array} \right.$$

(We may write  $I_{\hat{B}_{\bar{\mu}}}^0$ , but the ideal depends also on  $\langle \bigoplus_{\alpha < \mu_n} K_\alpha^n : n < \omega \rangle$

not just on  $\hat{B}_{\bar{\mu}}$  itself).

(3) For  $X, A \subseteq \hat{B}_{\bar{\mu}}$ ,

$$\text{recall } \langle A \rangle_{\hat{B}_{\bar{\mu}}} = \left\{ \sum_{n < n^*} a_n y_n : y_n \in A, a_n \in \mathbb{Z} \text{ and } n^* \in \mathbb{N} \right\},$$

$$\text{and let } c\ell_{\hat{B}_{\bar{\mu}}}(X) = \{x : (\forall n)(\exists y \in X)(x - y \in p^n \hat{B}_{\bar{\mu}})\}.$$

(4) Let  $J_{\bar{\mu}}^1$  be the ideal which  $J_{\bar{\mu}}^{0.5}$  generates, where

$$J_{\bar{\mu}}^{0.5} = \left\{ A \subseteq \prod_{n < \omega} \mu_n : \text{for some } n < \omega \text{ for no } m \in [n, \omega) \right.$$

and  $\beta < \gamma < \mu_m$  do we have :

for every  $k \in [m, \omega)$  there are  $\eta, \nu \in A$  such  
that:  $\eta(m) = \beta, \nu(m) = \gamma, \eta \upharpoonright m = \nu \upharpoonright m$   
and  $\eta \upharpoonright (m, k) = \nu \upharpoonright (m, k)$  }.

(5)

$$J_{\bar{\mu}}^0 = \left\{ A \subseteq \prod_{n < \omega} \mu_n : \text{for some } n < \omega \text{ and } k, \text{ the mapping } \eta \mapsto \eta \upharpoonright n \right.$$

is  $(\leq k)$ -to-one }.

(6)  $J_{\bar{\mu}}^2$  is the ideal of nowhere dense subsets of  $\prod_n \mu_n$  (under the following natural topology: a neighbourhood of  $\eta$  is  $U_{\eta, n} = \{\nu : \nu \upharpoonright n = \eta \upharpoonright n\}$  for some  $n$ ).

(7)  $J_{\bar{\mu}}^3$  is the ideal of meagre subsets of  $\prod_n \mu_n$ , i.e. subsets which are included in countable union of members of  $J_{\bar{\mu}}^2$ .

**Observation 5.2.** (1)  $I_{\bar{\mu}}^0, J_{\bar{\mu}}^0, J_{\bar{\mu}}^{0.5}$  are  $(< \aleph_1)$ -based, i.e. for  $I_{\bar{\mu}}^0$ : if  $A \subseteq \hat{B}_{\bar{\mu}}, A \notin I_{\bar{\mu}}^0$  then there is a countable  $A_0 \subseteq A$  such that  $A_0 \notin I_{\bar{\mu}}^0$ .

(2)  $I_{\bar{\mu}}^1, J_{\bar{\mu}}^0, J_{\bar{\mu}}^1, J_{\bar{\mu}}^2, J_{\bar{\mu}}^3$  are ideals,  $J_{\bar{\mu}}^3$  is  $\aleph_1$ -complete.

(3)  $J_{\bar{\mu}}^0 \subseteq J_{\bar{\mu}}^1 \subseteq J_{\bar{\mu}}^2 \subseteq J_{\bar{\mu}}^3$ .

(4) There is a function  $g$  from  $\prod_{n < \omega} \mu_n$  into  $\hat{B}_{\bar{\mu}}$  such that for every  $X \subseteq$

$$\prod_{n < \omega} \mu_n: \quad X \notin J_{\bar{\mu}}^1 \quad \Rightarrow \quad g''(X) \notin I_{\bar{\mu}}^1.$$

*Proof.* E.g. 4) Let  $g(\eta) = \sum_{n < \omega} p^n(*t_{\eta(n)}^n)$ .

Let  $X \subseteq \prod_{n < \omega} \mu_n, X \notin J_{\bar{\mu}}^1$ . Assume  $g''(X) \in \bar{I}_{\bar{\mu}}^1$ , so for some  $\ell^*$  and  $A_\ell \subseteq \hat{B}_{\bar{\mu}}, (\ell < \ell^*)$  we have  $A_\ell \in I_{\bar{\mu}}^0$ , and  $g''(X) \subseteq \bigcup_{\ell < \ell^*} A_\ell$ , so  $X = \bigcup_{\ell < \ell^*} X_\ell$ , where

$$X_\ell =: \{\eta \in X : g(\eta) \in A_\ell\}.$$

As  $J_{\bar{\mu}}^1$  is an ideal, for some  $\ell < \ell^*, X_\ell \notin J_{\bar{\mu}}^1$ . So by the definition of  $J_{\bar{\mu}}^1$ , for some infinite  $\Gamma \subseteq \omega$  for each  $m \in \Gamma$  we have  $\beta_m < \gamma_m < \mu_m$  and for every  $k \in [m, \omega)$  we have  $\eta_{m,k}, \nu_{m,k}$ , as required in the definition of  $J_{\bar{\mu}}^1$ . So  $g(\eta_{m,k}), g(\nu_{m,k}) \in A_\ell$  (for  $m \in \Gamma, k \in (m, \omega)$ ). Now

$$*t_{\gamma_m}^m - *t_{\beta_m}^m = g(\eta_{m,k}) - g(\nu_{m,k}) \pmod{p^k \hat{B}_{\bar{\mu}}},$$

but  $g(\eta_{m,k}) - g(\nu_{m,k}) \in \langle A_\ell \rangle_{\hat{B}_{\bar{\mu}}}$ . Hence

$$(\exists z \in \langle A_\ell \rangle_{\hat{B}_{\bar{\mu}}}) [*t_{\gamma_m}^m - *t_{\beta_m}^m = z \pmod{p^k \hat{B}_{\bar{\mu}}}],$$

as this holds for each  $k, *t_{\gamma_m}^m - *t_{\beta_m}^m \in \text{cl}(\langle A_\ell \rangle_{\hat{B}_{\bar{\mu}}})$ .

This contradicts  $A_\ell \in I_\mu^0$ . □<sub>5.2</sub>

**Definition 5.3.** Let  $I \subseteq \mathcal{P}(X)$  be downward closed (and for simplicity  $\{\{x\} : x \in X\} \subseteq I$ ). Let  $I^+ = \mathcal{P}(X) \setminus I$ . Let

$$\mathbf{U}_I^{<\kappa}(\mu) =: \min \{ |\mathcal{P}| : \mathcal{P} \subseteq [\mu]^{<\kappa}, \text{ and for every } f : X \longrightarrow \mu \text{ for some } Y \in \mathcal{P}, \text{ we have } \{x \in X : f(x) \in Y\} \in I^+ \}.$$

Instead of  $<\kappa^+$  in the superscript of  $\mathbf{U}$  we write  $\kappa$ . If  $\kappa > |\text{Dom}(I)|^+$ , we omit it (since then its value does not matter).

*Remark 5.4.* (1) If  $2^{<\kappa} + |\text{Dom}(I)|^{<\kappa} \leq \mu$  we can find  $F \subseteq$  partial functions from  $\text{Dom}(I)$  to  $\mu$  such that:

- (a):  $|F| = \mathbf{U}_I^{<\kappa}(\mu)$ ,
- (b):  $(\forall f : X \longrightarrow \mu)(\exists Y \in I^+)[f \upharpoonright Y \in F]$ .

- (2) Such functions (as  $\mathbf{U}_I^{<\kappa}(\mu)$ ) are investigated in **pcf** theory ([She94b], [She93b, §6], [She96a, §2], [She02]).
- (3) If  $I \subseteq J \subseteq \mathcal{P}(X)$ , then  $\mathbf{U}_I^{<\kappa}(\mu) \leq \mathbf{U}_J^{<\kappa}(\mu)$ , hence by 5.2(3), and the above

$$\mathbf{U}_{J_\mu^0}^{<\kappa}(\mu) \leq \mathbf{U}_{J_\mu^1}^{<\kappa}(\mu) \leq \mathbf{U}_{J_\mu^2}^{<\kappa}(\mu) \leq \mathbf{U}_{J_\mu^3}^{<\kappa}(\mu)$$

and by 5.2(4) we have  $\mathbf{U}_{J_\mu^1}^{<\kappa} \leq \mathbf{U}_{J_\mu^2}^{<\kappa}$ .

- (4) On  $\text{IND}_\theta(\bar{\kappa})$  (see 5.5 below) see [She02].

**Definition 5.5.**  $\text{IND}'_\theta(\langle \kappa_n : n < \omega \rangle)$  means that for every model  $M$  with universe  $\bigcup_{n < \omega} \kappa_n$  and  $\leq \theta$  functions, for some  $\Gamma \in [\omega]^{\aleph_0}$  and  $\eta \in \prod_{n < \omega} \kappa_n$  we have:

$$n \in \Gamma \quad \Rightarrow \quad \eta(n) \notin \text{cl}_M \{ \eta(\ell) : \ell \neq n \}.$$

*Remark 5.6.* Actually if  $\theta \geq \aleph_0$ , this implies that we can fix  $\Gamma$ , hence replacing  $\langle \kappa_n : n < \omega \rangle$  by an infinite subsequence we can have  $\Gamma = \omega$ .

**Theorem 5.7.** (1) If  $\mu_n \rightarrow (\kappa_n)_{2^\theta}^2$  and  $\text{IND}'_\theta(\langle \kappa_n : n < \omega \rangle)$  then  $\prod_{n < \omega} \mu_n$

is not the union of  $\leq \theta$  sets from  $J_\mu^1$ .

- (2) If  $\theta = \theta^{\aleph_0}$  and  $\neg \text{IND}'_\theta(\langle \mu_n : n < \omega \rangle)$  then  $\prod_{n < \omega} \mu_n$  is the union of  $\leq \theta$  members of  $J_\mu^1$ .

- (3) If  $\limsup_n \mu_n \geq 2$ , then  $\prod_{n < \omega} \mu_n \notin J_\mu^3$  (so also the other ideals defined above are not trivial by 5.2(3), (4)).

*Proof.* 1) Suppose  $\prod_{n < \omega} \mu_n$  is  $\bigcup_{i < \theta} X_i$ , and each  $X_i \in J_\mu^1$ . We define for each

$i < \theta$  and  $n < k < \omega$  a two-place relation  $R_i^{n,k}$  on  $\mu_n$ :

$$\beta R_i^{n,k} \gamma \text{ if and only if there are } \eta, \nu \in X_i \subseteq \prod_{\ell < k} \mu_\ell \text{ such that}$$

$$\eta \upharpoonright [0, n) = \nu \upharpoonright [0, n) \quad \text{and} \quad \eta \upharpoonright (n, k) = \nu \upharpoonright (n, k) \quad \text{and} \quad \eta(n) = \beta, \nu(n) = \gamma.$$

Note that  $R_i^{n,k}$  is symmetric and

$$n < k_1 < k_2 \ \& \ \beta R_i^{n,k_2} \gamma \ \Rightarrow \ \beta R_i^{n,k_1} \gamma.$$

As  $\mu_n \rightarrow (\kappa_n)_{2^\theta}^2$ , we can find  $A_n \in [\mu_n]^{\kappa_n}$  and a truth value  $\mathbf{t}_i^{n,k}$  such that for all  $\beta < \gamma$  from  $A_n$ , the truth value of  $\beta R_i^{n,k} \gamma$  is  $\mathbf{t}_i^{n,k}$ . If for some  $i$  the set

$$\Gamma_i =: \{n < \omega : \text{for every } k \in (n, \omega) \text{ we have } \mathbf{t}_i^{n,k} = \text{true}\}$$

is infinite, we get a contradiction to “ $X_i \in J_\mu^1$ ”, so for some  $n(i) < \omega$  we have  $n(i) = \sup(\Gamma_i)$ .

For each  $n < k < \omega$  and  $i < \theta$  we define a partial function  $F_i^{n,k}$  from  $\prod_{\substack{\ell < k, \\ \ell \neq n}} A_\ell$  into  $A_n$ :

$F(\alpha_0 \dots \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_k)$  is the first  $\beta \in A_n$  such that for some  $\eta \in X_i$  we have

$$\begin{aligned} \eta \upharpoonright [0, n) &= \langle \alpha_0, \dots, \alpha_{n-1} \rangle, & \eta(n) &= \beta, \\ \eta \upharpoonright (n, k) &= \langle \alpha_{n+1}, \dots, \alpha_{k-1} \rangle. \end{aligned}$$

So as  $\text{IND}'_\theta(\langle \kappa_n : n < \omega \rangle)$  there is  $\eta = \langle \beta_n : n < \omega \rangle \in \prod_{n < \omega} A_n$  such that for infinitely many  $n$ ,  $\beta_n$  is not in the closure of  $\{\beta_\ell : \ell < \omega, \ell \neq n\}$  by the  $F_i^{n,k}$ 's. As  $\eta \in \prod_{n < \omega} A_n \subseteq \prod_{n < \omega} \mu_n = \bigcup_{i < \theta} X_i$ , necessarily for some  $i < \theta$ ,  $\eta \in X_i$ . Let  $n \in (n(i), \omega)$  be such that  $\beta_n$  is not in the closure of  $\{\beta_\ell : \ell < \omega \text{ and } \ell \neq n\}$  and let  $k > n$  be such that  $\mathbf{t}_i^{n,k} = \text{false}$ . Now  $\gamma =: F_i^{n,k}(\beta_0, \dots, \beta_{n-1}, \beta_{n+1}, \dots, \beta_{k-1})$  is well defined  $\leq \beta_n$  (as  $\beta_n$  exemplifies that there is such  $\beta$ ) and is  $\neq \beta_n$  (by the choice of  $\langle \beta_\ell : \ell < \omega \rangle$ ), so by the choice of  $n(i)$  (so of  $n, k$  and earlier of  $\mathbf{t}_i^{n,k}$  and of  $A_n$ ) we get contradiction to “ $\gamma < \beta_n$  are from  $A_n$ ”.

2) Let  $M$  be an algebra with universe  $\sum_{n < \omega} \mu_n$  and  $\leq \theta$  functions (say  $F_i^n$  for  $i < \theta$ ,  $n < \omega$ ,  $F_i^n$  is  $n$ -place) exemplifying  $\neg \text{IND}'_\theta(\langle \mu_n : n < \omega \rangle)$ . Let

$$\Gamma =: \{ \langle (k_n, i_n) : n^* \leq n < \omega \rangle : n^* < \omega \text{ and } \bigwedge_n n < k_n < \omega \text{ and } i_n < \theta \}.$$

For  $\rho = \langle (k_n, i_n) : n^* \leq n < \omega \rangle \in \Gamma$  let

$$\begin{aligned} A_\rho =: \{ \eta \in \prod_{n < \omega} \mu_n : \text{for every } n \in [n^*, \omega) \text{ we have} \\ \eta(n) = F_{i_n}^{k_n-1}(\eta(0), \dots, \eta(n-1), \eta(n+1), \dots, \eta(k_n)) \}. \end{aligned}$$

So, by the choice of  $M$ ,  $\prod_{n < \omega} \mu_n = \bigcup_{\rho \in \Gamma} A_\rho$ . On the other hand, it is easy to check that  $A_\rho \in J_\mu^1$ . □<sub>5.7</sub>

**Theorem 5.8.** *If  $\mu = \sum_{n < \omega} \lambda_n$ ,  $\lambda_n^{\aleph_0} < \lambda_{n+1}$  and  $\mu < \lambda = \text{cf}(\lambda) < \mu^{+\omega}$*

*then  $\mathbf{U}_{J_{\langle \lambda_n : n < \omega \rangle}^{\aleph_0}}(\lambda) = \lambda$  and even  $\mathbf{U}_{J_{\langle \lambda_n : n < \omega \rangle}^{\aleph_3}}(\lambda) = \lambda$ .*

*Proof.* See [She93b, §6], [She96a, §2], and [She02] for considerably more.

**Lemma 5.9.** *Assume  $\lambda > 2^{\aleph_0}$  and*

- (\*)**(a):**  $\prod_{n < \omega} \mu_n < \mu$  and  $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$ ,
- (b):**  $\hat{B}_{\bar{\mu}} \notin I_{\bar{\mu}}^0$  and  $\lim_n \sup \mu_n$  is infinite,
- (c):**  $\mathbf{U}_{I_{\bar{\mu}}^0}^{\aleph_0}(\lambda) = \lambda$  (note  $I_{\bar{\mu}}^0$  is not required to be an ideal).

*Then there is no universal member in  $\mathfrak{K}_{\lambda}^{rs(p)}$ .*

*Proof.* Let  $S \subseteq \lambda$ ,  $\bar{C} = \langle C_{\delta} : \delta \in S \rangle$  guesses clubs of  $\lambda$ , chosen as in the proof of 3.3 (so  $\alpha \in \text{nacc}(C_{\delta}) \Rightarrow \text{cf}(\alpha) > 2^{\aleph_0}$ ). Instead of defining the relevant invariant we prove the theorem directly, but we could define it, somewhat clumsily (like [Shear, III, §3]).

Assume  $H \in \mathfrak{K}_{\lambda}^{rs(p)}$  is a pretender to universality; without loss of generality with the set of elements of  $H$  equal to  $\lambda$ .

Let  $\chi = \beth_7(\lambda)^+$ ,  $\bar{\mathfrak{A}} = \langle \mathfrak{A}_{\alpha} : \alpha < \lambda \rangle$  be an increasing continuous sequence of elementary submodels of  $(\mathcal{H}(\chi), \in, <_{\chi}^*)$ ,  $\bar{\mathfrak{A}} \upharpoonright (\alpha + 1) \in \mathfrak{A}_{\alpha+1}$ ,  $\|\mathfrak{A}_{\alpha}\| < \lambda$ ,  $\mathfrak{A}_{\alpha} \cap \lambda$  an ordinal,  $\mathfrak{A} = \bigcup_{\alpha < \lambda} \mathfrak{A}_{\alpha}$  and  $\{H, \langle \mu_n : n < \omega \rangle, \mu, \lambda\} \in \mathfrak{A}_0$ , so  $B_{\bar{\mu}}, \hat{B}_{\bar{\mu}} \in \mathfrak{A}_0$  (where  $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ , of course).

For each  $\delta \in S$ , let  $\mathcal{P}_{\delta} =: [C_{\delta}]^{\aleph_0} \cap \mathfrak{A}$ . Choose  $A_{\delta} \subseteq C_{\delta}$  of order type  $\omega$  almost disjoint from each  $a \in \mathcal{P}_{\delta}$ , and from  $A_{\delta_1}$  for  $\delta_1 \in \delta \cap S$ ; its existence should be clear as  $2^{\aleph_0} < \lambda < \mu^{\aleph_0}$ . So

(\*)<sub>0</sub>: every countable  $A \in \mathfrak{A}$  is almost disjoint to  $A_{\delta}$ .

By 5.2(2),  $I_{\bar{\mu}}^0$  is  $(< \aleph_1)$ -based so by 5.4(1) and the assumption (c) we have

(\*)<sub>1</sub>: for every  $f : \hat{B}_{\bar{\mu}} \rightarrow \lambda$  for some countable  $Y \subseteq \hat{B}_{\bar{\mu}}$ ,  $Y \notin I_{\bar{\mu}}^0$ , we have  $f \upharpoonright Y \in \mathfrak{A}$

(remember  $(\prod_{n < \omega} \mu_n)^{\aleph_0} = \prod_{n < \omega} \mu_n$ ).

Let  $B$  be  $\bigoplus \{G_{\alpha, i}^n : n < \omega, \alpha < \lambda, i < \sum_{k < \omega} \mu_k\}$ , where

$$G_{\alpha, i}^n = \langle x_{\alpha, i}^n \rangle_{G_{\alpha, i}^n} \cong \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

So  $B, \hat{B}, \langle (n, \alpha, i, x_{\alpha, i}^n) : n < \omega, \alpha < \lambda, i < \sum_{k < \omega} \mu_k \rangle$  are well defined. Let  $G$  be the subgroup of  $\hat{B}$  generated by:

$B \cup \{x \in \hat{B} : \text{for some } \delta \in S, x \text{ is in the closure of}$

$$\bigoplus \{G_{\alpha, i}^n : n < \omega, i < \mu_n, \alpha \text{ is the } n\text{th element of } A_{\delta}\}.$$

As  $\prod_{n < \omega} \mu_n < \mu < \lambda$ , clearly  $G \in \mathfrak{K}_{\lambda}^{rs(p)}$ , without loss of generality the set of elements of  $G$  is  $\lambda$  and let  $h : G \rightarrow H$  be an embedding. Let

$$E_0 =: \{\delta < \lambda : (\mathfrak{A}_{\delta}, h \upharpoonright \delta, G \upharpoonright \delta) \prec (\mathfrak{A}, h, G)\},$$

$$E =: \{\delta < \lambda : \text{otp}(E_0 \cap \delta) = \delta\}.$$

They are clubs of  $\lambda$ , so for some  $\delta \in S$ ,  $C_\delta \subseteq E$  (and  $\delta \in E$  for simplicity). Let  $\eta_\delta$  enumerate  $A_\delta$  increasingly.

There is a natural embedding  $g = g_\delta$  of  $B_{\bar{\mu}}$  into  $G$ :

$$g({}^*t_i^n) = x_{\eta_\delta(n),i}^n.$$

Let  $\hat{g}_\delta$  be the unique extension of  $g_\delta$  to an embedding of  $\hat{B}_{\bar{\mu}}$  into  $G$ ; those embeddings are pure, (in fact  $g''_\delta(\hat{B}_{\bar{\mu}}) \setminus g''_\delta(B_{\bar{\mu}}) \subseteq G \setminus G \cap \mathfrak{A}_\delta$ ). So  $h \circ \hat{g}_\delta$  is an embedding of  $\hat{B}_{\bar{\mu}}$  into  $H$ , not necessarily pure but still an embedding, so the distance function can become smaller but not zero and

$$h \circ \hat{g}_\delta(\hat{B}_{\bar{\mu}}) \setminus h \circ g_\delta(B_{\bar{\mu}}) \subseteq H \setminus \mathfrak{A}_\delta.$$

Remember  $\hat{B}_{\bar{\mu}} \subseteq \mathfrak{A}_0$  (as it belongs to  $\mathfrak{A}_0$  and has cardinality  $\prod_{n < \omega} \mu_n < \lambda$  and

$\lambda \cap \mathfrak{A}_0$  is an ordinal). By  $(*)_1$  applied to  $f = h \circ \hat{g}_\delta$  there is a countable  $Y \subseteq \hat{B}_{\bar{\mu}}$  such that  $Y \notin I_{\bar{\mu}}^0$  and  $f \upharpoonright Y \in \mathfrak{A}$ . But, from  $f \upharpoonright Y$  we shall below reconstruct some countable sets not almost disjoint to  $A_\delta$ , reconstruct meaning in  $\mathfrak{A}$ , in contradiction to  $(*)_0$  above.

As  $Y \notin I_{\bar{\mu}}^0$  we can find an infinite  $S^* \subseteq \omega \setminus m^*$  and for  $n \in S^*$ ,  $z_n \in \bigoplus_{\alpha < \mu_n} K_\alpha^n \setminus \{0\}$  and  $y_n^\ell \in \hat{B}_{\bar{\mu}}$  (for  $\ell < \omega$ ) such that:

$$(*)_2: z_n + y_{n,\ell} \in \langle Y \rangle_{\hat{B}_{\bar{\mu}}}, \quad \text{and}$$

$$(*)_3: y_{n,\ell} \in p^\ell \hat{B}_{\bar{\mu}}.$$

Without loss of generality  $p z_n = 0 \neq z_n$  hence  $p y_n^\ell = 0$ . Let

$$\nu_\delta(n) =: \min(C_\delta \setminus (\eta_\delta(n) + 1)), \quad z_n^* = (h \circ \hat{g}_\delta)(z_n) \quad \text{and} \quad y_{n,\ell}^* = (h \circ \hat{g}_\delta)(y_{n,\ell}).$$

Now clearly  $\hat{g}_\delta(z_n) = g_\delta(z_n) = x_{\eta_\delta(n),i}^n \in G \upharpoonright \nu_\delta(n)$ , hence  $(h \circ \hat{g}_\delta)(z_n) \notin H \upharpoonright \eta_\delta(n)$ , that is  $z_n^* \notin H \upharpoonright \eta_\delta(n)$ .

So  $z_n^* \in H_{\nu_\delta(n)} \setminus H_{\eta_\delta(n)}$  belongs to the  $p$ -adic closure of  $\text{Rang}(f \upharpoonright Y)$ . As  $H$ ,  $G$ ,  $h$  and  $f \upharpoonright Y$  belongs to  $\mathfrak{A}$ , also  $K$ , the closure of  $\text{Rang}(f \upharpoonright Y)$  in  $H$  by the  $p$ -adic topology belongs to  $\mathfrak{A}$ , and clearly  $|K| \leq 2^{\aleph_0}$ , hence

$$A^* = \{\alpha \in C_\delta : K \cap H_{\min(C_\delta \setminus (\alpha+1))} \setminus H_\alpha \text{ is not empty}\}$$

is a subset of  $C_\delta$  of cardinality  $\leq 2^{\aleph_0}$  which belongs to  $\mathfrak{A}$ , hence  $[A^*]^{\aleph_0} \subseteq \mathfrak{A}$  but  $A_\delta \subseteq A^*$  so  $A_\delta \in \mathfrak{A}$ , a contradiction.  $\square_{5.9}$

## 6. IMPLICATIONS BETWEEN THE EXISTENCE OF UNIVERSALS

**Theorem 6.1.** *Let  $\bar{n} = \langle n_i : i < \omega \rangle$ ,  $n_i \in [1, \omega)$ . Remember*

$$J_{\bar{n}}^2 = \{A \subseteq \prod_{i < \omega} n_i : A \text{ is nowhere dense}\}.$$

*Assume  $\lambda \geq 2^{\aleph_0}$ ,  $T_{J_{\bar{n}}^2}^{\aleph_0}(\lambda) = \lambda$  or just  $T_{J_{\bar{n}}^2}^{\aleph_0}(\lambda) = \lambda$  for every such  $\bar{n}$ , and*

$$n < \omega \quad \Rightarrow \quad \lambda_n \leq \lambda_{n+1} \leq \lambda_\omega = \lambda \quad \text{and}$$

$$\lambda \leq \prod_{n < \omega} \lambda_n \quad \text{and} \quad \bar{\lambda} = \langle \lambda_i : i \leq \omega \rangle.$$

- (1) If in  $\mathfrak{K}_\lambda^{fc}$  there is a universal member  
 then in  $\mathfrak{K}_\lambda^{rs(p)}$  there is a universal member.  
 (2) If in  $\mathfrak{K}_\lambda^{fc}$  there is a universal member for  $\mathfrak{K}_\lambda^{fc}$   
 then in

$$\mathfrak{K}_\lambda^{rs(p)} =: \{G \in \mathfrak{K}_\lambda^{rs(p)} : \lambda_n(G) \leq \lambda\}$$

there is a universal member (even for  $\mathfrak{K}_\lambda^{rs(p)}$ ).

( $\lambda_n(G)$  were defined in 1.1).

*Remark 6.2.* (1) Similarly for “there are  $M_i \in \mathfrak{K}_{\lambda_1}$  ( $i < \theta$ ) with  $\langle M_i : i < \theta \rangle$  being universal for  $\mathfrak{K}_\lambda$ ”.

(2) The parallel of 1.1 holds for  $\mathfrak{K}_\lambda^{fc}$ .

(3) By §5 only the case  $\lambda$  singular or  $\lambda = \mu^+$  &  $\text{cf}(\mu) = \aleph_0$  &  $(\forall \alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$  is of interest for 6.1.

*Proof.* 1) By 1.1, (2)  $\Rightarrow$  (1).

More elaborately, by part (2) of 6.1 below there is  $H \in \mathfrak{K}_\lambda^{rs(p)}$  which is universal in  $\mathfrak{K}_\lambda^{rs(p)}$ . Clearly  $|G| = \lambda$  so  $H \in \mathfrak{K}_\lambda^{rs(p)}$ , hence for proving part (1) of 6.1 it suffices to prove that  $H$  is a universal member of  $\mathfrak{K}_\lambda^{rs(p)}$ . So let  $G \in \mathfrak{K}_\lambda^{rs(p)}$ , and we shall prove that it is embeddable into  $H$ . By 1.1 there is  $G'$  such that  $G \subseteq G' \in \mathfrak{K}_\lambda^{rs(p)}$ . By the choice of  $H$  there is an embedding  $h$  of  $G'$  into  $H$ . So  $h \upharpoonright G$  is an embedding of  $G$  into  $H$ , as required.

2) Let  $T^*$  be a universal member of  $\mathfrak{K}_\lambda^{fc}$  (see §2) and let  $P_\alpha = P_\alpha^{T^*}$ .

Let  $\chi > 2^\lambda$ . Without loss of generality  $P_n = \{n\} \times \lambda_n$ ,  $P_\omega = \lambda$ . Let

$$B_0 = \bigoplus \{G_t^n : n < \omega, t \in P_n\},$$

$$B_1 = \bigoplus \{G_t^n : n < \omega \text{ and } t \in P_n\},$$

where  $G_t^n \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$ ,  $G_t^n$  is generated by  $x_t^n$ . Let  $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ ,  $\|\mathfrak{B}\| = \lambda$ ,  $\lambda + 1 \subseteq \mathfrak{B}$ ,  $T^* \in \mathfrak{B}$ , hence  $B_0, B_1 \in \mathfrak{B}$  and  $\hat{B}_0, \hat{B}_1 \in \mathfrak{B}$  (the torsion completion of  $B$ ). Let  $G^* = \hat{B}_1 \cap \mathfrak{B}$ .

Let us prove that  $G^*$  is universal for  $\mathfrak{K}_\lambda^{rs(p)}$  (by 1.1 this suffices). Let  $G \in \mathfrak{K}_\lambda^{rs(p)}$ , so by 1.1 without loss of generality  $B_0 \subseteq G \subseteq \hat{B}_0$ . We define  $R$ :

$$R = \{\eta : \eta \in \prod_{n < \omega} \lambda_n \text{ and for some } x \in G \text{ letting}$$

$$x = \sum \{a_i^n p^{n-k} x_i^n : n < \omega, i \in w_n(x)\} \text{ where}$$

$$w_n(x) \in [\lambda_n]^{< \aleph_0}, a_i^n p^{n-k} x_i^n \neq 0 \text{ we have}$$

$$\bigwedge_n \eta(n) \in w_n(x) \cup \{\ell : \ell + |w_n(x)| \leq n\}.$$

Lastly let  $M =: (R \cup \bigcup_{n < \omega} \{n\} \times \lambda_n, P_n, F_n)_{n < \omega}$  where  $P_n = \{n\} \times \lambda_n$  and  $F_n(\eta) = (n, \eta(n))$ , so clearly  $M \in \mathfrak{R}_\lambda^{fc}$ . Consequently, there is an embedding  $g : M \rightarrow T^*$ , so  $g$  maps  $\{n\} \times \lambda_n$  into  $P_n^{T^*}$  and  $g$  maps  $R$  into  $P_\omega^{T^*}$ . Let  $g(n, \alpha) = (n, g_n(\alpha))$  (i.e. this defines  $g_n$ ). Clearly  $g \upharpoonright (\cup P_n^M) = g \upharpoonright (\bigcup_n \{n\} \times \lambda_n)$  induces an embedding  $g^*$  of  $B_0$  to  $B_1$  (by mapping the generators into the generators).

The problem is why:

$$(*) : \text{if } x = \sum \{a_i^n p^{n-k} x_i^n : n < \omega, i \in w_n(x)\} \in G \\ \text{then } g^*(x) = \sum \{a_i^n p^{n-k} g^*(x_i^n) : n < \omega, i \in w_n(x)\} \in G^*.$$

As  $G^* = \hat{B}_1 \cap \mathfrak{B}$ , and  $2^{\aleph_0} + 1 \subseteq \mathfrak{B}$ , it is enough to prove  $\langle g''(w_n(x)) : n < \omega \rangle \in \mathfrak{B}$ . Now for notational simplicity  $\bigwedge_n [|w_n(x)| \geq n + 1]$  (we can add an element of  $G^* \cap \mathfrak{B}$  or just repeat the arguments). For each  $\eta \in \prod_{n < \omega} w_n(x)$  we know that

$$g(\eta) = \langle g(\eta(n)) : n < \omega \rangle \in T^* \quad \text{hence is in } \mathfrak{B}$$

(as  $T^* \in \mathfrak{B}$ ,  $|T^*| \leq \lambda$ ). Now by assumption there is  $A \subseteq \prod_{n < \omega} w_n(x)$  which is not nowhere dense such that  $g \upharpoonright A \in \mathfrak{B}$ , hence for some  $n^*$  and  $\eta^* \in \prod_{\ell < n^*} w_\ell(x)$ ,  $A$  is dense above  $\eta^*$  (in  $\prod_{n < \omega} w_n(x)$ ). Hence

$$\langle \{\eta(n) : \eta \in A\} : n^* \leq n < \omega \rangle = \langle w_n[x] : n^* \leq n < \omega \rangle,$$

but the former is in  $\mathfrak{B}$  as  $A \in \mathfrak{B}$ , and from the latter the desired conclusion follows.  $\square_{6.1}$

## 7. NON-EXISTENCE OF UNIVERSALS FOR TREES WITH SMALL DENSITY

For simplicity we deal below with the case  $\delta = \omega$ , but the proof works in general (as for  $\mathfrak{R}_\lambda^{fr}$  in §2) with minor changes. Section 1 hinted we should look at  $\mathfrak{R}_\lambda^{tr}$  not only for the case  $\bar{\lambda} = \langle \lambda : \alpha \leq \omega \rangle$  (i.e.  $\mathfrak{R}_\lambda^{tr}$ ), but in particular for

$$\bar{\lambda} = \langle \lambda_n : n < \omega \rangle \wedge \langle \lambda \rangle, \quad \lambda_n^{\aleph_0} < \lambda_{n+1} < \mu < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}.$$

Here we get for this class (embeddings are required to preserve levels), results stronger than the ones we got for the classes of Abelian groups we have considered.

**Theorem 7.1.** *Assume that*

- (a):  $\bar{\lambda} = \langle \lambda_\alpha : \alpha \leq \omega \rangle$ ,  $\lambda_n < \lambda_{n+1} < \lambda_\omega$ ,  $\lambda = \lambda_\omega$ , all are regulars,
- (b):  $D$  is a filter on  $\omega$  containing cobounded sets,
- (c):  $\text{tcf}(\prod \lambda_n / D) = \lambda$  (indeed, we mean  $=$ , we could just use  $\lambda \in \text{pcf}_D(\{\lambda_n : n < \omega\})$ ),
- (d):  $(\sum_{n < \omega} \lambda_n)^+ < \lambda < \prod_{n < \omega} \lambda_n$ .
- (e):  $\mathcal{P}(\mathbf{a}) / J_{< \lambda}(\{\lambda_n : n < \omega\})$  is infinite.

*Then there is no universal member in  $\mathfrak{K}_\lambda^{tr}$ .*

*Proof.* Clearly if  $n_i < n_{i+1} < \omega$  for  $i < \omega$  and  $\bar{\lambda}' = \langle \lambda_{n_i} : i < \omega \rangle \frown \langle \lambda \rangle$  and is  $\mathfrak{K}_\lambda^{tr}$  there is a universal member then in  $\mathfrak{K}_\lambda^{tr}$ , there is a universal member. So without loss of generality  $\lambda = \max \text{pcf}\{\lambda_n : n < \omega\}$  without loss of generality  $D - \{A : \max \text{proof}\{\lambda_n : n \in \omega \setminus A\} < \lambda\}$ . Let  $\mu = \sum_{n < \omega} \lambda_n$ .

We now notice that there is a sequence  $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \mu < \alpha < \lambda \rangle$  such that:

- (1)  $|\mathcal{P}_\alpha| < \lambda$ ,
- (2)  $a \in \mathcal{P}_\alpha \Rightarrow a$  is a closed subset of  $\alpha$  of order type  $\leq \mu$ ,
- (3)  $a \in \bigcup_{\alpha < \lambda} \mathcal{P}_\alpha$  &  $\beta \in \text{nacc}(a) \Rightarrow a \cap \beta \in \mathcal{P}_\beta$ ,
- (4) For all club subsets  $E$  of  $\lambda$ , there are stationarily many  $\delta$  for which there is an  $a \in \bigcup_{\alpha < \lambda} \mathcal{P}_\alpha$  such that

$$\text{cf}(\delta) = \aleph_0 \text{ \& } a \in \mathcal{P}_\delta \text{ \& } \text{otp}(a) = \mu \text{ \& } a \subseteq E.$$

[Why? If  $\lambda = \mu^{++}$ , then it is the successor of a regular, so we use [She91, §4], i.e.

$$\{\alpha < \lambda : \text{cf}(\alpha) \leq \mu\}$$

is the union of  $\leq \mu^+$  sets with squares.

If  $\lambda > \mu^{++}$ , then we can use [She93a, §1], which guarantees that there is a stationary  $S \in I[\lambda]$ . and note that we do not require  $\sup(a) = \alpha$  even if  $a \in \mathcal{P}_\alpha$  and  $\text{otp}(a) = \mu$

We can now find a sequence

$$\langle f_\alpha, g_{\alpha,a} : \alpha < \lambda \text{ \& } a \in \mathcal{P}_\alpha \rangle$$

such that:

- (a):  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  is a  $<_D$ -increasing cofinal sequence in  $\prod_{n < \omega} \lambda_n$ ,
- (b):  $g_{\alpha,a} \in \prod_{n < \omega} \lambda_n$ ,
- (c):  $\bigwedge_{\beta < \alpha} f_\beta <_D g_{\alpha,a} <_D f_{\alpha+1}$ ,
- (d):  $\lambda_n > |a|$  &  $\beta \in \text{nacc}(a) \Rightarrow g_{\beta, a \cap \beta}(n) < g_{\alpha,a}(n)$ .
- (e): for every  $f \in \prod \lambda_n$  for some  $\alpha$  we have  $f < f_\alpha$

[How? Choose  $\bar{f}$  by  $\text{tcf}(\prod_{n < \omega} \lambda_n / D) = \lambda + \max \text{pcf}\{\lambda_n : n < \omega\} = \lambda$ . Then choose  $g$ 's by induction, possibly throwing out some of the  $f$ 's; this is from [She94b, II, §1].]

Let  $T \in \mathfrak{K}_\lambda^{tr}$ .

We introduce for  $x \in \text{lev}_\omega(T)$  and  $\ell < \omega$  the notation  $F_\ell^T(x) = F_\ell(x)$  to denote the unique member of  $\text{lev}_\ell(T)$  which is below  $x$  in the tree order of  $T$ .

For  $a \in \bigcup_{\alpha < \lambda} \mathcal{P}_\alpha$ , let  $a = \{\alpha_{a,\xi} : \xi < \text{otp}(a)\}$  be an increasing enumeration and let  $a(\zeta) = \{\alpha_{a,\xi} : \xi < \zeta\}$ . We shall consider two cases. In the first one, we assume that the following statement (\*) holds. In this case, the proof is

easier, and maybe (\*) always holds for some  $D$ , but we do not know this at present.

(\*): There is a partition  $\langle A_n : n < \omega \rangle$  of  $\omega$  into sets not disjoint to any member of  $D$ .

Let  $A_n \in D^+$  be pairwise disjoint, for  $n < \omega$ , let  $D_n$  be the filter generated by  $D$  and  $A_n$ . Let for  $a \in \bigcup_{\alpha < \lambda} \mathcal{P}_\alpha$  with  $\text{otp}(a) = \mu$ , and for  $x \in \text{lev}_\omega(T)$ ,

$$\text{inv}(x, a, T) =: \langle \xi_n(x, a, T) : n < \omega \rangle,$$

where

$$\begin{aligned} \xi_n(x, a, T) =: \min \{ \xi \leq \text{otp}(a) : & \text{if } \xi < \text{otp}(a) \text{ then } \omega\xi + \omega \leq \text{otp}(a) \\ & \text{and for some } m < \omega \text{ we have} \\ & \langle F_\ell^T(x) : \ell < \omega \rangle <_{D_n} g_{\alpha', a'} \text{ where} \\ & \alpha' = \alpha_{a, \omega\xi+m} \text{ and } a' = a \cap \alpha' \}. \end{aligned}$$

Let

$$\begin{aligned} \text{INv}(a, T) =: \{ \text{inv}(x, a, T) : x \in T \ \& \ \text{lev}_T(x) = \omega \}, \\ \text{INV}(T) =: \{ c : \text{for every club } E \subseteq \lambda, \text{ for some } \delta \text{ and } a \in P \\ \text{we have } \text{otp}(a) = \mu \ \& \ a \subseteq E \ \& \ a \in P_\delta \\ \text{and for some } x \in T \text{ of } \text{lev}_T(x) = \omega, \ c = \text{inv}(x, a, T) \}. \end{aligned}$$

(Alternatively, we could have looked at the function giving each  $a$  the value  $\text{INv}(a, T)$ , and then divide by a suitable club guessing ideal as in the proof in §3, see Definition 3.6.)

Clearly

**Fact:**  $\text{INV}(T)$  has cardinality  $\leq \lambda$ .

The main point is the following

**Main Fact:** If  $\mathbf{h} : T^1 \rightarrow T^2$  is an embedding, then

$$\text{INV}(T^1) \subseteq \text{INV}(T^2).$$

*Proof of the Main Fact under (\*)* We define for  $n \in \omega$

$$E_n =: \{ \delta < \lambda_n : \delta > \bigcup_{\ell < n} \lambda_\ell \text{ and } (\forall x \in \text{lev}_{n+1}(T^1)) (F_n(\mathbf{h}(x)) < \delta \Leftrightarrow F_n(x) < \delta) \}.$$

We similarly define  $E_\omega$ , so  $E_n$  ( $n \in \omega$ ) and  $E_\omega$  are clubs (of  $\lambda_n$  and  $\lambda$  respectively). Now suppose  $c \in \text{INV}(T_1) \setminus \text{INV}(T_2)$ . Without loss of generality  $E_\omega$  is (also) a club of  $\lambda$  which exemplifies that  $c \notin \text{INV}(T_2)$ . For  $h \in \prod_{n < \omega} \lambda_n$ , let

$$h^+(n) =: \min(E_n \setminus (h(n) + 1)), \quad \text{and} \quad \beta[h] = \min\{\beta < \lambda : h < f_\beta\}.$$

(Note that  $h < f_{\beta[h]}$ , not just  $h <_D f_{\beta[h]}$  check?.) For a sequence  $\langle h_i : i < i^* \rangle$  of functions from  $\prod_{n < \omega} \lambda_n$ , we use  $\langle h_i : i < i^* \rangle^+$  for  $\langle h_i^+ : i < i^* \rangle$ . Now let

$$E^* =: \{ \delta < \lambda : \text{if } \alpha < \delta \text{ then } \beta[f_\alpha^+] < \delta \text{ and } \delta \in \text{acc}(E_\omega) \}.$$

Thus  $E^*$  is a club of  $\lambda$ . Since  $c \in \text{INV}(T_1)$ , there is  $\delta < \lambda$  and  $a \in P_\delta$  such that for some  $x \in \text{lev}_\omega(T_1)$  we have

$$a \subseteq E^* \ \& \ \text{otp}(a) = \mu \ \& \ c = \text{inv}(x, a, T_1).$$

Let for  $n \in \omega$ ,  $\xi_n =: \xi_n(x, a, T_1)$ , so  $c = \langle \xi_n : n < \omega \rangle$ . Also let for  $\xi < \mu$ ,  $\alpha_\xi =: \alpha_{a, \xi}$ , so  $a = \{ \alpha_\xi : \xi < \mu \}$  is an increasing enumeration. Now fix an  $n < \omega$  and consider  $\mathbf{h}(x)$ . Then we know that for some  $m$

$$\begin{aligned} (\alpha): \quad & \langle F_\ell^{T_1}(x) : \ell < \omega \rangle <_{D_n} g_{\alpha'} \text{ where } \alpha' = \alpha_{\omega \xi_n + m} \quad \text{and} \\ (\beta): \quad & \text{for no } \xi < \xi_n \text{ is there such an } m. \end{aligned}$$

Now let us look at  $F_\ell^{T_1}(x)$  and  $F_\ell^{T_2}(\mathbf{h}(x))$ . They are not necessarily equal, but

$$(\gamma): \quad \min(E_\ell \setminus (F_\ell^{T_1}(x) + 1)) = \min(E_\ell \setminus (F_\ell^{T_2}(\mathbf{h}(x)) + 1))$$

(by the definition of  $E_\ell$ ). Hence

$$(\delta): \quad \langle F_\ell^{T_1}(x) : \ell < \omega \rangle^+ = \langle F_\ell^{T_2}(\mathbf{h}(x)) : \ell < \omega \rangle^+.$$

Now note that by the choice of  $g$ 's

$$(\varepsilon): \quad (g_{\alpha_\varepsilon, a \cap \alpha_\varepsilon})^+ <_{D_n} g_{\alpha_{\varepsilon+1}, a \cap \alpha_{\varepsilon+1}}.$$

From  $(\delta)$  and  $(\varepsilon)$  it follows that  $\xi_n(\mathbf{h}(x), a, T^2) = \xi_n(x, a, T^1)$ . Hence  $c \in \text{INV}(T^2)$ . □<sub>Main Fact</sub>

Now it clearly suffices to prove:

**Fact A:** For each  $c = \langle \xi_n : n < \omega \rangle \in {}^\omega \mu$  we can find a  $T \in \mathfrak{R}_\lambda^{tr}$  such that  $c \in \text{INV}(T)$ .

*Proof of the Fact A in case (\*) holds* For each  $a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta$  with  $\text{otp}(a) = \mu$

we define  $x_{c,a} =: \langle x_{c,a}(\ell) : \ell < \omega \rangle$  by:

$$\text{if } \ell \in A_n, \text{ then } x_{c,a}(\ell) = g_{\alpha_a, \omega \xi_{n+1}}(\ell).$$

Let

$$T = \bigcup_{n < \omega} \prod_{\ell < n} \lambda_\ell \cup \{ x_{c,a} : a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta \ \& \ \text{otp}(a) = \mu \}.$$

We order  $T$  by  $\triangleleft$ .

It is easy to check that  $T$  is as required. □<sub>A</sub>

Now we are left to deal with the case that  $(*)$  does not hold. Let

$$\text{pcf}(\{\lambda_n : n < \omega\}) = \{\kappa_\alpha : \alpha \leq \alpha^*\}$$

be an enumeration in increasing order so in particular

$$\kappa_{\alpha^*} = \max \text{pcf}(\{\lambda_n : n < \omega\}).$$

Without loss of generality  $\kappa_{\alpha^*} = \lambda$  (by throwing out some elements if necessary; see beginning of the proof) and  $\lambda \cap \text{pcf}(\{\lambda_n : n < \omega\})$  has no last

element (this appears explicitly in [She94b], but is also straightforward from the pcf theorem). In particular,  $\alpha^*$  is a limit ordinal. Hence, without loss of generality

$$D = \{A \subseteq \omega : \lambda > \max \text{pcf}\{\lambda_n : n \in \omega \setminus A\}\}.$$

Let  $\langle \mathfrak{a}_{\kappa_\alpha} : \alpha \leq \alpha^* \rangle$  be a generating sequence for  $\text{pcf}(\{\lambda_n : n < \omega\})$ , i.e.

$$\max \text{pcf}(\mathfrak{a}_{\kappa_\alpha}) = \kappa_\alpha \quad \text{and} \quad \kappa_\alpha \notin \text{pcf}(\{\lambda_n : n < \omega\} \setminus \mathfrak{a}_{\kappa_\alpha}).$$

(The existence of such a sequence is part of the pcf theorem). Without loss of generality,

$$\mathfrak{a}_{\alpha^*} = \{\lambda_n : n < \omega\}.$$

Now note

**Observation 7.2.** *If  $\text{cf}(\alpha^*) = \aleph_0$ , then  $(*)$  holds.*

Why? Let  $\langle \alpha(n) : n < \omega \rangle$  be a strictly increasing cofinal sequence in  $\alpha^*$ . Let  $\langle B_n : n < \omega \rangle$  partition  $\omega$  into infinite pairwise disjoint sets and let

$$A_\ell =: \{k < \omega : \bigvee_{n \in B_\ell} [\lambda_k \in \mathfrak{a}_{\kappa_{\alpha(n)}} \setminus \bigcup_{m < n} \mathfrak{a}_{\kappa_{\alpha(m)}}]\}.$$

To check that this choice of  $\langle A_\ell : \ell < \omega \rangle$  works, recall that for all  $\alpha$  we know that  $\mathfrak{a}_{\kappa_\alpha}$  does not belong to the ideal generated by  $\{\mathfrak{a}_{\kappa_\beta} : \beta < \alpha\}$  and use the pcf calculus (i.e.  $n \in B_\ell \Rightarrow \lambda_n \in \text{pcf}(\mathfrak{a}_{\kappa_{\alpha(n)}}) \subseteq \text{pcf}\{\lambda_k : k \in A_\ell\}$  so  $\max \text{pcf}\{\lambda_k : k \in A_\ell\} \geq \max \text{pcf}\{\lambda_n : n \in B_\ell\} \geq \sum_{n \in B_\ell} \kappa_{\alpha(n)}$  but  $\text{pcf}(\{\lambda_n : n < \omega\} \setminus \sum_{n \in B_\ell} \kappa_{\alpha(n)}) = \{\lambda\}$  so we are done).  $\square$

Now let us go back to the general case, assuming  $\text{cf}(\alpha^*) > \aleph_0$ . Our problem is the possibility that

$$\mathcal{P}(\{\lambda_n : n < \omega\})/J_{<\lambda}[\{\lambda_n : n < \omega\}].$$

is finite. Let now  $A_\alpha =: \{n : \lambda_n \in \mathfrak{a}_\alpha\}$ , and

$$\begin{aligned} J_\alpha &=: \{A \subseteq \omega : \max(\text{pcf}\{\lambda_\ell : \ell \in A\}) < \kappa_\alpha\} \\ J'_\alpha &=: \{A \subseteq \omega : \max \text{pcf}(\{\lambda_\ell : \ell \in A\} \cap \mathfrak{a}_{\kappa_\alpha}) < \kappa_\alpha\}. \end{aligned}$$

For  $T \in \mathfrak{K}_\lambda^{tr}$ ,  $x \in \text{lev}_\omega(T)$ ,  $\alpha < \alpha^*$  and  $a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta$ :

$$\begin{aligned} \xi_\alpha^*(x, a, T) &=: \min \left\{ \xi : \bigvee_m [\langle F_\ell^T(x) : \ell < \omega \rangle <_{J'_\alpha} g_{\alpha', a'}] \text{ where} \right. \\ &\quad \left. \alpha' = \alpha_{a, \omega \xi + m} \text{ and } a' = a \cap \alpha' \right\}. \end{aligned}$$

Let

$$\text{inv}_\alpha(x, a, T) =: \langle \xi_{\alpha+n}^*(x, a, T) : n < \omega \rangle,$$

$$\text{INV}(a, T) =: \{\text{inv}_\alpha(x, a, T) : x \in T \ \& \ \alpha < \alpha^* \ \& \ \text{lev}_T(x) = \omega\},$$

and

$\text{INV}(T) = \{c \in \prod_{n < \omega} \lambda_n : \text{each } \xi \in \text{Rang}(i) \text{ is a succession ordinal and for every club } E^* \text{ of } \lambda \text{ for some } a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta \text{ with } \text{otp}(a) = \mu \text{ and } a \subseteq E^* \text{ for arbitrarily large } \alpha < \alpha^*, \text{ there is } x \in \text{lev}_\omega(T) \text{ such that } \text{inv}_\alpha(x, a, T) = c\}$ .

As before, the point is to prove the Main Fact.

*Proof of the Main Fact in general* Suppose  $\mathbf{h} : T^1 \rightarrow T^2$  and  $c \in \text{INV}(T^1) \setminus \text{INV}(T^2)$ . Let  $E'$  be a club of  $\lambda$  which witnesses that  $c \notin \text{INV}(T^2)$ . We define  $E_n, E_\omega$  as before, as well as  $E^*$  ( $\subseteq E_\omega$ ). Now let us choose  $a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta$  with  $a \subseteq E^*$  and  $\text{otp}(a) = \mu$  a witnessing  $c \in \text{INV}(T')$  for the club  $E^*$ . So  $a = \{\alpha_{a,\xi} : \xi < \mu\}$ , which we shorten as  $a = \{\alpha_\xi : \xi < \mu\}$ . For each  $\xi < \mu$ , as before, we know that

$$(g_{\alpha_\xi, a \cap \alpha_\xi})^+ <_{J_{\alpha^*}} g_{\alpha_{\xi+1}, a \cap \alpha_{\xi+1}}.$$

By the definition of  $<_{J_{\alpha^*}}$ , there are  $\beta_{\xi,\ell} < \alpha^*$  ( $\ell < \ell_\xi$ ) such that

$$\{\ell : g_{\alpha_\xi, a \cap \alpha_\xi}^+(\ell) \geq g_{\alpha_{\xi+1}, a \cap \alpha_{\xi+1}}(\ell)\} \subseteq \bigcup_{\ell < \ell_\xi} A_{\beta_{\xi,\ell}}.$$

Let  $c = \langle \xi_n : n < \omega \rangle$  and let

$$\Upsilon = \{\beta_{\xi,\ell} : \text{for some } n \text{ and } m \text{ we have } \xi = \omega\xi_n + m \text{ \& } \ell < \ell_\xi \text{ or}$$

$\xi = \omega(\xi_n - 1) + m \& \ell < \ell_\xi\}$ . Thus  $\Upsilon \subseteq \alpha^*$  is countable. Since  $\text{cf}(\alpha^*) > \aleph_0$ , the set  $\Upsilon$  is bounded in  $\alpha^*$ . Now we know that  $c$  appears as an invariant for  $a$  and arbitrarily large  $\delta < \alpha^*$ , for some  $x_{a,\delta} \in \text{lev}_\omega(T_1)$  we have  $c = \text{inv}_\alpha(x_{a,\delta}, a, T_1)$ . If  $\delta > \sup(\Upsilon)$ ,  $c \in \text{INV}(T^2)$  is exemplified by  $a, \delta, \mathbf{h}(x_{a,\delta})$ , just as before.  $\square$

We still have to prove that every  $c = \langle \xi_n : n < \omega \rangle$  appears as an invariant; i.e. the parallel of Fact A.

*Proof of Fact A in the general case:* Define for each  $a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta$  with  $\text{otp}(a) = \mu$  and  $\beta < \alpha^*$

$$x_{c,a,\beta} = \langle x_{c,a,\beta}(\ell) : \ell < \omega \rangle,$$

where

$$x_{c,a,\beta}(\ell) = \begin{cases} \alpha_{a,\omega\xi_n + \delta} & \text{if } \lambda_\ell \in \mathfrak{a}_{\beta+k} \setminus \bigcup_{k' < k} \mathfrak{a}_{\beta+k'} \\ 0 & \text{if } \lambda_\ell \notin \mathfrak{a}_{\beta+k} \text{ for any } k < \omega. \end{cases}$$

Form the tree as before. Now for any club  $E$  of  $\lambda$ , we can find  $a \in \bigcup_{\delta < \lambda} \mathcal{P}_\delta$  with  $\text{otp}(a) = \mu$ ,  $a \subseteq E$  such that  $\langle x_{c,a,\beta} : \beta < \alpha^* \rangle$  shows that  $c \in \text{INV}(T)$ .  $\square_{7.1}$

*Remark 7.3.* (1) Clearly, this proof shows not only that there is no one  $T$  which is universal for  $\mathfrak{R}_\lambda^{tr}$ , but that any sequence of  $\langle \prod_{n < \omega} \lambda_n$  trees will fail. This occurs generally in this paper, as we have tried to mention in each particular case.

- (2) The case “ $\lambda < 2^{\aleph_0}$ ” is included in the theorem, though for the Abelian group application the  $\bigwedge_{n < \omega} \lambda_n^{\aleph_0} < \lambda_{n+1}$  is necessary.

*Remark 7.4.* (1) If  $\mu^+ < \lambda = \text{cf}(\lambda) < \chi < \mu^{\aleph_0}$  and  $\chi^{[\lambda]} < \mu^{\aleph_0}$  (or at least  $T_{\text{id}^a(\bar{C})}(\chi) < \mu^{\aleph_0}$ ) we can get the results for “no  $M \in \mathfrak{K}_\chi^x$  is universal for  $\mathfrak{K}_\lambda^x$ ”, see §8 (and [She96c]).

We can below (and subsequently in §8) use  $J_m^3$  as in §6.

**Theorem 7.5.** (1) Assume that  $2^{\aleph_0} < \lambda_0$ ,  $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle^\wedge \langle \lambda \rangle$ ,  $\mu = \sum_{n < \omega} \lambda_n$ ,  $\lambda_n < \lambda_{n+1}$ ,  $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$ .  
 If, for simplicity,  $\bar{m} = \langle m_i : i < \omega \rangle$  and  $\prod m_i < \lambda_0$  with  $m_i \in [2, \lambda_0)$  and  $\mathbf{U}_{J_m^2}^{<\mu}(\lambda) = \lambda$  (remember

$$J_m^2 = \{A \subseteq \prod_{i < \omega} m_i : A \text{ is nowhere dense}\}$$

and definition 5.3),

then in  $\mathfrak{K}_\lambda^{\text{tr}}$  there is no universal member (hence also in  $\mathfrak{K}_{\langle \lambda : i \leq \omega \rangle}^{\text{tr}}$ ).

- (2) If we replace  $\mathbf{U}_{J_m^2}^{<\mu}(\lambda) = \lambda$  by  $\mathbf{U}_{J_m^2}^{<\mu}(\lambda) < \text{cov}(\mu, \mu, \aleph_1, 2)$  whee  $\Gamma \in (J_m^2)^+$ ,  $J_m' = J_m^2 \upharpoonright \Gamma$  and  $|\Gamma| < \mathbf{U}_{J_m'}^{<\Gamma}(\lambda)$  and  $\mu^* \leq \mu$ ,  $(\forall \mu_1)[\mu_1 < \mu \Rightarrow \mu_1^{\aleph_0} < \mu^*]$  then the conclusion still holds.

*Proof.* 1) Let  $\lambda^* = \mathbf{U}_{J_m^2}^{<\mu}(\lambda)$ . Let  $S \subseteq \lambda$ ,  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  be a club guessing sequence on  $\lambda$  with  $\text{otp}(C_\delta) \geq \mu$ . We can choose  $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle$ , satisfying  $\{J_m^2, \bar{C}, \langle \lambda_n : n < \omega \rangle, \mu, \lambda, \bar{m}, T^*\}$  belong to  $\mathfrak{A}_0$  ( $T^*$  is a candidate for the universal), and  $\mathfrak{A}_\alpha \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ ,  $\chi = \beth_7(\lambda)^+$ ,  $\|\mathfrak{A}_\alpha\| < \lambda$ ,  $\mathfrak{A}_\alpha$  increasingly continuous,  $\langle \mathfrak{A}_\beta : \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$ ,  $\mathfrak{A}_\alpha \cap \lambda$  is an ordinal, and

$$E =: \{\alpha : \mathfrak{A}_\alpha \cap \lambda = \alpha\}.$$

Choose  $\mathfrak{A} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ , satisfying  $\lambda^* + 1 \subseteq \mathfrak{A}$ ,  $\lambda^* = \|\mathfrak{A}\|$ , and  $\bigcup_{\alpha < \lambda} \mathfrak{A}_\alpha \prec \mathfrak{A}$

Note that  $\Gamma \subseteq \mathfrak{A}$ . Without loss of generality  $T^*$  satisfies

- (\*)<sub>1</sub>:  $\text{lev}_\alpha(T^*) \subseteq {}^\alpha \lambda$  for  $\alpha \leq \omega$  and  $x \leq_T y$  means  $x \leq y$  and for  $\eta \in T^*$  such that  $\text{lg}(\eta) = n$  let  $\text{Suc}_{T^*}(\eta) = \{\eta' \langle \alpha \rangle : \alpha < \lambda_n\}$
- (\*)<sub>2</sub>: for every  $\eta \in \text{lev}_\omega(T^*)$  and  $k < \omega$ , there are  $\lambda$  members  $\nu$  of  $\text{lev}_\omega(T^*)$  such that  $\nu \upharpoonright k = \eta \upharpoonright k$

NOTE: By  $\mathbf{U}_{J_m^2}^{<\mu}(\lambda) = \lambda^*$ ,

- (\*) : if  $x_\eta \in \text{lev}_\omega(T^*)$  for  $\eta \in \Gamma \subseteq \prod \bar{m}$  then for some  $A \in (J_m')^+$  the set  $\langle (\eta, x_\eta) : \eta \in A \rangle$  belongs to  $\mathfrak{A}$ , hence  $\{x_\eta(n) : \eta \in A \text{ and for some } \nu \in A \text{ we have } \text{lg}(\eta \cap \nu) = n\}$  belongs to  $\mathfrak{A}$ . Hence: if the mapping  $\eta \mapsto x_\eta$  for  $\eta \in \Gamma$  is continuous then  $\langle x_\rho : \rho \in \text{cl}_\Gamma(t) \rangle \in \mathfrak{A}$ .

We let

$$P^0 = P^0(\mathfrak{A}) = \left\{ \bar{x} : \bar{x} = \langle x_\rho : \rho \in t \rangle \in \mathfrak{A} \text{ and } x_\rho \in \text{lev}_{\ell g(\rho)}(T^*), t \in (J'_{\bar{m}})^* \text{ and} \right. \\ \left. \begin{array}{l} \text{the mapping } \rho \mapsto x_\rho \text{ preserves all of the relations} \\ \ell g(\rho_1 \cap \rho_2) = n \end{array} \right\}.$$

Assume  $\bar{x} = \langle x_\rho : \rho \in t \rangle \in P^0$ . For  $\delta \in S$  such that  $C_\delta \subseteq E$  let

$$\text{inv}(\bar{x}, C_\delta, T^*, \bar{\mathfrak{A}}) =: \{ \alpha \in C_\delta : (\exists \rho \in \text{Dom}(\bar{x}))(x_\rho \in \mathfrak{A}_{\min(C_\delta \setminus (\alpha+1))} \setminus \mathfrak{A}_\alpha) \}.$$

Let

$$\text{Inv}(C_\delta, T^*, \mathfrak{A}) =: \\ \{ a : \text{for some } \bar{x} \in P^0, a = \text{inv}(\bar{x}, C_\delta, T^*, \mathfrak{A}) \text{ is a countable set} \}.$$

Note:  $\text{inv}(\bar{x}, C_\delta, T, \mathfrak{A})$  has cardinality at most continuum, so  $\text{Inv}(C_\delta, T^*, \mathfrak{A})$  is a family of  $\leq 2^{\aleph_0} \times |\mathfrak{A}| = \lambda^*$  countable subsets of  $C_\delta$ .

We continue as before. Let  $\alpha_{\delta,\varepsilon}$  be the  $\varepsilon$ -th member of  $C_\delta$  for  $\varepsilon < \mu$ . So as  $\lambda^* < \text{cov}(\mu, \mu, \aleph_1, 2)$  we can find  $\gamma_n < \mu$  for  $n < \omega$  such that:  $a \in \mathfrak{A} \cap [\mu]^{<\mu} \Rightarrow \aleph_0 > |\{\gamma_n : n < \omega\} \cap a|$  finite; how? using coding of finite sequences. Hence we can find  $\gamma_n \in (\bigcup_{\ell < n} \lambda_\ell, \lambda_n)$  limit such that for each  $\delta \in S$

and  $a \in \text{Inv}(C_\delta, T^*, \bar{\mathfrak{A}})$  we have  $\{\gamma_n + \ell : n < \omega \text{ and } \ell < m_i\} \cap \{\text{otp}(\alpha \cap C_\delta) : \alpha \in a\}$  is bounded in  $\mu$ .

Let  $\Xi_n = \{ \rho \in \prod_{\bar{m}} : n = \sup\{\ell : \eta(\ell) \neq 0\} \}$ .

For each  $\delta \in S^* = \{ \delta \in S : C_\delta \subseteq E \}$  we choose by induction on  $n < \omega$  for each  $\rho \in \Xi_n$ , a member  $y_{\delta,\rho} \in \text{lev}_\omega(T^*) \cap (\mathfrak{A}_{\alpha_\delta, \gamma_{\ell+1}} \setminus \mathfrak{A}_{\alpha_\delta, \gamma_\ell})$  such that for  $\rho_1, \rho_2 \in \bigcup_{m \leq n} \Xi_m$  and  $k$  we have  $[\rho_1 \upharpoonright k = \rho_2 \upharpoonright k \Leftrightarrow y_{\delta, \rho_1 \upharpoonright k} = y_{\delta, \rho_2 \upharpoonright k}]$ . Let for  $\rho \in \prod_{\bar{m}}, x_\rho^\delta$  be the unique  $x \in \prod_{n < \omega} \lambda_n$  such that  $\nu \in \bigcup_n \Xi_n \& \rho \upharpoonright k = \nu \upharpoonright k \Rightarrow x \upharpoonright k = y_{\delta, \nu}$ , so  $x_\rho^\delta = y_{\delta, \rho}$  for  $\rho \in \bigcup_n \Xi_n$ .

Now we can find  $T$  such that  $\text{lev}_n(T) = \text{lev}_n(T^*)$  for  $n < \omega$  and

$$\text{lev}_\omega(T) = \{ x_\rho^\delta : \delta \in S^* \text{ and } \rho \in \prod_{\bar{m}} \bar{m} \}$$

So, if  $T^*$  is universal then there is an embedding  $f : T \rightarrow T^*$ , define  $\mathfrak{A}'_\alpha$  (for  $\alpha < \lambda$ ) as before such that  $\mathfrak{A}_\delta, \langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle \in \mathfrak{A}'_0$  and  $T, f \in \mathfrak{A}'_0$ , so  $\mathfrak{A}'_\alpha$  is closed under  $f$  and  $f^{-1}$  and hence

$$E' = \{ \alpha \in E : \mathfrak{A}'_\alpha \cap \lambda = \alpha \}$$

is a club of  $\lambda$ . By the choice of  $\bar{C}$  for some  $\delta \in S$  we have  $C_\delta \subseteq E'$ . Now use (\*) with  $x_\eta = f(x_\eta^\delta)$ . Thus we get  $A \in (J'_{\bar{m}})^+$  such that  $\{(\eta, x_\eta) : \eta \in A\} \subseteq b \in \mathfrak{A}, |b| \leq \mu^*$ . We can assume  $\{(\eta, x_\eta) : \eta \in A\} \in \mathfrak{A}$  so there is  $\nu \in \bigcup_{k \in \mathbb{N}} \prod_{i < k} m_i$  such that  $A$  is dense above  $\nu$ , hence as  $f$  is continuous,

$\langle (\eta, x_\eta) : \nu \triangleleft \eta \in \Gamma \rangle \in \mathfrak{A}$ . So  $\langle x_\eta : \eta \in \Gamma, \nu \trianglelefteq \eta \rangle \in P^0(\mathfrak{A})$ , and hence the set

$$\{ \alpha_{\delta, \gamma_\ell} : \nu \triangleleft \rho_\ell \}$$

is included in  $\text{inv}(\bar{x}, C_\delta, T^*, \mathfrak{A})$ . Hence

$$a = \{\alpha_{\delta, \gamma_\ell} : \nu \triangleleft \rho_\ell\} \in \text{Inv}(C_\delta, T^*, \mathfrak{A}),$$

contradicting

“ $\{\alpha_{\delta, \gamma_\ell} : \ell < \omega\}$  has finite intersection with any  $a \in \text{Inv}(C_\delta, T^*, \mathfrak{A})$ ”.

*Remark 7.6.* We can a priori fix a set of  $\aleph_0$  candidates and say more on their order of appearance, so that  $\text{Inv}(\bar{x}, C_\delta, T^*, \bar{\mathfrak{A}})$  has order type  $\omega$ . This makes it easier to phrase a true invariant, i.e.  $\langle \langle \eta_n, t_n \rangle : n < \omega \rangle$  is as above,  $\langle \eta_n : n < \omega \rangle$  lists  ${}^\omega > \omega$  with no repetition,  $\langle t_n \cap {}^\omega \omega : n < \omega \rangle$  are pairwise disjoint. If  $x_\rho \in \text{lev}_\omega(T^*)$  for  $\rho \in {}^\omega \omega$ ,  $\bar{T}^* = \langle \bar{T}_\zeta^* : \zeta < \lambda \rangle$  representation

$$\begin{aligned} & \text{inv}(\langle x_\rho : \rho \in {}^\omega \omega \rangle, C_\delta, \bar{T}^*) = \\ & \{\alpha \in C_\delta : \text{for some } n, (\forall \rho)[\rho \in t_n \cap {}^\omega \omega \Rightarrow x_\rho \in T_{\min(C_\delta \setminus (\alpha+1))}^* \setminus T_\alpha^*]\}. \end{aligned}$$

□<sub>7.5</sub>

By 7.5 and [She00] we get

*nice? Conclusion 7.7.* If  $\beth_\omega < \lambda_{n+1} < \mu = \sum_n \lambda_n$  and  $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$  and  $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle \wedge \lambda$  then in  $\mathfrak{K}_\lambda^{\text{tr}}$  there is no universal member.

+-+

*Remark 7.8.* (1) We can in 7.7 and earlier deal with simpler cardinals, but it is more cumbersome.

(2) We can give sufficient conditions for having  $T_\alpha \in \mathfrak{K}_\lambda^{\text{tr}}$  for  $\alpha < \lambda^{**}$ , such that into no  $T^* \in \mathfrak{K}_\lambda^{\text{tr}}$  can we embed  $> \mathbf{U}_{J_{m|\Gamma}^2}(\lambda)$  of then [see, more, see [She01] end of section 2]

(3) If we like to use e.g.  $|\Gamma| < 2^{\aleph_0}$ , we need to deal with a closure of a copy of  $\Gamma$ .

## 8. UNIVERSALS IN SINGULAR CARDINALS

In §3, §5, 7.5, we can in fact deal with “many” singular cardinals  $\lambda$ . This is done by proving a stronger assertion on some regular  $\lambda$ . Here  $\mathfrak{K}$  is a class of models.

**Lemma 8.1.** (1) *There is no universal member in  $\mathfrak{K}_{\mu^*}$  if for some  $\lambda < \mu^*$ ,  $\theta \geq 1$  we have:*

$\otimes_{\lambda, \mu^*, \theta}[\mathfrak{K}]$ : *not only there is no universal member in  $\mathfrak{K}_\lambda$  but if we assume:*

$$\langle M_i : i < \theta \rangle \text{ is given, } \|M_i\| \leq \mu^* < \prod_n \lambda_n, M_i \in \mathfrak{K},$$

*then there is a structure  $M$  from  $\mathfrak{K}_\lambda$  (in some cases of a simple form) not embeddable in any  $M_i$ .*

(2) *Assume*

$\otimes_1^\sigma$ :  $\langle \lambda_n : n < \omega \rangle$  is given,  $\lambda_n^{\aleph_0} < \lambda_{n+1}$ ,

$$\mu = \sum_{n < \omega} \lambda_n < \lambda = \text{cf}(\lambda) \leq \mu^* < \prod_{n < \omega} \lambda_n$$

and  $\mu^+ < \lambda$  or at least there is a club guessing  $\bar{C}$  as in  $(**)_{\lambda}^1$  (ii) of 3.4 for  $(\lambda, \mu)$ .

Then there is no universal member in  $\mathfrak{K}_{\mu^*}$  (and moreover  $\otimes_{\lambda, \mu^*, \theta}[\mathfrak{K}]$  holds) in the following cases

**(a)**: for torsion free groups, i.e.  $\mathfrak{K} = \mathfrak{K}_{\lambda}^{rtf}$  if  $\text{cov}(\mu^*, \lambda^+, \lambda^+, \lambda) < \prod_{n < \omega} \lambda_n$ , see notation 0.4 on cov)

**(b)**: for  $\mathfrak{K} = \mathfrak{K}_{\lambda}^{tcf}$ ,

**(c)**: for  $\mathfrak{K} = \mathfrak{K}_{\lambda}^{tr}$  as in 7.5 -  $\text{cov}(\mathbf{U}_{J_m^3}(\mu^*), \lambda^+, \lambda^+, \lambda) < \prod_{n < \omega} \lambda_n$ ,

**(d)**: for  $\mathfrak{K}_{\lambda}^{rs(p)}$ : like case (c) (for appropriate ideals), replacing  $tr$  by  $rs(p)$ .

*Remark 8.2.* (1) For 7.5 as  $\bar{m} = \langle \omega : i < \omega \rangle$  it is clear that the subtrees  $t_n$  are isomorphic. We can use  $m_i \in [2, \omega)$ , and use coding; anyhow it is immaterial since  ${}^\omega\omega, {}^\omega 2$  are similar.

(2) We can also vary  $\bar{\lambda}$  in 8.1  $\otimes_2$ , case (c).

(3) We can replace cov in  $\otimes_2(a),(c)$  by

$$\sup \text{pp}_{\Gamma(\lambda)}(\chi) : \text{cf}(\chi) = \lambda, \lambda < \chi \leq \mathbf{U}_{J_m^3}(\mu^*)\}$$

(see [She94a, 5.4], 2.4).

*Proof.* Should be clear, e.g.

*Proof of Part 2), Case (c)* Let  $\langle T_i : i < i^* \rangle$  be given,  $i^* < \prod_{n < \omega} \lambda_n$  such that

$$\|T_i\| \leq \mu^* \quad \text{and} \quad \mu^\otimes =: \text{cov}(\mathbf{U}_{J_m^3}(\mu^*), \lambda^+, \lambda^+, \lambda) < \prod_{n < \omega} \lambda_n.$$

By [She94a, 5.4] and pp calculus ([She94a, 2.3]),  $\mu^\otimes = \text{cov}(\mu^\otimes, \lambda^+, \lambda^+, \lambda)$ . Let  $\chi = \beth_7(\lambda)^+$ . For  $i < i^*$  choose  $\mathfrak{B}_i \prec (H(\chi) \in \langle \chi \rangle^*)$ ,  $\|\mathfrak{B}_i\| = \mu^\otimes$ ,  $T_i \in \mathfrak{B}_i$ ,  $\mu^\otimes + 1 \subseteq \mathfrak{B}_i$ . Let  $\langle Y_\alpha : \alpha < \mu^\otimes \rangle$  be a family of subsets of  $T_i$  exemplifying the Definition of  $\mu^\otimes = \text{cov}(\mu^\otimes, \lambda^+, \lambda^+, \lambda)$ .

Given  $\bar{x} = \langle x_\eta : \eta \in {}^\omega\omega \rangle$ ,  $x_\eta \in \text{lev}_\omega(T_i)$ ,  $\eta \mapsto x_\eta$  continuous (in our case this means  $\ell g(\eta_1 \cap \eta_2) = \ell g(x_{\eta_1} \cap x_{\eta_2}) =: \ell g(\max\{\rho : \rho \triangleleft \eta_1 \ \& \ \rho \triangleleft \eta_2\})$ ). Then for some  $\eta \in {}^{>\omega}\omega$ ,

$$\langle x_\rho : \eta \triangleleft \rho \in {}^\omega\omega \rangle \in \mathfrak{B}.$$

So given  $\langle \langle x_\eta^\zeta : \eta \in {}^\omega\omega \rangle : \zeta < \lambda \rangle$ ,  $x_\eta^\zeta \in \text{lev}_\omega(T_i)$  we can find  $\langle (\alpha_j, \eta_j) : j < j^* < \lambda \rangle$  such that:

$$\bigwedge_{\zeta < \lambda} \bigvee_j \langle x_\eta^\zeta : \eta_j \triangleleft \eta \in {}^\omega\omega \rangle \in Y_\alpha.$$

Closing  $Y_\alpha$  enough we can continue as usual.

□<sub>8.1</sub>

## 9. METRIC SPACES AND IMPLICATIONS

**Definition 9.1.** (1)  $\mathfrak{K}^{mt}$  is the class of metric spaces  $M$  (i.e.  $M = (|M|, d)$ ,  $|M|$  is the set of elements,  $d$  is the metric, i.e. a two-place function from  $|M|$  to  $\mathbb{R}^{\geq 0}$  such that  $d(x, y) = 0 \Leftrightarrow x = y$  and  $d(x, z) \leq d(x, y) + d(y, z)$  and  $d(x, y) = d(y, x)$ ).

An embedding  $f$  of  $M$  into  $N$  is a one-to-one function from  $|M|$  into  $|N|$  which is continuous, i.e. such that:

if in  $M$ ,  $\langle x_n : n < \omega \rangle$  converges to  $x$

then in  $N$ ,  $\langle f(x_n) : n < \omega \rangle$  converges to  $f(x)$ .

(2)  $\mathfrak{K}^{ms}$  is defined similarly but  $\text{Rang}(d) \subseteq \{2^{-n} : n < \omega\} \cup \{0\}$  and instead of the triangular inequality we require

$$d(x, y) = 2^{-i}, \quad d(y, z) = 2^{-j} \quad \Rightarrow \quad d(x, z) \leq 2^{-\min\{i-1, j-1\}}.$$

(3)  $\mathfrak{K}^{tr[\omega]}$  is like  $\mathfrak{K}^{tr}$  but  $P_\omega^M = |M|$  and embeddings preserve  $x E_n y$  (not necessarily its negation) are one-to-one, and remember  $\bigwedge_n x E_n y \Rightarrow x \upharpoonright n = y \upharpoonright n$ .

(4)  $\mathfrak{K}^{mt(c)}$  is the class of semi-metric spaces  $M = (|M|, d)$ , which means that for the constant  $c \in \mathbb{R}^+$  the triangular inequality is weakened to  $d(x, z) \leq cd(x, y) + cd(y, z)$  with embedding as in 9.1(1) (so for  $c = 1$  we get  $\mathfrak{K}^{mt}$ ).

(5)  $\mathfrak{K}^{mt[c]}$  is the class of pairs  $(A, d)$  such that  $A$  is a non-empty set,  $d$  a two-place symmetric function from  $A$  to  $\mathbb{R}^{\geq 0}$  such that  $[d(x, y) = 0 \Leftrightarrow x = y]$  and

$$d(x_0, x_n) \leq c \sum_{\ell < n} d(x_\ell, x_{\ell+1}) \quad \text{for any } n < \omega \text{ and } x_0, \dots, x_n \in A.$$

(6)  $\mathfrak{K}^{ms(c)}$ ,  $\mathfrak{K}^{ms[c]}$  are defined parallelly.

(7)  $\mathfrak{K}^{rs(p), \text{pure}}$  is defined like  $\mathfrak{K}^{rs(p)}$  but the embeddings are pure.

*Remark 9.2.* There are, of course, other notions of embeddings; isometric embeddings if  $d$  is preserved, co-embeddings if the image of an open set is open, bi-continuous means an embedding which is a co-embedding. The isometric embedding is the weakest, its case is essentially equivalent to the  $\mathfrak{K}_\lambda^{tr}$  case (as in 9.7(3)); for the open case there is a universal: discrete space. The universal for  $\mathfrak{K}_\lambda^{mt}$  under bicontinuous case exist in cardinality  $\lambda^{\aleph_0}$ , see [Kow57].

**Definition 9.3.** (1)  $\text{Univ}^0(\mathfrak{K}^1, \mathfrak{K}^2) = \{(\lambda, \kappa, \theta) : \text{there are } M_i \in \mathfrak{K}_\kappa^2 \text{ for } i < \theta \text{ such that any } M \in \mathfrak{K}_\lambda^1 \text{ can be embedded into some } M_i\}$ . We may omit  $\theta$  if it is 1. We may omit the superscript 0.

(2)  $\text{Univ}^1(\mathfrak{K}^1, \mathfrak{K}^2) = \{(\lambda, \kappa, \theta) : \text{there are } M_i \in \mathfrak{K}_\kappa^2 \text{ for } i < \theta \text{ such that any } M \in \mathfrak{K}_\lambda^1 \text{ can be represented as the union of } < \lambda \text{ sets } A_\zeta \text{ } (\zeta < \zeta^* < \lambda) \text{ such that each } M \upharpoonright A_\zeta \text{ can be embedded into some } M_i\}$  and is a  $\leq_{\mathfrak{K}^1}$ -submodel of  $M$ .

- (3) If above  $\aleph^1 = \aleph^2$  we write it just once; (naturally we usually assume  $\aleph^1 \subseteq \aleph^2$ ).

*Remark 9.4.* (1) We prove our theorems for  $Univ^0$ , we can get parallel things for  $Univ^1$ .

- (2) Many previous results of this paper can be rephrased using a pair of classes.  
 (3) We can make 9.5 below deal with pairs and/or function  $H$  changing cardinality.  
 (4)  $Univ^\ell$  has the obvious monotonicity properties.

**Proposition 9.5.** (1) Assume  $\aleph^1, \aleph^2$  has the same models as their members and every embedding for  $\aleph^2$  is an embedding for  $\aleph^1$ .

Then  $Univ(\aleph^2) \subseteq Univ(\aleph^1)$ .

- (2) Assume there is for  $\ell = 1, 2$  a function  $H_\ell$  from  $\aleph^\ell$  into  $\aleph^{3-\ell}$  such that:

- (a):  $\|H_1(M_1)\| = \|M_1\|$  for  $M_1 \in \aleph^1$ ,  
 (b):  $\|H_2(M_2)\| = \|M_2\|$  for  $M_2 \in \aleph^2$ ,  
 (c): if  $M_1 \in \aleph^1$ ,  $M_2 \in \aleph^2$ ,  $H_1(M_1) \in \aleph^2$  is embeddable into  $M_2$  then  $M_1$  is embeddable into  $H_2(M_2) \in \aleph^1$ .

Then  $Univ(\aleph^2) \subseteq Univ(\aleph^1)$ .

**Definition 9.6.** We say  $\aleph^1 \leq \aleph^2$  if the assumptions of 9.5(2) hold. We say  $\aleph^1 \equiv \aleph^2$  if  $\aleph^1 \leq \aleph^2 \leq \aleph^1$  (so larger means with fewer cases of universality).

**Theorem 9.7.** (1) The relation " $\aleph^1 \leq \aleph^2$ " is a quasi-order (i.e. transitive and reflexive).

- (2) If  $(\aleph^1, \aleph^2)$  are as in 9.5(1) then  $\aleph^1 \leq \aleph^2$  (use  $H_1 = H_2 =$  the identity).  
 (3) For  $c_1 > 1$  we have  $\aleph^{mt(c_1)} \equiv \aleph^{mt[c_1]} \equiv \aleph^{ms[c_1]} \equiv \aleph^{ms(c_1)}$ .  
 (4)  $\aleph^{tr[\omega]} \leq \aleph^{rs(p)}$ .  
 (5)  $\aleph^{tr[\omega]} \leq \aleph^{tr(\omega)}$ .  
 (6)  $\aleph^{tr(\omega)} \leq \aleph^{rs(p), pure}$ .

*Proof.* 1) Check.

2) Check.

3) Choose  $n(*) < \omega$  large enough and  $\aleph^1, \aleph^2$  any two of the four. We define  $H_1, H_2$  as follows.  $H_1$  is the identity. For  $(A, d) \in \aleph^\ell$  let  $H_\ell((A, d)) = (A, d^{[\ell]})$  where  $d^{[\ell]}(x, y) = \inf\{1/(n+n(*)) : 2^{-n} \geq d(x, y)\}$  (the result is not necessarily a metric space,  $n(*)$  is chosen so that the semi-metric inequality holds). The point is to check clause (c) of 9.5(2); so assume  $f$  is a function which  $\aleph^2$ -embeds  $H_1((A_1, d_1))$  into  $(A_2, d_2)$ ; but

$$H_1((A_1, d_1)) = (A_1, d_1), \quad H_2((A_2, d_2)) = (A_2, d_2^{[2]}),$$

so it is enough to check that  $f$  is a function which  $\aleph^1$ -embeds  $(A_1, d_1^{[1]})$  into  $(A_2, d_2^{[2]})$  i.e. it is one-to-one (obvious) and preserves limit (check).

4) For  $M = (A, E_n)_{n < \omega} \in \mathfrak{K}^{tr[\omega]}$ , without loss of generality  $A \subseteq {}^\omega \lambda$  and

$$\eta E_n \nu \quad \Leftrightarrow \quad \eta \in A \ \& \ \nu \in A \ \& \ \eta \upharpoonright n = \nu \upharpoonright n.$$

Let  $B^+ = \{\eta \upharpoonright n : \eta \in A \text{ and } n < \omega\}$ . We define  $H_1(M)$  as the (Abelian) group generated by

$$\{x_\eta : \eta \in A \cup B\} \cup \{y_{\eta,n} : \eta \in A, n < \omega\}$$

freely except

$$\begin{array}{lll} p^{n+1}x_\eta = 0 & \text{iff} & \eta \in B, \ell g(\eta) = n \\ y_{\eta,0} = x_\eta & \text{iff} & \eta \in A \\ py_{\eta,n+1} - y_{\eta,n} = x_{\eta \upharpoonright n} & \text{iff} & \eta \in A, n < \omega \\ p^{n+1}y_{\eta,n} = 0 & \text{iff} & \eta \in B, n < \omega. \end{array}$$

For  $G \in \mathfrak{K}^{rs(p)}$  let  $H_2(G)$  be  $(A, E_n)_{n < \omega}$  with:

$$A = G, \quad x E_n y \quad \text{iff} \quad G \models \text{“}p^n \text{ divides } (x - y)\text{”}.$$

$H_2(G) \in \mathfrak{K}^{tr[\omega]}$  as “ $G$  is separable” implies  $(\forall x)(x \neq 0 \Rightarrow (\exists n)[x \notin p^n G])$ . Clearly clauses (a), (b) of Definition 9.1(2) hold. As for clause (c), assume  $(A, E_n)_{n < \omega} \in \mathfrak{K}^{tr[\omega]}$ . As only the isomorphism type counts without loss of generality  $A \subseteq {}^\omega \lambda$ . Let  $B = \{\eta \upharpoonright n : n < \omega : \eta \in A\}$  and  $G = H_1((A, E_n)_{n < \omega})$  be as above. Suppose that  $f$  embeds  $G$  into some  $G^* \in \mathfrak{K}^{rs(p)}$ , and let  $(A^*, E_n^*)_{n < \omega}$  be  $H_2(G^*)$ . We should prove that  $(A, E_n)_{n < \omega}$  is embeddable into  $(A^*, E_n^*)_{n < \omega}$ .

Let  $f^* : A \rightarrow A^*$  be  $f^*(\eta) = x_\eta \in A^*$ . Clearly  $f^*$  is one to one from  $A$  to  $A^*$ ; if  $\eta E_n \nu$  then  $\eta \upharpoonright n = \nu \upharpoonright n$  hence  $G \models p^n \upharpoonright (x_\eta - x_\nu)$  hence  $(A^*, A_n^*)_{n < \omega} \models \eta E_n^* \nu$ . □<sub>9.7</sub>

*Remark 9.8.* In 9.7(4) we can prove  $\mathfrak{K}_\lambda^{tr[\omega]} \leq \mathfrak{K}_\lambda^{rs(p)}$ .

**Theorem 9.9.** (1)  $\mathfrak{K}^{mt} \equiv \mathfrak{K}^{mt(c)}$  for  $c \geq 1$ .

(2)  $\mathfrak{K}^{mt} \equiv \mathfrak{K}^{ms[c]}$  for  $c > 1$ .

*Proof.* 1) Let  $H_1 : \mathfrak{K}^{mt} \rightarrow \mathfrak{K}^{mt(c)}$  be the identity. Let  $H_2 : \mathfrak{K}^{mt(c)} \rightarrow \mathfrak{K}^{mt}$  be defined as follows:

$H_2((A, d)) = (A, d^{mt})$ , where

$$\begin{aligned} d^{mt}(y, z) = \\ \inf \left\{ \sum_{\ell=0}^n d(x_\ell, x_{\ell,n}) : n < \omega \ \& \ x_\ell \in A \text{ (for } \ell \leq n) \ \& \ x_0 = y \ \& \ x_n = z \right\}. \end{aligned}$$

Now

(\*)<sub>1</sub>:  $d^{mt}$  is a two-place function from  $A$  to  $\mathbb{R}^{\geq 0}$ , is symmetric,  $d^{mt}(x, x) = 0$  and it satisfies the triangular inequality.

This is true even on  $\mathfrak{K}^{mt(c)}$ , but here also

(\*)<sub>2</sub>:  $d^{mt}(x, y) = 0 \Leftrightarrow x = 0$ .

[Why? As by the Definition of  $\mathfrak{K}^{mt[c]}$ ,  $d^{mt}(x, y) \geq \frac{1}{c}d(x, y)$ . Clearly clauses (a), (b) of 9.5(2) hold.]

Next,

(\*)<sub>3</sub>: If  $M_1, N \in \mathfrak{K}^{mt}$ ,  $f$  is an embedding (for  $\mathfrak{K}^{mt}$ ) of  $M_1$  into  $N$  then  $f$  is an embedding (for  $\mathfrak{K}^{mt[c]}$ ) of  $H_1(M)$  into  $H_1(N)$

[why? as  $H_1(M) = M$  and  $H_2(N) = N$ ],

(\*)<sub>4</sub>: If  $M, N \in \mathfrak{K}^{mt[c]}$ ,  $f$  is an embedding (for  $\mathfrak{K}^{mt[c]}$ ) of  $M$  into  $N$  then  $f$  is an embedding (for  $\mathfrak{K}^{mt}$ ) of  $H_2(M)$  into  $H_1(N)$

[why? as  $H_\ell^*$  preserves  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} x_n \neq x$ ].

So two applications of 9.5 give the equivalence.

2) We combine  $H_2$  from the proof of (1) and the proof of 9.7(3).  $\square_{9.9}$

**Definition 9.10.** (1) If  $\bigwedge_n \mu_n = \aleph_0$  let

$$J^{mt} = J_{\bar{\mu}}^{mt} = \left\{ A \subseteq \prod_{n < \omega} \mu_n : \begin{array}{l} \text{for every } n \text{ large enough,} \\ \text{for every } \eta \in \prod_{\ell < n} \mu_\ell \\ \text{the set } \{\eta'(n) : \eta \triangleleft \eta' \in A\} \text{ is finite} \end{array} \right\}.$$

(2) Let  $T = \bigcup_{\alpha \leq \omega} \prod_{n < \alpha} \mu_n$ ,  $(T, d^*)$  be a metric space such that

$$\prod_{\ell < n} \mu_\ell \cap \text{closure} \left( \bigcup_{m < n} \prod_{\ell < m} \mu_\ell \right) = \emptyset;$$

now

$$I_{(T, d^*)}^{mt} = \left\{ A \subseteq \prod_{n < \omega} \mu_n : \begin{array}{l} \text{for some } n, \text{ the closure of } A \text{ (in } (T, d^*)) \\ \text{is disjoint to } \bigcup_{m \in [n, \omega)} \prod_{\ell < m} \mu_\ell \end{array} \right\}.$$

(3) Let  $H \in \mathfrak{K}^{rs(p)}$ ,  $\bar{H} = \langle H_n : n < \omega \rangle$ ,  $H_n \subseteq H$  pure and closed,  $n < m \Rightarrow H_n \subseteq H_m$  and  $\bigcup_{n < \omega} H_n$  is dense in  $H$ . Let

$$I_{H, \bar{H}}^{rs(p)} = \left\{ A \subseteq H : \begin{array}{l} \text{for some } n \text{ the closure of } \langle A \rangle_H \text{ intersected with} \\ \bigcup_{\ell < \omega} H_\ell \text{ is included in } H_n \end{array} \right\}.$$

**Proposition 9.11.** Suppose that  $2^{\aleph_0} < \mu$  and  $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$  and

(\*) <sub>$\lambda$</sub> :  $\mathbf{U}_{J_{\bar{\mu}}^{mt}}(\lambda) = \lambda$  or at least  $\mathbf{U}_{J_{\bar{\mu}}^{mt}}(\lambda) < \lambda^{\aleph_0}$  for some  $\bar{\mu} = \langle \mu_n : n < \omega \rangle$  such that  $\prod_{n < \omega} \mu_n < \lambda$ .

Then  $\mathfrak{K}_\lambda^{mt}$  has no universal member.

**Proposition 9.12.** (1)  $J^{mt}$  is  $\aleph_1$ -based.

(2) The minimal cardinality of a set which is not in the  $\sigma$ -ideal generated by  $J^{mt}$  is  $\mathfrak{b}$ .

(3)  $I_{(T, d^*)}^{mt}, I_{H, \bar{H}}^{rs(p)}$  are  $\aleph_1$ -based.

(4)  $J^{mt}$  is a particular case of  $I_{(T, d^*)}^{mt}$  (i.e. for some choice of  $(T, d^*)$ ).

(5)  $I_{\bar{\mu}}^0$  is a particular case of  $I_{H,\bar{H}}^{rs(p)}$ .

*Proof.* of 9.11. Let

$$T_\alpha = \{(\eta, \nu) \in {}^\alpha\lambda \times {}^\alpha(\omega + 1) : \text{for every } n \text{ such that } n + 1 < \alpha \\ \text{we have } \nu(n) < \omega\}$$

and for  $\alpha \leq \omega$  let  $T = \bigcup_{\alpha \leq \omega} T_\alpha$ . We define on  $T$  the relation  $<_T$ :

$$(\eta_1, \nu_1) \leq (\eta_2, \nu_2) \quad \text{iff} \quad \eta_1 \trianglelefteq \eta_2 \ \& \ \nu_1 \triangleleft \nu_2.$$

We define a metric:

if  $(\eta_1, \nu_1) \neq (\eta_2, \nu_2) \in T$  and  $(\eta, \nu)$  is their maximal common initial segment and  $(\eta, \nu) \in T$  then necessarily  $\alpha = \text{lg}((\eta, \nu)) < \omega$  and we let:

if  $\eta_1(\alpha) \neq \eta_2(\alpha)$  then

$$d((\eta_1, \nu_1), (\eta_2, \nu_2)) = 2^{-\sum\{\nu(\ell): \ell < \alpha\}},$$

if  $\eta_1(\alpha) = \eta_2(\alpha)$  (so  $\nu_1(\alpha) \neq \nu_2(\alpha)$ ) then

$$d((\eta_1, \nu_1), (\eta_2, \nu_2)) = 2^{-\sum\{\nu(\ell): \ell < \alpha\}} \times 2^{-\min\{\nu_1(\alpha), \nu_2(\alpha)\}}.$$

Now, for every  $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$ , and  $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ ,  $\eta_\delta \in {}^\omega\delta$ ,  $\eta_\delta$  increasing let  $M_{\bar{\eta}}$  be  $(T, d) \upharpoonright A_{\bar{\eta}}$ , where

$$A_{\bar{\eta}} = \bigcup_{n < \omega} T_n \cup \{(\eta_\delta, \nu) : \delta \in S, \nu \in {}^\omega\omega\}.$$

The rest is as in previous cases (note that  $\langle (\eta^\wedge \langle \alpha \rangle, \nu^\wedge \langle n \rangle) : n < \omega \rangle$  converges to  $(\eta^\wedge \langle \alpha \rangle, \nu^\wedge \langle \omega \rangle)$  and even if  $(\eta^\wedge \langle \alpha \rangle, \nu^\wedge \langle n \rangle) \leq (\eta_n, \nu_n) \in T_\omega$  then  $\langle (\eta_n, \nu_n) : n < \omega \rangle$  converge to  $(\eta^\wedge \langle \alpha \rangle, \nu^\wedge \langle \omega \rangle)$ ). □<sub>9.12</sub>

**Proposition 9.13.** *If  $\text{IND}_{\chi'}(\langle \mu_n : n < \omega \rangle)$ , then  $\prod_{n < \omega} \mu_n$  is not the union of  $\leq \chi$  members of  $I_{\bar{\mu}}^0$  (see Definition 5.5 and Theorem 5.7).*

*Proof.* Suppose that  $A_\zeta = \{ \sum_{n < \omega} p^n x_{\alpha_n}^n : \langle \alpha_n : n < \omega \rangle \in X_\zeta \}$  and  $\alpha_n < \mu_n$  are such that if  $\sum p^n x_{\alpha_n}^n \in A_\zeta$  then for infinitely many  $n$  for every  $k < \omega$  there is  $\langle \beta_n : n < \omega \rangle$ ,

$$(\forall \ell < k)[\alpha_\ell = \beta_\ell \Leftrightarrow \ell = n] \quad \text{and} \quad \sum_{n < \omega} p^n x_{\beta_n}^n \in A_\zeta \quad (\text{see §5}).$$

This clearly follows. □<sub>9.13</sub>

## 10. ON MODULES

Here we present the straight generalization of the one prime case like Abelian reduced separable  $p$ -groups. This will be expanded in [She01] (including the proof of 10.4). (see ??)

*Hypothesis 10.1.* (A):  $R$  is a ring,  $\bar{\epsilon} = \langle \epsilon_n : n < \omega \rangle$ ,  $\epsilon_n$  is a definition of an additive subgroup of  $R$ -modules by an existential positive formula (finitary or infinitary) decreasing with  $n$ , we write  $\epsilon_n(M)$  for this additive subgroup,  $\epsilon_\omega(M) = \bigcap_n \epsilon_n(M)$ .

(B):  $\mathfrak{K}$  is the class of  $R$ -modules.

(C):  $\mathfrak{K}^* \subseteq \mathfrak{K}$  is a class of  $R$ -modules, which is closed under direct summand, direct limit and for which there is  $M^*$ ,  $x^* \in M^*$ ,  $M^* = \bigoplus_{\ell \leq n} M_\ell^* \oplus M_n^{**}$ ,  $M_n^* \in \mathfrak{K}$ ,  $x_n^* \in \epsilon_n(M_n^*) \setminus \epsilon_{n+1}(M^*)$ ,  $x^* - \sum_{\ell < n} x_\ell^* \in \epsilon_n(M^*)$ .

**Definition 10.2.** For  $M_1, M_2 \in \mathfrak{K}$ , we say  $h$  is a  $(\mathfrak{K}, \bar{\epsilon})$ -homomorphism from  $M_1$  to  $M_2$  if it is a homomorphism and it maps  $M_1 \setminus \epsilon_\omega(M_1)$  into  $M_2 \setminus \epsilon_\omega(M_2)$ ; we say  $h$  is an  $\bar{\epsilon}$ -pure homomorphism if for each  $n$  it maps  $M_1 \setminus \epsilon_n(M_1)$  into  $M_2 \setminus \epsilon_n(M_2)$ .

**Definition 10.3.** (1) Let  $H_n \subseteq H_{n+1} \subseteq H$ ,  $\bar{H} = \langle H_n : n < \omega \rangle$ ,  $cl$  is a closure operation on  $H$ ,  $cl$  is a function from  $\mathcal{P}(H)$  to itself and

$$X \subseteq cl(X) = cl(cl(X)).$$

Define

$$I_{H, \bar{H}, cl} = \left\{ A \subseteq H : \text{for some } k < \omega \text{ we have } cl(A) \cap \bigcup_{n < \omega} H_n \subseteq H_k \right\}.$$

(2) We can replace  $\omega$  by any regular  $\kappa$  (so  $H = \langle H_i : i < \kappa \rangle$ ).

**Claim 10.4.** Assume  $|R| + \mu^+ < \lambda = cf(\lambda) < \mu^{\aleph_0}$ , then for every  $M \in \mathfrak{K}_\lambda$  there is  $N \in \mathfrak{K}_\lambda$  with no  $\bar{\epsilon}$ -pure homomorphism from  $N$  into  $M$ .

*Remark 10.5.* In the interesting cases  $cl$  has infinitary character. The applications here are for  $\kappa = \omega$ . For the theory, *pcf* is nicer for higher  $\kappa$ .

## 11. OPEN PROBLEMS

**Problem 11.1.** (1) If  $\mu^{\aleph_0} \geq \lambda$  then any  $(A, d) \in \mathfrak{K}_\lambda^{mt}$  can be embedded into some  $M' \in \mathfrak{K}_\lambda^{mt}$  with density  $\leq \mu$ .

(2) If  $\mu^{\aleph_0} \geq \lambda$  then any  $(A, d) \in \mathfrak{K}_\lambda^{ms}$  can be embedded into some  $M' \in \mathfrak{K}_\lambda^{ms}$  with density  $\leq \mu$ .

**Problem 11.2.** (1) Other inclusions on  $\text{Univ}(\mathfrak{K}^x)$  or show consistency of non inclusions (see §9).

(2) Is  $\mathfrak{K}^1 \leq \mathfrak{K}^2$  the right partial order? (see §9).

(3) By forcing reduce consistency of  $\mathbf{U}_{J_1}(\lambda) > \lambda + 2^{\aleph_0}$  to that of  $\mathbf{U}_{J_2}(\lambda) > \lambda + 2^{\aleph_0}$ .

**Problem 11.3.** (1) The cases with the weak *pcf* assumptions, can they be resolved in ZFC? (the *pcf* problems are another matter).

(2) Use [She00], [She02] to get ZFC results for large enough cardinals.

**Problem 11.4.** If  $\lambda_n^{\aleph_0} < \lambda_{n+1}$ ,  $\mu = \sum_{n < \omega} \lambda_n$ ,  $\lambda = \mu^+ < \mu^{\aleph_0}$  can  $(\lambda, \lambda, 1)$  belong to  $\text{Univ}(\mathfrak{K})$ ? For  $\mathfrak{K} = \mathfrak{K}^{tr}, \mathfrak{K}^{rs(p)}, \mathfrak{K}^{trf}$ ?

**Problem 11.5.** (1) If  $\lambda = \mu^+$ ,  $2^{<\mu} = \lambda < 2^\mu$  can  $(\lambda, \lambda, 1) \in \text{Univ}(\mathfrak{K}^{\text{or}} = \text{class of linear orders})$ ?

- (2) Similarly for  $\lambda = \mu^+$ ,  $\mu$  singular, strong limit,  $\text{cf}(\mu) = \aleph_0$ ,  $\lambda < \mu^{\aleph_0}$ .  
 (3) Similarly for  $\lambda = \mu^+$ ,  $\mu = 2^{<\mu} = \lambda^+ < 2^\mu$ .

**Problem 11.6.** (1) Analyze the existence of universal member from  $\mathfrak{K}_\lambda^{rs(p)}$ ,  $\lambda < 2^{\aleph_0}$ .

- (2) §4 for many cardinals, i.e. is it consistent that:  $2^{\aleph_0} > \aleph_\omega$  and for every  $\lambda < 2^{\aleph_0}$  there is a universal member of  $\mathfrak{K}_\lambda^{rs(p)}$ ?

**Problem 11.7.** (1) If there are  $A_i \subseteq \mu$  for  $i < 2^{\aleph_0}$ ,  $|A_i \cap A_j| < \aleph_0$ ,  $2^\mu = 2^{\aleph_0}$  find forcing adding  $S \subseteq [\omega]^\mu$  universal for  $\{(B, \triangleleft) : \omega > \omega \subseteq B \subseteq \omega^{\geq \omega}, |B| \leq \lambda\}$  under (level preserving) natural embedding.

**Problem 11.8.** For simple countable  $T$ ,  $\kappa = \kappa^{<\kappa} < \lambda \subseteq \kappa$  force existence of universal for  $T$  in  $\lambda$  still  $\kappa = \kappa^{<\kappa}$  but  $2^\kappa = \chi$ .

**Problem 11.9.** Make [She93c, §4], [She96b, §1] work for a larger class of theories more than simple.

See on some of these problems [DS04], [She01].

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