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STRONGLY MEAGER AND STRONG MEASURE ZERO SETS

TOMEK BARTOSZYŃSKI AND SAHARON SHELAH

ABSTRACT. In this paper we present two consistency results concerning the existence of large strong measure zero and strongly meager sets.

1. INTRODUCTION

Let $\mathcal{M}$ denote the collection of all meager subsets of 2^ω and let $\mathcal{N}$ be the collection of all subsets of 2^ω that have measure zero with respect to the standard product measure on 2^ω.

Definition 1.1. *Suppose that $X \subseteq 2^\omega$ and let $+$ denote the componentwise addition modulo 2. We say that X is strongly meager if for every $H \in \mathcal{N}$, $X + H = \{x + h : x \in X, h \in H\} \neq 2^\omega$.*

We say that X is a strong measure zero set if for every $F \in \mathcal{M}$, $X + F \neq 2^\omega$. Let $\mathcal{SM}$ denote the collection of strongly meager sets and let $\mathcal{SN}$ denote the collection of strong measure zero sets.

For a family of sets $\mathcal{J} \subseteq P(\mathbb{R})$ let
$\mathsf{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}$ *and* $\bigcup \mathcal{A} = 2^\omega\}$.
$\mathsf{non}(\mathcal{J}) = \min\{|X| : X \notin \mathcal{J}\}$.

Strong measure zero sets are usually defined as those subsets X of 2^ω such that for every sequence of positive reals $\{\varepsilon_n : n \in \omega\}$ there exists a sequence of basic open sets $\{I_n : n \in \omega\}$ with diameter of I_n smaller than ε_n and $X \subseteq \bigcup_n I_n$. The Galvin-Mycielski-Solovay theorem ([4]) guarantees that both definitions are yield the same families of sets.

Recall the following well–known facts. Any of the following sentences is consistent with ZFC,

(1) $\mathcal{SN} = [2^\omega]^{\leq\aleph_0}$, (Laver [7])
(2) $\mathcal{SN} = [2^\omega]^{\leq\aleph_1}$, (Corazza [3], Goldstern-Judah-Shelah [5])
(3) $\mathcal{SM} = [2^\omega]^{\leq\aleph_0}$. (Carlson, [2])
(4) $\mathsf{non}(\mathcal{SN}) = \mathfrak{d} = 2^{\aleph_0} > \aleph_1$, $\mathsf{cov}(\mathcal{M}) = \aleph_1$ and there exists a strong measure zero set of size $2^{\aleph_0}$. (Goldstern-Judah-Shelah [5])

The proofs of the above results as well as all other results quoted in this paper can be also found in [1].

In this paper we will show that the following statements are consistent with ZFC:

- for any regular $\kappa > \aleph_0$, $\mathcal{SM} = [2^\omega]^{<\kappa}$,
- $\mathcal{SM}$ is an ideal and $\mathsf{add}(\mathcal{SM}) \geq \mathsf{add}(\mathcal{M})$,

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- $\mathsf{non}(\mathcal{SN}) = 2^{\aleph_0} > \aleph_1$, $\mathfrak{d} = \aleph_1$ and there is a strong measure zero set of size $2^{\aleph_0}$.

2. $\mathcal{SM}$ may have large additivity

In this section we will show that $\mathcal{SM}$ can be an ideal with large additivity. Let

$$\mathfrak{m} = \min\{\gamma : \mathbf{MA}_\gamma \text{ fails}\}.$$

We will show that $\mathcal{SM} = [2^\omega]^{<\mathfrak{m}}$ is consistent with ZFC, provided $\mathfrak{m}$ is regular. In particular, the model that we construct will satisfy $\mathsf{add}(\mathcal{SM}) = \mathsf{add}(\mathcal{M})$.

Note that if $\mathcal{SM} = [2^\omega]^{<\mathfrak{m}}$ then $2^{\aleph_0} > \mathfrak{m}$, since Martin's Axiom implies the existence of a strongly meager set of size $2^{\aleph_0}$. Our construction is a generalization of the construction from [2].

To witness that a set is not strongly meager we need a measure zero set. The following theorem is crucial.

Theorem 2.1 (Lorentz)**.** *There exists a function $K \in \omega^{\mathbb{R}}$ such that for every $\varepsilon > 0$, if $A \in [2^\omega]^{\geq K(\varepsilon)}$ then for all except finitely many $k \in \omega$ there exists $C \subseteq 2^k$ such that*

(1) $|C| \cdot 2^{-k} \leq \varepsilon$,
(2) $(A{\restriction}k) + C = 2^k$.

Proof Proof of this lemma can be found in [8] or [1]. □

Definition 2.2. *For each $n \in \omega$ let $\{C^n_m : n, m \in \omega\}$ be an enumeration of all clopen sets in 2^ω of measure $\leq 2^{-n}$. For a real $r \in \omega^\omega$ and $n \in \omega$ define an open set*

$$H^r_n = \bigcup_{m>n} C^m_{r(m)}.$$

It is clear that H^r_n is an open set of measure not exceeding 2^{-n}. In particular, $H^r = \bigcap_{n\in\omega} H^r_n$ is a Borel measure zero set of type G_δ.

Theorem 2.3. *Let $\kappa > \aleph_0$ be a regular cardinal. It is consistent with ZFC that $\mathbf{MA}_{<\kappa} + \mathcal{SM} = [2^\omega]^{<\kappa}$ holds. In particular, it is consistent that $\mathcal{SM}$ is an ideal and $\mathsf{add}(\mathcal{SM}) = \mathsf{add}(\mathcal{M}) > \aleph_1$.*

Proof Fix κ such that $\mathsf{cf}(\kappa) = \kappa > \aleph_0$. Let $\lambda > \kappa$ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$. Start with a model $\mathbf{V} \models \mathsf{ZFC} + 2^{\aleph_0} = \lambda$.

Suppose that $\mathcal{P}$ is a forcing notion of size $< \kappa$. We can assume that there is $\gamma < \kappa$ such that $\mathcal{P} = \gamma$ and $\leq, \perp \subseteq \gamma \times \gamma$.

Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \lambda\}$ be a finite support iteration such that for each $\alpha < \lambda$,

(1) $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbf{C}$, if α is limit,
(2) there is $\gamma = \gamma_\alpha$ such that $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq (\gamma, \leq, \perp)$ is a ccc forcing notion.

By passing to a dense subset we can assume that if $p \in \mathcal{P}_\lambda$ then $p : \mathsf{dom}(p) \longrightarrow \kappa$, where $\mathsf{dom}(p)$ is a finite subset of λ.

By bookkeeping we can guarantee that $\mathbf{V}^{\mathcal{P}_\lambda} \models \mathbf{MA}_{<\kappa}$. In particular, $\mathbf{V}^{\mathcal{P}_\lambda} \models [2^\omega]^{<\kappa} \subseteq \mathcal{SM}$.

It remains to show that no set of size κ is strongly meager.

Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}_\lambda} \cap 2^\omega$ is a set of size κ. Find limit ordinal $\alpha < \lambda$ such that $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}_\alpha}$. As usual we can assume that $\alpha = 0$. Let c be the Cohen real

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added at the step $\alpha = 0$. We will show that $\mathbf{V}^{\mathcal{P}_\lambda} \models X + H^c = 2^\omega$, which will end the proof.

Suppose that the above assertion is false. Let $p \in \mathcal{P}_\lambda$ and let $\dot{z}$ be a $\mathcal{P}_\lambda$-name for a real such that

$$p \Vdash_\lambda \dot{z} \notin X + H^{\dot{c}}.$$

Let $X = \{x_\xi : \xi < \kappa\}$ and for each ξ find $p_\xi \geq p$ and $n_\xi \in \omega$ such that

$$p_\xi \Vdash_\lambda \dot{z} \notin x_\xi + H^{\dot{c}}_{n_\xi}.$$

Let $Y \subseteq \kappa$ be a set of size κ such that

(1) $n_\xi = \widetilde{n}$ for $\xi \in Y$,
(2) $\{\mathsf{dom}(p_\xi) : \xi \in Y\}$ form a Δ-system with root $\widetilde{\Delta}$,
(3) $p_\xi \restriction \widetilde{\Delta} = \widetilde{p}$, for $\xi \in Y$,
(4) $p_\xi(0) = \widetilde{s}$, with $|\widetilde{s}| = \ell > \widetilde{n}$, for $\xi \in Y$.

Fix a subset $X' = \{x_{\xi_j} : j < K(2^{-\ell})\} \subseteq Y$ and let $\widetilde{m} \in \omega$ be such that $C^\ell_{\widetilde{m}} + X' = 2^\omega$.

Define condition $p^\star$ as

$$p^\star(\beta) = \begin{cases} p_{\xi_j} & \text{if } \alpha \neq \beta \ \&\ \beta \in \mathsf{dom}(p_{\xi_j}),\ j < K(2^{-\ell}) \\ \widetilde{s}^\frown\widetilde{m} & \text{if } \alpha = \beta \end{cases} \quad \text{for } \beta < \lambda.$$

On one hand $p^\star \Vdash_\lambda C^\ell_{\widetilde{m}} \subseteq H^{\dot{c}}_{\widetilde{n}}$, so $p^\star \Vdash_\lambda X' + H^{\dot{c}}_{\widetilde{n}} = 2^\omega$. On the other hand, $p^\star \geq p_{\xi_j}$, $j \leq K(2^{-\ell})$, so $p^\star \Vdash_\lambda \dot{z} \notin X' + H^{\dot{c}}_{\widetilde{n}}$. Contradiction.

To finish the proof we show that $\mathbf{V}^{\mathcal{P}_\lambda} \models \mathsf{add}(\mathcal{M}) = \kappa$. First note that $\mathbf{MA}_{<\kappa}$ implies that $\mathsf{add}(\mathcal{M}) \geq \kappa$ in $\mathbf{V}^{\mathcal{P}_\lambda}$. The other inequality is a consequence of the general theory. Recall that (see [1])

(1) $\mathsf{add}(\mathcal{M}) = \min\{\mathsf{cov}(\mathcal{M}), \mathfrak{b}\}$

Suppose that $F \subset \omega^\omega$ is an unbounded family of size $\geq \kappa$.

2. if $\mathcal{P}$ is a forcing notion of cardinality $< \kappa$ then F remains unbounded in $\mathbf{V}^{\mathcal{P}}$.
3. if $\{\mathcal{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \lambda\}$ is a finite support iteration such that $\Vdash_\alpha |\mathcal{Q}_\alpha| < \kappa$ then $\mathbf{V}^{\mathcal{P}_\lambda} \models F$ is unbounded..

From the results quoted above follows that $\mathsf{add}(\mathcal{M}) \leq \mathfrak{b} \leq \kappa$ in $\mathbf{V}^{\mathcal{P}_\lambda}$, which ends the proof. $\square$

3. Strong measure zero sets

In this section we will discuss models with strong measure zero sets of size $2^{\aleph_0}$.
We start with the definition of forcing that will be used in our construction.

Definition 3.1. *The infinitely equal forcing notion* **EE** *is defined as follows:* $p \in$ **EE** *if the following conditions are satisfied:*

(1) $p : \mathsf{dom}(p) \longrightarrow 2^{<\omega}$,
(2) $\mathsf{dom}(p) \subseteq \omega,\ |\omega \setminus \mathsf{dom}(p)| = \aleph_0$,
(3) $p(n) \in 2^n$ *for all* $n \in \mathsf{dom}(p)$.

For $p, q \in$ **EE** *and* $n \in \omega$ *we define:*

(1) $p \geq q \iff p \supseteq q$, *and*
(2) $p \geq_n q \iff p \geq q$ *and the first* n *elements of* $\omega \setminus \mathsf{dom}(p)$ *and* $\omega \setminus \mathsf{dom}(q)$ *are the same.*

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It is easy to see (see [1]) that **EE** is proper (satisfies axiom A), and strongly ω^ω bounding, that is if $p \Vdash \tau \in \omega$ and $n \in \omega$ then there is $q \geq_n p$ and a finite set $F \subseteq \omega$ such that $q \Vdash \tau \in F$.

In [5] it is shown that a countable support iteration of **EE** and rational perfect set forcing produces a model where there is a strong measure zero set of size $2^{\aleph_0}$. In particular, one can construct (consistently) a strong measure zero of size $2^{\aleph_0}$ without Cohen reals. The remaining question is whether such a construction can be carried out without unbounded reals.

Theorem 3.2 ([5]). *Suppose that $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ is a countable support iteration of proper, strongly ω^ω-bounding forcing notions. Then*

$$\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathcal{SN} \subseteq [\mathbb{R}]^{\leq \aleph_1}. \quad \square$$

The theorem above shows that using countable support iteration we cannot build a model with a strong measure zero set of size $> \mathfrak{d}$. Since countable support iteration seems to be the universal method for constructing models with $2^{\aleph_0} = \aleph_2$ the above result seems to indicate that a strong measure zero set of size $> \mathfrak{d}$ cannot be constructed at all. Strangely it is not the case.

Theorem 3.3. *It is consistent that* $\mathsf{non}(\mathcal{SN}) = 2^{\aleph_0} > \mathfrak{d} = \aleph_1$ *and there are strong measure zero sets of size* $2^{\aleph_0}$.

Proof Suppose that $\mathbf{V} \models \mathsf{CH}$ and $\kappa = \kappa^{\aleph_0} > \aleph_1$. Let $\mathcal{P}$ be a countable support product of κ copies of **EE**. The following facts are well-known (see [6])

(1) $\mathcal{P}$ is proper,
(2) $\mathcal{P}$ satisfies $\aleph_2$-cc,
(3) $\mathcal{P}$ is ω^ω-bounding,
(4) for $f \in \mathbf{V}[G] \cap \omega^\omega$ there exists a countable set $A \subseteq \kappa$, $A \in \mathbf{V}$ such that $f \in \mathbf{V}[G{\restriction}A]$.

It follows from (3) that $\mathbf{V}^{\mathcal{P}} \models \mathfrak{d} = \aleph_1$. Moreover, (1) and (2) imply that $2^{\aleph_0} = \kappa$ in $\mathbf{V}^{\mathcal{P}}$.

For a set $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}}$ let $\mathsf{supp}(X) \subseteq \kappa$ be a set such that $X \in \mathbf{V}[G{\restriction}\mathsf{supp}(X)]$. Note that $\mathsf{supp}(X)$ is not determined uniquely, but we can always choose it so that $|\mathsf{supp}(X)| = |X| + \aleph_0$.

Lemma 3.4. *Suppose that $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}}$ and $\mathsf{supp}(X) \neq \kappa$. Then $\mathbf{V}^{\mathcal{P}} \models X \in \mathcal{SN}$*

Note that this lemma finishes the proof. Clearly the assumptions of the lemma are met for all sets of size $< \kappa$ and also for many sets of size κ.

Proof We will use the following characterization (see [1]):

Lemma 3.5. *The following conditions are equivalent.*

(1) *$X \subseteq 2^\omega$ has strong measure zero.*
(2) *For every $f \in \omega^\omega$ there exists $g \in (2^{<\omega})^\omega$ such that $g(n) \in 2^{f(n)}$ for all n and*

$$\forall x \in X\ \exists n\ x{\restriction}f(n) = g(n). \quad \square$$

Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}} \cap 2^\omega$ is given and $\mathsf{supp}(X) \neq \kappa$. Let $\alpha^\star \in \kappa \setminus \mathsf{supp}(X)$. We will check condition (2) of the previous lemma.

Fix $f \in \mathbf{V}^{\mathcal{P}} \cap \omega^\omega$. Since $\mathcal{P}$ is ω^ω-bounding we can assume that $f \in \mathbf{V}$. Consider a condition $p \in \mathcal{P}$. Fix $\{k_n : n \in \omega\}$ such that $k_n \geq f(n)$ and $k_n \notin \mathsf{dom}(p(\alpha^\star))$ for

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$n \in \omega$. Let $p_f \geq p$ be any condition such that $\omega \setminus \{k_n : n \in \omega\} \subseteq \mathsf{dom}(p_f(\alpha^\star))$. We will check that

$$p_f \Vdash_{\mathcal{P}} \forall x \in X \; \exists n \; x{\restriction}f(n) = \dot{G}(\alpha^\star)(k_n){\restriction}f(n),$$

where $\dot{G}$ is the canonical name for the generic object. Take $x \in X$ and $r \geq p_f$. Find n such that $k_n \notin \mathsf{dom}\big(r(\alpha^\star)\big)$. Let $r' \geq r$ and s be such that

(1) $\mathsf{supp}(r') \subseteq \mathsf{supp}(X)$
(2) $r' \geq r{\restriction}\mathsf{supp}(X)$,
(3) $r' \Vdash_{\mathcal{P}} x{\restriction}k_n = s$.

Let

$$r''(\beta) = \begin{cases} r'(\beta) & \text{if } \beta \neq \alpha^\star \\ r'(\alpha^\star) \cup \{(k_n, s)\} & \text{if } \beta = \alpha^\star \end{cases} .$$

It is easy to see that $r'' \Vdash x{\restriction}f(n) = \dot{G}(\alpha^\star)(k_n){\restriction}f(n)$. Since f and x were arbitrary we are done. □

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Department of Mathematics and Computer Science, Boise State University, Boise, Idaho 83725 U.S.A.

Email address: tomek@math.idbsu.edu, http://math.idbsu.edu/~tomek

Department of Mathematics, Hebrew University, Jerusalem, Israel

Email address: shelah@sunrise.huji.ac.il, http://math.rutgers.edu/~shelah/