

# Ladder gaps over stationary sets

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## Abstract

For a stationary set  $S \subseteq \omega_1$  and a ladder system  $C$  over  $S$ , a new type of gaps called  $C$ -Hausdorff is introduced and investigated. We describe a forcing model of ZFC in which, for some stationary set  $S$ , for every ladder  $C$  over  $S$ , every gap contains a subgap that is  $C$ -Hausdorff. But for every ladder  $E$  over  $\omega_1 \setminus S$  there exists a gap with no subgap that is  $E$ -Hausdorff.

A new type of chain condition, called polarized chain condition, is introduced. We prove that the iteration with finite support of polarized c.c.c posets is again a polarized c.c.c poset.

## 1 Introduction

We first review some notations and definitions related to Hausdorff gaps. In fact we follow here the terminology given by M. Scheepers in his monograph [3] on Hausdorff gaps, but since we restrict ourselves to  $(\omega_1, \omega_1^*)$  gaps our nomenclature is somewhat simpler. The collection of all infinite subsets of

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$\omega$  is denoted  $[\omega]^\omega$ , and for  $a, b \in [\omega]^\omega$ ,  $a \subseteq^* b$  means that  $a \setminus b$  is finite. In this case  $X(a, b)$  (the “excess” number) is defined to be the least  $k$  such that  $a \setminus b \subseteq k$ . Thus  $a \setminus X(a, b) \subseteq b$ , but if  $X(a, b) > 0$  then  $X(a, b) - 1 \in a \setminus b$ .

A pre-gap is a pair of sequences  $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$  where  $I, J \subseteq \omega_1$  are uncountable and  $a_i, b_j \in [\omega]^\omega$  are such that

$$a_{i_0} \subseteq^* a_{i_1} \subseteq^* b_{j_1} \subseteq^* b_{j_0}$$

whenever  $i_0 < i_1$  are in  $I$  and  $j_0 < j_1$  in  $J$ . In most cases  $I = J = \omega_1$ . Given a pre-gap as above, and uncountable subsets  $I' \subseteq I$  and  $J' \subseteq J$ , the restriction  $g \upharpoonright (I', J')$  of  $g$  is the pre-gap  $\{(a_i \mid i \in I'), (b_j \mid j \in J')\}$ . We write  $g \upharpoonright I$  for  $g \upharpoonright (I, I)$ .

An interpolation for a pre-gap  $g$  is a set  $x \subseteq \omega$  such that

$$a_i \subseteq^* x \subseteq^* a_j$$

for every  $i$  and  $j$ . A pre-gap with no interpolation is called a gap. A famous construction of Hausdorff produces gaps in ZFC (which are now called Hausdorff gaps). Specifically, a Hausdorff gap is a pre-gap  $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$  such that for every  $\alpha \in \omega_1$  and  $n \in \omega$  the set

$$\{\beta \in \alpha \mid a_\beta \setminus n \subseteq b_\alpha\}$$

is finite.

A *special* (or Kunen) gap is a pre-gap  $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$  such that for some  $n_0 \in \omega$ :

1.  $a_\alpha \setminus n_0 \subseteq b_\alpha$  for every  $\alpha \in \omega_1$ , and
2. for all  $\alpha < \beta < \omega_1$ ,

$$(a_\alpha \cup a_\beta) \setminus n_0 \not\subseteq b_\alpha \cap b_\beta$$

(equivalently,  $a_\alpha \setminus n_0 \not\subseteq b_\beta$  or  $a_\beta \setminus n_0 \not\subseteq b_\alpha$ ).

The interest in these definitions arises from the fact (not too difficult to prove) that these Hausdorff and Kunen pre-gaps are gaps and remain gaps as long as  $\omega_1$  is not collapsed: they have no interpolation in any extension in which  $\omega_1$  remains uncountable.

In this paper we define two additional types of “special” gaps:  $S$ -Hausdorff gaps where  $S \subseteq \omega_1$  is a stationary set, and  $C$ -Hausdorff gaps, where  $C$  is a ladder system over  $S$ .

The motivation for this work is the desire to find an example with gaps of the phenomenon in which  $\omega_1$  is “split” in a certain behavior on a stationary set  $S \subset \omega_1$  and an opposite behavior on its complement  $\omega_1 \setminus S$ .

**Definition 1.1** *Let  $S \subseteq \omega_1$  be a stationary set. A pre-gap  $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$  is  $S$ -Hausdorff iff for some closed unbounded (club) set  $D \subseteq \omega_1$ , for every  $\delta \in S \cap D$  and for every sequence of ordinals  $(i_n \in \delta \mid n \in \omega)$  increasing and cofinal in  $\delta$*

$$\lim_{n \rightarrow \infty} X(a_{i_n}, b_\delta) = \infty. \quad (1)$$

*That is, for every  $k$ , there is only a finite number of  $n \in \omega$  for which  $a_{i_n} \setminus k \subseteq b_\delta$ . Since, for  $\delta < \delta'$ ,  $b_{\delta'} \subseteq^* b_\delta$ , it follows that eventually  $X(a_{i_n}, b_\delta) \leq X(a_{i_n}, b_{\delta'})$ , and hence (1) holds for every  $\delta' \geq \delta$  in  $\omega_1$ . If the pre-gap  $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$  is defined only on uncountable sets  $I, J \subseteq \omega_1$ , we can still define it to be  $S$ -Hausdorff if for some closed unbounded set  $D \subseteq \omega_1$ , for every  $\delta \in S \cap D$ , for every  $j \in J \setminus \delta$ , and for every increasing sequence of ordinals  $(i_n \in \delta \cap I \mid n \in \omega)$  cofinal in  $\delta$ ,*

$$\lim_{n \rightarrow \infty} X(a_{i_n}, b_j) = \infty \quad (2)$$

Clearly, every Hausdorff gap is an  $\omega_1$ -Hausdorff gap, and the closed unbounded set  $D$  can be taken to be  $\omega_1$ . The converse of this also holds, in the sense that every  $\omega_1$ -Hausdorff gap contains a Hausdorff gap. For suppose that  $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$  is some  $\omega_1$ -Hausdorff gap, and let  $D \subseteq \omega_1$  be the closed unbounded set given by the definition of  $g$  as an  $\omega_1$ -Hausdorff gap. Define  $I' \subseteq I$  such that every two members of  $I'$  contain a point from  $D$  in between. We claim that  $g' = \{(a_i \mid i \in I'), (b_j \mid j \in J)\}$ , is a Hausdorff gap. Indeed, if  $\alpha \in J$  then for every  $n \in \omega$  the set  $E = \{\beta \in \alpha \cap I' \mid a_\beta \setminus n \subseteq b_\alpha\}$  is necessarily finite. For if not, then let  $\delta$  be an accumulation point of  $E$ , and let  $\beta_i \in E$ , for  $i \in \omega$ , be increasing and converging to  $\delta$ . Necessarily  $\delta \in D$ , and  $a_{\beta_i} \setminus n \subseteq b_\alpha$  shows that (2) does not hold.

**Proposition 1.2** *If  $S \subseteq \omega_1$  is stationary, then any  $S$ -Hausdorff pre-gap is a gap. (So that any pre-gap containing an  $S$ -Hausdorff pre-gap is a gap.)*

**Proof.** Assume that this not so and let  $x$  be an interpolation of an  $S$ -Hausdorff pre-gap  $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$ . Then there is a fixed

$n_0 \in \omega$  such that for unbounded sets of indices  $I' \subseteq I$   $J' \subseteq J$ , for every  $\alpha \in I'$ ,  $\beta \in J'$

$$a_\alpha \setminus n_0 \subseteq x \setminus n_0 \subseteq b_\beta.$$

And as a consequence

$$a_\alpha \setminus n_0 \subseteq b_\beta \tag{3}$$

holds. Since  $g$  is assumed to be  $S$ -Hausdorff there exists a closed unbounded set  $D \subseteq \omega_1$  as in the definition. We may assume that every  $\delta \in D$  is an accumulation point of  $I'$ . Take now a limit  $\delta \in S \cap D$  and any sequence  $i_n \in I' \cap \delta$  increasing to  $\delta$ . Take any  $j \in J' \setminus \delta$ . Then equation (3) implies that  $X(a_{i_n}, b_\delta) \leq n_0$ , which is a contradiction.  $\dashv$

A stronger notion than that of being  $S$ -Hausdorff can be defined if the rate at which the sequences in (1) tend to infinity is uniform. For this we must recall the definition of a scale or ladder system on a stationary set.

If  $S \subseteq \omega_1$  is a stationary set, then a ladder (system) over  $S$  is a sequence  $C = (c_\alpha \mid \alpha \in S \text{ is a limit ordinal})$  such that every  $c_\alpha = (c_\alpha(n) \mid n \in \omega)$  is an increasing, cofinal in  $\alpha$   $\omega$ -sequence.

**Definition 1.3** *For a ladder system  $C$  over  $S$ , we say that a pre-gap  $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$  is  $C$ -Hausdorff iff for some closed-unbounded set  $D \subseteq \omega_1$  for all  $\delta \in S \cap D$  and  $j \in J \setminus \delta$  there is  $k \in \omega$  such that for every  $n \geq k$  in  $\omega$ , if  $i \in I \cap (\delta \setminus c_\delta(n))$ , then  $X(a_i, b_j) > n$ .*

Every  $C$ -Hausdorff gap (where  $C$  is a ladder system over a stationary set  $S$ ) is  $S$ -Hausdorff. Our aim in this paper is to prove the following consistency result.

**Theorem 1.4** *Assume G.C.H for simplicity. Suppose that  $\kappa$  is a cardinal such that  $\text{cf}(\kappa) > \aleph_1$ . Let  $S$  be a stationary co-stationary subset of  $\omega_1$ . Then there is a c.c.c poset of size  $\kappa$  such that in every generic extension made via  $P$   $2^{\aleph_0} = \kappa$  and the following hold.*

1. *For every ladder system  $C$  over  $S$ , every gap contains a subgap that is  $C$ -Hausdorff.*
2. *For every ladder system  $E$  over  $\omega_1 \setminus S$  there is a gap  $g$  with no subgap that is  $E$ -Hausdorff.*

## 2 Gaps introduced by forcing

Gaps can be created by forcing with finite conditions (a method due to Hechler [1]). These gaps are not  $S$ -Hausdorff for any stationary set, as we are going to see.

If  $f \in 2^n$  ( $f$  is a function defined on  $n$  with range included in  $\{0, 1\}$ ) then  $f$  is a characteristic function and we let  $[f] = \{k \mid f(k) = 1\}$  be the subset of  $n$  represented by  $f$ .

Let  $(I, <_I)$  be any ordering isomorphic to  $\omega_1 + \omega_1^*$ . For example take  $I = (\omega_1 \times \{0\}) \cup (\omega_1 \times \{1\})$  with  $\langle \alpha, 0 \rangle <_I \langle \beta, 0 \rangle <_I \langle \beta, 1 \rangle <_I \langle \alpha, 1 \rangle$  whenever  $\alpha < \beta < \omega_1$ .

Define the poset  $P$  by  $p \in P$  iff  $p$  is a finite function defined on  $I$  and such that:

1. For some  $n$  (called the “height” of  $p$ )  $p(i) \in 2^n$  for every  $i \in \text{dom}(p)$ . (The height of the empty function is defined to be 0.)
2. For every  $\alpha \in \omega_1$ ,  $\langle \alpha, 0 \rangle \in \text{dom}(p)$  iff  $\langle \alpha, 1 \rangle \in \text{dom}(p)$ , and in this case  $[p(\langle \alpha, 0 \rangle)] \subseteq [p(\langle \alpha, 1 \rangle)]$ .

The intuition behind this definition is that for  $\alpha \in \omega_1$ ,  $p(\langle \alpha, 0 \rangle)$  will “grow” to become  $a_\alpha$ , and  $p(\langle \alpha, 1 \rangle)$  will finally become  $b_\alpha$ , as  $p$  runs over the generic filter. So that  $(\langle a_\alpha \mid \alpha \in \omega_1 \rangle, \langle b_\alpha \mid \alpha \in \omega_1 \rangle)$  will be the generic gap with the additional property that  $a_\alpha \subseteq b_\alpha$  for every  $\alpha$ . The ordering of  $P$  reflects this intuition as follows.

For  $p_1, p_2 \in P$  define  $p_1 \leq p_2$  ( $p_2$  extends  $p_1$ ) iff

1.  $d_1 = \text{dom}(p_1) \subseteq d_2 = \text{dom}(p_2)$ , and for every  $i \in d_1$ ,  $p_1(i) \subseteq p_2(i)$  (so  $\text{height}(p_1) \leq \text{height}(p_2)$ ).
2. For every  $i, j \in \text{dom}(p_1)$ , if  $i <_I j$  then

$$[p_2(i)] \setminus [p_1(i)] \subseteq [p_2(j)].$$

It is easy to see that any condition in  $P$  has extensions with arbitrarily large height and with domains that extend arbitrarily over  $I$ . In fact, given  $i \in \text{dom}(p)$  and  $k \in \omega$  above height  $p$ , we can require that the extension  $p'$  puts  $k$  in  $[p'(i)]$ .

If  $\alpha \in \omega_1$ , we can write  $\alpha \in \text{dom}(p)$  instead of  $\langle \alpha, 0 \rangle \in \text{dom}(p)$  (which is equivalent to  $\langle \alpha, 1 \rangle \in \text{dom}(p)$ ). So  $\text{dom}(p)$  has two meanings, and the context decides if it means a set of ordinals or a set of pairs.

Suppose that  $A \subseteq I$  is such that  $\langle \alpha, 0 \rangle \in A$  iff  $\langle \alpha, 1 \rangle \in A$ . Let  $P_A$  be the subposet of  $P$  consisting of all conditions  $p$  such that  $\text{dom}(p) \subseteq A$ . If  $p \in P$  then  $p \upharpoonright A \in P_A$  and  $p \upharpoonright A \leq p$ . We prove some additional properties of this restriction map taking  $p$  to  $p \upharpoonright A$ .

In the definition of  $p \leq q$  what really counts is the restriction of  $q$  to the domain of  $p$ . That is,  $p \leq q$  iff  $p \leq q \upharpoonright \text{dom}(p)$ . It follows that  $p \leq q$  implies that  $p \upharpoonright A \leq q \upharpoonright A$ . It also follows that if  $p$  and  $q$  are conditions such that for  $C = \text{dom}(p) \cap \text{dom}(q)$ ,  $p \upharpoonright C = q \upharpoonright C$ , then  $p$  and  $q$  are compatible. In fact, in this case,  $p \cup q$  is the minimal extension of  $p$  and  $q$ .

Suppose that  $\text{dom}(p) = \text{dom}(q)$ . Then  $p$  and  $q$  are compatible in  $P$  iff  $p \leq q$  or  $q \leq p$ .

For compatible conditions  $p$  and  $q$ , we define a canonical extension  $p \vee q$  of both  $p$  and  $q$ . However,  $P$  is not a lattice and  $p \vee q$  is not the minimum of all extensions of  $p$  and  $q$ . To define it, we first make an observation. Consider  $C = \text{dom}(p) \cap \text{dom}(q)$ . Then  $p \upharpoonright C$  and  $q \upharpoonright C$  are comparable in  $P_C$  (since they are compatible and have the same domain), and hence we can assume without loss of generality that  $q \geq p \upharpoonright A$  and  $n = \text{height}(q) \geq m = \text{height}(p)$  where  $A = \text{dom}(q)$  (the restriction on the heights is needed only in case  $p \upharpoonright A = \emptyset$  since it follows from  $q \geq p \upharpoonright A$  otherwise). Then  $r = p \vee q$  is defined as follows on  $\text{dom}(p) \cup \text{dom}(q)$ , and it will be evident that  $p \vee q$  is an extension of  $p$  and  $q$ .

For  $i \in \text{dom}(q)$  define  $r(i) = q(i)$ . For  $i \in \text{dom}(p) \setminus A$  define  $r(i) \in 2^n$  by the following two conditions:

$$p(i) \subseteq r(i). \tag{4}$$

$$[r(i)] \setminus [p(i)] = \bigcup \{ [q(k)] \setminus m \mid k <_I i \text{ and } k \in A \cap \text{dom}(p) \}. \tag{5}$$

This definition makes sense since  $A \cap \text{dom}(p) \subseteq \text{dom}(q)$ .

It is clear that  $r \in P$ ,  $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$  and  $r \upharpoonright A = q$ . We prove that  $r \geq p$ . Clause 1 in the definition of extension is obvious, and we have to check clause 2. Suppose that  $i, j \in \text{dom}(p)$  and  $i <_I j$ . We have to show that

$$[r(i)] \setminus [p(i)] \subseteq [r(j)]. \tag{6}$$

So consider any  $a \in [r(i)] \setminus [p(i)]$ .

**Case 1:**  $i \in A$ . Then  $r(i) = q(i)$ . If  $j \in A$  as well, then (6) follows from our assumption that  $q \geq p \upharpoonright A$ , and since  $r(i) = q(i)$ ,  $r(j) = q(j)$  in this case. If,

on the other hand,  $j \notin A$ , then

$$[r(j)] \setminus [p(j)] = \bigcup \{[q(k)] \setminus m \mid k <_I j \text{ and } k \in A \cap \text{dom}(p)\}$$

by the definition of  $r$ . Since  $i \in A \cap \text{dom}(p)$ ,  $i <_I j$ , and  $a \in q(i) \setminus m$ ,  $a \in [r(j)] \setminus [p(j)]$  as required.

**Case 2:**  $i \notin A$ . Then  $i \in \text{dom}(p) \setminus A$  and (5) implies that for some  $k \in A \cap \text{dom}(p)$  such that  $k <_I i$ ,  $a \in [q(k)] \setminus m$ . Then  $k <_I j$ , both indices are in  $\text{dom}(p)$ , and  $k \in A$ , which brings us back to Case 1.  $\dashv$

This argument has the following corollary.

**Corollary 2.1** *Suppose that  $p_1, p_2 \in P$  and  $C = \text{dom}(p_1) \cap \text{dom}(p_2)$  are such that  $p_1 \upharpoonright C \geq p_2 \upharpoonright C$  and  $\text{height}(p_1) \geq \text{height}(p_2)$ . Then  $p_1 \vee p_2$  can be formed (an extension of  $p_1$  and  $p_2$ ).*

**Proof.** Define  $A = \text{dom}(p_1)$ . Then  $p_1 \geq p_2 \upharpoonright A$  (because  $p_1 \geq p_1 \upharpoonright C \geq p_2 \upharpoonright C = p_2 \upharpoonright A$ ). So  $r = p_1 \vee p_2$  can be formed.  $\dashv$

**Lemma 2.2**  *$P$  satisfies the c.c.c. In fact if  $\{p_\alpha \mid \alpha \in S\} \subseteq P$  where  $S \subseteq \omega_1$  is stationary, then for some stationary set  $S' \subseteq S$ , every finite set of conditions in  $\{p_\alpha \mid \alpha \in S'\}$  is compatible. (This is Talayaco's condition [4].)*

**Proof.** If  $p, q \in P$  have the same height and for  $C = \text{dom}(p) \cap \text{dom}(q)$  it happens that  $p \upharpoonright C = q \upharpoonright C$ , then  $p \cup q$  is an extension of  $p$  and  $q$ . Hence a  $\Delta$ -system argument works here.  $\dashv$

If  $G \subset P$  is some generic filter over  $P$ , define for every  $\alpha \in \omega_1$   $a_\alpha = \bigcup \{[p(\langle \alpha, 0 \rangle)] \mid p \in G\}$ , and  $b_\alpha = \bigcup \{[p(\langle \alpha, 1 \rangle)] \mid p \in G\}$ . A standard density argument shows that  $g$  is a pre-gap, and we denote it as  $g$ .

**Lemma 2.3** *The generic pre-gap  $g$  is a gap.*

**Proof.** Suppose that  $x \in V^P$  is a name, forced to be an interpolation for the generic pre-gap  $g$ . For every  $\alpha \in \omega_1$  find a condition  $p_\alpha \in P$  and a number  $n_\alpha \in \omega$  such that

$$p_\alpha \Vdash_P a_\alpha \setminus n_\alpha \subseteq x \setminus n_\alpha \subseteq b_\alpha. \quad (7)$$

Then for some stationary set  $S \subseteq \omega_1$ , and some fixed  $n \in \omega$ ,  $n = n_\alpha$  for every  $\alpha \in S$ , and the sets  $\text{dom}(p_\alpha)$  form a  $\Delta$ -system with core  $C \subset I$  (a finite set). We also assume that  $p_\alpha \upharpoonright C$  is fixed for  $\alpha \in S$ . For  $\alpha < \beta$ , both in  $S$  and above the ordinals involved in  $C$ , consider  $p_\alpha$  and  $p_\beta$ . Pick any  $k \geq n$  such

that  $k \geq \text{height}(p_\alpha)$  as well. Let  $i = \langle \alpha, 0 \rangle$ , and  $j = \langle \beta, 1 \rangle$ . We shall find an extension  $r$  of  $p_\alpha$  and  $p_\beta$  such that  $r(i)(k) = 1$  and  $r(j)(k) = 0$ . Then  $r \Vdash k \in a_\alpha \wedge k \notin b_\beta$ . But this contradicts (7).

To define  $r$ , define first an extension  $p'_\alpha \geq p_\alpha$  by requiring that  $p'_\alpha(i)(k) = 1$  and  $[p'_\alpha(\langle \gamma, 0 \rangle)] = [p_\alpha(\langle \gamma, 0 \rangle)]$  for every  $\langle \gamma, 0 \rangle \in C$ . This is possible since  $i$  is never  $<_I$  below  $\langle \gamma, 0 \rangle \in C$ . Now  $p'_\alpha$  extends  $p_\beta \upharpoonright C$  and hence  $r = p'_\alpha \vee p_\beta$  can be formed. Since the only members of  $C$  below  $j$  (in  $<_I$ ) are of the form  $\langle \gamma, 0 \rangle$ , it follows that  $[r(j)] = [p_\beta(j)]$ . Thus  $r(j)(k) = 0$ .  $\dashv$

The following lemma implies that if  $G$  is a  $(V, P)$ -generic filter,  $g$  the generic gap, and  $U \in V[G]$  is any stationary subset of  $\omega_1$  in the extension, then no uncountable restriction of  $g$  is  $U$ -Hausdorff.

**Lemma 2.4** *The following holds in  $V^P$  for the generic gap  $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$ . If  $J, K \subseteq \omega_1$  are unbounded, then there is a club set  $D_0 \subseteq \omega_1$  such that for every  $\delta \in D_0$  and  $k \in K \setminus \delta$  there are  $m \in \omega$  and a sequence  $j(n) \in \delta \cap J$  increasing and cofinal in  $\delta$  such that  $a_{j(n)} \setminus m \subseteq b_k$  for all  $n \in \omega$ .*

**Prof.** Let  $J, K \in V^P$  be names forced by every condition in  $P$  to be unbounded subsets of  $\omega_1$ . Define in  $V^P$  the following set  $D_0 \subseteq \omega_1$ :  $\delta \in D_0$  if and only if  $\delta \in \omega_1$  is a limit ordinal such that:

*for all  $k \in K \setminus \delta$  there is some  $m \in \omega$  and an increasing, cofinal in  $\delta$  sequence  $j(n) \in \delta \cap J$  with  $a_{j(n)} \setminus m \subseteq b_k$ .*

We want to prove that  $D_0$  contains a closed unbounded subset of  $\omega_1$ , and assume that it does not. So  $R = \omega_1 \setminus D_0$  is (forced by some condition to be) stationary in  $V^P$ , and hence the set, defined in  $V$ , of ordinals that are potentially in  $R$  is stationary in  $V$ . Namely, the set  $R_0 \subset \omega_1$  of ordinals forced by some condition to be in  $R$  is stationary. For every  $\delta \in R_0$  pick a condition  $p_\delta$  that forces  $\delta \notin D_0$ . By extending  $p_\delta$  we can find some  $k_\delta \geq \delta$  such that

$$p_\delta \Vdash_P k_\delta \in K \text{ shows that } \delta \notin D_0.$$

By extending  $p_\delta$  again, we can find some  $j_\delta \in \omega_1 \setminus \delta$  forced by  $p_\delta$  to be in  $J$  (which is possible since  $J$  is supposed to be unbounded in  $\omega_1$ ). If necessary, a further extension ensures that both  $j_\delta$  and  $k_\delta$  are in the domain of  $p_\delta$ . Now there exists some  $m = m_\delta \in \omega$  such that  $p_\delta \Vdash a_{j_\delta} \setminus m \subseteq b_{k_\delta}$  (the height of  $p_\delta$  will do). We can extend  $p_\delta$  once again and find  $f(\delta) < \delta$  such that

$$p_\delta \Vdash_P \text{ there is no } j \in J, f(\delta) < j < \delta, \text{ for which } a_j \setminus m \subseteq b_{k_\delta}. \quad (8)$$



We may assume that, for a stationary set  $T \subseteq R_0$ , the domains of  $p_\alpha$ , for  $\alpha \in T$ , form a  $\Delta$  system, that they all have the same height, say  $n$ , and the same restriction to the core. We also assume that the functions  $p_\alpha(\langle j_\alpha, 0 \rangle) : n \rightarrow \{0, 1\}$  do not depend on  $\alpha$ , and that  $f(\alpha)$  and  $m = m_\alpha$  are fixed on  $T$  ( $m \leq n$ ). Now by Talayaco's chain condition for  $P$ , there is a stationary  $T' \subseteq T$  such that for every  $\alpha, \beta \in T'$ ,  $p_\alpha \vee p_\beta$  is a common extension. Pick some  $\alpha \in T'$  that is an accumulation point of  $T'$  (and such that for every  $\beta < \alpha$ ,  $j_\beta < \alpha$ ). Then find  $\beta < \alpha$ ,  $\beta \in T'$  such that  $f(\alpha) < \beta$ . Then (as we shall see)

$$p_\beta \vee p_\alpha \Vdash a_{j_\beta} \subseteq a_{j_\alpha}, \quad j_\beta \in J, \quad \text{and } \alpha > j_\beta > f(\alpha).$$

Yet

$$p_\alpha \Vdash a_{j_\alpha} \setminus m \subseteq b_{k_\alpha},$$

and this is a contradiction to (8). Why does  $p_\beta \vee p_\alpha$  force  $a_{j_\beta} \subseteq a_{j_\alpha}$ ? Because the functions  $p_\alpha(\langle j_\alpha, 0 \rangle)$  and  $p_\beta(\langle j_\beta, 0 \rangle)$  are the same, they describe  $a_{j_\alpha} \cap n$  and  $a_{j_\beta} \cap n$ , so  $p_\beta \vee p_\alpha$  forces  $a_{j_\beta} \subseteq a_{j_\alpha}$ .  $\dashv$

### 3 Specializing pre-gaps on a ladder system

**Theorem 3.1** *For every ladder system  $C$  over a stationary set  $S \subseteq \omega_1$ , and gap  $g$ , there is a c.c.c forcing notion  $Q = Q_{g,C}$  such that in  $V^Q$  a restriction of  $g$  to some uncountable set is  $C$ -Hausdorff (and hence  $S$ -Hausdorff). In fact  $Q$  satisfies a stronger property than c.c.c, the polarized chain condition, which we shall define later.*

**Proof.** Fix for the proof a ladder system  $C = \langle c_\delta \mid \delta \in S \rangle$  over a stationary set  $S \subseteq \omega_1$  consisting of limit ordinals, and a pre-gap  $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$ . The forcing poset  $Q = Q_{g,C}$  defined below is designed to make an uncountable restriction of  $g$  into a  $C$ -Hausdorff gap.

Define  $p \in Q$  iff  $p = (w, s)$  where

1.  $w \in [\omega_1]^{<\aleph_0}$  (i.e. a finite subset of  $\omega_1$ ), and
2.  $s \in [S]^{<\aleph_0}$ .

If  $p \in Q$  then we write  $p = (w^p, s^p)$  for the two components of  $p$ .

The ordering  $p \leq q$  ( $q$  extends  $p$ ) is defined by

- a.  $w^p \subseteq w^q$ ,  $s^p \subseteq s^q$ , and
- b. If  $\delta \in s^p$  and  $i \in w^p$  are such that  $\delta \leq i$ , then for every  $j \in (w^q \setminus w^p) \cap \delta$ ,

$$a_j \setminus | c_\delta \cap j | \not\subseteq b_i.$$

Or, equivalently,  $X(a_j, b_i) > |c_\delta \cap j|$ . It is easy to check that this is indeed an ordering defined on  $Q$ .

If  $G$  is generic over  $Q$ , define  $W = \bigcup \{w \mid \exists s(w, s) \in G\}$ . We will prove that if  $g$  is a gap then  $Q$  satisfy the c.c.c. So  $\omega_1$  is preserved. Clearly, if  $p = (w, s)$  is a condition, then for any  $\sigma \in S$ ,  $(w, s \cup \{\sigma\})$  extends  $p$ , and if  $j \in \omega_1$  and  $j > \max(w)$ , then  $(w \cup \{j\}, s)$  extends  $p$ . (If, however,  $j < \max(w)$ , then  $(w \cup \{j\}, s)$  may be incompatible with  $(w, s)$ .) It follows that  $W$  is unbounded in  $\omega_1$  and  $\{(a_i \mid i \in W), (b_j \mid j \in W)\}$  is  $C$ -Hausdorff.

So the generic filter over  $Q$  selects an unbounded in  $\omega_1$  restriction of  $g$  that is  $C$ -Hausdorff.

If  $p = (w, s)$  is a condition then for every  $\alpha \in \omega_1$  the restriction  $p \upharpoonright \alpha = (w \cap \alpha, s \cap \alpha)$  is defined. Clearly  $p \upharpoonright \alpha \leq p$ .

If  $p = (w, s)$  and  $q = (v, r)$  are conditions in  $Q$  then define  $p \cup q = (w \cup v, s \cup r)$ . If  $p$  and  $q$  are compatible in  $Q$ , then  $p \cup q \in Q$  is the least upper bound of  $p$  and  $q$ .

The following lemma describes a situation in which the compatibility of  $p_1$  and  $p_2$  can be deduced. This is the situation resulting when  $p_1$  and  $p_2$  come from a  $\Delta$ -system, with core fixed below  $\gamma$ , and such that  $p_1$  is bounded by some  $\alpha$  such that the domain of  $p_2$  has empty intersection with the ordinal interval  $[\gamma, \alpha]$ . The proof is straightforward.

**Lemma 3.2** *Suppose that*

1.  $p_1 = (w_1, s_1)$  and  $p_2 = (w_2, s_2)$  are in  $P$ .
2.  $\gamma < \alpha < \omega_1$  are such that
  - (a)  $w_1 \subseteq \alpha$ , and  $p_1 \upharpoonright \gamma$  is compatible with  $p_2$ .
  - (b)  $w_2 \cap \alpha \subset \gamma$ , and  $s_2 \cap (\alpha + 1) \subset \gamma$ .  $p_2 \upharpoonright \alpha = p_2 \upharpoonright \gamma$  is compatible with  $p_1$ .

3. Define

$$A = \bigcap \{a_i \mid i \in w_1 \setminus \gamma\}$$

$$B = \bigcup \{b_j \mid j \in w_2 \setminus \gamma\}$$

and suppose that there is  $n \in A \setminus B$  such that, for every  $\delta \in s_2 \setminus \alpha$ ,  $n > |c_\delta \cap \alpha|$ .

Then  $p_1$  and  $p_2$  are compatible.

**Proof.** Form  $p = p_1 \cup p_2$  and prove that  $p_1, p_2 \leq p$ .  $p_1 \leq p$  is immediate. As for  $p_2 \leq p$ , observe that  $X(a_i, b_j) > |c_\delta \cap \alpha|$  for every  $i \in w_1 \setminus \gamma$ ,  $j \in w_2 \setminus \gamma$ , and  $\delta \in s_2 \setminus \gamma$ .  $\dashv$

The following simple lemma is used in proving that  $Q$  is a c.c.c poset.

**Lemma 3.3** *Suppose  $g = \{(A_i \mid i \in I), (B_j \mid j \in J)\}$  is a pre-gap such that for every  $i \in I$  and  $j \in J$ ,  $i < j$  implies that  $A_i \subseteq B_j$ . Then  $g$  is not a gap.*

**Proof.** By throwing away a countable set of indices from  $J$  we can assume for every  $n \in \omega$  that if  $n \notin B_j$  for some  $j$ , then  $n \notin B_j$  for uncountably many  $j$ 's. Define then  $x = \bigcup_{i \in I} A_i$ . Then  $x \subseteq B_j$  for every  $j$ , because otherwise there are some  $i \in I$ ,  $j \in J$ , and  $n \in \omega$  such that  $n \in A_i \setminus B_j$ . But then we may find uncountably many indices  $j'$  such that  $n \notin B_{j'}$  and in particular there is such  $j' > i$ . Thence  $A_i \not\subseteq B_{j'}$ , contradicting our assumption.  $\dashv$

**Theorem 3.4** *Suppose that the domain of our ladder system  $C$ , namely  $S$ , is co-stationary.*

1. *If  $g$  is a gap then  $Q = Q_{g,C}$  satisfies the c.c.c.*
2. *Suppose that  $T_1, T_2 \subseteq \omega_1 \setminus S$  are stationary sets and  $\bar{p} = (p_\delta^\ell \mid \delta \in T_\ell)$ , for  $\ell = 1, 2$ , are two sequences of conditions in  $Q$  such that, for some fixed  $p^* \in Q$ ,  $p^* \geq p_\delta^1 \upharpoonright \delta$ ,  $p_\mu^2 \upharpoonright \mu$ , for every  $\delta \in T_1$  and  $\mu \in T_2$ , is such that  $p^*$  is compatible with every  $p_\delta^1$  and with every  $p_\mu^2 \upharpoonright \mu$ . Then there are stationary subsets  $T'_1 \subseteq T_1$  and  $T'_2 \subseteq T_2$  such that, for every  $\alpha_1 \in T'_1$  and  $\alpha_2 \in T'_2$ , if  $\alpha_1 < \alpha_2$  then  $p_{\alpha_1}^1$  and  $p_{\alpha_2}^2$  are compatible in  $Q$ .*

**Proof.** We prove 2 since the proof of 1 is similar. For any condition  $p = (w, s)$  define  $\text{dom}(p) = w \cup s$ . Suppose that  $\text{dom}(p^*) \subseteq \gamma$ . Then  $\text{dom}(p_\delta^1) \cap \delta \subseteq \gamma$ , and  $\text{dom}(p_\mu^2) \cap \mu \subseteq \gamma$ , for every  $\delta \in T_1$  and  $\mu \in T_2$ . We may assume that if  $i < j$  then  $\text{dom}(p_i^\ell) \subset \cap (\text{dom}(p_j^m) \setminus \gamma)$  for  $\ell, m \in \{1, 2\}$ .

Since  $\delta \in T_\ell$  implies that  $\delta \notin S$ , it follows for  $p_\delta^\ell = (w, s)$  that the ladder sequence  $c_i$  for any  $i \in s$  is bounded below  $\delta$ . So the finite union

$$\delta \cap \bigcup \{c_i \mid i \in s^{p_\delta^\ell} \setminus \delta\}$$

is bounded below  $\delta$ . Using Fodor's lemma we may even assume that this intersection is bounded below  $\gamma$  (extend  $\gamma$  if necessary) and has a fixed finite cardinality.

For every  $\delta \in T_1$  define

$$A_\delta = \bigcap \{a_i \mid i \in w^{p_\delta^1} \setminus \gamma\}$$

Similarly, for  $\delta \in T_2$  define

$$B_\delta = \bigcup \{b_i \mid i \in w^{p_\delta^2} \setminus \gamma\}.$$

Clearly, any interpolation for  $G = \{(A_\delta \mid \delta \in T_1), \{B_\delta \mid \delta \in T_2\}\}$  is also an interpolation for  $g$ , and hence  $G$  is a gap.

Let  $k \in \omega$  be such that for every  $\delta \in T_\ell$ , if  $p_\delta^\ell = (w, s)$  and  $\alpha \in s \setminus \gamma$ , then  $|c_\alpha \cap \delta| < k$ .

Now we find a stationary set  $T'_1 \subset T_1$  such that for every  $n \in \omega$  if  $n \in A_\delta$  for some  $\delta \in T'_1$  then  $n \in A_\delta$  for a stationary set of  $\delta$ 's in  $T'_1$ . Simply throw away countably many non-stationary sets from  $T_1$ . Similarly, find a stationary  $T'_2 \subseteq T_2$  such that if  $n \notin B_\delta$  for some  $\delta \in T'_2$  then  $n \notin B_\delta$  for a stationary set of  $\delta \in T'_2$ .

Now lemma 3.3 gives  $\alpha_1 \in T'_1$  and  $\alpha_2 \in T'_2$  with  $\alpha_1 < \alpha_2$  such that  $A_{\alpha_1} \setminus k \not\subseteq B_{\alpha_2}$ . If we pick  $n \in A_{\alpha_1} \setminus B_{\alpha_2}$  such that  $n \geq k$  then there are stationary sets  $T''_1 \subseteq T'_1$  and  $T''_2 \subseteq T'_2$  such that  $n \in A_{\alpha_1} \setminus B_{\alpha_2}$  for every  $\alpha_1 \in T''_1$  and  $\alpha_2 \in T''_2$ . Hence if  $\alpha_1 \in T''_1$ ,  $\alpha_2 \in T''_2$ , and  $\alpha_1 < \alpha_2$ , then  $p_{\alpha_1}^1$  and  $p_{\alpha_2}^2$  are compatible in  $Q$  by lemma 3.2.  $\dashv$

### 3.1 Polarized chain condition

Theorem 3.4 shows that the poset  $Q_{g,C}$  for a gap  $g$  and ladder  $C$  over a stationary co-stationary set  $S$  satisfies some kind of a chain condition, suitable for two sequences indexed by stationary subsets of  $\omega_1 \setminus S$ . We formulate this condition in general and later prove that the iteration with finite support preserves this condition.

**Definition 3.5** *Let  $T \subseteq \omega_1$  be a stationary set. A c.c.c poset  $P$  satisfies the polarized chain condition (p.c.c) for  $T$  if it satisfies the following requirement. Suppose that*

1.

$$\bar{p}^\ell = (p_\delta^\ell \mid \delta \in T_\ell) \text{ for } \ell = 1, 2$$

are two sequences of conditions in  $P$ , where  $T_\ell \subseteq T$  are stationary for  $\ell = 1, 2$ .

2.  $p^* \in P$  is such that for each  $\ell = 1, 2$

$$p^* \Vdash_P \{\delta \in T_\ell \mid p_\delta^\ell \in G_P\} \text{ is stationary in } \omega_1,$$

where  $G_P$  is the name of the generic filter over  $P$ .

Then there are stationary sets  $T'_\ell \subseteq T_\ell$  for  $\ell = 1, 2$  such that  $p_{\alpha_1}^1$  and  $p_{\alpha_2}^2$  are compatible in  $P$  whenever  $\alpha_1 < \alpha_2$  are in  $T'_1$  and  $T'_2$  respectively.

We want to prove that if  $g$  is a gap and  $C$  a ladder over a stationary set  $S$  such that  $T = \omega_1 \setminus S$  is also stationary, then  $Q = Q_{g,C}$  satisfies the p.c.c. for  $T$ . The problem is that if  $p^*$  is as in the p.c.c. definition then it is not necessarily of the form to which theorem 3.4 is applicable, and so we need some argument to deduce that  $Q$  is p.c.c.

Recall that every club (closed unbounded) subset of  $\omega_1$  in a generic extension of  $V$  made via a c.c.c poset contains a club subset in  $V$ . The following property of c.c.c posets is also needed.

**Lemma 3.6** *Let  $P$  be a c.c.c poset. Suppose that  $T \subseteq \omega_1$  is stationary, and  $\langle p_\alpha \mid \alpha \in T \rangle$  is a sequence of conditions in  $P$  indexed along  $T$ . Then there exists some  $p_\alpha$  such that*

$$p_\alpha \Vdash \{\beta \in T \mid p_\beta \in G\} \text{ is stationary.}$$

*In fact, the set of these  $\alpha$ 's is stationary in  $\omega_1$ .*

**Prof.** Assume that this is not the case and, for some club  $D \subseteq \omega_1$ , for every  $\alpha \in T \cap D$  there is a club set  $C_\alpha$  (necessarily in  $V$ ) and an extension  $p'_\alpha$  of  $p_\alpha$  such that

$$p'_\alpha \Vdash \beta \in C_\alpha \cap T \rightarrow p_\beta \notin G. \tag{9}$$

Let  $C = \{\beta \in \omega_1 \mid (\forall \alpha < \beta) \beta \in C_\alpha\}$  be the diagonal intersection of these club sets. Then  $C$  is closed unbounded in  $\omega_1$ . Take a maximal antichain (surely countable) from the set of extensions  $\{p'_\alpha \mid \alpha \in T \cap D\}$ , and let  $\alpha_0$  be an index in  $T \cap C \cap D$  higher than all indexes of this countable antichain.

Then  $p'_{\alpha_0}$  is compatible with some  $p'_\alpha$  with  $\alpha < \alpha_0$ . But  $\alpha_0 \in C_\alpha$  which leads to a contradiction since  $p'_{\alpha_0}$  forces that  $p_{\alpha_0} \in G$ , and  $p'_\alpha$  forces that  $p_{\alpha_0} \notin G$  (by 9).  $\dashv$

Now we prove that  $Q$  is p.c.c. for  $T = \omega_1 \setminus S$ .

**Lemma 3.7** *If  $C$  is a ladder system over  $S$ , and  $T = \omega_1 \setminus S$  is stationary, then, for any gap  $g$ ,  $Q_{g,c}$  is p.c.c. over  $T$ .*

**Proof.** Suppose that  $T_1, T_2 \subseteq T$  are stationary, and  $\bar{p}^1, \bar{p}^2$  are two sequences of conditions indexed along  $T_1$  and  $T_2$ . Let  $q^* \in Q$  be such that for  $\ell = 1, 2$

$$q^* \Vdash_Q \{\delta \in T_\ell \mid p_\delta^\ell \in G\} \text{ is stationary in } \omega_1. \quad (10)$$

We claim first that we may assume that  $q^* \leq p_\delta^1$  for every  $\delta \in T_1$ . This can be achieved as follows. First, observe that the set of  $\delta \in T_1$  for which  $p_\delta^1$  and  $q^*$  are compatible is stationary. Then use Fodor's theorem on this stationary set to fix  $p_\delta^1 \upharpoonright \delta$ . Rename  $T_1$  to the the resulting stationary set. Redefine  $p_\delta^1$  as  $p_\delta^1 \cup q^*$ , and finally apply lemma 3.6 to obtain  $\delta_0$  such that

$$p_{\delta_0}^1 \Vdash_Q \{\delta \in T_1 \mid p_\delta^1 \in G\} \text{ is stationary in } \omega_1.$$

Now  $q^*$  is as required. Since  $q^*$  extends the original  $q^*$ , it still satisfies (10) with respect to  $T_2$ . Repeat this procedure for  $T_2$ , and obtain two sequences to which Theorem 3.4 is applicable.  $\dashv$

We note here for a possible future use a stronger form of polarized chain condition (strong-p.c.c) which is not used in this paper.

**Definition 3.8** *Let  $T \subseteq \omega_1$  be stationary. A poset  $P$  is said to satisfy the strong-p.c.c over  $T$  if whenever two sequences are given*

$$\bar{p}^\ell = (p_\delta^\ell \mid \delta \in T_\ell) \text{ for } \ell = 1, 2$$

*of conditions in  $P$ , where  $T_\ell \subseteq T$  are stationary for  $\ell = 1, 2$ , and for some  $p^* \in P$ , for every  $\ell = 1, 2$ ,*

$$p^* \Vdash_P \{\delta \in T_\ell \mid p_\delta^\ell \in G_P\} \text{ is stationary in } \omega_1,$$

*then there are stationary subsets  $T'_\ell \subseteq T_\ell$  for  $\ell = 1, 2$ , and conditions  $q_\delta \geq p_\delta^2$  for  $\delta \in T'_2$  such that:*

*For every  $\delta \in T'_2$  and  $q \in P$  such that  $q_\delta \leq q$  there exists  $\alpha < \delta$  such that for every  $\beta$  that satisfies  $\alpha < \beta \in T'_1 \cap \delta$*

*$q$  and  $p_\beta^1$  are compatible in  $P$ .*

## 4 Iteration of p.c.c posets

Our aim in this section is to prove that the iteration with finite support of p.c.c posets is again p.c.c. It is well known (by Martin and Solovay [2]) that since each of the iterands satisfies the countable chain condition the iteration is again c.c.c, but we have to prove the preservation of the polarized property.

Our posets are separative (and if not, they can be made separative). A poset is separative iff  $p \not\leq q$  implies that some extension of  $q$  is incompatible with  $p$ .

**Lemma 4.1** *Suppose that  $P$  is a p.c.c. poset, and that  $Q \in V^P$  is (forced by every condition in  $P$  to be) a p.c.c. poset. Then the iteration  $P * Q$  satisfies the polarized chain condition too.*

**Proof.** Suppose that  $(p_\delta^\ell, q_\delta^\ell) \in P * Q$  are given for  $\delta \in T_\ell \subseteq T$  and for  $\ell = 1, 2$ , such that for some condition  $(p, q) \in P * Q$

$$(p, q) \Vdash \{\delta \in T_\ell \mid (p_\delta^\ell, q_\delta^\ell) \in G_{P*Q}\} \text{ is stationary} \quad (11)$$

for  $\ell = 1, 2$ . Then

$$p \Vdash_P \{\delta \in T_\ell \mid p_\delta^\ell \in G_P\} \text{ is stationary.}$$

Let  $G \subset P$  be  $V$ -generic, with  $p \in G$ . In  $V[G]$  form the interpretations  $q[G]$  (interpretation of  $q$ ) and  $Q[G]$  (interpretation of  $Q$ ). Then  $q[G] \in Q[G]$ . Define the sets

$$T'_\ell = \{\delta \in T_\ell \mid p_\delta^\ell \in G\}, \quad \ell = 1, 2$$

(which are stationary) and define the sequences

$$\langle q_\delta^\ell[G] \mid \delta \in T'_\ell \rangle, \quad \text{for } \ell = 1, 2.$$

Then in  $V[G]$

$$q[G] \Vdash_{Q[G]} \{\delta \in T'_\ell \mid q_\delta^\ell[G] \in H\} \text{ is stationary}$$

where  $H$  is the name for the  $V[G]$  generic filter over  $Q[G]$ . (This follows from (11) and since forcing with  $P * Q$  is equivalent to the iteration of forcing with  $P$  and then with  $Q[G]$ .)

Since  $Q[G]$  satisfies the polarized chain condition for  $T$ , there are stationary sets  $T''_\delta \subseteq T'_\delta$  such that:

if  $\delta_1 \in T_1''$ ,  $\delta_2 \in T_2''$ , and  $\delta_1 < \delta_2$ , then  $q_{\delta_1}^1[G]$  and  $q_{\delta_2}^2[G]$  are compatible in  $Q[G]$ .

Back in  $V$ , let  $S_1$  and  $S_2$  be  $V^P$  names of  $T_1''$  and  $T_2''$  respectively, forced to have these properties. The following short lemma will be applied to  $S_1$  and to  $S_2$ .

**Lemma 4.2** *Suppose that  $\langle p_\delta \mid \delta \in T \rangle$  is a sequence in  $P$ ,  $S$  is a name of a subset of  $\omega_1$  and  $p \in P$  a condition such that*

$$p \Vdash_P S \subseteq \{\alpha \in T \mid p_\delta \in G\} \text{ and } S \text{ is stationary in } \omega_1.$$

*Then there is a stationary subset  $T^* \subseteq T$ , and conditions  $p_\delta^*$  extending both  $p_\delta$  and  $p$  for each  $\delta \in T^*$  such that  $p_\delta^* \Vdash \delta \in S$ .*

**Proof.** Define  $T^*$  by the condition that  $\delta \in T^*$  iff  $\delta \in T$  and there is a common extension of  $p$  and  $p_\delta$  that forces  $\delta \in S$ . We must prove that  $T^*$  is stationary. If  $C \subseteq \omega_1$  is any closed unbounded set, find  $p' \geq p$  and  $\delta \in C$  such that  $p' \Vdash \delta \in S$ . Then  $\delta \in T$  and  $p' \Vdash p_\delta \in G$ . Hence  $p_\delta \leq p'$  (because  $P$  is separative). So  $\delta \in T^*$ .  $\dashv$

Apply the lemma to  $S_1$  and find a stationary set  $T_1^* \subseteq T_1$  and conditions  $p_\delta^{*1} \geq p_\delta^1, p$ , for  $\delta \in T_1^*$  such that

$$p_\delta^{*1} \Vdash \delta \in S_1.$$

Then (lemma 3.6) find an extension of  $p$ , denoted  $p^*$ , such that

$$p^* \Vdash \{\delta \in T_1^* \mid p_\delta^{*1} \in G\} \text{ is stationary.}$$

Apply the same argument to  $S_2$ , and find a stationary set  $T_2^* \subseteq T_2$  and conditions  $p_\delta^{*2} \geq p_\delta^2, p^*$  for  $\delta \in T_2^*$  such that  $p_\delta^{*2} \Vdash \delta \in S_2$ . Now  $p^{**} \geq p^*$  can be found such that

$$p^{**} \Vdash \{\delta \in T_2^* \mid p_\delta^{*2} \in G\} \text{ is stationary.}$$

Since  $P$  satisfies the p.c.c., there are stationary sets  $T_1^{**} \subseteq T_1^*$  and  $T_2^{**} \subseteq T_2^*$  such that for every  $\delta_1 < \delta_2$  in  $T_1^{**}$  and  $T_2^{**}$  (respectively)  $p_{\delta_1}^{*1}$  and  $p_{\delta_2}^{*2}$  are compatible in  $P$ , say by some condition  $p'$  extending both. But then  $p' \Vdash \delta_1 \in S_1$  and  $\delta_2 \in S_2$ . It follows that  $(p_{\delta_1}^1, q_{\delta_1}^1)$  and  $(p_{\delta_2}^2, q_{\delta_2}^2)$  are compatible in  $P * Q$  showing that  $P * Q$  satisfies the p.c.c. The point is that

$$p' \Vdash_P q_{\delta_1}^1 \text{ and } q_{\delta_2}^2 \text{ are compatible in } Q$$

and hence for some  $q' \in V^P$ ,  $p' \Vdash_P q' \geq q_{\delta_1}^1, q_{\delta_2}^2$ . That is,  $(p'q') \geq (p_{\delta_1}^1, q_{\delta_1}^1), (p_{\delta_2}^2, q_{\delta_2}^2)$ .  $\dashv$



**Theorem 4.3** *Let  $T$  be a stationary subset of  $\omega_1$ . An iteration with finite support of p.c.c. for  $T$  posets is again p.c.c. for  $T$ .*

**Proof.** The theorem is proved by induction on the length,  $\delta$ , of the iteration. For  $\delta$  a successor ordinal, this is essentially Lemma 4.1. So we assume that  $\delta$  is a limit ordinal, and  $\langle P_\alpha \mid \alpha \leq \delta \rangle$  is a finite support iteration, where  $P_{\alpha+1} = P_\alpha * Q_\alpha$  is obtained with  $Q_\alpha \in V^{P_\alpha}$  a p.c.c poset for  $T$ . Thus conditions in  $P_\delta$  are finite functions  $p$  defined on a finite subset  $\text{dom}(p) \subset \delta$ , and are such that for every  $\alpha \in \text{dom}(p)$ ,  $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \in Q_\alpha$ . It is well-known that  $P_\delta$  satisfies the c.c.c, and we must prove the polarized property.

Suppose that  $\bar{p}^\ell = \langle p_i^\ell \mid i \in T_\ell \rangle$  for  $\ell = 1, 2$  are two sequences of conditions in  $P_\delta$ , where  $T_\ell \subseteq T$  are stationary, and suppose also that  $p^* \in P_\delta$  is such that

$$p^* \Vdash_{P_\delta} \{i \in T_\ell \mid p_i^\ell \in G\} \text{ is stationary for } \ell = 1, 2. \quad (12)$$

We may assume that  $p^*$  is compatible with every  $p_i^\ell$  (just throw away those conditions that are not). The case  $\text{cf}(\delta) > \omega_1$  is trivial, because the support of all conditions is bounded by some  $\delta' < \delta$  to which induction is applied). So there are two cases to consider.

**Case 1:**  $\text{cf}(\delta) = \omega$ . Let  $\langle \delta_n \mid n \in \omega \rangle$  be an increasing  $\omega$ -sequence converging to  $\delta$ . For every  $p_i^\ell$  there is  $n \in \omega$  such that  $\text{dom}(p_i^\ell) \subseteq \delta_n$ . It follows from (12) that for some specific  $n \in \omega$ , for some extension  $p^{**} \geq p^*$

$$p^{**} \Vdash \{i \in T_\ell \mid p_i^\ell \in P_{\delta_n} \cap G\} \text{ is stationary for } \ell = 1, 2.$$

Now we can apply the inductive assumption to  $P_{\delta_n}$ .

**Case 2:**  $\text{cf}(\delta) = \omega_1$ . Let  $\langle \delta_\alpha \mid \alpha \in \omega_1 \rangle$  be an increasing, continuous, and cofinal in  $\delta$  sequence. Intersecting  $T_1$  and  $T_2$  with a suitable closed unbounded set, we may assume that for every  $\alpha < \beta$   $\alpha \in T_1$  and  $\beta \in T_2$ ,  $\text{dom}(p_\alpha^1) \subset \beta$ .

We claim that we may without loss of generality assume that, for some  $\gamma < \delta$ ,  $\text{dom}(p_\alpha^\ell) \cap \delta_\alpha$  for all  $\alpha \in T_\ell$ . We get this in two steps.

In the first step, find a stationary  $T'_1 \subseteq T_1$  such that the sets  $\text{dom}(p_\alpha^1) \cap \delta_\alpha$ , for  $\alpha \in T'_1$ , are bounded by some  $\gamma < \delta$ . For each  $\alpha \in T'_1$  let  $p_\alpha^{1*}$  be a common extension of  $p_\alpha^1$  and  $p^*$ . Then (use lemma 3.6) find an extension  $p^{**} \geq p^*$  such that

$$p^{**} \Vdash \{\alpha \in T'_1 \mid p_\alpha^1 \in G\} \text{ is stationary.}$$

Since  $p^{**}$  extends  $p^*$ ,  $p^{**} \Vdash \{i \in T_2 \mid p_i^2 \in G\}$  is stationary. We can again assume that each  $p_i^2$  is compatible with  $p^{**}$  and get  $T'_2 \subseteq T_2$  stationary such that  $\text{dom}(p_\alpha^2) \cap \delta_\alpha$  is bounded by some  $\gamma' < \delta$  (we rename  $\gamma$  to be the

maximum of  $\gamma$  and  $\gamma'$ ). Rename the stationary sets as  $T_1, T_2$  and we have our assumption.

Apply induction to  $P_\gamma$  and to the conditions  $p_\alpha^\ell \upharpoonright \gamma$ . This yields two stationary subsets which are as required.  $\dashv$

## 5 The model

**Theorem 5.1** *Assuming the consistency of ZFC, the following property is consistent with ZFC. There is a stationary co-stationary set  $S \subseteq \omega_1$  such that*

1. *For every ladder system  $C$  over  $S$ , every gap contains a  $C$ -Hausdorff subgap.*
2. *For every ladder system  $H$  over  $T = \omega_1 \setminus S$  there is a gap  $g$  with no subgap that is  $H$ -Hausdorff.*

To obtain the required generic extension we assume that  $\kappa$  is a cardinal in  $V$  (the ground model) such that  $\text{cf}(\kappa) > \omega_1$  and even  $\kappa^{\aleph_1} = \kappa$ . We shall obtain a generic extension  $V[G]$  in which  $2^{\aleph_0} = \kappa$  and the two required properties of the theorem hold. For this we define a finite support iteration of length  $\kappa$ , iterating posets  $P$  as in section 2, which introduce generic gaps, and posets of the form  $Q_{g,C}$ , as in section 3.1, which are designed to introduce a  $C$ -Hausdorff subgap to  $g$ .

We denote this iteration  $\langle P_\alpha \mid \alpha < \kappa \rangle$ . So  $P_{\alpha+1} \simeq P_\alpha * R(\alpha)$ , where the  $\alpha$ -th iterand  $R(\alpha)$  is either some  $P$  or some  $Q_{g,C}$ . The rules to determine  $R(\alpha)$  are specified below. For any limit ordinal  $\delta \leq \kappa$ ,  $P_\delta$  is the finite support iteration of the posets  $\langle P_\alpha \mid \alpha < \delta \rangle$ . We define  $P_\kappa$  as our final poset, and we shall prove that in  $V^{P_\kappa}$  the two properties of the theorem hold.

Recall that  $P$  satisfies Talayaco's condition and is hence a p.c.c. poset, and each  $Q_{g,C}$  is p.c.c. over  $T = \omega_1 \setminus S$  (by lemma 3.7).

Since the iterand posets satisfy the p.c.c over  $T$ , each  $P_\alpha$  is a p.c.c. poset over  $T$  (and in particular a c.c.c poset). It follows that every ladder system and every gap in  $V^{P_\kappa}$  are already in some  $V^{P_\alpha}$  for  $\alpha < \kappa$ . It is obvious that if  $g \in V^{P_\alpha}$  is forced by  $p \in P_\kappa$  to be a gap in  $V^{P_\kappa}$ , then  $p \upharpoonright \alpha$  forces it to be a gap already in  $V^{P_\alpha}$ .

To determine the iterands, we assume a standard bookkeeping scheme which ensures two things:

1. For every ladder system  $C$  over  $S$  and gap  $g$  in  $V^{P_\kappa}$ , there exists a stage  $\alpha < \kappa$  so that  $C, g \in V^{P_\alpha}$ , and  $R(\alpha)$  is  $Q_{g,C}$ .
2. For some unbounded set of ordinals  $\alpha \in \kappa$  the iterand  $R(\alpha)$  is  $P$ , producing a generic gap  $g$ , and the subsequent iterand  $R(\alpha + 1)$  is  $Q_{g,C}$  for some ladder sequence  $C$  over  $S$ .

The first item ensures that, in  $V^{P_\kappa}$ , for every ladder system  $C$  over  $S$ , every gap contains a  $C$ -Hausdorff subgap. (A  $C$ -Hausdorff subgap in  $V_{\alpha+1}$  remains  $C$ -Hausdorff at every later stage and in the final model).

Suppose now that  $H$  is a ladder over  $T = \omega_1 \setminus S$ . Then  $H$  appears in some  $V^{P_\alpha}$  such that  $R(\alpha)$  is the poset  $P$ , and  $R(\alpha + 1)$  is the poset  $Q_{g,C}$  where  $g$  is the generic gap introduced by  $R(\alpha)$ , and  $C$  is some ladder sequence over  $S$ . We want to prove that  $g$  is a gap that has no  $H$ -Hausdorff subgap in  $V^{P_\kappa}$ . We first prove that  $g$  remains a gap in  $V^{P_\kappa}$ . It is clearly a gap in  $V^{P_{\alpha+1}}$  by Lemma 2.3. Since  $g$  is  $C$ -Hausdorff in  $V^{P_{\alpha+2}}$ , it remains a gap in  $V^{P_\kappa}$  (by Lemma 1.2).

This generic gap  $g$  satisfies the conclusion of lemma 2.4 in  $V^{P_{\alpha+1}}$ :

If  $J, K \subseteq \omega_1$  are unbounded, then there is a club set  $D_0 \subseteq \omega_1$  such that for every  $\delta \in D_0$  and  $k \in K \setminus \delta$  there are  $m \in \omega$  and a sequence  $j(n) \in \delta \cap J$  increasing and cofinal in  $\delta$  such that  $a_{j(n)} \setminus m \subset b_k$  for all  $n \in \omega$ .

Since  $P_\kappa \simeq P_{\alpha+1} * R$ , where the remainder  $R \simeq P_\kappa / P_{\alpha+1}$  is interpreted in  $V^{P_{\alpha+1}}$  as a finite support iteration of p.c.c. posets over  $T$ , we can view  $P_\kappa$  as a two-stage iteration in which the second stage is a p.c.c. poset over  $T$ . Thus, for simplicity of expression, we can assume that  $V^{P_{\alpha+1}}$  is the ground model. The following lemma then ends the proof.

**Lemma 5.2** *Suppose in the ground model  $V$  a ladder system  $H$  over a stationary set  $T \subseteq \omega_1$ , and a gap  $g$  that has the property quoted above (the conclusion of lemma 2.4). Suppose also a poset  $R$  that is p.c.c. over  $T$ . Then in  $V^R$  the gap  $g$  contains no  $H$ -Hausdorff subgap.*

**Proof.** Let  $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$  and assume (for the sake of a contradiction) some condition  $q'$  in  $R$  forces that  $g' = \{(a_\alpha \mid \alpha \in A), (b_\beta \mid \beta \in B)\}$  is a  $H$ -Hausdorff subgap, where  $A$  and  $B$  are names forced by  $q'$  to be unbounded in  $\omega_1$ . Since every club subset of  $\omega_1$  in a c.c.c. generic extension

contains a club subset in the ground model, we may assume that the club,  $D$ , which appears in definition 1.3 (of  $g'$  being  $H$ -Hausdorff) is in  $V$ .

For every  $\delta \in T \cap D$  define two conditions in  $R$  (extending the given condition  $q$ ):

1.  $p_\delta \in R$  is such that for some  $\alpha(\delta) \in \omega_1 \setminus \delta$ ,  $p_\delta \Vdash_R \alpha(\delta) \in A$ . (This is possible since  $A$  is forced to be unbounded.)
2.  $q_\delta \in R$  extending  $p_\delta$  is such that, for some  $\beta(\delta) \in \omega_1 \setminus \delta$ ,  $q_\delta \Vdash_R \beta(\delta) \in B$ . Moreover, as  $g'$  is forced to be  $H$ -Hausdorff, we can assume that for some  $m_\delta \in \omega$ ,

$$q_\delta \Vdash_R \text{ for every } n \geq m_\delta, \text{ if } i \in A \cap (\delta \setminus c_\delta(n)), \text{ then } X(a_i, b_{\beta(\delta)}) > n.$$

By Lemma 3.6 some condition forces that  $q_\delta \in G$  (and hence  $p_\delta \in G$ ) for a stationary set of indices  $\delta \in T \cap D$ . Since  $R$  is p.c.c. for  $T$ , there are stationary subsets  $T_1, T_2 \subseteq T$  such that any  $p_{\delta_1}$  is compatible with  $q_{\delta_2}$  if  $\delta_1 \in T_1$ ,  $\delta_2 \in T_2$  and  $\delta_1 < \delta_2$ .

Consider now the two unbounded sets  $J = \{\alpha(\delta) \mid \delta \in T_1\}$ , and  $K = \{\beta(\delta) \mid \delta \in T_2\}$ . Apply the conclusion of lemma 2.4 quoted above to  $J$  and  $K$ , and let  $D_0$  be the club set that appears there. Pick any  $\delta \in D \cap T_2 \cap D_0$ . Consider  $k = \beta(\delta)$ . Then  $k \in K \setminus \delta$ , and so there are  $m \in \omega$  and a sequence  $j(n) \in \delta \cap J$  cofinal in  $\delta$  such that

$$a_{j(n)} \setminus m \subset b_k \text{ for all } n \in \omega. \tag{13}$$

Yet every  $j(n)$  is of the form  $\alpha(\delta_n)$  for some  $\delta_n \in T_1 \cap \delta$ , and the  $\delta_n$ 's tend to  $\delta$ . So (13) can be written as

$$X(a_{\alpha(\delta_n)}, b_k) \leq m. \tag{14}$$

It follows from the definition of  $T_1$  and  $T_2$  that  $p_{\delta_n}$  and  $q_\delta$  are compatible in  $R$ . If  $q'$  is a common extension, then  $q'$  forces that for every  $n \geq m_\delta$  if  $i = \alpha(\delta_n) \geq c_\delta(n)$  then  $X(a_i, b_k) > n$ . It suffices now to take  $n \geq \max\{m, m_\delta\}$  to get the contradiction to (14).

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