On inverse γ-systems and the number of $L_{\infty\lambda}$-equivalent, non-isomorphic models for λ singular

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Abstract

Suppose λ is a singular cardinal of uncountable cofinality κ. For a model $\mathcal{M}$ of cardinality λ, let $\mathrm{No}(\mathcal{M})$ denote the number of isomorphism types of models $\mathcal{N}$ of cardinality λ which are $L_{\infty\lambda}$-equivalent to $\mathcal{M}$. In [She85] Shelah considered inverse κ-systems $\mathcal{A}$ of abelian groups and their certain kind of quotient limits $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$. In particular Shelah proved in [She85, Fact 3.10] that for every cardinal μ there exists an inverse κ-system $\mathcal{A}$ such that $\mathcal{A}$ consists of abelian groups having cardinality at most μ^κ and $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big) = \mu$. Later in [She86, Theorem 3.3] Shelah showed a strict connection between inverse κ-systems and possible values of No (under the assumption that $\theta^\kappa < \lambda$ for every $\theta < \lambda$): if $\mathcal{A}$ is an inverse κ-system of abelian groups having cardinality $< \lambda$, then there is a model $\mathcal{M}$ such that $\mathrm{card}(\mathcal{M}) = \lambda$ and $\mathrm{No}(\mathcal{M}) = \mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big)$. The following was an immediate consequence (when $\theta^\kappa < \lambda$ for every $\theta < \lambda$): for every nonzero $\mu < \lambda$ or $\mu = \lambda^\kappa$ there is a model $\mathcal{M}_\mu$ of cardinality λ with $\mathrm{No}(\mathcal{M}_\mu) = \mu$. In this paper we show: for every nonzero $\mu \leq \lambda^\kappa$ there is an inverse κ-system $\mathcal{A}$ of abelian groups having cardinality $< \lambda$ such that $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big) = \mu$ (under the assumptions $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for all $\theta < \lambda$ when $\mu > \lambda$), with the obvious new consequence concerning the possible value of No. Specifically, the case $\mathrm{No}(\mathcal{M}) = \lambda$ is possible when $\theta^\kappa < \lambda$ for every $\theta < \lambda$.[1]

1 Introduction

Suppose λ is a cardinal. For a model $\mathcal{M}$ we let $\mathrm{card}(\mathcal{M})$ denote the cardinality of the universe of $\mathcal{M}$. When $\mathcal{M}$ and $\mathcal{N}$ are models of the same vocabulary and they satisfy the same sentences of the infinitary language $L_{\infty\lambda}$, we write $\mathcal{M} \equiv_{\infty\lambda} \mathcal{N}$. For any model $\mathcal{M}$ of cardinality λ we define $\mathrm{No}(\mathcal{M})$ to be the cardinality of the set

$$\big\{\mathcal{N}/\cong \;\big|\; \mathrm{card}(\mathcal{N}) = \lambda \text{ and } \mathcal{N} \equiv_{\infty\lambda} \mathcal{M}\big\},$$

where $\mathcal{N}/\cong$ is the equivalence class of $\mathcal{N}$ under the isomorphism relation. Our principal purpose is to study the possible values of $\mathrm{No}(\mathcal{M})$ for models $\mathcal{M}$ of singular cardinality with uncountable cofinality.

When $\mathcal{M}$ is countable, $\mathrm{No}(\mathcal{M}) = 1$ by [Sco65]. This result extends to structures of cardinality λ when λ is a singular cardinal of countable cofinality [Cha68].

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If $V = L$, λ is an uncountable regular cardinal which is not weakly compact, and $\mathcal{M}$ is a model of cardinality λ, then $\mathrm{No}(\mathcal{M})$ has either the value 1 or 2^λ. For $\lambda = \aleph_1$ this result was first proved in [Pal77a]. Later in [She81] Shelah extended this result to all other regular non-weakly compact cardinals. The possibility $\mathrm{No}(\mathcal{M}) = \aleph_0$ is consistent with ZFC + GCH in case $\lambda = \aleph_1$, as remarked in [She81]. The values $\mathrm{No}(\mathcal{M}) \in \omega \setminus \{0, 1\}$ are proved to be consistent with ZFC + GCH in the forthcoming paper of the authors [SV97] (number 646 in Shelah's publications).

The case $\mathcal{M}$ has cardinality of a weakly compact cardinal is dealt with in [She82] by Shelah. The result is that for κ weakly compact there is for every $1 \le \mu \le \kappa$ a model $\mathcal{M}_\mu$ such that $\mathrm{No}(\mathcal{M}_\mu) = \mu$. There is in preparation by the authors a paper where the question for κ weakly compact is revisited.

The case $\mathcal{M}$ is of singular cardinality λ with uncountable cofinality κ was first treated in [She85], where the relations of $\mathcal{M}$ have infinitely many places. Later in [She86] Shelah improved the result by showing that if $\theta^\kappa < \lambda$ for every $\theta < \lambda$ and $0 < \mu < \lambda$ then $\mathrm{No}(\mathcal{M}) = \mu$ is possible for a model $\mathcal{M}$ having cardinality λ and relations of finitely many places only. The main idea in those papers was to transform the problem of possible values of $\mathrm{No}(\mathcal{M})$ into a question concerning possible cardinalities of "quotient limit" $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ of an inverse system $\mathcal{A}$ of groups [She86, Theorem 3.3]:

Theorem 1 (λ *cardinal with* $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$)*If* $\theta^\kappa < \lambda$ *for every* $\theta < \lambda$ *and* $\mathcal{A}$ *is an inverse* κ*-system of abelian groups having cardinality* $< \lambda$*, then there is a model* $\mathcal{M}$ *of cardinality* λ (*with relations having finitely many places only*) *such that* $\mathrm{No}(\mathcal{M}) = \mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big)$.

Actually the groups in [She86, Theorem 3.3] are not limited to be abelian. However, abelian groups suffice for the present purposes.

The recent paper fills a gap left open since the paper [She86]. We present a uniform way to construct inverse κ-system of abelian groups having a quotient limit of desired cardinality. The most important new case is that the cardinality of a quotient limit can be λ for some inverse system (in other cases, where the result below can be applied, the Singular Cardinal Hypothesis fails). The result of this paper is:

Theorem 2 (λ *cardinal with* $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$)*For every nonzero* $\mu \le \lambda$ *there is an inverse* κ*-system* $\mathcal{A} = \langle G_i, h_{i,j} \mid i < j < \kappa \rangle$ *of abelian groups satisfying that* $\mathrm{card}(G_i) < \lambda$ *for every* $i < \kappa$ *and* $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big) = \mu$. *The same conclusion holds also for the values* $\lambda < \mu \le \lambda^\kappa$ *under the assumption that* $2^\kappa < \lambda$ *and* $\theta^{<\kappa} < \lambda$ *for every* $\theta < \lambda$.

So the general method used here to find new possibilities for the values of $\mathrm{No}(\mathcal{M})$ is the same as in [She86]. As an immediate consequence of the last theorem we get:

Theorem 3 *Suppose* λ *is a singular cardinal of uncountable cofinality* κ. *For each nonzero* $\mu \le \lambda^\kappa$ *there is a model* $\mathcal{M}$ (*with relations having finitely many places only*) *satisfying* $\mathrm{card}(\mathcal{M}) = \lambda$ *and* $\mathrm{No}(\mathcal{M}) = \mu$, *provided that* $\theta^\kappa < \lambda$ *for every* $\theta < \lambda$.

We give all necessary definitions concerning inverse κ-systems $\mathcal{A}$ of abelian groups and their special kind of quotient limits $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ in the next section.

2 Preliminaries

Definition 2.1 *Suppose γ is a limit ordinal and for every $i < j < \gamma$, G_i is a group and $h_{i,j}$ is a homomorphism from G_j into G_i. The family $\mathcal{A} = \langle G_i, h_{i,j} \mid i < j < \gamma \rangle$ is called an inverse γ-system when the equation $h_{i,j} \circ h_{j,k} = h_{i,k}$ holds for every $i < j < k < \gamma$. As in [She85] we assume that all the groups G_i, $i < \gamma$, are additive abelian groups.*

To simplify our notation we make an agreement that the letters i, j, k, and l always denote ordinals smaller than γ. Hence "for all $i < j$" means "for all ordinals i and j with $i < j < \gamma$" and so on.

The main objects of our study are the following two sets:

$$\mathrm{Gr}(\mathcal{A}) = \Big\{ \langle \boldsymbol{a}^{i,j} \mid i < j < \gamma \rangle \mid \boldsymbol{a}^{i,j} \in G_i \ \text{ and for all } k > j,\ \boldsymbol{a}^{i,k} = \boldsymbol{a}^{i,j} + h_{i,j}(\boldsymbol{a}^{j,k}) \Big\};$$

$$\mathrm{Fact}(\mathcal{A}) = \Big\{ \langle \boldsymbol{a}^{i,j} \mid i < j < \gamma \rangle \mid \text{for some } \bar{y} \in \prod_{k<\gamma} G_k,\ \boldsymbol{a}^{i,j} = \bar{y}^i - h_{i,j}(\bar{y}^j) \Big\}.$$

We consider $\mathrm{Gr}(\mathcal{A})$ and $\mathrm{Fact}(\mathcal{A})$ as additive abelian groups where the group operation $+$ and the unit element $\mathbf{0}$ are pointwise defined. The factor group $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ is well-defined since $\mathrm{Fact}(\mathcal{A}) \subseteq \mathrm{Gr}(\mathcal{A})$ by the requirements $h_{i,j} \circ h_{j,k} = h_{i,k}$ for all $i < j < k$. For any inverse γ-system $\mathcal{A}$, the group $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ is called the quotient limit of $\mathcal{A}$.

Definition 2.2 *We let $\gamma \star \gamma$ be the set $\{(i,j) \in \gamma \times \gamma \mid i < j\}$. For every subset I of $\gamma \star \gamma$ we define*

$$I^{1st} = \{ i < \gamma \mid (i,j) \in I \text{ for some } j < \gamma \}$$

and for each $i \in I^{1st}$,

$$I[i] = \{ j < \gamma \mid (i,j) \in I \}.$$

We also say that

> *I is cobounded if $\gamma \smallsetminus I^{1st}$ and $\gamma \smallsetminus I[i]$, for all $i \in I^{1st}$, are bounded subsets of γ;*
>
> *I is coherent if I^{1st} is unbounded in γ and for every $i \in I^{1st}$, $I[i] = I^{1st} \smallsetminus (i+1)$;*
>
> *I is eventually coherent if it is unbounded and for every $i \in I^{1st}$, $I^{1st} \smallsetminus I[i]$ is a bounded subset of γ.*

Remark. Suppose I is an eventually coherent subset of $\gamma \star \gamma$ and S is a subset of $I^{1\text{st}}$. If $\mathrm{card}(S) < \mathrm{cf}(\gamma)$, then $I^{1\text{st}} \smallsetminus (\bigcap_{i \in S} I[i])$ is a bounded subset of γ. If S is unbounded in γ, then $I \cap (S \times S)$ is an eventually coherent subset of I.

In [She86, Claim 1.12] Shelah proved (note the remark given after the following lemma) that if two sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ from $\mathrm{Gr}(\mathcal{A})$ agree on a coherent set of indices, then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A})$. The following slight improvement of this condition has an essential role in the proof of Theorem 2.

Lemma 2.3 *Suppose $\mathcal{A}$ is an inverse γ-system, and $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{Gr}(\mathcal{A})$. Then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A})$ holds if there is an eventually coherent subset I of $\gamma \star \gamma$ such that $\boldsymbol{a}^{i,j} = \boldsymbol{b}^{i,j}$ for all $(i,j) \in I$.*

Proof. We shall need an eventually coherent subset J of I having the property that $\langle J[i] \mid i \in J^{1\text{st}} \rangle$ is a decreasing chain of end segments of $J^{1\text{st}}$. Let S be an unbounded subset of I having the order type $\mathrm{cf}(\gamma)$. Define a subset J of I by $J^{1\text{st}} = S$ and for all $j \in S$,

$$J[j] = S \cap \bigcap_{i \in S \cap (j+1)} \big(I[i] \smallsetminus (i^* + 1) \big),$$

where i^* is the supremum of the bounded subset $I^{1\text{st}} \setminus I[i]$ of γ. The set J is well-defined since I is eventually coherent and $\text{card}\big(S \cap (j+1)\big) < \text{cf}(\gamma)$ for all $j < \gamma$. Now J is also eventually coherent, and furthermore, for all $i \in J^{1\text{st}}$, $J[i] = S \setminus \min(J[i])$ and for all $j \in J^{1\text{st}} \setminus i$, $\min(J[i]) \leq \min(J[j])$.

Define for every $i < \gamma$, i' to be $\min\big(J^{1\text{st}} \setminus (i+1)\big)$ and $i'' = \min(J[i'])$. Then the following are satisfied for all $i < j$:

$$i < i' < i'',\ j < j' < j'',\ i' \leq j',\ i'' \leq j'', \text{ and also } i' < j'';$$

$$j'' \in I^{1\text{st}} \text{ and } (i', i''), (j', j''), (i', j'') \in I.$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are in $\text{Gr}(\mathcal{A}^T_R)$ we have

$$\begin{aligned} \boldsymbol{a}^{i,j''} &= \boldsymbol{a}^{i,j} + h_{i,j}(\boldsymbol{a}^{j,j''}), \\ \boldsymbol{b}^{i,j''} &= \boldsymbol{b}^{i,j} + h_{i,j}(\boldsymbol{b}^{j,j''}). \end{aligned}$$

Therefore the following equations hold:

$$\text{(A)} \qquad \begin{aligned} \boldsymbol{a}^{i,j} - \boldsymbol{b}^{i,j} &= (\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''}) - \big(h_{i,j}(\boldsymbol{a}^{j,j''}) - h_{i,j}(\boldsymbol{b}^{j,j''})\big) \\ &= (\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''}) - h_{i,j}(\boldsymbol{a}^{j,j''} - \boldsymbol{b}^{j,j''}). \end{aligned}$$

Because of $i < i' < j''$ we also have that

$$\begin{aligned} \boldsymbol{a}^{i,j''} &= \boldsymbol{a}^{i,i'} + h_{i,i'}(\boldsymbol{a}^{i',j''}), \\ \boldsymbol{b}^{i,j''} &= \boldsymbol{b}^{i,i'} + h_{i,i'}(\boldsymbol{b}^{i',j''}). \end{aligned}$$

Since $(i', j'') \in I$, $\boldsymbol{a}^{i',j''} = \boldsymbol{b}^{i',j''}$ holds. Hence we get

$$\text{(B)} \qquad \boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''} = \boldsymbol{a}^{i,i'} - \boldsymbol{b}^{i,i'}.$$

Moreover, $i < i' < i''$ yields

$$\begin{aligned} \boldsymbol{a}^{i,i''} &= \boldsymbol{a}^{i,i'} + h_{i,i'}(\boldsymbol{a}^{i',i''}), \\ \boldsymbol{b}^{i,i''} &= \boldsymbol{b}^{i,i'} + h_{i,i'}(\boldsymbol{b}^{i',i''}). \end{aligned}$$

Now $(i', i'') \in I$ implies that $\boldsymbol{a}^{i',i''} = \boldsymbol{b}^{i',i''}$, and consequently

$$\boldsymbol{a}^{i,i'} - \boldsymbol{b}^{i,i'} = \boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''}.$$

This equation together with (A) and (B) implies that for all $i < j$

$$\boldsymbol{a}^{i,j} - \boldsymbol{b}^{i,j} = (\boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''}) - h_{i,j}(\boldsymbol{a}^{j,j''} - \boldsymbol{b}^{j,j''}).$$

So the sequence $\bar{y} = \langle \boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''} \mid i < \gamma \rangle \in \prod_{i<\gamma} G_i$ exemplifies that $\boldsymbol{a} - \boldsymbol{b} \in \text{Fact}(\mathcal{A})$, and we have $\boldsymbol{a} \equiv \boldsymbol{b} \mod \text{Fact}(\mathcal{A})$. $\blacksquare_{2.3}$

Remark. In [She86, Claim 1.12] the groups of an inverse system $\mathcal{A}$ need not to be abelian groups. Hence instead of the factor group $\text{Gr}(\mathcal{A})/\text{Fact}(\mathcal{A})$ a partition $\text{Gr}(\mathcal{A})/{\approx_\mathcal{A}}$ with a special kind of equivalence relation $\approx_\mathcal{A}$ were considered there. However, it is straightforward to prove, by means of the preceding proof, also the more general case of Lemma 2.3 where "equivalent modulo Fact($\mathcal{A}$)" is replaced by $\approx_\mathcal{A}$.

In the next section we shall need a notion of a tree, so we shortly describe our notation.

Definition 2.4 *Suppose $T = \langle T, \lhd \rangle$ is a tree of height γ. For every $i < \gamma$, T_i is the i^{th} level of the tree. When $i < j < \gamma$ and $\eta \in T_j$, then $\eta \restriction i$ denotes the unique element $\nu \in T_i$ for which $\nu \lhd \eta$ holds. For each $i < \gamma$ and $\nu \in T_i$, $T_j[\nu]$ is the set $\{\eta \in T_j \mid \nu \lhd \eta\}$. The set of all γ-branches of T, i.e., the set $\{t \in \prod_{i<\gamma} T_i \mid \text{for all } i < j,\ t(i) \lhd t(j)\}$, is denoted by* $\text{Br}_\gamma(T)$.

3 The inverse γ-system of free R-modules

In this section we define special kind of inverse γ-systems $\mathcal{A}_R^T$ and prove a result concerning cardinalities of their quotient limit $\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)$ (Conclusion 3.12). A direct consequence of the result will be Theorem 2.

Definition 3.1 *Suppose γ is a limit ordinal, R is a ring, and T is a tree of height γ. We define an inverse γ-system $\mathcal{A}_R^T = \langle G_i, h_{i,j} \mid i < j < \gamma\rangle$ by the following stipulations:*

a) *for each $i < \gamma$, G_i is the R-module freely generated by $\{x_{\nu,l} \mid \nu \in T_i \text{ and } i < l < \gamma\}$;*

b) *for every $i < j < \gamma$, $h_{i,j}$ is the homomorphism from G_j into G_i determined by the values $h_{i,j}(x_{\eta,l}) = x_{\eta\restriction i,l} - x_{\eta\restriction i,j}$, for all $\eta \in T_j$ and $l > j$. (It is easy to check that the equations $h_{i,k} = h_{i,j} \circ h_{j,k}$ are satisfied for all $i < j < k$).*

We consider $\mathrm{Gr}(\mathcal{A}_R^T)$, $\mathrm{Fact}(\mathcal{A}_R^T)$, and $\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)$ as R-modules where the operations $+, \cdot$, and the unit element $\mathbf{0}$ for addition are pointwise defined.

For each $t \in \mathrm{Br}_\gamma(T)$, we define $\boldsymbol{t}$ to be the sequence $\langle x_{t(i),j} \mid i < j < \gamma\rangle$. Directly by the definitions of G_i and $h_{i,j}$, $\boldsymbol{t}$ belongs to $\mathrm{Gr}(\mathcal{A}_R^T)$ for every $t \in \mathrm{Br}_\gamma(T)$. We let $\langle \boldsymbol{t}\rangle_R^{t\in\mathrm{Br}_\gamma(T)}$ be the submodule of $\mathrm{Gr}(\mathcal{A}_R^T)$ generated by the elements $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$. When $\mathrm{Br}_\gamma(T)$ is empty $\langle \boldsymbol{t}\rangle_R^{t\in\mathrm{Br}_\gamma(T)}$ is the trivial submodule $\{\mathbf{0}\}$.

Remark. Each G_i is nonempty when T has height γ. Hence $\prod_{i<\gamma} G_i$ is nonempty, and also

$$\mathrm{Fact}(\mathcal{A}_R^T) = \Big\{\langle \bar{y}^i - h_{i,j}(\bar{y}^j) \mid i < j < \gamma\rangle \mid \bar{y} \in \prod_{i<\gamma} G_i\Big\}$$

is nonempty. So $\mathrm{Gr}(\mathcal{A}_R^T) \supseteq \mathrm{Fact}(\mathcal{A}_R^T)$ is nonempty for every ring R and tree T of height γ.

Observe also that the inverse γ-system $\mathcal{A}_R^T$ is the same as used in [She85, Claim 3.8] when R is the trivial ring $\{0,1\}$ and T consists of μ many disjoint γ-branches. So the proof given in this section offers an alternative proof for [She85, Claim 3.8], and even more information, namely that $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big)$ must be exactly μ not only $\geq \mu$.

Definition 3.2 *Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}_R^T)$ and $i < j < \gamma$. By the definition of G_i and the requirement $\boldsymbol{a}^{i,j} \in G_i$, we define $a_{\nu,l}^{i,j}$ for $\nu \in T_i$ and $l > i$, to be the coefficients from R (with only finitely many of them nonzero) which satisfy the equation*

$$\boldsymbol{a}^{i,j} = \sum\nolimits_{l>i}^{\nu\in T_i} a_{\nu,l}^{i,j} \cdot x_{\nu,l}.$$

The finite set $\{(\nu,l) \in T_i \times \big(\gamma \setminus (i+1)\big) \mid a_{\nu,l}^{i,j} \neq 0\}$ is called the support of $\boldsymbol{a}^{i,j}$, and it is denoted by $\mathrm{supp}(\boldsymbol{a}^{i,j})$.

Suppose S is a subset of γ, $e \in G_i$, and $e_{\nu,l} \in R$ for every $\nu \in T_i$ and $l > i$ are elements such that

$$e = \sum\nolimits_{l>i}^{\nu\in T_i} e_{\nu,l} \cdot x_{\nu,l}.$$

Then we write $e\restriction S$ for the following element of G_i:

$$\sum\nolimits_{l\in S\setminus(i+1)}^{\nu\in T_i} e_{\nu,l} \cdot x_{\nu,l}.$$

The following simple lemma has an important corollary.

Lemma 3.3

a) *The restriction* $h_{i,j}(e)\restriction j$ *equals 0 for every* $i<j$ *and* $e \in G_j$.

b) *For every* $\boldsymbol{a} \in \mathrm{Fact}(\mathcal{A}_R^T) \setminus \{\mathbf{0}\}$, *there are* $i<j<\gamma$ *such that* $\boldsymbol{a}^{i,j}\restriction j \neq 0$.

Proof. a) Straightforwardly by the definitions of G_j and $h_{i,j}$.
b) By the definition of $\mathrm{Fact}(\mathcal{A}_R^T)$, let $\bar{y} \in \prod_{i<\gamma} G_i$ be such that for all $i<j$, $\boldsymbol{a}^{i,j} = \bar{y}^i - h_{i,j}(\bar{y}^j)$. In addition to that let $y^i_{\nu,l} \in R$, for $i<\gamma$, $\nu \in T_i$ and $l>i$, be such that

$$\bar{y}^i = \sum\nolimits_{l>i}^{\nu\in T_i} y^i_{\nu,l}\cdot x_{\nu,l}.$$

Since $\boldsymbol{a} \neq \mathbf{0}$ there must be $i<\gamma$ with $\bar{y}^i \neq 0$. Define j to be $\min\{l>i \mid y^i_{\nu,l} \neq 0 \text{ for some } \nu \in T_i\}+1$. Then $\bar{y}^i\restriction j$ is nonzero and because $h_{i,j}(\bar{y}^j)\restriction j = 0$, we have $\boldsymbol{a}^{i,j}\restriction j = \bar{y}^i\restriction j - h_{i,j}(\bar{y}^j)\restriction j = \bar{y}^i\restriction j \neq 0$. ■

Corollary 3.4 *The elements* $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$, *are independent over* $\mathrm{Fact}(\mathcal{A}_R^T)$, *i.e.,*

$$\langle \boldsymbol{t}\rangle_R^{t\in\mathrm{Br}_\gamma(T)} \cap \mathrm{Fact}(\mathcal{A}_R^T) = \{\mathbf{0}\}.$$

Hence $\mathcal{A}_R^T$ *satisfies* $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big) \geq \mathrm{card}\big(\langle \boldsymbol{t}\rangle_R^{t\in\mathrm{Br}_\gamma(T)}\big)$.

Proof. Directly by the definition of $\boldsymbol{t}$, $\boldsymbol{t}^{i,j} = x_{t(i),j}$ and hence $\boldsymbol{t}^{i,j}\restriction j = 0$, for all $t \in \mathrm{Br}_\gamma(T)$ and $i<j$. So for any nonzero $\boldsymbol{a} = \sum_{1\leq m\leq n} d_m \cdot \boldsymbol{t_m}$, where $n<\omega$, $d_m \in R \setminus \{0\}$, and $t_m \in \mathrm{Br}_\gamma(T)$, the restrictions $\boldsymbol{a}^{i,j}\restriction j$ are equal to 0 for all $i<j$. So by the preceding lemma $\boldsymbol{a}$ can not be in $\mathrm{Fact}(\mathcal{A}_R^T)$. ■

Next we derive equations of weighty significance.

Lemma 3.5 *Suppose* $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}_R^T)$ *and* $i<j<k<\gamma$. *Then the following equations are satisfied for all* $\nu \in T_i$:

(A) $\quad b^{i,k}_{\nu,l} = b^{i,j}_{\nu,l} \quad$ *when* $i<l<j$;

(B) $\quad b^{i,k}_{\nu,j} = b^{i,j}_{\nu,j} - \sum\nolimits_{l>j}^{\eta\in T_j[\nu]} b^{j,k}_{\eta,l}$;

(C) $\quad b^{i,k}_{\nu,l} = b^{i,j}_{\nu,l} + \sum\nolimits_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l} \quad$ *when* $l>j$.

Proof. By dividing the sum into groups we get that

$$\begin{aligned}\boldsymbol{b}^{i,j} &= \sum\nolimits_{l>i}^{\nu\in T_i} b^{i,j}_{\nu,l}\cdot x_{\nu,l}\\ &= \sum_{\nu\in T_i}\Big(\sum_{i<l<j} b^{i,j}_{\nu,l}\cdot x_{\nu,l} + b^{i,j}_{\nu,j}\cdot x_{\nu,j} + \sum_{l>j} b^{i,j}_{\nu,l}\cdot x_{\nu,l}\Big).\end{aligned}$$

Similarly the following equation is satisfied,

$$\boldsymbol{b}^{i,k} = \sum_{\nu\in T_i}\Big(\sum_{i<l<j} b^{i,k}_{\nu,l}\cdot x_{\nu,l} + b^{i,k}_{\nu,j}\cdot x_{\nu,j} + \sum_{l>j} b^{i,k}_{\nu,l}\cdot x_{\nu,l}\Big).$$

From the definition of $h_{i,j}$ we may infer that

$$\begin{aligned} h_{i,j}(\boldsymbol{b}^{j,k}) &= \textstyle\sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l}\cdot h_{i,j}(x_{\eta,l}) \\ &= \textstyle\sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l}\cdot (x_{\eta\restriction i,l} - x_{\eta\restriction i,j}) \\ &= \textstyle\sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l}\cdot x_{\eta\restriction i,l} - \sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l}\cdot x_{\eta\restriction i,j} \\ &= \textstyle\sum_{\nu\in T_i}\Big(\sum_{l>j}\big(\sum_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l}\big)\cdot x_{\nu,l} - \big(\sum_{(\eta\in T_j[\nu],\ l>j)} b^{j,k}_{\eta,l}\big)\cdot x_{\nu,j}\Big). \end{aligned}$$

So the equations (A), (B), and (C) for all $i<j<k$ follow by comparing the coefficients of each generator $x_{\nu,l}$ in the equation $\boldsymbol{b}^{i,k} = \boldsymbol{b}^{i,j} + h_{i,j}(\boldsymbol{b}^{j,k})$. $\blacksquare_{3.5}$

Lemma 3.6 *Suppose* $\boldsymbol{a}\in \mathrm{Gr}(\mathcal{A}^T_R)$.

a) *For all* $i<j<k$, $\boldsymbol{a}^{i,j}\restriction j = \boldsymbol{a}^{i,k}\restriction j$.

b) $(\mathrm{cf}(\gamma)>\aleph_0)$ *For every* $i<\gamma$, *the union* $\bigcup_{i<j<\gamma}\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)$ *is of finite cardinality (where* $\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) = \mathrm{supp}(\boldsymbol{a}^{i,j})\cap(T_i\times j)$ *of course).*

c) $(\mathrm{cf}(\gamma)>\aleph_0)$ *There is* $\boldsymbol{b}\in\mathrm{Gr}(\mathcal{A}^T_R)$ *satisfying the following conditions:*

$\boldsymbol{a}\equiv\boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}^T_R)$,
$I = \{(i,j)\in\gamma\star\gamma \mid \boldsymbol{b}^{i,j}\restriction j = 0\}$ *is cobounded (in fact* $I^{1st}=\gamma$*), and*
for every $(i,j)\in I$, $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\}$.

Proof. a) The claim holds directly by Lemma 3.5(A).

b) Suppose the union is infinite. Since $\mathrm{cf}(\gamma)>\aleph_0$ there is some $k<\gamma$ for which already $\bigcup_{j<k}\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)$ is infinite. By (a), $\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)\subseteq\mathrm{supp}(\boldsymbol{a}^{i,k})$ for each $j<k$. Consequently $\bigcup_{j<k}\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)\subseteq\mathrm{supp}(\boldsymbol{a}^{i,k})$ contrary to the finiteness of $\mathrm{supp}(\boldsymbol{a}^{i,k})$.

c) By (a) and (b) there must be for every $i<\gamma$ a bound $i^*\in\gamma\setminus(i+1)$ such that for every $j\ge i^*$, $\boldsymbol{a}^{i,i^*}\restriction i^* = \boldsymbol{a}^{i,j}\restriction j$. Define an element $\boldsymbol{c}\in\mathrm{Fact}(\mathcal{A}^T_R)$ by

$$\boldsymbol{c}^{i,j} = \boldsymbol{a}^{i,i^*}\restriction i^* - h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*),$$

for all $i<j$. Let $\boldsymbol{b}$ be $\boldsymbol{a}-\boldsymbol{c}$. Then $\boldsymbol{a}\equiv\boldsymbol{b}\mod\mathrm{Fact}(\mathcal{A}^T_R)$ and for every $i<\gamma$ and $j\ge i^*$

$$\begin{aligned} \boldsymbol{b}^{i,j} &= \boldsymbol{a}^{i,j} - \boldsymbol{c}^{i,j} \\ &= \boldsymbol{a}^{i,j}\restriction(\gamma\setminus j) + \boldsymbol{a}^{i,j}\restriction j - \boldsymbol{a}^{i,i^*}\restriction i^* + h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*) \\ &= \boldsymbol{a}^{i,j}\restriction(\gamma\setminus j) + h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*). \end{aligned}$$

It follows from Lemma 3.3(a) that $\boldsymbol{b}^{i,j}\restriction j = 0$ for all $i<\gamma$ and $j\ge i^*$, and thus I is cobounded.

Now suppose, contrary to the last claim in (c), that $b^{i,j}_{\nu,l}\ne 0$ for some $i<\gamma$, $j\ge i^*$, $\nu\in T_i$, and $l>j$. Let k be $\max\{i^*,j^*,l+1\}$. Then both $\boldsymbol{b}^{i,k}\restriction k$ and $\boldsymbol{b}^{j,k}\restriction k$ are 0. By Lemma 3.5(C) the following equation holds:

$$\sum\nolimits_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l} = b^{i,k}_{\nu,l} - b^{i,j}_{\nu,l}.$$

Since $b^{i,j}_{\nu,l}\ne 0$ and $l<k$ implies $b^{i,k}_{\nu,l}=0$ the sum $\sum_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l}$ must be nonzero. So there is $\eta\in T_j[\nu]$ with $b^{j,k}_{\eta,l}\ne 0$. This contradicts the facts $l<k$ and $\boldsymbol{b}^{j,k}\restriction k$ equals 0. $\blacksquare$

Lemma 3.7 *Suppose $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}_R^T)$ and I is a subset of $\{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j}\}$. Then for all $(i,j) \in I$, $\nu \in T_i$, and $k \in I[i] \cap I[j]$,*

$$b^{i,j}_{\nu,j} = \sum\nolimits_{\eta \in T_j[\nu]} b^{j,k}_{\eta,k} = b^{i,k}_{\nu,k}.$$

Proof. Since (i,k) and (j,k) are in I, both $b^{i,k}_{\nu,j}$ and $b^{j,k}_{\eta,l}$ are equal to 0 for all $\eta \in T_j$ when $l \neq k$. Hence Lemma 3.5(B) can be reduced to the form $b^{i,j}_{\nu,j} = \sum_{\eta \in T_j[\nu]} b^{j,k}_{\eta,k}$. Now $(i,j) \in I$ guarantees that $b^{i,j}_{\nu,k} = 0$. Thus the reduced form together with Lemma 3.5(C) (applied for $l = k$) yield $b^{i,j}_{\nu,j} = b^{i,k}_{\nu,k}$. ■

Lemma 3.8 *Suppose $\boldsymbol{b}$ is an element of $\mathrm{Gr}(\mathcal{A}_R^T)$.*

a) *If $\boldsymbol{b}$ is not in $\mathrm{Fact}(\mathcal{A}_R^T)$ and I is an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\}$ for all $(i,j) \in I$, then there is an eventually coherent subset J of I with $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\} \neq 0$ whenever $(i,j) \in J$.*

b) $(\mathrm{cf}(\gamma) > \aleph_0)$ *If J is an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\} \neq 0$ for all $(i,j) \in J$, then there are a bound $n^* < \omega$ and an eventually coherent subset K of J such that* $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ *for all $(i,j) \in K$.*

Proof. a) Since $\boldsymbol{b} \not\equiv \boldsymbol{0} \mod \mathrm{Fact}(\mathcal{A}_R^T)$ it follows by Lemma 2.3 that there is no subset of $\{(i,j) \in I \mid \boldsymbol{b}^{i,j} = 0\}$ which would be eventually coherent. Hence there is an unbounded subset S of I^{1st} such that for each $i \in S$ there is $j_i \in I[i]$ with $\boldsymbol{b}^{i,j_i}$ nonzero. Fix any $i \in S$. Since $\boldsymbol{b}^{i,j_i} = \boldsymbol{b}^{i,j_i}\restriction\{j_i\} \neq 0$, let ν_i be an element of T_i with $b^{i,j_i}_{\nu_i,j_i} \neq 0$. By Lemma 3.7, $b^{i,k}_{\nu_i,k} = b^{i,j_i}_{\nu_i,j_i} \neq 0$ for all $k \in I[i] \cap I[j_i]$. Because I was eventually coherent, we have shown that $J = I \cap (S \times S)$ is an eventually coherent set as wanted in the claim.

b) First of all we claim that for each $i \in J^{\mathrm{1st}}$ the union $\bigcup_{j \in J[i]} \mathrm{supp}(\boldsymbol{b}^{i,j})$ is of finite cardinality. Observe that for every $(i,j) \in J$, $\mathrm{supp}(\boldsymbol{b}^{i,j}) = \mathrm{supp}(\boldsymbol{b}^{i,j}) \cap (T_i \times \{j\})$.

Assume, contrary to this subclaim, that $i \in J^{\mathrm{1st}}$, $\langle j_m \mid m < \omega\rangle$ is an increasing sequence of ordinals in $J[i]$, and $\{\nu_m \mid m < \omega\}$ is a set of distinct elements from T_i such that $b^{i,j_m}_{\nu_m,j_m}$ nonzero for every $m < \omega$. Since J is eventually coherent and γ is of uncountable cofinality let $k < \gamma$ be the minimal element in $J[i] \cap \bigcap_{m<\omega} J[j_m]$. Now for each $m < \omega$, the pairs (i, j_m), (i,k), and (j_m, k) are in J, and by Lemma 3.7, the equation $b^{i,j_m}_{\nu_m,j_m} = b^{i,k}_{\nu_m,k} \neq 0$ holds. So the infinite set $\{(\nu_m, k) \mid m < \omega\}$ is a subset of $\mathrm{supp}(\boldsymbol{b}^{i,k})$, a contradiction.

It follows from the subclaim that for each $i \in J^{\mathrm{1st}}$, the finite ordinal

$$n_i = \mathrm{card}\Big(\bigcup_{j \in J[i]} \mathrm{supp}(\boldsymbol{b}^{i,j})\Big) + 1$$

satisfies $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n_i$ for all $j \in J[i]$. Since J^{1st} is uncountable, there are $n^* < \omega$ and an unbounded subset S of J^{1st} such that $n_i = n^*$ for all $i \in S$. So n^* and the set $K = J \cap (S \times S)$ meet the requirements of the claim. ■

Lemma 3.9 $(\mathrm{cf}(\gamma) > \mathrm{card}(R))$ *Suppose $\boldsymbol{b}$ is in $\mathrm{Gr}(\mathcal{A}_R^T)$ and I is an eventually coherent subset of $\{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j} \neq 0\}$. Then there are $d \in R$, $t \in \mathrm{Br}_\gamma(T)$, and an eventually coherent subset J of I for which $b^{i,j}_{t(i),j} = d \neq 0$ whenever $(i,j) \in J$.*

Proof. We define by induction on $\alpha < \mathrm{cf}(\gamma)$ the following objects:

an increasing sequence $\langle i_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle$ of ordinals in $I^{1\mathrm{st}}$ with limit γ;

an increasing sequence $\langle \nu_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle \in \prod_{\alpha<\mathrm{cf}(\gamma)} T_{i_\alpha}$;

subsets K_α of $I[i_\alpha]$ such that $I^{1\mathrm{st}} \smallsetminus K_\alpha$ are bounded in γ;

elements $d_\alpha \in R \smallsetminus \{0\}$ such that for every $k \in K_\alpha$, $b^{i_\alpha,k}_{\nu_\alpha,k} = d_\alpha$.

This suffices since $\mathrm{card}(R) < \mathrm{cf}(\gamma)$ implies that there are $d \in R$ and $H \subseteq \mathrm{cf}(\gamma)$ unbounded in $\mathrm{cf}(\gamma)$ such that $d_\alpha = d$ for every $\alpha \in H$. Moreover, the claim is satisfied by $t \in \mathrm{Br}_\gamma(T)$ and $J \subseteq I$ defined as follows. For every $i < \gamma$, $t(i) = \nu_{\beta_i}\restriction i$, where $\beta_i = \min\{\alpha < \mathrm{cf}(\gamma) \mid i_\alpha \geq i\}$, and $J = \bigcup_{\alpha\in H} \big(\{i_\alpha\} \times (S \cap K_\alpha)\big)$, where S is $\{i_\alpha \mid \alpha \in H\}$.

Let $\langle \gamma_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle$ be an increasing sequence with limit γ. Define

$$i_\alpha = \min\big((I^{1\mathrm{st}} \cap \bigcap_{\beta<\alpha} K_\beta) \smallsetminus \gamma_\alpha\big)$$

and

$$j = \min\big(I^{1\mathrm{st}} \cap I[i_\alpha] \cap \bigcap_{\beta<\alpha} I[i_\beta]\big),$$

where both $\bigcap_{\beta<\alpha} K_\beta$ and $\bigcap_{\beta<\alpha} I[i_\beta]$ are equal to γ when $\alpha = 0$. This pair (i_α, j) is well-defined since I is eventually coherent, $\alpha < \mathrm{cf}(\gamma)$, and when $\alpha > 0$, $I^{1\mathrm{st}} \smallsetminus K_\beta$ is bounded for each $\beta < \alpha$ by the induction hypothesis.

If $\alpha = 0$, then $(i_0, j) \in I$ guarantees that $\boldsymbol{b}^{i_0,j}\restriction\{j\} = \boldsymbol{b}^{i_0,j} \neq 0$. Hence we can find $\nu_0 \in T_{i_0}$ with $b^{i_0,j}_{\nu_0,j} \neq 0$.

When $\alpha > 0$ we define elements $\eta_\beta \in T_{i_\alpha}[\nu_\beta]$ for each $\beta < \alpha$ as follows. Fix $\beta < \alpha$. Since $i_\alpha \in K_\beta$ we get by the induction hypothesis that $b^{i_\beta,i_\alpha}_{\nu_\beta,i_\alpha} = d_\beta \neq 0$. Furthermore $(i_\beta, i_\alpha) \in I$ (because $K_\beta \subseteq I[i_\beta]$), $(i_\beta, j) \in I$, and $(i_\alpha, j) \in I$ together with Lemma 3.7 yield

$$\sum\nolimits_{\eta\in T_{i_\alpha}[\nu_\beta]} b^{i_\alpha,j}_{\eta,j} = b^{i_\beta,i_\alpha}_{\nu_\beta,i_\alpha} \neq 0.$$

Therefore we can find $\eta_\beta \in T_{i_\alpha}[\nu_\beta]$ for which $b^{i_\alpha,j}_{\eta_\beta,j} \neq 0$.

If $\alpha > 0$ is a successor ordinal define ν_α to be $\eta_{\alpha-1}$. When α is a limit ordinal, the finiteness of the support $\mathrm{supp}(\boldsymbol{b}^{i_\alpha,j})$ ensures that there are $\nu_\alpha \in T_{i_\alpha}$ and an unbounded subset H of α such that $\eta_{\beta'} = \nu_\alpha$ for all $\beta' \in H$. By the induction hypothesis $\nu_\beta \lhd \nu_{\beta'}$ for all $\beta < \beta' < \alpha$. Hence $\nu_\beta \lhd \nu_{\beta'} \lhd \eta_{\beta'} = \nu_\alpha$ holds for every $\beta < \alpha$ and $\beta' = \min(H \smallsetminus \beta)$.

Let d_α be $b^{i_\alpha,j}_{\nu_\alpha,j}$. By Lemma 3.7, every $k \in I[i_\alpha] \cap I[j]$ satisfies that $b^{i_\alpha,k}_{\nu_\alpha,k} = b^{i_\alpha,j}_{\nu_\alpha,j} = d_\alpha$. Hence i_α, ν_α, and d_α together with the set $K_\alpha = I[i_\alpha] \cap I[j]$ meet the requirements given at the beginning of the proof. $\blacksquare_{3.9}$

Corollary 3.10 $(\mathrm{cf}(\gamma) > \aleph_0)$ *If* $\mathrm{Br}_\gamma(T)$ *is empty, then* $\mathrm{Gr}(\mathcal{A}^T_R) = \mathrm{Fact}(\mathcal{A}^T_R)$.

Proof. Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}^T_R) \smallsetminus \mathrm{Fact}(\mathcal{A}^T_R)$. By Lemma 3.6(c) together with Lemma 3.8(a) there is $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}^T_R)$ such that $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}^T_R)$ and the set $\{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j} \neq 0\}$ is eventually coherent. By Lemma 3.9 there is a γ-branch through the tree T, i.e., $\mathrm{Br}_\gamma(T) \neq \emptyset$. Observe that the assumption $\mathrm{card}(R) < \mathrm{cf}(\gamma)$ is not needed, as can be seen from the proof of Lemma 3.9. $\blacksquare$

Lemma 3.11 ($\mathrm{cf}(\gamma) > \max\{\aleph_0, \mathrm{card}(R)\}$) *The elements* $\boldsymbol{t}$*,* $t \in \mathrm{Br}_\gamma(T)$*, generate* $\mathrm{Gr}(\mathcal{A}_R^T)$ *modulo* $\mathrm{Fact}(\mathcal{A}_R^T)$*.*

Proof. We show that for every $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}_R^T)$ with $\boldsymbol{a} \notin \mathrm{Fact}(\mathcal{A}_R^T)$ we can find $n < \omega$, $d_1, \dots, d_n \in R \setminus \{0\}$ and $t_1, \dots, t_n \in \mathrm{Br}_\gamma(T)$ satisfying

(A) $$\boldsymbol{a} \equiv \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \mod \mathrm{Fact}(\mathcal{A}_R^T).$$

Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}_R^T) \setminus \mathrm{Fact}(\mathcal{A}_R^T)$. By Lemma 3.6(c) and Lemma 3.8(a) let $\boldsymbol{b}$ be an element of $\mathrm{Gr}(\mathcal{A}_R^T)$ and I_1 an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}_R^T)$ and for each $(i,j) \in I_1$, $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j} {\restriction} \{j\} \neq 0$. Furthermore, we may assume by Lemma 3.8(b) that $n^* < \omega$ is a bound for which $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ hold for all $(i,j) \in I_1$.

By Lemma 3.9 there are $d_1 \in R$, $t_1 \in \mathrm{Br}_\gamma(T)$, and an eventually coherent set $J_1 \subseteq I_1$ having the property that $b^{i,j}_{t_1(i),j} = d_1 \neq 0$ whenever $(i,j) \in J_1$. Since $d_1 \cdot \boldsymbol{t_1} \in \mathrm{Gr}(\mathcal{A}_R^T)$, the sequence $\boldsymbol{c} = \boldsymbol{b} - d_1 \cdot \boldsymbol{t_1}$ is in $\mathrm{Gr}(\mathcal{A}_R^T)$. If $\boldsymbol{c}$ is in $\mathrm{Fact}(\mathcal{A}_R^T)$, then $\boldsymbol{b} \equiv d_1 \cdot \boldsymbol{t_1} \mod \mathrm{Fact}(\mathcal{A}_R^T)$, and because of $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}_R^T)$, also (A) holds for $n = 1$.

Suppose $1 \le n < \omega$ and objects $d_m \in R \setminus \{0\}$, $t_m \in \mathrm{Br}_\gamma(T)$, and $J_m \subseteq J_1$ for $m \le n$ are already defined. Assume also that these objects satisfy the following conditions:

1) $J_{m'} \supseteq J_m$ for all $1 \le m' \le m \le n$;
2) for all $1 \le m' < m \le n$ and $i \in (J_m)^{\mathrm{1st}}$, $t_{m'}(i) \neq t_m(i)$;
3) for every $1 \le m \le n$ and $(i,j) \in J_m$, $b^{i,j}_{t_m(i),j} = d_m \neq 0$;
4) $\boldsymbol{c} = \boldsymbol{b} - \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \notin \mathrm{Fact}(\mathcal{A}_R^T)$.

Clearly $\boldsymbol{c}^{i,j} = \boldsymbol{c}^{i,j} {\restriction} \{j\}$ and $\mathrm{card}(\mathrm{supp}(\boldsymbol{c}^{i,j})) \le \mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ for all $(i,j) \in J_n$. Again by Lemma 3.8(a), there is an eventually coherent set $I_{n+1} \subseteq J_n$ such that for each $(i,j) \in I_{n+1}$, $\boldsymbol{c}^{i,j} \neq 0$. Moreover, by Lemma 3.9, there are $d_{n+1} \in R$, $t_{n+1} \in \mathrm{Br}_\gamma(T)$, and an eventually coherent set $J_{n+1} \subseteq \{(i,j) \in I_{n+1} \mid c^{i,j}_{t_{n+1}(i),j} = d_{n+1} \neq 0\}$.

The properties (2), (3) and (4) above imply that $c^{i,j}_{t_m(i),j} = b^{i,j}_{t_m(i),j} - d_m = 0$ for every $m \le n$ and $(i,j) \in J_m$. On the other hand, $c^{i,j}_{t_{n+1}(i),j}$ is nonzero for each $(i,j) \in J_{n+1}$. Thus $t_{n+1}(i)$ can not be in $\{t_m(i) \mid 1 \le m \le n\}$ if $i \in (J_{n+1})^{\mathrm{1st}}$. So for all $(i,j) \in J_{n+1}$, $x_{t_{n+1}(i),j} \notin \{x_{t_m(i),j} \mid 1 \le m \le n\}$, and consequently $b^{i,j}_{t_{n+1}(i),j} = c^{i,j}_{t_{n+1}(i),j}$. Thus also J_{n+1}, t_{n+1}, and d_{n+1} satisfy the properties (1), (2), and (3) (but not necessarily (4)).

We claim that there must be $n < n^*$ such that

(B) $$\boldsymbol{b} - \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \in \mathrm{Fact}(\mathcal{A}_R^T).$$

Assume, contrary to this subclaim, that the process introduced above has been carried out n^* many times and objects J_m, t_m, d_m for $i \le m \le n^*$ are defined. In addition to that suppose they satisfy the conditions (1), (2), and (3). Define $i = \min\left((J_{n^*})^{\mathrm{1st}}\right)$ and $j = \min(J_{n^*}[i])$. Then for every $m \le n^*$, $(i,j) \in J_m$ yields $b^{i,j}_{t_m(i),j} = d_m \neq 0$. This contradicts the condition $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$, since the set $\{(t_m(i), j) \mid m \le n^*\} \subseteq \mathrm{supp}(\boldsymbol{b}^{i,j})$ is of cardinality n^*.

Now suppose $n < \omega$ is a finite ordinal satisfying (B). Then $\boldsymbol{b} \equiv \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \mod \mathrm{Fact}(\mathcal{A}_R^T)$, and because $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}_R^T)$ also (A) is satisfied. ■

Conclusion 3.12 *For any ordinal γ of uncountable cofinality, ring R with* $\mathrm{card}(R) < \mathrm{cf}(\gamma)$, *and tree T of height γ, the inverse γ-system $\mathcal{A}^T_R = \langle G_i, h_{i,j} \mid i < j < \gamma\rangle$ has the properties that*

$$\mathrm{card}(G_i) = \max\{\mathrm{card}(\gamma), \mathrm{card}(T_i), \mathrm{card}(R)\}$$

for all $i < \gamma$, and

$$\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}^T_R)/\mathrm{Fact}(\mathcal{A}^T_R)\big) = \mathrm{card}\big(\langle \boldsymbol{t}\rangle_R^{t\in \mathrm{Br}_\gamma(T)}\big).$$

Proof of Theorem 2. Remember that λ and κ were cardinals with $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$. We wanted to study possible cardinalities μ of the quotient limit $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$, where $\mathcal{A}$ is an inverse κ-system consisting of abelian groups having cardinality $< \lambda$. Now Conclusion 3.12 gives a complete solution to this problem because of $\lambda > \mathrm{cf}(\lambda) = \kappa = \mathrm{cf}(\kappa) > \aleph_0$. Namely, in order to meet the requirements $\mathrm{card}(G_i) < \lambda$ for all $i < \kappa$, it is needed only to ensure that R and the i^{th} level of T are small enough. On the other hand, a suitable choice of R and T yields any desired value for $\mu = \mathrm{card}\big(\mathrm{Gr}(\mathcal{A}^T_R)/\mathrm{Fact}(\mathcal{A}^T_R)\big)$. We briefly describe methods to choose suitable R and T for every nonzero $\mu \leq \lambda^\kappa$.

For any R, $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}^T_R)/\mathrm{Fact}(\mathcal{A}^T_R)\big)$ equals 1 when $\mathrm{Br}_\kappa(T)$ is empty. So $\mu = 1$ is possible since obviously there exists a tree of height κ without κ-branches and having levels of cardinality $< \lambda$ when λ singular of cofinality κ. Also all the finite values $\mu > 1$ are possible by taking T with only one κ-branch and R with $\mathrm{card}(R) = \mu$.

Furthermore the case of infinite $\mu < \lambda$ is satisfied by any R with $\mathrm{card}(R) < \min\{\kappa, \mu\}$ and T with exactly μ many κ-branches. The value $\mu = \lambda$ is possible for any R with $\mathrm{card}(R) < \kappa$ because a suitable tree can be constructed, for example, as follows. Let $\langle \lambda_i \mid i < \kappa\rangle$ be an increasing sequence of ordinals $< \lambda$ with limit λ. Then the tree

$$T = \{t{\restriction}\alpha \mid \alpha < \kappa, t \in \prod_{i<\kappa} \lambda_i, \text{and } t(i) \text{ is nonzero only for finitely many } i < \kappa\},$$

ordered by inclusion, satisfies $\mathrm{card}(\mathrm{Br}_\kappa(T)) = \lambda$ and $\mathrm{card}(T_i) = \lambda_i < \lambda$ for each $i < \kappa$.

Also the cardinalities μ of the quotient limit, when $\lambda < \mu \leq \lambda^\kappa$, are possible for any ring of cardinality $< \kappa$. Existence of a suitable tree is proved for example in [She89, Fact 10] under the assumption that $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for every $\theta < \lambda$ (other sources for a proof are given in [She94, Analytical Guide §10]). ■

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