

Stationary Sets and Infinitary Logic

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Abstract

Let K_λ^0 be the class of structures $\langle \lambda, <, A \rangle$, where $A \subseteq \lambda$ is disjoint from a club, and let K_λ^1 be the class of structures $\langle \lambda, <, A \rangle$, where $A \subseteq \lambda$ contains a club. We prove that if $\lambda = \lambda^{<\kappa}$ is regular, then no sentence of $L_{\lambda+\kappa}$ separates K_λ^0 and K_λ^1 . On the other hand, we prove that if $\lambda = \mu^+$, $\mu = \mu^{<\mu}$, and a forcing axiom holds (and $\aleph_1^L = \aleph_1$ if $\mu = \aleph_0$), then there is a sentence of $L_{\lambda\lambda}$ which separates K_λ^0 and K_λ^1 .

One of the fundamental properties of $L_{\omega_1\omega}$ is that although every countable ordinal itself is definable in $L_{\omega_1\omega}$, the class of all countable well-ordered structures is not. In particular, the classes

$$\begin{aligned} K^0 &= \{ \langle \omega, R \rangle : R \text{ well-orders } \omega \} \\ K^1 &= \{ \langle \lambda, R \rangle : \langle \omega, R \rangle \text{ contains a copy of the rationals} \} \end{aligned}$$

cannot be separated by any $L_{\omega_1\omega}$ -sentence. In this paper we consider infinite quantifier languages $L_{\kappa\lambda}$, $\lambda > \omega$. Here well-foundedness is readily definable, but we may instead consider the class

$$T_\lambda = \{ \langle \lambda, R \rangle : \langle \lambda, R \rangle \text{ is a tree with no branches of length } \lambda \}.$$

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If $\lambda = \lambda^{<\lambda}$, then a result of Hyttinen [1] implies that T_λ cannot be defined in $L_{\lambda+\lambda}$.

The main topic of this paper is the question whether the classes

$$\begin{aligned} K_\lambda^0 &= \{ \langle \lambda, <, A \rangle : A \text{ is disjoint from a club of } \lambda \} \\ K_\lambda^1 &= \{ \langle \lambda, <, A \rangle : A \text{ contains a club of } \lambda \} \end{aligned}$$

can be separated in $L_{\lambda+\lambda}$ and related languages. Note that a set $A \subseteq \lambda$ contains a club if and only if the tree $T(A)$ of continuously ascending sequences of elements of A has a branch of length λ . We show (Theorem 1) that the classes K_λ^0 and K_λ^1 cannot be separated by a sentence of $L_{\lambda+\kappa}$, if $\lambda = \lambda^{<\kappa}$ is regular. The proof of this result uses forcing in a way which seems to be new in the model theory of infinitary languages. It follows from this result that the class

$$S_\lambda = \{ \langle \lambda, <, A \rangle : A \text{ is stationary on } \lambda \},$$

that separates K_λ^0 and K_λ^1 , is undefinable in $L_{\lambda+\kappa}$, if $\lambda = \lambda^{<\kappa}$ is regular. We complement this result by showing (Theorem 10) that if either $\lambda = \mu^+$ and $\mu = \mu^{<\mu} > \omega$ or $\lambda = \omega_1$ and additionally a forcing axiom holds, then there is a sentence of $L_{\lambda\lambda}$ which defines S_λ and thereby separates K_λ^0 and K_λ^1 .

Hyttinen [1] actually proves more than undefinability of T_λ in $L_{\lambda+\lambda}$. He shows that T_λ is undefinable - assuming $\lambda = \lambda^{<\lambda}$ - in $PC(L_{\lambda+\lambda})$. We show (Theorems 5 and 6) that the related statement that S_{ω_1} is definable in $PC(L_{\omega_2\omega_1})$ is independent of ZFC+CH.

1 The case $\lambda = \lambda^{<\mu}$.

Theorem 1 *If $\lambda = \lambda^{<\kappa}$ is regular, then the classes K_λ^0 and K_λ^1 cannot be separated by a sentence of $L_{\lambda+\kappa}$.*

Proof. Assume $\lambda = \lambda^{<\kappa}$ is regular and $\psi \in L_{\lambda+\kappa}$. Let \mathcal{P} be the forcing notion for adding a Cohen subset to λ . Thus $p \in \mathcal{P}$ if p is a mapping $p : \alpha_p \rightarrow 2$ for some $\alpha_p < \lambda$. A condition p extends another condition q , in symbols $p \geq q$, if $\alpha_p \geq \alpha_q$ and $p|_{\alpha_q} = q$. Let G be \mathcal{P} -generic and $g = \bigcup G$. Thus

$$V[G] \models g^{-1}(1) \text{ is bi-stationary on } \lambda.$$

Now either ψ or $\neg\psi$ is true in $\langle \lambda, <, g^{-1}(1) \rangle$ in $V[G]$. We may assume, by symmetry, that it is ψ . Let $p \in G$ such that

$$p|_{-\mathcal{P}} \langle \lambda, <, \tilde{g}^{-1}(1) \rangle \models \psi,$$

where \tilde{g} is the canonical name for g . It is easy to use $\lambda = \lambda^{<\kappa}$ and regularity of λ to construct an elementary chain $\langle M_\xi : \xi < \lambda \rangle$ such that

- (i) $M_\xi \prec \langle H(\text{beth}_7(\lambda)), \in, <^* \rangle$, where $<^*$ is a well-ordering of $H(\text{beth}_7(\lambda))$.
- (ii) $\lambda + 1 \cup \{p\} \cup \{\mathcal{P}\} \cup TC(\{\psi\}) \subseteq M_0$.
- (iii) $\langle M_\eta : \eta < \xi \rangle \in M_{\xi+1}$.
- (iv) $M_\nu = \bigcup_{\xi < \nu} M_\xi$ for limit ν .
- (v) $(M_\xi)^{<\kappa} \subseteq M_{\xi+1}$.
- (vi) $|M_\xi| = \lambda$.

Let $M = \bigcup_{\xi < \lambda} M_\xi$. Note, that $M^{<\kappa} \subseteq M$ because λ is regular. We shall construct two \mathcal{P} -generic sets, G^0 and G^1 , over M . For this end, list open dense $D \subseteq \mathcal{P}$ with $D \in M$ as $\langle D_\xi : \xi < \lambda \rangle$. Define $G^l = \{p_\xi^l : \xi < \lambda\}$ so that $p_0^l = p$, $p_{\xi+1}^l \geq p_\xi^l$ with $p_{\xi+1}^l \in D_\xi \cap M$, $p_{\xi+1}^l(\alpha_{p_\xi^l}) = l$, and $p_\nu^l = \bigcup_{\xi < \nu} p_\xi^l$ for limit ν . Clearly, G^l is \mathcal{P} -generic over M and

$$M[G^l] \models [\langle \lambda, <, (g^l)^{-1}(1) \rangle \models \psi],$$

where $g^l = \bigcup G^l$. Note also that $M[G^l]^{<\kappa} \subseteq M[G^l]$, because $M^{<\kappa} \subseteq M$ and \mathcal{P} is $<\kappa$ -closed.

Lemma 2 *If $\varphi(\vec{x}) \in L_{\lambda+\kappa}$ such that $TC(\{\varphi(\vec{x})\}) \subseteq M$, $X \in M$, and $\vec{a} \in \lambda^{<\kappa}$, then*

$$\langle \lambda, <, X \rangle \models \varphi(\vec{a}) \iff M[G^l] \models [\langle \lambda, <, X \rangle \models \varphi(\vec{a})].$$

Proof. Easy induction on $\varphi(\vec{x})$. \square

By the lemma, $\langle \lambda, <, (g^l)^{-1}(1) \rangle \models \psi$. By construction, $\langle \lambda, <, (g^l)^{-1}(1) \rangle \in K_\lambda^l$. Now we can finish the proof. Suppose $K_\lambda^0 \subseteq \text{Mod}(\psi)$ and $K_\lambda^1 \cap \text{Mod}(\psi) = \emptyset$. This contradicts the fact that $\langle \lambda, <, (g^1)^{-1}(1) \rangle \in K_\lambda^1 \cap \text{Mod}(\psi)$. Suppose $K_\lambda^1 \subseteq \text{Mod}(\psi)$ and $K_\lambda^0 \cap \text{Mod}(\psi) = \emptyset$. This contradicts $\langle \lambda, <, (g^0)^{-1}(1) \rangle \in K_\lambda^0 \cap \text{Mod}(\psi)$. \square

Corollary 3 *If $\lambda = \lambda^{<\kappa}$ is regular, then there is no $\varphi \in L_{\lambda+\kappa}$ such that for all $A \subseteq \lambda$: $\langle \lambda, <, A \rangle \models \varphi \iff A$ is stationary.*

Theorem 1 gives a new proof of the result, referred to above, that if $\lambda = \lambda^{<\lambda}$, then T_λ is not definable in $L_{\lambda\lambda}$. Our proof does not give the stronger result that T_λ is not definable in $PC(L_{\lambda\lambda})$, and there is a good reason: S_{ω_1} may be $PC(L_{\omega_1\omega_1})$ -definable, even if $2^{\aleph_0} = \aleph_1$. This is the topic of the next section.

2 An application of Canary trees.

A tree \mathcal{C} is a *Canary tree* if \mathcal{C} has cardinality $\leq 2^\omega$, \mathcal{C} has no uncountable branches, but if a stationary subset of ω_1 is killed by forcing which does not add new reals, then this forcing adds an uncountable branch to \mathcal{C} . By [4], this is equivalent to the statement that

- (\star) For every co-stationary $A \subseteq \omega_1$ there is a mapping f with $\text{Rng}(f) \subseteq \mathcal{C}$ such that for all increasing closed sequences s, s' of elements of A , if s is an initial segment of s' , then $f(s) <_{\mathcal{C}} f(s')$.

Theorem 4 (i) $\text{Con}(ZF) \rightarrow \text{Con}(ZFC + CH + \text{there is a Canary tree})$ [3]

(ii) $V=L \rightarrow \text{there are no Canary trees}$ [6].

Thus the non-existence of Canary trees is consistent with CH, relative to the consistency of ZF. This result was first proved in [3] by the method of forcing.

Theorem 5 *Assuming CH and the existence of a Canary tree, there is a $\Phi \in PC(L_{\omega_2\omega_1})$ such that for all $A \subseteq \omega_1$: $\langle \omega_1, <, A \rangle \models \Phi \iff A$ is stationary.*

Proof. Let \mathcal{C} be a Canary tree. It is easy to construct a $PC(L_{\omega_2\omega_1})$ -sentence Ψ such that the following conditions are equivalent for all $A \subseteq \omega_1$:

- (i) $\langle \omega_1, <, A \rangle \models \Psi$
- (ii) There is a mapping f with $\text{Rng}(f) \subseteq \mathcal{C}$ such that for all increasing closed sequences s, s' of elements of A , if s is an initial segment of s' , then $f(s) <_{\mathcal{C}} f(s')$.

We allow predicate symbols with ω -sequences of variables in the $PC(L_{\omega_2\omega_1})$ -sentence Ψ . Now the claim follows from the property (\star) of Canary trees.

□

Theorem 6 *Con(ZF) implies Con(ZFC + CH + there is no $\Phi \in PC(L_{\omega_2\omega_1})$ such that for all $A \subseteq \omega_1$: $\langle \omega_1, <, A \rangle \models \Phi \iff A$ is stationary).*

Proof. We start with a model of GCH and add \aleph_2 Cohen subsets to ω_1 . In the extension GCH continues to hold. Suppose there is in the extension a $\Phi \in PC(L_{\omega_2\omega_1})$ such that for all $A \subseteq \omega_1$:

$$\langle \omega_1, <, A \rangle \models \Phi \iff A \text{ is stationary.}$$

Since the forcing to add \aleph_2 Cohen subsets of ω_1 satisfies the \aleph_2 -c.c., Φ belongs to the extension of the universe by \aleph_1 of the subsets. By first adding all but one of the subsets we can work in $V[A]$ where A is a Cohen subset of ω_1 and Φ is in V . Note that A is a bi-stationary subset of ω_1 . Let \mathcal{P} be in V the forcing for adding a Cohen generic subset of ω_1 and let \tilde{A} be the \mathcal{P} -name for A . Let p force $\langle \omega_1, <, \tilde{A} \rangle \models \Phi$. By arguing as in the proof of Theorem 1, we can construct in V a model M of cardinality \aleph_1 containing \mathcal{P} such that $M^\omega \subseteq M$,

$$M \models [p] \dashv \langle \omega_1, <, \tilde{A} \rangle \models \Phi,$$

and, furthermore, we can extend p to a \mathcal{P} -generic set $H \subseteq \omega_1$ over M such that H is non-stationary. Thus $M[H]$ satisfies

$$\langle \omega_1, <, H \rangle \models \Phi. \tag{1}$$

Now (1) is true in V , because $M[H]^\omega \subseteq M[H]$. Since \mathcal{P} is countably closed, we have (1) in $V[A]$, whence H is stationary in $V[A]$, contrary to the fact that H is non-stationary in V . \square

3 An application to the topological space ${}^{\omega_1}\omega_1$.

Let \mathcal{N}_1 denote the generalized Baire space consisting of all functions $f : \omega_1 \rightarrow \omega_1$, with the sets

$$N_s = \{f \in \mathcal{N}_1 : f \upharpoonright \text{Dom}(s) = s\},$$

where $s \in {}^{<\omega_1}\omega_1$, as basic open sets. We call open sets Σ_1^0 and closed sets Π_1^0 . A set of the form $\bigcup_{\xi < \omega_1} A_\xi$, where each A_ξ is in $\bigcup_{\beta < \alpha} \Pi_\beta^0$, is called Σ_α^0 . Respectively, a set of the form $\bigcap_{\xi < \omega_1} A_\xi$, where each A_ξ is in $\bigcup_{\beta < \alpha} \Sigma_\beta^0$, is called Π_α^0 . In \mathcal{N}_1 it is natural to define Borel sets as follows: A subset of \mathcal{N}_1 is *Borel* if it is Σ_α^0 or Π_α^0 for some $\alpha < \omega_2$. A set $A \subseteq \mathcal{N}_1$ is Π_1^1 if there is an

open set $B \subseteq \mathcal{N}_1 \times \mathcal{N}_1$ such that $\forall f(f \in A \iff \forall g((f, g) \in B))$. A set is Σ_1^1 if its complement is Π_1^1 .

Let CUB be the set of characteristic functions of closed unbounded subsets of ω_1 , and NON-STAT the set of characteristic functions of non-stationary subsets of ω_1 . Clearly, CUB and NON-STAT are disjoint Σ_1^1 . It was proved in [4] that, assuming CH, CUB and NON-STAT are Π_1^1 if and only if there is a Canary tree. Another result on [4] says that the sets CUB and NON-STAT cannot be separated by any Π_3^0 or Σ_3^0 set.

Theorem 7 *Assuming CH, the sets CUB and NON-STAT cannot be separated by a Borel set.*

Proof. Let $\{s_\alpha : \alpha < \omega_1\}$ enumerate all $s \in {}^{<\omega_1}\omega_1$. Let $C = \bigcup_{\alpha < \omega_2} C_\alpha$, where

$$\begin{aligned} C_0 &= \{0, 1\} \times \mathcal{N}_1 \\ C_\delta &= \{2, 3\} \times \omega_1 \left(\bigcup_{\alpha < \delta} C_\alpha \right). \end{aligned}$$

Now we define a Borel set B_c for each $c \in C$ as follows:

$$\begin{aligned} B_{(0,f)} &= \bigcup_{\alpha < \omega_1} N_{s_{f(\alpha)}} \quad , \quad B_{(1,f)} = \bigcap_{\alpha < \omega_1} \mathcal{N}_1 \setminus N_{s_{f(\alpha)}}, \\ B_{(2,f)} &= \bigcup_{\alpha < \omega_1} B_{f(\alpha)} \quad , \quad B_{(3,f)} = \bigcap_{\alpha < \omega_1} B_{f(\alpha)}. \end{aligned}$$

Clearly, every Borel subset X of \mathcal{N}_1 is of the form B_c for some $c \in C$. Then we call c a *Borel code* of X .

Assume A is a Borel set which separates CUB and NON-STAT. Let c be a Borel code of A . Let \mathcal{P} be the forcing notion for adding a Cohen subset to ω_1 . Let G be \mathcal{P} -generic and $g = \bigcup G$. Thus

$$V[G] \models g^{-1}(1) \text{ is bi-stationary.}$$

Now either $g^{-1}(1) \in B_c$ or $g^{-1}(1) \in B_c$ in $V[G]$. We may assume, by symmetry, that $g^{-1}(1) \in B_c$. Let $p \in G$ such that

$$p \Vdash_{-\mathcal{P}} \tilde{g}^{-1}(1) \in B_c,$$

where \tilde{g} is the canonical name for g . Let $M \prec \langle H(\text{beth}_7(\omega_1)), \in, <^* \rangle$, where $<^*$ is a well-ordering of $H(\text{beth}_7(\lambda))$, such that $\omega_1 + 1 \cup \{p\} \cup \{\mathcal{P}\} \cup TC(\{c\}) \subseteq M$, $M^{<\omega_1} \subseteq M$ and $|M| = \omega_1$.

We shall construct two \mathcal{P} -generic sets, G^0 and G^1 , over M . For this end, list open dense $D \subseteq \mathcal{P}$ with $D \in M$ as $\langle D_\xi : \xi < \omega_1 \rangle$. Define $G^l = \{p_\xi^l : \xi < \omega_1\}$ so that $p_0^l = p$, $p_{\xi+1}^l \geq p_\xi^l$ with $p_{\xi+1}^l \in D_\xi \cap M$, $p_{\xi+1}^l(\alpha_{p_\xi^l}) = l$, and $p_\nu^l = \bigcup_{\xi < \nu} p_\xi^l$ for limit ν . Clearly, G^l is \mathcal{P} -generic over M and

$$M[G^l] \models (g^l)^{-1}(1) \in B_c,$$

where $g^l = \bigcup G^l$. Note also that $M[G^l]^{<\omega} \subseteq M[G^l]$, because $M^{<\omega} \subseteq M$ and \mathcal{P} is ω -closed.

Lemma 8 *If $c \in C$ such that $TC(\{c\}) \subseteq M$, and $f \in M$, then*

$$f \in B_c \iff M[G^l] \models [f \in B_c].$$

Proof. Easy induction on c . \square

By the lemma, $(g^l)^{-1}(1) \in B_c$. By construction, $(g^0)^{-1}(1) \in \text{NON-STAT}$ and $(g^1)^{-1}(1) \in \text{CUB}$. Now we can finish the proof. Suppose $\text{CUB} \subseteq A$ and $\text{NON-STAT} \cap A = \emptyset$. This contradicts the fact that $(g^0)^{-1}(1) \in \text{NON-STAT} \cap A$. Suppose $\text{NON-STAT} \subseteq A$ and $\text{CUB} \cap A = \emptyset$. This contradicts the fact that $(g^1)^{-1}(1) \in \text{CUB} \cap A$. \square

4 The case $\lambda^\mu > \lambda$.

Let μ be a cardinal. Sets $A, B \subseteq \mu$ are called *almost disjoint (on μ)* if $\text{sup}(A \cap B) < \mu$. An *almost disjoint λ -sequence* of subsets of μ is a sequence $\mathcal{B} = \langle B_\alpha : \alpha < \lambda \rangle$ such that for all $\alpha \neq \beta$, $|B_\alpha| = \mu$ and the sets B_α and B_β are almost disjoint. The sequence \mathcal{B} is said to be *definable on L_λ* if there is a sequence $\langle \delta_\alpha : \alpha < \lambda \rangle$ such that $\limsup_{\alpha < \lambda} \delta_\alpha = \lambda$ and the predicates $x \in B_y \wedge y < \delta_\alpha$ and $x = \delta_y \wedge x < \alpha \wedge y < \alpha$ are definable on every structure $\langle L_\alpha, \in \rangle$, where $\alpha < \lambda$, that is, there is a first order formula $\varphi_0(x, y)$ of the language of set theory such that for $x, y < \alpha < \lambda$:

$$x \in B_y \wedge y < \delta_\alpha \iff \langle L_\alpha, \in \rangle \models \varphi_0(x, y).$$

Lemma 9 *If $\aleph_1^L = \aleph_1$, then there is an almost disjoint ω_1 -sequence of subsets of ω_1 , which is definable on L_{ω_1} .*

Proof. There is a set $\{B_i : i < \omega_1^L\}$ of almost disjoint subsets of ω in L . Since $\aleph_1^L = \aleph_1$, this set is really of cardinality \aleph_1 . Let $\theta(x, y)$ be a Σ_1 -formula of set theory such that for all α and $x, y \in L_\alpha$, $x <_L y \iff L_\alpha \models \theta(x, y)$, where $<_L$ is the canonical well-ordering of L . The claim follows easily. \square

Theorem 10 *Suppose*

- (i) $\lambda = \mu^+$.
- (ii) *There is an almost disjoint λ -sequence $\mathcal{B} = \langle B_\alpha : \alpha < \lambda \rangle$ of subsets of μ which is definable on L_λ .*
- (iii) *For all club subsets C of λ there is a subset X of μ such that for all $\alpha < \lambda$ we have*

$$\alpha \in C \iff \sup(B_\alpha \setminus C) < \mu.$$

Then there is a sentence $\varphi \in L_{\lambda\lambda}$ so that for all $A \subseteq \lambda$:

$$\langle \lambda, <, A \rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. Suppose φ_0 defines the almost disjoint sequence, as above. We define a sequence of formulas of $L_{\lambda\lambda}$. The variable vectors \vec{x} in these formulas are always sequences of the form $\langle x_i : i < \mu \rangle$. Let Φ be the conjunction of a large but finite number of axioms of $ZFC + V = L$. If $\psi(\vec{z})$ is a formula of set theory, let $\psi'(\vec{z}, \vec{x}, \vec{u}, \vec{v})$ be the result of replacing every quantifier $\forall y \dots$ in Φ by $\forall y (\bigvee_{i < \mu} y = x_i \rightarrow \dots)$, every quantifier $\exists y \dots$ in Φ by $\exists y (\bigvee_{i < \mu} y = x_i \wedge \dots)$, and $y \in z$ everywhere in Φ by $\bigvee_{i < \mu} (y = u_i \wedge z = v_i)$. The following formulas pick μ from $\langle \lambda, < \rangle$:

$$\begin{aligned} \varphi_{\approx\mu}(y) &\iff \exists \vec{x} ((\bigwedge_{i < j < \mu} x_i < x_j) \wedge \forall z (z < y \leftrightarrow \bigvee_{i < \mu} z = x_i)), \\ \varphi_{\in\mu}(y) &\iff \forall u (\varphi_{\approx\mu}(u) \rightarrow y < u), \\ \psi_{\in\mu}(\vec{y}) &\iff \bigwedge_{i < \mu} \varphi_{\in\mu}(y_i) \varphi_{B,1}(x, \vec{u}, \vec{v}, z, y) \end{aligned}$$

The following formulas are needed to refer to well-founded models of set theory:

$$\begin{aligned} \varphi_{uni}(\vec{x}, z) &\iff \bigvee_{i < \mu} z = x_i \\ \varphi_{eps}(\vec{x}, \vec{u}, \vec{v}, z, y) &\iff \varphi_{uni}(\vec{x}, z) \wedge \varphi_{uni}(\vec{x}, y) \wedge \bigvee_{i < \mu} (z = u_i \wedge y = v_i) \\ \varphi_{wf}(\vec{x}, \vec{u}, \vec{v}) &\iff \Phi'(\vec{x}, \vec{u}, \vec{v}) \wedge \forall \vec{y} ((\bigwedge_{i < \mu} \varphi_{uni}(\vec{x}, y_i)) \rightarrow \\ &\quad \bigvee_{i < \mu} \neg \varphi_{eps}(\vec{x}, \vec{u}, \vec{v}, y_{i+1}, y_i)) \\ \varphi_{cor}(\vec{x}, \vec{u}, \vec{v}, z) &\iff \forall s (s < z \leftrightarrow \bigvee_{i < \mu} (s = u_i \wedge z = v_i)) \end{aligned}$$

Let

$$\varphi_B(z, y) \iff \exists \vec{x} \exists \vec{u} \exists \vec{v} (\varphi_{wf}(\vec{x}, \vec{u}, \vec{v}) \wedge \varphi_{cor}(\vec{x}, \vec{u}, \vec{v}, z) \wedge \varphi_{cor}(\vec{x}, \vec{u}, \vec{v}, y) \wedge \phi'_0(z, y, \vec{x}, \vec{u}, \vec{v})).$$

The point is that if $\alpha \in \mu$ and $\beta \in \lambda$, then $\alpha \in B_\beta$ if and only if $\langle \lambda, < \rangle \models \varphi_B(\alpha, \beta)$. The following formula says that the element y of μ is in the subset of λ coded by \vec{x} :

$$\varphi_\varepsilon(y, \vec{x}) \iff \exists u (\varphi_{\in \mu}(u) \wedge \forall z ((\varphi_B(z, y) \wedge \bigwedge_{i < \mu} z \neq x_i) \rightarrow z < u),$$

Finally, if:

$$\begin{aligned} \varphi_{ub}(\vec{x}) &\iff \forall y \exists z (y < z \wedge \varphi_\varepsilon(z, \vec{x})), \\ \varphi_{cl}(\vec{x}) &\iff \forall y (\forall z (z < y \rightarrow \exists u (z < u \wedge u < y \wedge \varphi_\varepsilon(u, \vec{x}))) \rightarrow \varphi_\varepsilon(y, \vec{x})) \\ \varphi_{cub}(\vec{x}) &\iff \varphi_{ub}(\vec{x}) \wedge \varphi_{cl}(\vec{x}) \\ \varphi_{stat} &\iff \forall \vec{x} ((\psi_{\in \mu}(\vec{x}) \wedge \varphi_{cub}(\vec{x})) \rightarrow \exists y (A(y) \wedge \varphi_\varepsilon(y, \vec{x}))), \end{aligned}$$

then $\langle \lambda, <, A \rangle \models \varphi_{stat}$ if and only if A is stationary. \square

Corollary 11 *If $2^{\aleph_0} > \aleph_1$, $\aleph_1^L = \aleph_1$ and MA, then there is a $\varphi \in L_{\omega_1 \omega_1}$ such that for all $A \subseteq \omega_1$:*

$$\langle \omega_1, <, A \rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. We choose $\lambda = \omega_1$ and $\mu = \omega_0$ in Theorem 10. Condition (ii) holds by Lemma 9. Condition (iii) is a consequence of MA + \neg CH by [2]. \square

Note. The proof of Corollary 11 shows that we actually get the following stronger result: If $2^{\aleph_0} > \aleph_1$, $\aleph_1^L = \aleph_1$ and MA, then the full second order extension $L_{\omega_1 \omega_1}^{II}$ of $L_{\omega_1 \omega_1}$ is reducible to $L_{\omega_1 \omega_1}$ in expansions of $\langle \omega_1, < \rangle$. Then, in particular, T_{\aleph_1} is $PC(L_{\omega_1 \omega_1})$ -definable. This kind of reduction cannot hold on all models. For example, ω_1 -like dense linear orders with a first element are all $L_{\infty \omega_1}$ -equivalent, but not $L_{\omega \omega}^{II}$ -equivalent.

For $\alpha < \lambda = \mu^+$, let $\langle a_i^\alpha : i < \mu \rangle$ be a continuously increasing sequence of subsets of α with $\alpha = \bigcup_{i < \mu} a_i^\alpha$ and $|a_i^\alpha| < \mu$. Define $f_\alpha : \mu \rightarrow \mu$ by

$$f_\alpha(i) = otp(a_i^\alpha).$$

Let D_μ be the club-filter on μ . Define for $f, g \in {}^\mu \mu$;

$$f \sim_{D_\mu} g \iff \{i : f(i) = g(i)\} \in D_\mu.$$

Lemma 12 f_α/D_μ is independent of the choice of the sequence $\langle a_i^\alpha : i < \mu \rangle$.

Theorem 13 *Suppose*

(i) $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$.

(ii) For every club $C \subseteq \lambda$ there is some $X \subseteq \mu \times \mu$ such that

$$\begin{aligned} \alpha \in C &\rightarrow \{i < \mu : (i, f_\alpha(i)) \in X\} \text{ contains a club} \\ \alpha \notin C &\rightarrow \{i < \mu : (i, f_\alpha(i)) \notin X\} \text{ contains a club.} \end{aligned}$$

Then there is a sentence $\varphi \in L_{\lambda\lambda}$ such that for all $A \subseteq \lambda$:

$$\langle \lambda, <, A \rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. This is like the proof of Theorem 10. One uses Lemma 12 to refer to the functions f_α . We leave the details to the reader. \square

The *Generalized Martin's Axiom for μ* (GMA_μ) from [5] is the following principle:

Suppose \mathcal{P} is a forcing notion with the properties:

(GMA1) Every descending sequence of length $< \mu$ in \mathcal{P} has a greatest lower bound.

(GMA2) If $p_\alpha \in \mathcal{P}$ for $\alpha < \mu^+$, then there is a club $C \subseteq \mu^+$ and a regressive function $f : \mu^+ \rightarrow \mu^+$ such that if $\alpha \in C$ and $\text{cf}(\alpha) = \mu$, then the set

$$A = \{p_\beta : \text{cf}(\beta) = \mu, f(\alpha) = f(\beta)\}$$

is well-met (i.e. $p, q \in A \rightarrow p \vee q \in A$).

Then for any dense open sets $D_\alpha \subseteq \mathcal{P}$, $\alpha < \kappa$, where $\kappa < 2^\mu$, there is a filter in \mathcal{P} which meets every D_α .

Proposition 14 *Suppose $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$, and GMA_μ . Then for every club $C \subseteq \lambda$ there is some $X \subseteq \mu \times \mu$ such that*

$$\begin{aligned} \alpha \in C &\rightarrow \{i < \mu : (i, f_\alpha(i)) \in X\} \text{ contains a club} \\ \alpha \notin C &\rightarrow \{i < \mu : (i, f_\alpha(i)) \notin X\} \text{ contains a club.} \end{aligned}$$

Proof. Let a club $C \subseteq \lambda$ be given. For $\alpha < \beta < \lambda$, let $C_{\alpha\beta} \in D_\mu$ so that $f_\alpha|C_{\alpha\beta} < f_\beta|C_{\alpha\beta}$. Let \mathcal{P} consist of conditions

$$p = (B^p, f^p, \mathbf{c}^p, g^p),$$

where

- (i) $B^p \subseteq \lambda$. $|B^p| < \mu$.
- (ii) f^p is a partial mapping with $\text{Dom}(f^p) \subseteq \mu \times \mu$, $|\text{Dom}(f^p)| < \mu$, and $\text{Rng}(f^p) \subseteq \{0, 1\}$.
- (iii) If $\alpha \in B^p$, then $\{i < \mu : (i, f_\alpha(i)) \in \text{Dom}(f^p)\}$ is an ordinal j_α^p .
- (iv) $\mathbf{c}^p = \langle c_\alpha^p : \alpha \in B^p \rangle$, where c_α^p is a closed subset of j_α^p . We denote $\max(c_\alpha^p)$ by δ^p .
- (v) If $\alpha \in B^p \cap C$ and $i \in C_\alpha^p$, then $f^p(i, f_\alpha(i)) = 1$. If $\alpha \in B^p \setminus C$ and $i \in C_\alpha^p$, then $f^p(i, f_\alpha(i)) = 0$.
- (vi) g^p is a partial mapping with $\text{Dom}(g^p) \subseteq [B^p]^2$ and $\text{Rng}(g^p) \subseteq \mu$.
- (vii) If $\alpha < \beta \in \text{Dom}(g^p)$, then $\emptyset \neq c_\alpha^p \setminus g(\alpha, \beta) \subseteq C_{\alpha\beta}$.

The partial ordering “ q extends p ” is defined as follows:

$$\begin{aligned} p \leq q \iff & B^p \subseteq B^q, f^p \subseteq f^q, g^p \subseteq g^q, \\ & \forall \alpha \in B^p (c_\alpha^p \text{ is an initial segment of } c_\alpha^q), \\ & \text{and if } \delta^p < \delta^q, \text{ then } \text{Dom}(g^q) \supseteq [B^p]^2. \end{aligned}$$

We show now that \mathcal{P} satisfies conditions (GMA1) and (GMA2).

Lemma 15 \mathcal{P} satisfies (GMA1).

Proof. Let $p_o \leq \dots \leq p_i \leq \dots (i < \gamma)$ in \mathcal{P} with $\gamma < \mu$. We may assume $\delta^{p_0} < \delta^{p_1} < \dots$. Let $\delta = \sup\{\delta^{p_i} : i < \gamma\}$. Let $B = \bigcup_{i < \gamma} B^{p_i}$. We extend $\bigcup_i f^{p_i}$ to f by defining

$$f(\delta, f_\alpha(\delta)) = \begin{cases} 1 & \text{if } \alpha \in B \cap C \\ 0 & \text{if } \alpha \in B \setminus C. \end{cases}$$

We have to check that this definition is coherent, i.e., if $\alpha \in B \cap C$ and $\beta \in B \setminus C$, then $f_\alpha(\delta) \neq f_\beta(\delta)$. Suppose $\alpha \in B^{p_i}$ and $\beta \in B^{p_{i'}}$ with $\alpha < \beta$

and $i < i'$. Since $\delta^{p_i} < \delta^{p_{i'}}$, $g(\alpha, \beta)$ is defined and $c_\alpha^{p_i} \setminus g(\alpha, \beta) \subseteq C_{\alpha\beta}$. Hence $\delta \in C_{\alpha\beta}$, whence $f_\alpha(\delta) < f_\beta(\delta)$.

Let $\mathbf{c} = \langle c_\alpha : \alpha \in B \rangle$ where $c_\alpha = \bigcup_i c_\alpha^{p_i} \cup \{\delta\}$. Let $j = \bigcup_i j^{p_i} \cup \{\delta\}$. Now the condition $p = (B, f, \mathbf{c}, g)$ is the needed l.u.b. of $(p_i)_{i < \mu}$. \square

Lemma 16 \mathcal{P} satisfies (GMA2).

Proof. Suppose p_α , $\alpha < \lambda$, are in \mathcal{P} . Let h be a one-one mapping from \mathcal{P} to odd ordinals $< \lambda$. By $\mu^{<\mu} = \mu$ there is a club $C \subseteq \lambda$ such that if $\alpha \in C$, $\text{cf}(\alpha) = \mu$, and $B^p \subseteq \alpha$, then $h(p) < \alpha$, and if $\alpha < \beta$, $\alpha, \beta \in C$, then $B^{p_\alpha} \subseteq \beta$. Choose a regressive function g from the complement of C to the even ordinals that is one-one on ordinals of cofinality μ . Suppose $\text{cf}(\alpha) = \mu$. Let $f(\alpha) = g(\alpha)$ if $\alpha \notin C$, and $f(\alpha) = h(p_\alpha|\alpha)$ if $\alpha \in C$. Suppose now $\alpha < \beta$, $\text{cf}(\alpha) = \text{cf}(\beta) = \mu$, and $f(\alpha) = f(\beta)$. W.l.o.g. $\alpha, \beta \in C$. Thus $h(p_\alpha|\alpha) = h(p_\beta|\beta)$, whence $p_\alpha|\alpha = p_\beta|\beta$. It follows that p_α and p_β have a l.u.b. \square

Let

$$D_{\alpha\beta} = \{p \in \mathcal{P} : \alpha \in B^p \text{ and } \delta^p \geq \beta\}$$

where $\alpha < \lambda$, $\beta < \mu$. We show that $D_{\alpha\beta}$ is dense open. Suppose therefore $p \in \mathcal{P}$ is given. We construct $q \in D_{\alpha\beta}$ with $p \leq q$. Let $B^q = B^p \cup \{\alpha\}$. Let

$$E = \bigcap \{C_{\xi\eta} : \xi, \eta \in B^q, \xi < \eta\} (\in D_\mu).$$

Let $\delta^q \in E \setminus \beta$. Define $\mathbf{c}^q = \langle c_\xi^q : \xi \in B^q \rangle$ by

$$c_\xi^q = \begin{cases} c_\xi^p \cup \langle \delta^q \rangle, & \text{if } \xi \neq \alpha \\ \langle \delta^q \rangle, & \text{if } \xi = \alpha. \end{cases}$$

Let

$$f^q = f^p \cup \begin{cases} \{((j, f_\alpha(j)), 1) : j^p \leq j \leq \delta^q\}, & \text{if } \alpha \in C \\ \{((j, f_\alpha(j)), 0) : j^p \leq j \leq \delta^q\}, & \text{if } \alpha \notin C. \end{cases}$$

Let $g^q(\xi, \eta) = \delta^p$ for $(\xi, \eta) \in [M^p]^2 \setminus \text{Dom}(g^p)$. Let $q = (B^q, f^q, g^q, \delta^q)$. Then $q \in D_{\alpha\beta}$, and $p \leq q$.

Let G be a filter that meets every $D_{\alpha\beta}$. Let

$$\begin{aligned} B &= \bigcup \{B^p : p \in G\} \\ f &= \bigcup \{f^p : p \in G\} \\ c_\alpha &= \bigcup \{c_\alpha^p : p \in G\} \end{aligned}$$

Then $B = \lambda$ and each c_α is a club of μ . Let $X = \{(\alpha, \beta) \in \mu \times \mu : f(\alpha, \beta) = 1\}$. Suppose $\alpha \in C$ and $i \in c_\alpha$. Then $f(i, f_\alpha(i)) = 1$ whence $(i, f_\alpha(i)) \in X$. Suppose $\alpha \notin C$ and $i \in c_\alpha$. Then $f(i, f_\alpha(i)) = 0$ whence $(i, f_\alpha(i)) \notin X$. \square

Corollary 17 *Suppose $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$, and GMA_μ . Then there is a sentence $\varphi \in L_{\lambda\lambda}$ such that for all $A \subseteq \lambda$:*

$$\langle \lambda, <, A \rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. The claim follows from Theorem 13 and Proposition 14. \square

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