Paper Sh:586, version 2011-11-28_12. See https://shelah.logic.at/papers/586/ for possible updates.

A POLARIZED PARTITION RELATION AND FAILURE OF GCH AT SINGULAR STRONG LIMIT
SH586

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Abstract. The main result is that for λ strong limit singular failing the continuum hypothesis (i.e. $2^\lambda > \lambda^+$), a polarized partition theorem holds.

§ 0. Introduction

In the present paper we show a polarized partition theorem for strong limit singular cardinals λ failing the continuum hypothesis. Let us recall the following definition.

Definition 0.1. For ordinal numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ and a cardinal θ, the polarized partition symbol

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \to \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}^{1,1}_{\theta}$$

means:
if d is a function from $\alpha_1 \times \beta_1$ into θ *then* for some $A \subseteq \alpha_1$ of order type α_2 and $B \subseteq \beta_1$ of order type β_2, the function $d \restriction A \times B$ is constant.

We address the following problem of Erdös and Hajnal:

(*) if μ is strong limit singular of uncountable cofinality, $\theta < \mathrm{cf}(\mu)$ does

$$\begin{pmatrix} \mu^+ \\ \mu \end{pmatrix} \to \begin{pmatrix} \mu \\ \mu \end{pmatrix}^{1,1}_{\theta} \qquad ?$$

The particular case of this question for $\mu = \aleph_{\omega_1}$ and $\theta = 2$ was posed by Erdös, Hajnal and Rado (under the assumption of GCH) in [EHR65, Problem 11, p.183]). Hajnal said that the assumption of GCH in [EHR65] was not crucial, and he added that the intention was to ask the question "in some, preferably nice, Set Theory".

Baumgartner and Hajnal have proved that if μ is weakly compact then the answer to **(*)** is "yes" (see [BH95]), also if μ is strong limit of cofinality $\aleph_0$. But for a weakly compact μ we do not know if for every $\alpha < \mu^+$:

Date: November 28, 2011.

1991 *Mathematics Subject Classification.* Primary ; Secondary:

Key words and phrases. ?

Research partially supported by "Basic Research Foundation" administered by The Israel Academy of Sciences and Humanities. The author thanks Alice Leonhardt for the beautiful typing.

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$$\binom{\mu^+}{\mu} \to \binom{\alpha}{\mu}^{1,1}_{\theta}.$$

The first time I heard the problem (around 1990) I noted that **(*)** holds when μ is a singular limit of measurable cardinals. This result is presented in Theorem 1.2. It seemed likely that we can combine this with suitable collapses, to get "small" such μ (like $\aleph_{\omega_1}$) but there was no success in this direction.

In September 1994, Hajnal reasked me the question putting great stress on it. Here we answer the problem **(*)** using methods of [She94]. But instead of the assumption of GCH (postulated in [EHR65]) we assume $2^\mu > \mu^+$. The proof seems quite flexible but we did not find out what else it is good for. This is a good example of the major theme of [She94]:

Thesis 0.2. Whereas CH and GCH are good (helpful, strategic) assumptions having many consequences, and, say, $\neg$CH is not, the negation of GCH at singular cardinals (i.e. for μ strong limit singular $2^\mu > \mu^+$ or, really the strong hypothesis: $\mathrm{cf}(\mu) < \mu \quad \Rightarrow \quad \mathrm{pp}(\mu) > \mu^+$) is a good (helpful, strategic) assumption.

Foreman pointed out that the result presented in Theorem 1.1 below is preserved by μ^+-closed forcing notions. Therefore, if

$$\mathbf{V} \models \binom{\lambda^+}{\lambda} \to \binom{\lambda}{\lambda}^{1,1}_{\theta}$$

then

$$V^{\mathrm{Levy}(\lambda^+,2^\lambda)} \models \binom{\lambda^+}{\lambda} \to \binom{\lambda}{\lambda}^{1,1}_{\theta}.$$

Consequently, the result is consistent with $2^\lambda = \lambda^+$ & λ is small. (Note that although our final model may satisfy the Singular Cardinals Hypothesis, the intermediate model still violates SCH at λ, hence needs large cardinals, see [Jec03].) For λ not small we can use Theorem 1.2).

Before we move to the main theorem, let us recall an open problem important for our methods:

Question 0.3.

(1) Let $\kappa = \mathrm{cf}(\mu) > \aleph_0, \mu > 2^\theta$ and $\lambda = \mathrm{cf}(\lambda) \in (\mu, \mathrm{pp}^+(\mu))$. Can we find $\theta < \mu$ and $\mathrm{Ga} \in [\mu \cap \mathrm{Reg}]^\theta$ such that: $\lambda \in \mathrm{pcf}(\mathrm{Ga}), \mathrm{Ga} = \bigcup\limits_{i<\kappa} \mathrm{Ga}_i$, Ga_i bounded in μ and $\sigma \in \mathrm{Ga}_i \Rightarrow \bigwedge\limits_{\alpha<\sigma} |\alpha|^\theta < \sigma$?

For this it is enough to show:

(2) If $\mu = \mathrm{cf}(\mu) > 2^{<\theta}$ but $\bigvee\limits_{\alpha<\mu} |\alpha|^{<\theta} \geq \mu$ then we can find $\mathrm{Ga} \in [\mu \cap \mathrm{Reg}]^{<\theta}$ such that $\lambda \in \mathrm{pcf}(\mathrm{Ga})$.

As shown in [She94]

Theorem 0.4. *If μ is strong limit singular of cofinality $\kappa > \aleph_0$, $2^\mu > \lambda = \mathrm{cf}(\lambda) > \mu$ then for some strictly increasing sequence $\langle \lambda_i : i < \kappa \rangle$ of regulars with limit μ, $\prod\limits_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa$ has true cofinality λ. If $\kappa = \aleph_0$, it still holds for $\lambda = \mu^{++}$.*

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[More fully, by [She94, Ch.II,§5], we know $\mathrm{pp}(\mu) =^+ 2^\mu$ and by [She94, Ch.III,1.6(2)], we know $\mathrm{pp}^+(\mu) = \mathrm{pp}^+_{J^{\mathrm{bd}}_\kappa}(\mu)$. Note that for $\kappa = \aleph_0$ we should replace J^{bd}_κ by a possibly larger ideal, using [She96, 1.1,6.5] but there is no need here.]

Remark 0.5. Note the problem is pp = cov problem, see more [She96, §1]; so if $\kappa = \aleph_0$, $\lambda < \mu^{+\omega_1}$ the conclusion of 0.4 holds; we allow to increase J^{bd}_κ, even "there are $< \mu^+$ fixed points $< \lambda^+$" suffices.

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§ 1. Main result

Theorem 1.1. *Suppose μ is strong limit singular satisfying $2^\mu > \mu^+$. Then*

(1) $\begin{pmatrix} \mu^+ \\ \mu \end{pmatrix} \to \begin{pmatrix} \mu+1 \\ \mu \end{pmatrix}^{1,1}_\theta$ *for any $\theta < \mathrm{cf}(\mu)$,*

(2) *if d is a function from $\mu^+ \times \mu$ to θ and $\theta < \mathrm{cf}(\mu)$ then for some sets $A \subseteq \mu^+$ and $B \subseteq \mu$ we have: $\mathrm{otp}(A) = \mu + 1$, $\mathrm{otp}(B) = \mu$ and the restriction $d \restriction A \times B$ does not depend on the first coordinate.*

Proof. 1) It follows from part (2), (as if $d(\alpha, \beta) = d'(\beta)$ for $\alpha \in A$, $\beta \in B$, where $d' : B \to \theta$, and $|B| = \mu$, $\theta < \mathrm{cf}(\mu)$ then there is $B' \subseteq B$, $|B'| = \mu$ such that $d' \restriction B'$ is constant and hence $d \restriction (A \times B')$ is constant as required).
2) Let $d : \mu^+ \times \mu \to \theta$. Let $\kappa = \mathrm{cf}(\mu)$ and $\bar\mu = \langle \mu_i : i < \kappa \rangle$ be a continuous strictly increasing sequence such that $\mu = \sum_{i<\kappa} \mu_i$, $\mu_0 > \kappa$. We can find a sequence $\bar C = \langle C_\alpha : \alpha < \mu^+ \rangle$ such that:

(A) $C_\alpha \subseteq \alpha$ is closed, $\mathrm{otp}(C_\alpha) < \mu$,

(B) $\beta \in \mathrm{nacc}(C_\alpha) \quad \Rightarrow \quad C_\beta = C_\alpha \cap \beta$,

(C) if C_α has no last element then $\alpha = \sup(C_\alpha)$, (so α is a limit ordinal) and any member of $\mathrm{nacc}(C_\alpha)$ is a successor ordinal,

(D) if $\sigma = \mathrm{cf}(\sigma) < \mu$ then the set
$$S_\sigma =: \{\delta < \mu^+ : \mathrm{cf}(\delta) = \sigma \ \ \& \ \ \delta = \sup(C_\delta) \ \ \& \ \ \mathrm{otp}(C_\delta) = \sigma\}$$
is stationary

(possible by [She93, §1]); we could have added

(E) for every $\sigma \in \mathrm{Reg} \cap \mu^+$ and a club E of μ^+, for stationary many $\delta \in S_\sigma$, E separates any two successive members of C_δ.

Let c be a symmetric two place function from μ^+ to κ such that for each $i < \kappa$ and $\beta < \mu^+$ the set

$\boxplus_1$ (a) the set $a^\beta_i =: \{\alpha < \beta : c(\alpha, \beta) \le i\}$ has cardinality $\le \mu_i$

(b) $\alpha < \beta < \gamma \Rightarrow c(\alpha, \gamma) \le \max\{c(\alpha, \beta), c(\beta, \gamma)\}$

(c) $\alpha \in C_\beta$ and $\mu_i \ge |C_\beta| \Rightarrow c(\alpha, \beta) \le i$

(as in [She79], easily constructed by induction on β).

Let $\bar\lambda = \langle \lambda_i : i < \kappa \rangle$ be a strictly increasing sequence of regular cardinals with limit μ such that $\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa$ has true cofinality μ^{++} (exists by 0.4 with $\lambda = \mu^{++} \le 2^\mu$). As we can replace $\bar\lambda$ by any subsequence of length κ, without loss of generality $(\forall i < \kappa)(\lambda_i > 2^{\mu_i^+})$. Lastly, let $\chi = \beth_8(\mu)^+$ and $<^*_\chi$ be a well ordering of $\mathscr{H}(\chi)$(=: $\{x$: the transitive closure of x is of cardinality $< \chi\}$).

Now we choose by induction on $\alpha < \mu^+$ sequences $\bar M_\alpha = \langle M_{\alpha,i} : i < \kappa \rangle$ such that:

(i) $M_{\alpha,i} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,

(ii) $\|M_{\alpha,i}\| = 2^{\mu_i^+}$ and ${}^{\mu_i^+}(M_{\alpha,i}) \subseteq M_{\alpha,i}$ and $2^{\mu_i^+} + 1 \subseteq M_{\alpha,i}$,

(iii) $d, c, \bar C, \bar\lambda, \bar\mu, \alpha \in M_{\alpha,i}$, $\langle M_{\beta,j} : \beta < \alpha, j < \kappa \rangle$ belongs to $M_{\alpha,i}$,

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(iv) $\bigcup_{\beta \in a_i^\alpha} M_{\beta,i} \subseteq M_{\alpha,i}$ and

(v) $\langle M_{\alpha,j} : j < i \rangle \in M_{\alpha,i}$,

(vi) $\bigcup_{j<i} M_{\alpha,j} \subseteq M_{\alpha,i}$,

(vii) $\langle M_{\beta,i} : \beta \in a_i^\alpha \rangle$ belongs to $M_{\alpha,i}$.

There is no problem to carry out the construction. Note that actually the clause (vii) follows from (i)–(vi), as a_i^α is defined from c, α, i, see $\boxplus_1$.

Our demands imply that

$\boxplus_2$ (a) $\beta \in a_i^\alpha \Rightarrow M_{\beta,i} \prec M_{\alpha,i}$

(b) $j < i \Rightarrow M_{\alpha,j} \prec M_{\alpha,i}$

(c) $a_i^\alpha \subseteq M_{\alpha,i}$, hence $\alpha \subseteq \bigcup_{i<\kappa} M_{\alpha,i}$.

For $\alpha < \mu^+$ let $f_\alpha \in \prod_{i<\kappa} \lambda_i$ be defined by $f_\alpha(i) = \sup(\lambda_i \cap M_{\alpha,i})$. Note that $f_\alpha(i) < \lambda_i$ as $\lambda_i = \mathrm{cf}(\lambda_i) > 2^{\mu_i^+} = \|M_{\alpha,i}\|$. Also, if $\beta < \alpha$ then for every $i \in [c(\beta,\alpha), \kappa)$ we have $\beta \in M_{\alpha,i}$ and hence $\bar{M}_\beta \in M_{\alpha,i}$. Therefore, as also $\bar{\lambda} \in M_{\alpha,i}$, we have $f_\beta \in M_{\alpha,i}$ and $f_\beta(i) \in M_{\alpha,i} \cap \lambda_i$.

Consequently

$\boxplus_3$ $(\forall i \in [c(\beta,\alpha), \kappa))(f_\beta(i) < f_\alpha(i))$ and thus $f_\beta <_{J_\kappa^{\mathrm{bd}}} f_\alpha$.

Since $\{f_\alpha : \alpha < \mu^+\} \subseteq \prod_{i<\kappa} \lambda_i$ has cardinality μ^+ and $\prod_{i<\kappa} \lambda_i / J_\kappa^{\mathrm{bd}}$ is μ^{++}-directed, there is $f^* \in \prod_{i<\kappa} \lambda_i$ such that

$(*)_1$ $(\forall \alpha < \mu^+)(f_\alpha <_{J_\kappa^{\mathrm{bd}}} f^*)$.

Let, for $\alpha < \mu^+$, $g_\alpha \in {}^\kappa\theta$ be defined by $g_\alpha(i) = d(\alpha, f^*(i))$. Since $|{}^\kappa\theta| < \mu < \mu^+ = \mathrm{cf}(\mu^+)$, there is a function $g^* \in {}^\kappa\theta$ such that

$(*)_2$ the set $A^* = \{\alpha < \mu^+ : g_\alpha = g^*\}$ is unbounded in μ^+.

Now choose, by induction on $\zeta < \mu^+$, models N_ζ such that:

(a) $N_\zeta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,

(b) the sequence $\langle N_\zeta : \zeta < \mu^+ \rangle$ is increasing continuous,

(c) $\|N_\zeta\| = \mu$ and ${}^{\kappa>}(N_\zeta) \subseteq N_\zeta$ if ζ is not a limit ordinal,

(d) $\langle N_\xi : \xi \leq \zeta \rangle \in N_{\zeta+1}$,

(e) $\mu + 1 \subseteq N_\zeta$

(f) $\bigcup_{\substack{\alpha<\zeta \\ i<\kappa}} M_{\alpha,i} \subseteq N_\zeta$

(g) $\langle M_{\alpha,i} : \alpha < \mu^+, i < \kappa \rangle, \langle f_\alpha : \alpha < \mu^+ \rangle, g^*, A^*$ and d belong to the first model N_0.

Let $E =: \{\zeta < \mu^+ : N_\zeta \cap \mu^+ = \zeta\}$. Clearly, E is a club of μ^+, and thus we can find an increasing sequence $\langle \delta_i : i < \kappa \rangle$ such that

$(*)_3$ $\delta_i \in S_{\mu_i^+} \cap \mathrm{acc}(E) (\subseteq \mu^+)$, (see clause (D) in the beginning of the proof).

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For each $i < \kappa$ choose a successor ordinal $\alpha^*_i \in \mathrm{nacc}(C_{\delta_i}) \setminus \bigcup\{\delta_j + 1 : j < i\}$. Take any $\alpha^* \in A^* \setminus \bigcup\limits_{i<\kappa} \delta_i$.
We choose by induction on $i < \kappa$ an ordinal j_i and sets A_i, B_i such that:

(α) $j_i < \kappa$ such that $\mu_{j_i} > \lambda_i$ (so $j_i > i$) and j_i strictly increasing in i,
(β) $f_{\delta_i} \restriction [j_i, \kappa) < f_{\alpha^*_{i+1}} \restriction [j_i, \kappa) < f_{\alpha^*} \restriction [j_i, \kappa) < f^* \restriction [j_i, \kappa)$,
(γ) for each $i_0 < i_1$ we have: $c(\delta_{i_0}, \alpha^*_{i_1}) < j_{i_1}$, and $c(\alpha^*_{i_0}, \alpha^*_{i_1}) < j_{i_1}$, and $c(\alpha^*_{i_1}, \alpha^*) < j_{i_1}$ and $c(\delta_{i_1}, \alpha^*) < j_{i_1}$,
(δ) $A_i \subseteq A^* \cap (\alpha^*_i, \delta_i)$,
(ϵ) $\mathrm{otp}(A_i) = \mu^+_i$,
(ζ) $A_i \in M_{\delta_i, j_i}$,
(η) $B_i \subseteq \lambda_{j_i}$,
(θ) $\mathrm{otp}(B_i) = \lambda_{j_i}$,
(ι) $B_\varepsilon \in M_{\alpha^*_i, j_i}$ for $\varepsilon < i$ and $B_i \in \cup\{M_{\alpha^*_i, j} : j < \kappa\}$
(κ) for every $\alpha \in \bigcup\limits_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\}$ and $\zeta \le i$ and $\beta \in B_\zeta \cup \{f^*(j_\zeta)\}$ we have $d(\alpha, \beta) = g^*(j_\zeta)$.

If we succeed then $A = \bigcup\limits_{\varepsilon<\kappa} A_\varepsilon \cup \{\alpha^*\}$ and $B = \bigcup\limits_{\zeta<\kappa} B_\zeta$ are as required. During the induction in stage i concerning (ι) we already know $\varepsilon < i \Rightarrow \bigvee\limits_{j<\kappa} B_\varepsilon \in M_{\alpha^*_i, j}$. So assume that the sequence $\langle (j_\varepsilon, A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle$ has already been defined.

We can find $j_i(0) < \kappa$ satisfying requirements (α), (β), (γ) and (ι) and such that $\bigwedge\limits_{\varepsilon<i} \lambda_{j_\varepsilon} < \mu_{j_i(0)}$. Then by "$j_1(0)$ satisfies clause (γ)" for each $\varepsilon < i$ we have $\delta_\varepsilon \in a^{\alpha^*_i}_{j_i(0)}$ and hence $M_{\delta_\varepsilon, j_\varepsilon} \prec M_{\alpha^*_i, j_i(0)}$ (for $\varepsilon < i$). But $A_\varepsilon \in M_{\delta_\varepsilon, j_\varepsilon}$ (by clause (ζ)) and $B_\varepsilon \in M_{\alpha^*_i, j_i(0)}$ (for $\varepsilon < i$), so $\{A_\varepsilon, B_\varepsilon : \varepsilon < i\} \subseteq M_{\alpha^*_i, j_i(0)}$. Since ${}^{\kappa>}(M_{\alpha^*_i, j_i(0)}) \subseteq M_{\alpha^*_i, j_i(0)}$ (see (ii)), the sequence $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle$ belongs to $M_{\alpha^*_i, j_i(0)}$. We know that for $\gamma_1 < \gamma_2$ in $\mathrm{nacc}(C_{\delta_i})$ we have $c(\gamma_1, \gamma_2) \le i$ (remember clause (B) and the choice of c). As $j_i(0) > i$ and so $\mu_{j_i(0)} \ge \mu^+_i$, the sequence

$$\bar{M}^* =: \langle M_{\alpha, j_i(0)} : \alpha \in \mathrm{nacc}(C_{\delta_i}) \rangle$$

is $\prec$-increasing and $\bar{M}^* \restriction \alpha \in M_{\alpha, j_i(0)}$ for $\alpha \in \mathrm{nacc}(C_{\delta_i})$ and $M_{\alpha^*_i, j_i(0)}$ appears in it. Also, as $\delta_i \in \mathrm{acc}(E)$, there is an increasing sequence $\langle \gamma_\xi : \xi < \mu^+_i \rangle$ of members of $\mathrm{nacc}(C_{\delta_i})$ such that $\gamma_0 = \alpha^*_i$ and $(\gamma_\xi, \gamma_{\xi+1}) \cap E \neq \emptyset$, say $\beta_\xi \in (\gamma_\xi, \gamma_{\xi+1}) \cap E$. Each element of $\mathrm{nacc}(C_{\delta_i})$ is a successor ordinal, so every γ_ξ is a successor ordinal. Each model $M_{\gamma_\xi, j_i(0)}$ is closed under sequences of length $\le \mu^+_i$ by clause (ii), and hence $\langle \gamma_\zeta : \zeta < \xi \rangle \in M_{\gamma_\xi, j_i(0)}$ (by choosing the right $\bar{C}$ and δ_i's we could have managed to have $\alpha^*_i = \min(C_{\delta_i})$, $\{\gamma_\xi : \xi < \mu^+_i\} = \mathrm{nacc}(C_\delta)$, without using this amount of closure).
For each $\xi < \mu^+_i$, recalling $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle \in M_{\alpha^*_i, j_i(\delta)}$ we know that

$$(\mathscr{H}(\chi), \in, <^*_\chi) \models \text{“}(\exists x \in A^*)[x > \gamma_\xi \text{ and } (\forall \varepsilon < i)(\forall y \in B_\varepsilon)(d(x, y) = g^*(j_\varepsilon))]\text{”}$$

because $x = \alpha^*$ satisfies it. As all the parameters, i.e. A^*, γ_ξ, d, g^* and $\langle B_\varepsilon : \varepsilon < i \rangle$, belong to N_{β_ξ} (remember clauses (e) and (c); note that $B_\varepsilon \in M_{\alpha^*_i, j_i(0)}$, $\alpha^*_i < \beta_\xi$), there is an ordinal $\beta^*_\xi \in (\gamma_\xi, \beta_\xi) \subseteq (\gamma_\xi, \gamma_{\xi+1})$ satisfying the demands on x. Now, necessarily for some $j_i(1,\xi) \in (j_i(0), \kappa)$ we have $\beta^*_\xi \in M_{\gamma_{\xi+1}, j_i(1,\xi)}$. Hence for some $j_i < \kappa$ the set

$$A_i := \{\beta^*_\xi : \xi < \mu_i^+ \text{ and } j_i(1,\xi) = j_i\}$$

has cardinality μ_i^+. Clearly $A_i \subseteq A^*$ (as each $\beta^*_\xi \in A^*$). Now, the sequence $\langle M_{\gamma_\xi, j_i} : \xi < \mu_i^+\rangle^\frown\langle M_{\delta_i, j_i}\rangle$ is $\prec$-increasing, and hence $A_i \subseteq M_{\delta_i, j_i}$. Since $\mu^+_{j_i} > \mu_i^+ = |A_i|$ we have $A_i \in M_{\delta_i, j_i}$. Note that at the moment we know that the set A_i satisfies the demands (δ)–(ζ). By the choice of $j_i(0)$, as $j_i > j_i(0)$, clearly $M_{\delta_i, j_i} \prec M_{\alpha^*, j_i}$, and hence $A_i \in M_{\alpha^*, j_i}$. Similarly, $\langle A_\varepsilon : \varepsilon \le i\rangle \in M_{\alpha^*, j_i}$, $\alpha^* \in M_{\alpha^*, j_i}$ and

$$\sup(M_{\alpha^*, j_i} \cap \lambda_{j_i}) = f_{\alpha^*}(j_i) < f^*(j_i).$$

Consequently, $\bigcup\limits_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\} \subseteq M_{\alpha^*, j_i}$ (by the induction hypothesis or the above) and it belongs to M_{α^*, j_i}. Since $\bigcup\limits_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\} \subseteq A^*$, clearly

$$(\mathscr{H}(\chi), \in, <^*_\chi)\text{``}(\forall x \in \bigcup_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\})\big(d(x, f^*(j_i)) = g^*(j_i)\big)\text{''}.$$

Note that

$$\bigcup_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\},\ g^*(j_i),\ d,\ \lambda_{j_i} \in M_{\alpha^*, j_i}$$

and

$$f^*(j_i) \in \lambda_{j_i} \setminus \sup(M_{\alpha^*, j_i} \cap \lambda_{j_i}).$$

Hence the set

$$B_i =: \{y < \lambda_{j_i} : (\forall x \in \bigcup_{\varepsilon \le i} A_\varepsilon \cup \{\alpha^*\})(d(x,y) = g^*(j_i))\}$$

has to be unbounded in λ_{j_i}. It is easy to check that j_i, A_i, B_i satisfy clauses (α)–(κ).

Thus we have carried out the induction step, finishing the proof of the theorem. $\square_{1.1}$

Theorem 1.2. *Suppose μ is singular limit of measurable cardinals. Then*

(1) $\begin{pmatrix} \mu^+ \\ \mu \end{pmatrix} \to \begin{pmatrix} \mu \\ \mu \end{pmatrix}^{1,1}_\theta$ *if $\theta = 2$ or at least $\theta < \mathrm{cf}(\mu)$*

(2) *Moreover, if $\alpha^* < \mu^+$ and $\theta < \mathrm{cf}(\mu)$ then* $\begin{pmatrix} \mu^+ \\ \mu \end{pmatrix} \to \begin{pmatrix} \alpha^* \\ \mu \end{pmatrix}^{1,1}_\theta$

(3) *If $\theta < \mu$, $\alpha^* < \mu^+$ and d is a function from $\mu^+ \times \mu$ to θ then for some $A \subseteq \mu^+$, $\mathrm{otp}(A) = \alpha^*$, and $B = \bigcup\limits_{i < \mathrm{cf}(\mu)} B_i \subseteq \mu, |B| = \mu$ we have:*

$$d \restriction A \times B_i \quad \textit{is constant for each } i < \mathrm{cf}(\mu).$$

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Proof. Easily 3) $\Rightarrow$ 2) $\Rightarrow$ 1), so we shall prove part 3).
Let $d : \mu^+ \times \mu \to \theta$. Let $\kappa =: \mathrm{cf}(\mu)$. Choose sequences $\langle \lambda_i : i < \kappa \rangle$ and $\langle \mu_i : i < \kappa \rangle$ such that $\langle \mu_i : i < \kappa \rangle$ is increasing continuous, $\mu = \sum\limits_{i<\kappa} \mu_i$, $\mu_0 > \kappa + \theta$, each λ_i is measurable and $\mu_i < \lambda_i < \mu_{i+1}$ (for $i < \kappa$). Let D_i be a λ_i-complete uniform ultrafilter on λ_i. For $\alpha < \mu^+$ define $g_\alpha \in {}^\kappa\theta$ by: $g_\alpha(i) = \gamma$ iff $\{\beta < \lambda_i : d(\alpha, \beta) = \gamma\} \in D_i$ (as $\theta < \lambda_i$ it exists). The number of such functions is $\theta^\kappa < \mu$ (as μ is necessarily strong limit), so for some $g^* \in {}^\kappa\theta$ the set $A =: \{\alpha < \mu^+ : g_\alpha = g^*\}$ is unbounded in μ^+. For each $i < \kappa$ we define an equivalence relation e_i on μ^+:

$$\alpha e_i \beta \quad \underline{\text{iff}} \quad (\forall \gamma < \lambda_i)[d(\alpha, \gamma) = d(\beta, \gamma)].$$

So the number of e_i-equivalence classes is $\leq {}^{\lambda_i}\theta < \mu$. Hence we can find $\langle \alpha_\zeta : \zeta < \mu^+ \rangle$ an increasing continuous sequence of ordinals $< \mu^+$ such that:

- $(*)$ for each $i < \kappa$ and e_i-equivalence class X we have:
  <u>either</u> $X \cap A \subseteq \alpha_0$
  <u>or</u> for every $\zeta < \mu^+$, $(\alpha_\zeta, \alpha_{\zeta+1}) \cap X \cap A$ has cardinality μ.

Let $\alpha^* = \bigcup\limits_{i<\kappa} a_i$, $|a_i| = \mu_i$, $\langle a_i : i < \kappa \rangle$ pairwise disjoint. Now we choose by induction on $i < \kappa$, A_i, B_i such that:

- (a) $A_i \subseteq \bigcup\{(\alpha_\zeta, \alpha_{\zeta+1}) : \zeta \in a_i\} \cap A$ and each $A_i \cap (\alpha_\zeta, \alpha_{\zeta+1})$ is a singleton,
- (b) $B_i \in D_i$,
- (c) if $\alpha \in A_i, \beta \in B_j, j \leq i$ then $d(\alpha, \beta) = g^*(j)$.

Now, in stage i, $\langle (A_\varepsilon, B_\varepsilon) : \varepsilon < i \rangle$ are already chosen. Let us choose A_i. For each $\zeta \in a_i$ choose $\beta_\zeta \in (\alpha_\zeta, \alpha_{\zeta+1}) \cap A$ such that if $i > 0$ then for some $\beta' \in A_0$, $\beta_\zeta e_i \beta'$, and let $A_i = \{\beta_\zeta : \zeta \in a_i\}$. Now clause (a) is immediate, and the relevant part of clause (c), i.e. $j < i$, is O.K.

Next, as $\bigcup\limits_{j \leq i} A_j \subseteq A$, the set

$$B_i =: \bigcap_{j \leq i} \bigcap_{\beta \in A_j} \{\gamma < \lambda_i : d(\beta, \gamma) = g^*(i)\}$$

is the intersection of $\leq \mu_i < \lambda_i$ sets from D_i and hence $B_i \in D_i$. Clearly clauses (b) and the remaining part of clause (c) (i.e. $j = i$) holds. So we can carry the induction and hence finish the proof. $\square_{1.2}$

References

[BH95] James E. Baumgartner and Andras Hajnal, *Polarized partition relations*, preprint (1995).

[EHR65] Paul Erdős, Andras Hajnal, and Richard Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hung. **16** (1965), 93–196.

[Jec03] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.

[She79] Saharon Shelah, *On successors of singular cardinals*, Logic Colloquium '78 (Mons, 1978), Stud. Logic Foundations Math., vol. 97, North-Holland, Amsterdam-New York, 1979, pp. 357–380. MR 567680

Paper Sh:586, version 2011-11-28_12. See https://shelah.logic.at/papers/586/ for possible updates.

[She93] ______, *Advances in cardinal arithmetic*, Finite and infinite combinatorics in sets and logic (Banff, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 411, Kluwer Acad. Publ., Dordrecht, 1993, arXiv: 0708.1979, pp. 355–383. MR 1261217

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She96] ______, *Further cardinal arithmetic*, Israel J. Math. **95** (1996), 61–114, arXiv: math/9610226. MR 1418289

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