# Universal graphs at the successor of a singular cardinal (revised after proof reading)

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**Abstract**

The paper is concerned with the existence of a universal graph at the successor of a strong limit singular $\mu$ of cofinality $\aleph_0$. Starting from the assumption of the existence of a supercompact cardinal, a model is built in which for some such $\mu$ there are $\mu^{++}$ graphs on $\mu^+$ that taken jointly are universal for the graphs on $\mu^+$, while $2^{\mu^+} >> \mu^{++}$.

The paper also addresses the general problem of obtaining a framework for consistency results at the successor of a singular strong limit starting from the assumption that a supercompact cardinal $\kappa$ exists. The result on the existence of universal graphs is obtained as a specific application of a more general method. [1]

# 0 Introduction

The question of the existence of a universal graph of a certain cardinality and with certain properties has been the subject of much research in mathematics ([FuKo], [Kj], [KoSh 492], [Rd], [Sh 175a], [Sh 500]). By universality we mean here that every other graph of the same size embeds into the universal graph. In the presence of $GCH$ it follows from the classical results in model theory ([ChKe]) that such a graph exists at every uncountable cardinality, and it is well known that the random graph ([Rd]) is universal for countable graphs (although the situation is not so simple when certain requirements on the graphs are imposed, see [KoSh 492]). When the assumption of $GCH$ is dropped, it becomes much harder to construct universal objects, and it is in fact usually rather easy to obtain negative consistency results by adding Cohen subsets to the universe (see [KjSh 409] for a discussion of this). For some classes of graphs there are no universal objects as soon as $GCH$ fails sufficiently ([Kj], [Sh 500](§2)), while for others there can exist consistently a small family of the class that acts jointly as a universal object for the class at the given cardinality ([Sh 457], [DjSh 614]). Much of what is known in the absence of $GCH$ is known about successors of regular cardinals ([Sh 457], [DjSh 614]). In [Sh 175a] there is a positive consistency result concerning

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the existence of a universal graph at the successor of singular $\mu$ where $\mu$ is not a strong limit. In this paper we address the issue of the existence of a universal graph at the successor of a singular strong limit and obtain a positive consistency result regarding the existence of a small family of such graphs that act jointly as universal for the graphs of the same size.

In addressing this specific problem, the paper also offers a step towards the solution of a more general problem of doing iterated forcing in connection with the successor of a singular. This is the case because the result about universal graphs is obtained as an application of a more general method. The method relies on an iteration of $(<\kappa)$-directed-closed $\theta \geq \kappa^+$-cc forcing, followed by the Prikry forcing for a normal ultrafilter $\mathcal{D}$ built by the iteration. The cardinal $\kappa$ here is supercompact in the ground model. The idea is that the Prikry forcing for $\mathcal{D}$ can be controlled by the iteration, as $\mathcal{D}$ is being built in the process as the union of an increasing sequence of normal filters that appear during the iteration. Apart from building $\mathcal{D}$, the iteration also takes care of the particular application it is aimed at by predicting the $\mathcal{D}$-names of the relevant objects and taking care of them (in our application, these objects are graphs on $\kappa^+$). The iteration is followed by the Prikry forcing for $\mathcal{D}$, so changing the cofinality of $\kappa$ to $\aleph_0$. Before doing the iteration we prepare $\kappa$ by rendering its supercompactness indestructible by $(<\kappa)$-directed-closed forcing through the use of Laver's diamond ([La]). Unlike the most common use of the indestructibility of $\kappa$ where one uses the fact that $\kappa$ is indestructible without necessarily refering back to how this indestructibility is obtained, we must use Laver's diamond itself for the definition of the iteration. We note that the result has an unusual feature in which the iteration is not constructed directly, but the existence of such an iteration is proved and used.

Some of the ideas connected to the forcing scheme discussed in this paper were pursued by A. Mekler and S. Shelah in [MkSh 274], and by M. Gitik and S. Shelah in [GiSh 597], both in turn relaying on M. Magidor's independence proof for $SCH$ at $\beth_\omega$ [Ma 1], [Ma 2] and Laver's indestructibility method, [La]. In [MkSh 274]§3 the idea of guessing Prikry names of an object after the final collapse is present, while [GiSh 597] considers densities of box topologies, and for the particular forcing used there presents a scheme similar to the one we use (although the iteration is different). The latter paper also reduced the strength of a large cardinal needed for the iteration to a hyper-measurable. The difference between [GiSh 597] and our results is that the individual forcing used in [GiSh 597] is basically Cohen forcing, while our interest here is to give a general axiomatic framework under which the scheme can be applied for many types of forcing notions.

The investigation of the consistent existence of universal objects also has relevance in model theory. The idea here is to classify theories in model theory by the size of their universality spectrum, and much research has been done to confirm that this classification is interesting from the model-theoretic point of view ([GrSh 174], [KjSh 409], [Sh 500], [DjSh 614]). The results here sound a word of caution to this programme. Our construction builds $\mu^{++}$ graphs on $\mu^+$ that are universal for the graphs on $\mu^+$, while $2^{\mu^+} >> \mu^{++}$ and $\mu$ is a strong limit singular of cofinality $\aleph_0$. In this model we naturally obtain club guessing on $S^{\mu^+}_{\aleph_0}$ for order type $\mu$, and this will prevent the prototype of a stable unsuperstable theory $\mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ from having a small universal family, see [Sh 457], [KjSh 447]. Hence the universality spectrum at such $\mu^+$ classifies the prototype of a simple unstable theory (the theory of a random graph), as less complicated than the prototype of a stable unsuperstable theory, contrary to the expectation. A possible conclusion is that in order to obtain a classification into as few as possible nicely defined classes one should concentrate the investigation of the universality spectrum as a dividing line for unstable theories only on the case $\lambda^+$ with $\lambda = \lambda^{<\lambda}$, as the case of the successor of a singular is too sensitive to the set theory involved. Perhaps just working with $\lambda^+$ where $\lambda = \lambda^{|T|}$ is a reasonable (as this rules out this particular example), or simply admitting the possibility of $\mathrm{Th}({}^\omega\omega, E_n)_{n<\omega}$ and the theory of the random graph being incompatible is a possible approach.

There are several further questions that this paper brings to mind. From the point of view of model theory it would be interesting to determine which other first order theories fit the scheme of this paper and from the point of view of graph theory one would like to improve the result on the existence of $\mu^{++}$ jointly universal graphs to having just one universal graph. Set-theoretically, we would like to be able to replace $\mu$ an unspecified singular strong limit by $\mu = \beth_\omega$, as well as to investigate singulars of different cofinality than $\aleph_0$. We did not concentrate here on obtaining the right consistency strength for our results, suggesting another question that may be addressed in the future work.

The paper is organised as follows. The major issue is to define the iteration used in the second step of the above scheme, which is done in certain generality in §1. We give there a sufficient condition for a one step forcing to fit the general scheme, so obtaining an axiomatic version of the method. In §2 we give the application to the existence of $\mu^{++}$ universal graphs of size $\mu^+$ for $\mu$ the successor of a strong limit singular of cofinality $\aleph_0$.

Most of our notation is entirely standard, with the possible exception of

**Notation 0.1.** For $\alpha$ and ordinal and a regular cardinal $\kappa < \alpha$, we let

$$S^\alpha_\kappa \stackrel{\rm def}{=} \{\beta < \alpha : \operatorname{cf}(\beta) = \kappa\}.$$

# 1 The general framework for forcing

**Definition 1.1.** Suppose that $\kappa$ is a strongly inaccessible cardinal $> \aleph_0$. A function $h : \kappa \to \mathcal{H}(\kappa)$ is called *Laver's diamond on* $\kappa$ iff for every $x$ and $\lambda$, there is an elementary embedding $\mathbf{j} : V \to M$ with

(1) $\operatorname{crit}(\mathbf{j}) = \kappa$ and $\mathbf{j}(\kappa) > \lambda$,

(2) ${}^\lambda M \subseteq M$,

(3) $(\mathbf{j}(h))(\kappa) = x$.

**Theorem 1.2.** Laver ([La]) Suppose that $\kappa$ is a supercompact cardinal. Then there is a Laver's diamond on $\kappa$.

**Hypothesis 1.3.** We work in a universe $V$ that satisfies

(1) $\kappa$ is a supercompact cardinal, $\theta = \operatorname{cf}(\theta) \geq \kappa^+$ and $GCH$ holds at and above $\kappa$,

(2) $\Upsilon^\theta = \Upsilon$ & $\chi = \Upsilon^+$ and

(3) $h : \kappa \to \mathcal{H}(\kappa)$ is a Laver's diamond.

**Remark 1.4.** (1) It is well known that the consistency of the above hypothesis follows from the consistency of the existence of a supercompact cardinal. We in fact only use the $\chi$-supercompactness of $\kappa$.
(2) With minor changes, one may replace $\chi = \Upsilon^+$ by $\chi$ being strongly inaccessible $> \theta$.

**Definition 1.5.** Laver ([La]) We define

$$\bar{R} = \langle R^+_\alpha, \underset{\sim}{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle,$$

an iteration done with Easton supports, and a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \kappa \rangle$ of cardinals, where $\underset{\sim}{R}_\alpha$ and $\lambda_\alpha$ are defined by induction on $\alpha < \kappa$ as follows.

If

(1) $h(\alpha) = (\underset{\sim}{P}, \lambda)$, where $\lambda$ and $\alpha$ are cardinals and $\underset{\sim}{P}$ is a $R^+_\alpha$-name of $(< \alpha)$-directed-closed forcing, and

(2) $(\forall \beta < \alpha)\, [\lambda_\beta < \alpha]$,

we let $\underset{\sim}{R}_\alpha \stackrel{\text{def}}{=} \underset{\sim}{P}$ and $\lambda_\alpha \stackrel{\text{def}}{=} \lambda$. Otherwise, let $\underset{\sim}{R}_\alpha \stackrel{\text{def}}{=} \{\emptyset\}$ and $\lambda_\alpha \stackrel{\text{def}}{=} \sup_{\beta<\alpha} \lambda_\beta$.

The extension in $R^+_\alpha$ is defined by letting

$$p \leq q \iff [\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)\ \&\ (\forall i \in \mathrm{Dom}(q))(q \restriction i \Vdash \text{“}p(i) \leq q(i)\text{”})],$$

(where $p$ denotes the weaker condition).

**Remark 1.6.** The forcing $R^+_\kappa$ used in this section is Laver's forcing from [La] which makes the supercompactness of $\kappa$ indestructible under any $(< \kappa)$-directed-closed forcing.

**Convention 1.7.** Definitions 1.9 and 1.12, Claim 1.13 and Observation 1.14 take place in $V_1 \stackrel{\text{def}}{=} V^{R^+_\kappa}$.

**Observation 1.8.** $\kappa^+ \leq \mathrm{cf}(\theta) = \theta < \chi = \Upsilon^+$, $2^\sigma = \sigma^+$ for $\sigma \geq \kappa$ and $\Upsilon^\theta = \Upsilon$ still hold in $V_1$, as $\mathrm{Rang}(h) \subseteq \mathcal{H}(\kappa)$, and $\kappa$ is still supercompact.

**Definition 1.9.** By induction on $i^* < \chi$ we define the family $\mathcal{K}_\theta^{i^*}$ as the family of all sequences

$$\bar{Q} = \langle P_i, \underset{\sim}{Q}_i, \underset{\sim}{A}_i : i < i^* = \lg(\bar{Q}) \rangle$$

which satisfy the following inductive definition, and we let $\mathcal{K}_\theta \stackrel{\text{def}}{=} \bigcup_{i<\chi} = \mathcal{K}_\theta^i$.

(1) $P_i \subseteq \mathcal{H}(\chi)$ (and each $P_i$ is a forcing notion, which will follow from the rest of the definition),

(2) each $P_i$ satisfies the $\chi$-cc and for $i \leq j$ the forcing notion $P_i$ is completely embedded into $P_j$ by the identity function,

(3) $\underset{\sim}{Q}_i$ is a $P_i$-name of a forcing notion (hence a partial order with the least element $\emptyset_{\underset{\sim}{Q}_i}$) which is a member of $\underset{\sim}{\mathcal{H}}(\chi)$ (hence of cardinality $\leq |\Upsilon|$),

(4) If $\mathrm{cf}(i) \geq \kappa$, then $P_i = \bigcup_{j<i} P_j$,

(5) $\underset{\sim}{A}_i$ is a canonical $P_{i+1}$-name of a subset of $\kappa$,

(6) Letting $G_i$ be $P_i$-generic over $V_1$, then in $V_1[G_i]$,

(a) we let $\mathrm{NUF} \stackrel{\text{def}}{=} \{\mathcal{D} : \mathcal{D}$ a normal ultrafilter on $\kappa\}$, and

(b) for every $\mathcal{D} \in \mathrm{NUF}$ we are given a $(< \kappa)$-directed-closed forcing notion $Q^i_{\mathcal{D}} \in \mathcal{H}(\chi)^{V_1[G_i]}$ whose minimal element is denoted by $\emptyset_{Q^i_{\mathcal{D}}}$,

(7) With the notation of (6), we have that $\underset{\sim}{Q}_i[G_i]$ is

$$\{\emptyset\} \cup \mathrm{NUF} \cup \left\{ \{\mathcal{D}\} \times Q^i_{\mathcal{D}} : \mathcal{D} \in \mathrm{NUF} \right\},$$

(8) The order on $\underset{\sim}{Q}_i[G_i]$ is given by letting

$$x \leq y \text{ iff } [x = y \text{ or } x = \emptyset \text{ or } (x = \mathcal{D} \in \mathrm{NUF} \ \& \ y \in \{x\} \times Q^i_{\mathcal{D}}) \text{ or}$$

$$x = (\mathcal{D}, x^*), y = (\mathcal{D}, y^*) \text{ for some } \mathcal{D} \in \mathrm{NUF} \text{ and } Q^i_{\mathcal{D}} \models \text{“}x^* \leq y^*\text{”}],$$

(9) We have (we adopt the usual meaning of "canonical" below, see [Sh -f], I. 5.12. for the exact definition)

$$P_i \stackrel{\text{def}}{=} \left\{ p : \begin{array}{l} \text{(i) } p \text{ is a function with domain } \subseteq i \\ \text{(ii) } j \in \mathrm{Dom}(p) \implies p(j) \text{ is a canonical } P_j\text{-name} \\ \quad \text{of a member of } \underset{\sim}{Q}_j \\ \text{(iii) } |S\mathrm{Dom}(p)| < \kappa \text{ (see below)} \end{array} \right\},$$

ordered by letting

$$p \leq q \iff [\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q) \;\&\; (\forall i \in \mathrm{Dom}(q))(q \restriction i \Vdash p(i) \leq q(i))],$$

where

(*Definition 1.9 continues below*)

**Notation 1.10.** (A) For $i < i^*$, and $p \in P_i$, we let the essential domain of $p$ be

$$S\mathrm{Dom}(p) \stackrel{\text{def}}{=} \left\{ j \in \mathrm{Dom}(p) : \neg \left[ p \restriction j \Vdash_{P_j} \begin{array}{c} \text{“}p(j) \in \{\emptyset\} \cup \mathrm{N\underset{\sim}{U}F} \cup \\ \{(\underset{\sim}{\mathcal{D}}, \emptyset_{Q^j_{\underset{\sim}{\mathcal{D}}}}) : \underset{\sim}{\mathcal{D}} \in \mathrm{N\underset{\sim}{U}F}\}\text{”} \end{array} \right] \right\},$$

(B) For $i < i^*$ and $p \in P_i$ we call $p$ *purely full in* $P_i$ iff:

$S\mathrm{Dom}(p) = \emptyset$ and for every $j < i$ we have

$$p \restriction j \Vdash_{P_j} \text{“}p(j) \in \mathrm{N\underset{\sim}{U}F}\text{”}.$$

If $i$ is clear from the context we may say that $p$ is *purely full in its domain.*

(C) Suppose that $i < i^*$ and $p \in P_i$ is purely full in $P_i$, we define

$$P_i/p \stackrel{\text{def}}{=} \left\{ q \in P_i : \begin{array}{l} p \leq q \;\&\; \text{for each } j < i, \\ q \restriction j \Vdash \text{“}q(j) \text{ is of the form } (\underset{\sim}{\mathcal{D}}, x) \text{ for some } x\text{”} \end{array} \right\},$$

with the order inherited from $P_i$.

(*Definition 1.9 continues:*)

(10) For every $i \leq i^*$ and $p \in P_i$ which is purely full in $P_i$ we have that $P_i/p$ satisfies $\theta$-cc and $P_i/p \in \mathcal{H}(\chi)$.

**Observation 1.11.** If $\bar{Q} \in \mathcal{K}_\theta$ and $i < \lg(\bar{Q})$, <u>then</u> $P_{i+1} = P_i * \underset{\sim}{Q}_i$.

**Definition 1.12.** (1) We define the family $\mathcal{K}_\theta^+$ as the family of all sequences

$$\bar{Q} = \langle P_i, \underset{\sim}{Q}_i, \underset{\sim}{A}_i : i < \chi \rangle$$

such that

$$i < \chi \implies \bar{Q} \restriction i \in \mathcal{K}_\theta.$$

We let $P_\chi \stackrel{\text{def}}{=} \bigcup_{i<\chi} P_i$.

(2) Suppose that $\bar{Q} \in \mathcal{K}_\theta^+$ and $\langle p_i : i < \chi \rangle$ with $p_i \in P_{\zeta_i}$ are purely full in $P_{\zeta_i}$ and increasing in $P_\chi$, where $\zeta_i \stackrel{\text{def}}{=} \min\{\zeta : p_i \in P_\zeta\}$ (so if $i < j$ then $p_i = p_j \restriction \zeta_i$). We define

$$P_\chi / \cup_{i<\chi} p_i \stackrel{\text{def}}{=} \{q \in P_\chi : (\forall i < \chi)[q \restriction \zeta_i \in P_{\zeta_i}/p_i]\},$$

with the order inherited from $P_\chi$.

**Claim 1.13.** (1) If $\bar{Q} \in \mathcal{K}_\theta$, <u>then</u> for all $i \leq \lg(\bar{Q})$, we have that $P_i$ is $(< \kappa)$-directed-closed.

(2) Similarly for $\bar{Q} \in \mathcal{K}_\theta^+$.

**Proof of the Claim.** (1) Given a directed family $\{p_\alpha : \alpha < \alpha^* < \kappa\}$ of conditions in $P_i$, we shall define a common extension $p$ of this family. Let us first let $\mathrm{Dom}(p) \stackrel{\text{def}}{=} \bigcup_{\alpha<\alpha^*} \mathrm{Dom}(p_\alpha)$. For $j \in \mathrm{Dom}(p)$, we define $p(j)$ by induction on $j$. We work in $V_1^{P_j}$ and assume that $\{p_\alpha \restriction j : \alpha < \alpha^*\} \subseteq G_{P_j}$.

If $j \notin \bigcup_{\alpha<\alpha^*} S\mathrm{Dom}(p_\alpha)$, then notice that there is at most one $\mathcal{D} \neq \emptyset$ such that for some (possibly more than one) $\alpha < \alpha^*$ we have $p_\alpha \restriction j \Vdash$ "$p_\alpha(j) = \underset{\sim}{\mathcal{D}}$", as the family is directed. If there is such $\mathcal{D}$, we let $p(j) \stackrel{\text{def}}{=} \mathcal{D}$, otherwise we let $p(j) = \emptyset$.

If $j \in \bigcup_{\alpha<\alpha^*} S\mathrm{Dom}(p_\alpha)$, similarly to the last paragraph, we conclude that there is a name $\underset{\sim}{\mathcal{D}}$ such that

$$[\alpha < \alpha^* \ \& \ j \in S\mathrm{Dom}(p_\alpha)] \implies p_\alpha \restriction j \Vdash \text{“}p_\alpha(j) \in \{\underset{\sim}{\mathcal{D}}\} \times \underset{\sim}{Q}^j_{\mathcal{D}}\text{”}.$$

As $\underset{\sim}{Q}^j_{\mathcal{D}}$ is forced to be $(<\kappa)$-directed-closed (see (6)(b) of Definition 1.9), we can find in $V_1^{P_j}$ a condition $q$ such that $q \geq p_j(\alpha)$ for all $\alpha < \alpha^*$ such that $j \in S\mathrm{Dom}(p_\alpha)$. Let $p(j) \stackrel{\text{def}}{=} (\underset{\sim}{\mathcal{D}}, q)$ for some such $q$.
(2) Follows from (1) as $\chi = \mathrm{cf}(\chi) > \kappa$. $\bigstar_{1.13}$

**Observation 1.14.** Suppose that $\bar{Q} \in \mathcal{K}^+_\theta$, $i < j < \chi$ and $p \in P_i$, $q \in P_j$ are purely full in their respective domains, while $p \leq q$. Then

(1) $\mathrm{Dom}(p) = i \subseteq j = \mathrm{Dom}(q)$ and $\alpha \in \mathrm{Dom}(p) \implies p(\alpha) = q(\alpha)$.

(2) Suppose that $r \in P_i/p$.

Then defining $r + q \in P_j$ by letting $\mathrm{Dom}(r+q) = \mathrm{Dom}(q)$ and letting for $\alpha \in \mathrm{Dom}(r)$

$$(r+q)(\alpha) \stackrel{\text{def}}{=} \begin{cases} r(\alpha) & \text{if } \alpha \in \mathrm{Dom}(p) \\ q(\alpha) & \text{otherwise,} \end{cases}$$

we obtain a condition in $P_j/q$.

(3) For $r_1, r_2 \in P_i/p$ we have that

($\alpha$) $r_1$ and $r_2$ are incompatible in $P_i/p$ iff $r_1 + q$ and $r_2 + q$ are incompatible in $P_j/q$,

($\beta$) $r_1 \leq_{P_i/p} r_2 \iff r_1 + q \leq_{P_j/q} r_2 + q$.

(4) $P_i/p \lessdot_f P_j/q$ where $f(r) \stackrel{\text{def}}{=} r + q$. We also write $f = f_{p,q}$.

(5) Suppose that the sequence $\bar{p} = \langle p_i : i < \chi \rangle$ satisfies that each $p_i \in P_{\zeta_i}$ is purely full in $P_{\zeta_i}$, and the sequence $\bar{p}$ is increasing in $P_\chi$, where

$$\zeta_i \stackrel{\text{def}}{=} \min\{\zeta : p_i \in P_\zeta\}$$

and $\langle \zeta_i : i < \chi \rangle$ is strictly increasing. Then $P^* = P_\chi / \cup_{i<\chi} p_i$ is isomorphic to the limit of a $(<\kappa)$-supported iteration of $(<\kappa)$-directed-closed $\theta$-cc forcing.

(6) For every $r \in P_\chi$, there is $q \geq r$ with $S\mathrm{Dom}(q) = S\mathrm{Dom}(r)$ and $p$ purely full in some $P_i$, such that $q \in P_i/p$.

**Convention 1.15.** This Convention applies to Observation 1.14(5) above.
(a) Justified by Observation 1.14(5), in the case that each $\zeta_i = \xi_i + 1$ in the sequel we may abuse the notation and write

$$P^* \approx \lim\langle P_{\xi_i}/(p_i \restriction \xi_i), \underset{\sim}{Q}^{\xi_i}_{p_i(\xi_i)} : i < \chi\rangle,$$

even though this is not literally an iteration of forcing (since the iterands are not specified at each coordinate). We do this to emphasize the sequence $\langle \underset{\sim}{Q}^{\xi_i}_{p_i(\xi_i)} : i < \chi\rangle$, whose importance will become clear in the Main Claim 1.18.
(b) Since $f_{p,q}$ are usually clear form the context we simplify the notation by not mentioning these functions explicitly.

**Claim 1.16.** Suppose that $\bar{Q} \in \mathcal{K}^+_\theta$ and $\underset{\sim}{t}$ is a $P_\chi$-name of an ordinal, while $p \in P_\chi$ is purely full in its domain.

Then for some $j < \chi$ and $q$ we have $p \leq q \in P_j$, and $q$ is purely full in $P_j$, and above $q$ we have that $\underset{\sim}{t}$ is a $P_j$-name (i.e $\underset{\sim}{t}$ is a $P_j/q$-name).

**Proof of the Claim.** Given $p \in P_\chi$ purely full in its domain, and suppose that the conclusion fails. Let $i < \chi$ be such that $p \in P_i$. We shall choose by induction on $\zeta < \theta$ ordinals $i_\zeta$ and $\gamma_\zeta$ and conditions $p_\zeta$ and $r_\zeta$ such that

(i) $i_\zeta \in [i, \chi)$ and $\langle i_\zeta : \zeta < \theta\rangle$ is increasing continuous,

(ii) $p_\zeta \in P_{i_\zeta}$ is purely full in $P_{i_\zeta}$, with $p_0 = p$ and $p_\zeta \leq p_\xi$ for $\zeta \leq \xi$,

(iii) $p_\zeta \leq r_\zeta$ with $r_\zeta \Vdash_{P_\chi}$ "$\underset{\sim}{t} = \gamma_\zeta$",

(iv) $r_\zeta$ is incompatible with every $r_\xi$ for $\xi < \zeta$,

(v) $p_\zeta \overset{\mathrm{def}}{=} \cup_{\xi<\zeta} p_\xi$ for $\zeta$ a limit,

(vi) $r_\zeta \in P_{i_{\zeta+1}}/p_{\zeta+1}$.

We now explain how to do this induction.

Given $p_\zeta$ and $i_\zeta$. Since we are assuming that $\underset{\sim}{t}$ is not a $P_{i_\zeta}$-name above $p_\zeta$, it must be possible to find $r_\zeta$ and $\gamma_\zeta$ as required. Having chosen $r_\zeta$, (by extending $r_\zeta$ if necessary), we can choose $p_{\zeta+1}$ as required in item (vi) above, see Observation 1.14(6). This determines $i_{\zeta+1}$. Note that $i_{\zeta+1} < \chi$ as $P_\chi \overset{\text{def}}{=} \bigcup_{j<\chi} P_j$.

However, completing the induction we arrive at a contradiction, as letting $p^* \overset{\text{def}}{=} \cup_{\zeta<\theta} p_\zeta$ we obtain a condition purely full in its domain. Hence $P \overset{\text{def}}{=} P_{\sup_{\zeta<\theta} i_\zeta}/p^*$ has $\theta$-cc, but $\{r_\zeta + p^* : \zeta < \theta\}$ forms a set of $\theta$ pairwise incompatible conditions in $P$. $\bigstar_{1.16}$

**Convention 1.17.** Now we go back to $V$, i.e. the Main Claim 1.18 takes place in $V$.

**Main Claim 1.18.** Suppose

($\alpha$) $\underset{\sim}{\bar{Q}} = \langle \underset{\sim}{P}_i, \underset{\sim}{Q}_i, \underset{\sim}{A}_i : i < \chi\rangle$ is an $R^+_\kappa$-name for a member of $\underset{\sim}{\mathcal{K}}^+_\theta$,

($\beta$) $\mathbf{j} : V \to M$ is an elementary embedding such that ${}^\Upsilon M \subseteq M$, $\text{crit}(\mathbf{j}) = \kappa$, $\chi < \mathbf{j}(\kappa)$ and

$$(\mathbf{j}(h))(\kappa) = (\underset{\sim}{P}_\chi, \chi)$$

(such a choice is possible by the definition of Laver's diamond) .

Considering $\mathbf{j}(\langle R^+_\alpha, \underset{\sim}{R}_\alpha, \lambda_\alpha : \alpha < \kappa\rangle)$ in $M$ (as for $\beta < \kappa$ we know that $\langle R^+_\alpha, \underset{\sim}{R}_\alpha, \lambda_\alpha : \alpha < \beta\rangle \in \mathcal{H}(\chi)$), by its definition we see that

$$\mathbf{j}(\langle R^+_\alpha, \underset{\sim}{R}_\alpha, \lambda_\alpha : \alpha < \kappa\rangle) = \langle R^+_\alpha, \underset{\sim}{R}_\alpha, \lambda_\alpha : \alpha < \mathbf{j}(\kappa)\rangle$$

and $\underset{\sim}{R}_\kappa = \underset{\sim}{P}_\chi$ while $\lambda_\kappa = \chi$. Hence $\mathbf{j}(R^+_\kappa) = R^+_\kappa * \underset{\sim}{P}_\chi * \underset{\sim}{R}^*$ for some $R^+_\kappa * \underset{\sim}{P}_\chi$-name $\underset{\sim}{R}^* \in M$ for a forcing notion, which is forced to be $(< \chi)$-directed-closed.

We also let

$$\underset{\sim}{\bar{Q}}' = \langle \underset{\sim}{P}'_i, \underset{\sim}{Q}'_i, \underset{\sim}{A}'_i : i < \mathbf{j}(\chi)\rangle \overset{\text{def}}{=} \mathbf{j}(\langle \underset{\sim}{P}_i, \underset{\sim}{Q}_i, \underset{\sim}{A}_i : i < \chi\rangle).$$

Then in $V^{R^+_\kappa}$, the following holds: we can find $\bar{\alpha} = \langle \alpha_i : i < \chi\rangle$, $\bar{p}^* = \langle p^*_i : i < \chi\rangle$ and $\bar{q}^* = \langle q^*_i = ({}^1q_i, {}^2q_i) : i < \chi\rangle$ such that

(a) $\langle \alpha_i : i < \chi \rangle$ is strictly increasing continuous and each $\alpha_i < \chi$,

(b) $p^*_i \in P_{\alpha_i+1}$ is purely full in $P_{\alpha_i+1}$,

(c) $\bar{p}^*$ is increasing in $P_\chi$,

(d) For every $i < \chi$, we have $\bar{q}^* \restriction i \in M^{R^+_\kappa}$, and in $M^{R^+_\kappa}$ we have

$$(p^*_i, {}^1q_i, {}^2q_i) \in P_\chi * \underset{\sim}{R}^* * \underset{\sim}{P}'_{\mathbf{j}(\alpha_i+1)},$$

while $(p^*_i, {}^1q_i) \in P_\chi * \underset{\sim}{R}^*$,

(e) In $M^{R^+_\kappa}$ we have that for $\gamma < \chi$

$$\langle (p^*_i, {}^1q_i, {}^2q_i) : i < \gamma \rangle \text{ is increasing in } P_\chi * \underset{\sim}{R}^* * \underset{\sim}{P}'_{\sup_{i<\gamma} \mathbf{j}(\alpha_i+1)},$$

(f) In $M^{R^+_\kappa}$, it is forced by $(p^*_{i+1}, {}^1q_{i+1})$ that ${}^2q_{i+1}$ is an upper bound to

$$\{\mathbf{j}(r) : r \in \underset{\sim}{G}_{P_{\alpha_i} * \underset{\sim}{Q}^{\alpha_i+1}_{p^*_i(\alpha_i)}}\},$$

(g) If $\underset{\sim}{B}$ is an $R^+_\kappa$-name of a $P_{\alpha_i+1}$-name of a subset of $\kappa$, <u>then</u> for some $R^+_\kappa * \underset{\sim}{P}_\chi$-name $\underset{\sim}{\mathbf{t}}_B$ for a truth value (i.e. an ordinal $\in \{0,1\}$, 1 standing for "true" and 0 for "false"):

(1) In $V$ we have that $(\emptyset_{R^+_\kappa}, p^*_{i+1})$ forces $\underset{\sim}{\mathbf{t}}_B$ to be a $P_{\underset{\sim}{\alpha}_{i+1}+1}/p^*_{i+1}$-name,

(2) $M \models [(\emptyset_{R^+_\kappa}, p^*_{i+1}, q^*_{i+1}) \Vdash$ "$\kappa \in \mathbf{j}(\underset{\sim}{B})$ iff $\underset{\sim}{\mathbf{t}}_B = 1$"$]$.

(h) In $M^{R^+_\kappa}$, either

$$(p^*_{i+1}, q^*_{i+1}) \Vdash \text{“}\kappa \in \mathbf{j}(\underset{\sim}{A}_{\alpha_i})\text{”},$$

or $p^*_i \Vdash_{P_\chi}$ that

$$\text{“ there is no } q = ({}^1q, {}^2q) \geq_{\underset{\sim}{R}^* * \underset{\sim}{P}'_{\mathbf{j}(\alpha_i)+1}} q^*_i \text{ with}$$
$$ {}^1q \Vdash_{\underset{\sim}{R}^*} \text{“}{}^2q(\mathbf{j}(\alpha_i)) \geq_{\underset{\sim}{P}'_{\mathbf{j}(\alpha_i)+1}} \{\mathbf{j}(r) : r \in \underset{\sim}{G}_{P_{\alpha_i} * \underset{\sim}{Q}^{\alpha_i}_{p^*_i(\alpha_i)}}\}$$
$$\text{and } \kappa \in \mathbf{j}(\underset{\sim}{A}_{\alpha_i})\text{””}.$$

[Note that $\mathbf{j}(\underset{\sim}{A}_{\alpha_i})$ is a $P'_{\mathbf{j}(\alpha_i)+1}$-name for a subset of $\mathbf{j}(\kappa)$.]

(i) If $\mathrm{cf}(i) \geq \theta$, <u>then</u> in $V^{R^+_\kappa * \underset{\sim}{P}_{\alpha_i}}$ we have $p^*_i(\alpha_i) \in \mathrm{NUF}$ and specifically

$$p^*_i(\alpha_i) = \left\{ \underset{\sim}{B}[G_{P_{\alpha_i}}] : \begin{array}{c} \underset{\sim}{B} \text{ is a } P_{\alpha_i}/(p^*_i \restriction \alpha_i)\text{-name for a subset of } \kappa \\ \text{and } \underset{\sim}{\mathbf{t}}_{\underset{\sim}{B}}[G_{P_{\alpha_i}}] = 1 \end{array} \right\}.$$

**Proof of the Main Claim.** Consider $\langle R^+_i, \underset{\sim}{R}_i, \lambda_i : \kappa < i < \mathbf{j}(\kappa)\rangle$ over $R^+_\kappa * \underset{\sim}{P}_\chi$ in $M$. By the inductive definition of $R_i$ (see Definition 1.5), which is preserved by $\mathbf{j}$, we have that $\underset{\sim}{R}_i$ is a name for the trivial forcing whenever $i$ is not such that $(\forall \beta < i)\lambda_\beta < i$. Since $(\mathbf{j}(h))(\kappa) = (\underset{\sim}{P}_\chi, \chi)$ we have that for every $i$ satisfying $\kappa < i < \chi$, $\underset{\sim}{R}_i$ is a name for the trivial forcing. For $\chi \leq i < \mathbf{j}(\kappa)$, we have that $\underset{\sim}{R}_i$ is a name for a $(< \chi)$-directed-closed forcing in $M$, so in $V$ as well, as ${}^{<\chi}M \subseteq M$. Similarly we conclude that $\underset{\sim}{P}'_{\mathbf{j}(\zeta)}$ names a $(< \chi)$-directed-closed forcing notion, for all $\zeta < \chi$. This observation will be used repeatedly and in particular will enable us to use the master condition idea in the induction below. In particular, we can conclude that $\underset{\sim}{R}_\chi$ is $(< \chi)$-directed-closed. By the choice of $\mathbf{j}$,

$$\Vdash_{\mathbf{j}(R^+_\kappa)} \text{“each } P'_i/p \text{ is } (< \chi)\text{-directed-closed for } p \in P'_i \text{ purely full in } P'_i.\text{”}$$

Now we choose $(\alpha_i, p^*_i, q^*_i) \in M^{R^+_\kappa}$ by an induction on $i$ carried in $V$. We start with $\alpha_0 = 0$, $p^*_0 \in P_1$ any condition purely full in $P_1$, and $q^*_0 = \emptyset$.

<u>Choice of $p^*_{i+1}, q^*_{i+1}$ and $\alpha_{i+1}$.</u>

Given $p^*_i$ and $\alpha_i$ in $V^{R^+_\kappa}$. We have (recall Convention 1.15) that

$$p^*_i \restriction \alpha_i \Vdash_{P_{\alpha_i}} \text{“}|\underset{\sim}{Q}^{\alpha_i}_{p^*_i(\alpha_i)}| \leq \Upsilon \ \&\ \underset{\sim}{G}_{\underset{\sim}{Q}^{\alpha_i}_{p^*_i(\alpha_i)}} \subseteq \underset{\sim}{P}_{\alpha_i+1}/p^*_i.\text{”}$$

Hence in $M$, letting $\underset{\sim}{X}_i \overset{\text{def}}{=} \{\mathbf{j}(r) : r \in \underset{\sim}{G}_{P_{\alpha_i} * \underset{\sim}{Q}^{\alpha_i}_{p^*_i(\alpha_i)}}\}$ we have

$$(\emptyset_{R^+_{\mathbf{j}(\kappa)}}, \mathbf{j}(p^*_i \restriction \alpha_i)) \Vdash_{P'_{\mathbf{j}(\alpha_i)}} \text{“}\underset{\sim}{X}_i \subseteq \underset{\sim}{P}'_{\mathbf{j}(\alpha_i)+1} \text{ is directed and } \mathbf{j}(p^*_i(\alpha_i)) \Vdash |\underset{\sim}{X}_i| \leq \Upsilon.\text{”}$$

In $V_1$, we have that the forcing $P_{\alpha_i+1}/p^*_i$ is a $\theta$-cc forcing notion of size $\leq \Upsilon$, hence there are $\leq \Upsilon^\theta \cdot \Upsilon = \Upsilon$ canonical $P_{\alpha_i+1}/p^*_i$-names for a subset of $\kappa$. Let us enumerate them in a limit order type as $\langle \underset{\sim}{B}^{i+1}_\zeta : \zeta < \zeta^*(i+1) \leq \Upsilon\rangle$, with

$$\underset{\sim}{B}^{i+1}_0 = \underset{\sim}{A}_i. \qquad (*)$$

This choice for $i+1$ helps us to fulfill clause (h) for $i$. By induction on $\zeta < \zeta^*(i+1)$ we choose $p^{i+1}_\zeta$ purely full in its domain, increasing continuous with $\zeta$, $q^{i+1}_\zeta = ({}^1q^{i+1}_\zeta, {}^2q^{i+1}_\zeta)$ increasing with $\zeta$, $\alpha^{i+1}_\zeta$ increasing with $\zeta$ and $\underset{\sim}{\mathbf{t}}_{\underset{\sim}{B}^{i+1}_\zeta}$ as follows.

Let $p^{i+1}_0 \overset{\text{def}}{=} p^*_i$, $\alpha^{i+1}_0 \overset{\text{def}}{=} \alpha_i$ and $q^{i+1}_0 \overset{\text{def}}{=} q^*_i$.

Coming to $\zeta+1$, let $G$ be a $R^+_\kappa * \underset{\sim}{P}_\chi$ generic over $V$ such that $(\emptyset_{R^+_\kappa}, p^{i+1}_\zeta) \in G$. In $M[G]$ we ask "the $\zeta$-question":

Is it true that there is no $q$ satisfying the following condition $(**)_q$, which means

$(\alpha)$ $q = ({}^1q, {}^2q) \geq_{\underset{\sim}{R}^* * \underset{\sim}{P}'_{\mathbf{j}(\alpha^{i+1}_\zeta)+1}} \{q^{i+1}_\xi : \xi \leq \zeta\}$ and

$(\beta)$ ${}^1q \Vdash_{\underset{\sim}{R}^*}$ "${}^2q \geq \underset{\sim}{X}_i$ & $\kappa \in \mathbf{j}(\underset{\sim}{B}^{i+1}_\zeta)$" & ${}^2q \in \underset{\sim}{P}'_{\mathbf{j}(\alpha^{i+1}_\zeta)+1}/(\mathbf{j}(p^{i+1}_\zeta) \restriction \mathbf{j}(\alpha^{i+1}_\zeta)+1)$?

(Here ${}^2q \geq \underset{\sim}{X}_i$ means that ${}^2q$ is above every condition in $\underset{\sim}{X}_i$, which then guarantees that clause (f) is satisfied).

<u>Case 1</u>. If the answer is positive, i.e. for no $q$ do we have that $(**)_q$ holds in $M$, we define $\underset{\sim}{\mathbf{t}}_{\underset{\sim}{B}^{i+1}_\zeta} \overset{\text{def}}{=} 0$ (hence a $R^+_\kappa * \underset{\sim}{P}_\chi$-name for a truth value), and define

$$q^{i+1}_{\zeta+1} = ({}^1q^{i+1}_{\zeta+1}, {}^2q^{i+1}_{\zeta+1})$$

to be any $R^+_\kappa * \underset{\sim}{P}_\chi$-name for a condition in $\underset{\sim}{R}^* * \underset{\sim}{P}'_{\mathbf{j}(\chi)}$ such that

$$(\emptyset_{R^+_\kappa}, p^{i+1}_\zeta) \Vdash \text{“}{}^1q^{i+1}_{\zeta+1} \geq_{\underset{\sim}{R}^*} {}^1q^{i+1}_\xi\text{”}$$

for every $\xi \leq \zeta$, and

$$(\emptyset_{R^+_\kappa}, p^{i+1}_\zeta, {}^1q^{i+1}_{\zeta+1}) \Vdash_{\underset{\sim}{P}'_{\mathbf{j}(\alpha^{i+1}_\zeta)+1}} \text{“}{}^2q^{i+1}_{\zeta+1} \geq \cup\{{}^2q^{i+1}_\xi : \xi \leq \zeta\} \ \&\ {}^2q^{i+1}_{\zeta+1} \geq \underset{\sim}{X}_i\text{”}.$$

The choice of ${}^1q^{i+1}_{\zeta+1}$ is possible by the induction hypothesis and the fact that

$$\Vdash_{R^+_\kappa * \underset{\sim}{P}_\chi} \text{“}\underset{\sim}{R}^* \text{ is } (<\chi)\text{-directed-closed”}.$$

Let us verify that the choice of ${}^2q^{i+1}_{\zeta+1}$ is possible. Working in $M$ we have that $(\emptyset_{R^+_\kappa}, p^{i+1}_\zeta, {}^1q^{i+1}_{\zeta+1})$ forces $\underset{\sim}{X}_i$ to be a $(<\kappa)$-directed subset of $\underset{\sim}{P}'_{\mathbf{j}(\chi)}$ of size $<\chi$. Hence if $\zeta = 0$ we can choose ${}^2q^{i+1}$ to be forced to be above $\underset{\sim}{X}_i$. We can similarly choose ${}^2q^{i+1}_{\zeta+1}$ for $\zeta > 0$.

Case 2. If the answer to the $\zeta$ question is negative, so there is $q$ satisfying $(**)_q$, we let $\mathbf{t}_{\underset{\sim}{B}^{i+1}_\zeta} \stackrel{\text{def}}{=} 1$ and choose $q^{i+1}_{\zeta+1} = ({}^1q^{i+1}_{\zeta+1}, {}^2q^{i+1}_{\zeta+1})$ in $M$ exemplifying the negative answer.

At any rate, $\mathbf{t}_{\underset{\sim}{B}^{i+1}_\zeta}$ is a $R^+_\kappa * \underset{\sim}{P}_\chi$-name for an ordinal. By Claim 2.19, in $V^{R^+_\kappa}$ there is $\alpha^{i+1}_{\zeta+1} \geq \alpha^{i+1}_\zeta$ and a purely full in its domain $p^{i+1}_{\zeta+1} \geq p^{i+1}_\zeta$ with $p^{i+1}_{\zeta+1} \in P_{\alpha^{i+1}_{\zeta+1}}$ such that $\mathbf{t}_{\underset{\sim}{B}^{i+1}_\zeta}$ is a $P_{\alpha^{i+1}_{\zeta+1}}/p^{i+1}_{\zeta+1}$-name.

For $\zeta$ limit, let $\alpha^{i+1}_\zeta \stackrel{\text{def}}{=} \sup_{\xi<\zeta} \alpha^{i+1}_\xi$, $p^{i+1}_\zeta \stackrel{\text{def}}{=} \cup_{\xi<\zeta} p^{i+1}_\xi$, and $q^{i+1}_\zeta$ not defined.

At the end, we let $\alpha_{i+1} \stackrel{\text{def}}{=} \sup_{\zeta<\zeta^*(i+1)} \alpha^{i+1}_{\zeta+1}$ and $p^*_{i+1}$ any purely full condition in $P_{\alpha_{i+1}+1}$ with $p^*_{i+1} \geq \bigcup_{\zeta<\zeta^*(i+1)} p^{i+1}_\zeta$, and $q^*_{i+1}$ such that

$$(\emptyset_{R^+_\kappa}, p^*_{i+1}) \Vdash \text{“}q^*_{i+1} \geq_{\underset{\sim}{R}^* * \underset{\sim}{P}'_{\mathbf{j}(\alpha_{i+1})+1}} \{q^{i+1}_\zeta : \zeta < \zeta^*(i+1)\}\text{”}.$$

Choice of $p^*_i, q^*_i$ and $\alpha_i$ for $i < \chi$ limit. We let $\alpha_i \stackrel{\text{def}}{=} \sup_{j<i} \alpha_j$ and choose $p^*_i \in P_{\alpha_i+1}$ purely full so that $p^*_i \geq \cup_{j\leq i} p^*_j$, and if $\mathrm{cf}(i) \geq \theta$, then

$$p^*_i(\alpha_i) \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} \underset{\sim}{B}[G_{P_{\alpha_i}}] : & \underset{\sim}{B} \text{ is a } P_{\alpha_i}/(p^*_i \restriction \alpha_i)\text{-name for a subset of } \kappa \\ & \text{and } \mathbf{t}_{\underset{\sim}{B}}[G_{P_{\alpha_i}}] = 1 \end{array} \right\}.$$

Recall that ${}^{\chi>}M \subseteq M$ so $\langle(\alpha_j, p^*_j, q^*_j) : j < i\rangle \in M$. Condition (f) is satisfied by the definition of the order in $P_{\alpha_i}$ (and $\mathbf{j}(P_{\alpha_i})$). It follows by the construction and standard arguments about elementary embeddings and master conditions that

$$p^*_i \restriction \alpha_i \Vdash_{P_{\alpha_i}} \text{“}p^*_i(\alpha_i) \in \mathrm{N\underset{\sim}{U}F}\text{”}.$$

Then we can choose $q^*_i$ so that $(\emptyset_{R^+_\kappa}, p^*_i, q^*_i) \geq (\emptyset_{R^+_\kappa}, p^*_j, q^*_j)$ for all $j < i$ and $q^*_i \geq \{\mathbf{j}(r) : r \in G_{P_{\alpha_i}}\}$, which is again possible by the observation at the beginning of the proof. $\bigstar_{1.18}$

**Conclusion 1.19.** In $V_1$, if $\bar{Q} \in \mathcal{K}^+_\theta$, $\langle p^*_i : i < \chi\rangle$ and $\langle\alpha_i : i < \chi\rangle$ are as guaranteed by Main Claim 1.18, letting $\underset{\sim}{\mathcal{D}}_i \stackrel{\text{def}}{=} p^*_i(\alpha_i)$, it follows by Observation 1.14(5) that

$$\bar{P} = \langle P_{\alpha_i}/(p^*_i \restriction \alpha_i), \underset{\sim}{Q}^{\alpha_i}_{\underset{\sim}{\mathcal{D}}_i} : i < \chi\rangle$$

is an iteration (see Convention 1.15(a)) with $(<\kappa)$-supports of $(<\kappa)$-directed-closed $\theta$-cc forcing. In addition, there is a club $C$ of $\chi$ with the property that in $V_1^{P_\chi}$

$$\langle \mathcal{D}_i : i \in C \;\&\; \mathrm{cf}(i) \geq \theta \rangle$$

is an increasing sequence of normal filters over $\kappa$, with

$$[i \in C \;\&\; \mathrm{cf}(i) \geq \theta] \implies P_{\alpha_i}/(p^*_i \restriction \alpha_i) \Vdash \text{“}\underset{\sim}{\mathcal{D}}_i \text{ is an ultrafilter over } \kappa\text{”}.$$

If $\delta < \chi$ satisfies $\mathrm{cf}(\delta) > \kappa$ then $\cup_{i<\delta} p^*_i$ forces over $P_{\alpha_\delta}$ that $\bigcup_{i<\delta} \underset{\sim}{\mathcal{D}}_i$ us an ultrafilter over $\kappa$ which is generated by $\mathrm{cf}(\delta)$ sets.

**Definition 1.20.** (In $V^{R^+_\kappa}$) Given $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i, \underset{\sim}{A}_i : i < \chi \rangle \in \mathcal{K}^+_\theta$.

We say that $\bar{Q}$ is *fitted* iff there is a continuous increasing sequence $\langle \alpha_i : i < \chi \rangle$ of ordinals $< \chi$, and a sequence $\langle p^*_i : i < \chi \rangle$ of conditions each purely full in its domain with $p^*_i \in P_{\alpha_i+1}$, such that letting $\underset{\sim}{\mathcal{D}}_i \overset{\mathrm{def}}{=} p^*_{i(\alpha_i)}$,

$$\langle P_{\alpha_i+1}/(p^*_i \restriction \alpha_i), \underset{\sim}{Q}^{\alpha_i}_{\underset{\sim}{\mathcal{D}}_i} : i < \chi \rangle$$

is an iteration with $(<\kappa)$-supports of $(<\kappa)$-directed-closed $\theta$-cc forcing, <u>and</u>

$$\mathrm{cf}(i) \geq \theta \implies \Vdash_{P_{\alpha_i+1}/(p^*_i \restriction \alpha_i)} \text{“}\underset{\sim}{A}_i \in \underset{\sim}{\mathcal{D}}_i\text{”}.$$

**Crucial Claim 1.21.** (In $V^{R^+_\kappa}$) The following is a sufficient condition for $\underset{\sim}{\bar{Q}} \in \mathcal{K}^+_\theta$ to be fitted:

There is a pair $(\mathbf{R}, \mathbf{h})$ such that:

(1) $\mathbf{R}$ is a function such that for every forcing $\mathbb{P}$ with $|\mathbb{P}| \leq \Upsilon$ in $\mathcal{H}(\chi)$ and a $\mathbb{P}$-name $\underset{\sim}{\mathcal{D}}$ of a normal ultrafilter on $\kappa$ we have that $\mathbf{R}[\mathbb{P}, \underset{\sim}{\mathcal{D}}]$ is well defined and is a $\mathbb{P}$-name of a forcing notion of cardinality $\leq \Upsilon$,

(2) for every purely full in its domain $p \in P_\chi$ and $i \in \mathrm{Dom}(p)$, we have that

$$p \restriction i \Vdash \text{“}\underset{\sim}{Q}^i_{p(i)} = \mathbf{R}[P_i/(p \restriction i), p(i)]\text{”},$$

(3) $\mathbf{h}$ is a function such that for every forcing $\mathbb{P}$ with $|\mathbb{P}| \leq \Upsilon$ in $\mathcal{H}(\chi)$ satisfying the $\theta$-cc and a $\mathbb{P}$-name $\underset{\sim}{\mathcal{D}}$ of a normal ultrafilter on $\kappa$, $\mathbf{h}(\mathbb{P}, \underset{\sim}{\mathcal{D}})$ is well defined and is a $\mathbb{P}$-name $\underset{\sim}{\mathbf{h}}_{[\mathbb{P}, \underset{\sim}{\mathcal{D}}]}$ of a function $\underset{\sim}{\mathbf{h}}_{[\mathbb{P}, \underset{\sim}{\mathcal{D}}]} : \mathbf{R}[\mathbb{P}, \underset{\sim}{\mathcal{D}}] \to \underset{\sim}{\mathcal{D}}$

such that for every purely full in its domain (if this makes sense for $\mathbb{P}$) $p \in P_\chi$ and $i \in \mathrm{Dom}(p)$ it is forced by $p \restriction i$ that:

"for every inaccessible $\kappa' < \kappa$ and every $(< \kappa')$-directed family $\mathbf{g}$ of conditions in $\mathbf{R}[P_i/(p \restriction i), p(i)]$ of size $< \kappa$, such that

$$r \in \mathbf{g} \implies \kappa' \in \underset{\sim}{\mathbf{h}}_{[P_i/(p \restriction i), p(i)]}(r),$$

there is $q \geq \mathbf{g}$ such that $q \Vdash \kappa' \in \underset{\sim}{A}_i$."

**Remark 1.22.** The condition in Claim 1.21 is sufficient for the present application in §2. It may be weakened if needed for some future application. Really, the condition to use instead of it is that in item (h) of Main Claim 1.18, for all $i$ of cofinality $< \theta$, we are "in the good case", i.e. the first case of item (h). However, we wish to have a criterion which can be used without the knowledge of the proof of the Main Claim 1.18, and the condition in Claim 1.21 is one such criterion.

**Proof of the Crucial Claim.** By Conclusion 1.19 it suffices to show that under the assumptions of this Claim, in the proof of Main Claim 1.18 we can choose $\langle \alpha_i : i < \chi \rangle$, $\langle p_i^* : i < \chi \rangle$ and $\langle q_i^* : i < \chi \rangle$ so that for every $i$ with $\mathrm{cf}(i) \geq \theta$, the answer to "the 0th question" in the choice of $q_1^{i+1}$ is negative, i.e. there is $q$ such that $(**)_q$ holds. The proof is by induction on such $i$. We use the notation of Main Claim 1.18.

Given $i$ with $\mathrm{cf}(i) \geq \theta$. Hence we have

$$p_i^*(\alpha_i) = \left\{ \underset{\sim}{B}[G_i] : \begin{array}{l} \underset{\sim}{B} \text{ a } P_{\alpha_i}/(p_i^* \restriction \alpha_i)\text{-name} \\ \text{for a subset of } \kappa \text{ and } \underset{\sim}{\mathbf{t}}_{\underset{\sim}{B}} = 1 \end{array} \right\} \stackrel{\text{def}}{=} \underset{\sim}{\mathcal{D}}_i.$$

In $M$ we have

$$(\emptyset_{R_\kappa^+}, p_i^*, q_i^*) \Vdash \left( \begin{array}{l} \{\mathbf{j}(r)(\mathbf{j}(\alpha_i)) : \mathbf{j}(r) \in \underset{\sim}{X}_i\} \text{ is } (< \kappa) - \text{directed of size } < \\ \mathbf{j}(\kappa), \kappa \text{ is inaccessible and } (\forall r)[\kappa \in \mathbf{j}(\underset{\sim}{\mathbf{h}}_{[P/(p^* \restriction \alpha_i), p_i^*(\alpha_i)]}(r))] \end{array} \right).$$

(The last statement is true by the definition of $\underset{\sim}{\mathcal{D}}_i$ and $\underset{\sim}{t}_{\underset{\sim}{B}}$, no matter what $\underset{\sim}{\mathbf{h}}_{[P/p^* \restriction \alpha_i, p_i^*(\alpha_i)]}(r)$ is forced to be.)

By the assumption (3) and elementarity, applying $\mathbf{j}$ we have that the answer to the "0th question" is negative. $\bigstar_{1.21}$

**Definition 1.23.** (In $V^{R^+_\kappa}$) Given $\theta = \mathrm{cf}(\theta) \in (\kappa, \chi)$. We define $\mathcal{K}^*_\theta$ in the same way as $\mathcal{K}^+_\theta$, but with a freedom of choice for $Q_0$. Namely, to obtain the definition of $\mathcal{K}^*_\theta$ from that of $\mathcal{K}^+_\theta$, we

(A) In item (6) of Definition 1.9, require $i > 0$,

(B) We let $Q_0$ be any $(< \kappa)$-directed-closed cardinal preserving forcing notion in $\mathcal{H}(\chi)$ that also preserves $\Upsilon^\theta = \Upsilon$.

**Claim 1.24.** (In $V^{R^+_\kappa}$) Main Claim 1.18, Conclusion 1.19, Definition 1.20 and Claim 1.21 hold with $\mathcal{K}^+_\theta$ replaced by $\mathcal{K}^*_\theta$.

**Proof of the Claim.** As in $V^{R^+_\kappa * \underset{\sim}{Q}_0}$, $\kappa$ is still indestructibly supercompact and $\Upsilon^\theta = \Upsilon$. $\bigstar_{1.24}$

**Discussion 1.25.** (1) In the present application, we need to make sure that cardinals are not collapsed, so we have $\theta = \kappa^+$ and $Q_{\mathcal{D}}$ is chosen to have a strong version of $\kappa^+$-cc which is preserved by iterations with $(< \kappa)$-supports.

(2) Clearly, Claim 1.21 remains true if we replace the word "inaccessible" by e.g "strongly inaccessible", "weakly compact", "measurable".

(3) As we shall see in section 2, the point of dealing with a fitted member of $\mathcal{K}^+_\theta$ is to be able to control the Prikry names in the forcing that will be performed after the iteration extracted from $\mathcal{K}^+_\theta$, namely the Prikry forcing over $\cup_{i<\delta}\mathcal{D}_i$ for some $\delta$. The point of $\underset{\sim}{A}_i$ is to give us a control of this ultrafilter in the appropriate universe.

# 2 Universal graphs

**Theorem 2.1.** Assume that it is consistent that there is a supercompact cardinal.

Then it is consistent to have a singular strong limit cardinal $\kappa$ of cofinality $\omega$ with $2^{\kappa^+} > \kappa^{++}$, on which there are $\kappa^{++}$ graphs of size $\kappa^+$ which are universal for the graphs of size $\kappa^+$.

**Proof.** We start with a universe $V$ in which $\kappa$, $\Upsilon$ and $\chi$ satisfy Hypothesis 1.3, with $\theta = \kappa^+$ (in particular $\kappa^+ < \Upsilon = \Upsilon^{\kappa^+}$). Let $R^+_\kappa$ be the forcing described in Definition 1.5. We work in $V^{R^+_\kappa}$, which we start calling $V$ from this point on. As we shall not use $h$ and $R^+_\kappa$ any more, we free the notation $h$ and $\underset{\sim}{R}_\alpha$ to be used with a different meaning in this section.

**Definition 2.2.** Let $Q_0$ be the Cohen forcing which makes $2^{\kappa^+} = \Upsilon$ by adding $\Upsilon$ distinct $\kappa^+$-branches $\{\eta_\alpha : \alpha < \Upsilon\}$ to $({}^{\kappa^+>}2)^V$ by conditions of size $\leq \kappa$, so no cardinal is collapsed and in the resulting universe

- each $\eta_\alpha \in {}^{\kappa^+}2$,
- $\alpha < \beta < \Upsilon \implies \eta_\alpha \neq \eta_\beta$ and
- $\zeta < \kappa^+ \implies |\{\eta_\alpha \restriction \zeta : \alpha < \Upsilon\}| \leq \kappa^+$.

Let $\bar\eta = \langle \eta_\alpha : \alpha < \kappa^+ \rangle$ be fixed for the rest of the section, and let us let $V_0 \stackrel{\text{def}}{=} V[G_{Q_0}]$.

**Notation 2.3.** If $\mathcal{D}$ is a normal ultrafilter on a measurable cardinal $\kappa$, let $\Pr(\mathcal{D})$ denote the Prikry forcing for $\mathcal{D}$.

**Discussion 2.4.** The idea of the proof is to embed "$\mathcal{D}$-named graphs" into a universal graph. We use an iteration of forcing to achieve this. As we intend to perform a Prikry forcing at the end of iteration, we need to control the names of graphs that appear after the Prikry forcing, so one worry is that there would be too many names to take care of by the bookkeeping. Luckily, we shall not be dealing with all such names, but only with those for which we are sure that they will actually be used at the end. This is achieved by building the ultrafilter that will serve for the Prikry forcing, as the union of filters that appear during the iteration. To this end, for every relevant $\mathcal{D}$ we also force a set $\underset{\sim}{A}$ that will in some sense be a "diagonal intersection" of $\mathcal{D}$, so its membership in the intended ultrafilter will guarantee that that ultrafilter contains $\mathcal{D}$ as a subset.

**Definition 2.5.** Suppose $V' \supseteq V_0$ is a universe in which $2^{\kappa^+} = \Upsilon$, $\bar\eta$ is fixed as per Definition 2.2, while $\kappa$ is measurable and $\mathcal{D}$ is a normal ultrafilter over $\kappa$. Working in $V'$, we define a forcing notion $Q \stackrel{\text{def}}{=} Q^{V'}_{\mathcal{D},\kappa,\bar\eta}$, as follows.

Let $\bar{\underset{\sim}{M}} = \langle \underset{\sim}{M}_\alpha = \langle \kappa^+, \underset{\sim}{R}_\alpha \rangle : \alpha < \Upsilon \rangle$ list without repetitions all canonical $\Pr(\mathcal{D})$-names for graphs on $\kappa^+$. By canonical in this context we mean names of the form

$$\underset{\sim}{\tau} \subseteq \bigcup_{\zeta < \xi < \kappa^+} \mathcal{A}_{\zeta,\xi} \times \{(\check{\zeta,\xi})\}$$

where each $\mathcal{A}_{\zeta,\xi}$ is a maximal antichain in $\Pr(\mathcal{D})$. Then $\underset{\sim}{\tau}_G$ is a subset of $[\kappa^+]^2$ and we identify it with a graph $\underset{\sim}{g} = g(\underset{\sim}{\tau})$ on $\kappa^+$ by letting $\{\zeta, \xi\}$ form an edge iff $\zeta < \xi$ and for some $p \in \mathcal{A}_{\zeta,\xi} \cap G$ we have $(p, (\check{\zeta,\xi})) \in \underset{\sim}{\tau}$ or $\xi < \zeta$ and for some $p \in \mathcal{A}_{\zeta,\xi} \cap G$ we have $(p, (\check{\xi,\zeta})) \in \underset{\sim}{\tau}$. (Note that if $p \in \Pr(\mathcal{D})$ and $\underset{\sim}{\sigma}$ is a $\Pr(\mathcal{D})$-name such that $p \Vdash$ "$\underset{\sim}{\sigma}$ is a graph on $\kappa^+$", then there is a canonical name $\underset{\sim}{\tau}$ as above such that $p \Vdash \underset{\sim}{\sigma} = \underset{\sim}{\tau}$.) In the list $\bar{M}$ we understand that $M_\alpha$ is the model with universe $\kappa^+$ where $R_\alpha$ is the graph relation obtained by some graph $g(\underset{\sim}{\tau})$ as above. For definiteness we pick the first such list in the canonical well-order of $\mathcal{H}(\chi)$. Elements of $Q$ are of the form

$$p = \langle A^p, B^p, u^p, \bar{f}^p = \langle f^p_\alpha : \alpha \in u^p \rangle \rangle,$$

where

(i) $A^p \in [\kappa]^{<\kappa}$,

(ii) $B^p \in \mathcal{D} \cap \mathcal{P}([\kappa \setminus (\mathrm{Sup}(A^p))])$,

(iii) $u^p \in [\Upsilon]^{<\kappa}$,

(iv) For $\alpha \in u^p$, we have that $f^p_\alpha$ is a partial one-to-one function from $\kappa^+$ with $|\mathrm{Dom}(f^p_\alpha)| < \kappa$, mapping $\zeta \in \mathrm{Dom}(f^p_\alpha)$ to an element of $\{\eta_\alpha \restriction \zeta\} \times \kappa$,

(v) For $\alpha, \beta \in u^p$, for every $x', x''$, if

$$f^p_\alpha(x') = f^p_\beta(x') \neq f^p_\alpha(x'') = f^p_\beta(x''),$$

<u>then</u> for every $w \in [A^p]^{<\aleph_0}$

$$\langle w, B^p \rangle \Vdash_{\Pr(\mathcal{D})} \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(x', x'') \text{ iff } \underset{\sim}{M}_\beta \models \underset{\sim}{R}_\beta(x', x'')\text{”}.$$

In addition, for every $w \in [A^p]^{<\aleph_0}$ and every $\alpha \in u^p$ and $x', x'' \in \mathrm{Dom}(f^\alpha_p)$, the condition $\langle w, B^p \rangle$ decides in the Prikry forcing for $\mathcal{D}$ if $\underset{\sim}{M}_\alpha$ satisfies $\underset{\sim}{R}_\alpha(x', x'')$.

We define the order on $Q$ by letting $p \leq q$ (here $q$ is the stronger condition) iff

(a) $A^p$ is an initial segment of $A^q$,

(b) $A^q \setminus A^p \subseteq B^p$,

(c) $B^p \supseteq B^q$,

(d) $u^p \subseteq u^q$,

(e) For $\alpha \in u^p$, we have $f^p_\alpha \subseteq f^q_\alpha$.

**Claim 2.6.** Suppose that $Q = Q^{V'}_{\mathcal{D},\kappa,\bar{\eta}}$ is defined as in Definition 2.5. Then in $V'$:

(1) $Q$ is a separative partial order.

(2) Suppose that $G$ is $Q$-generic over $V'$, and let in $V'[G]$

$$A^* \stackrel{\text{def}}{=} \bigcup\{A : (\exists B, u, \bar{f})[\langle A, B, u, \bar{f}\rangle \in G]\}.$$

Then $A^*$ is an unbounded subset of $\kappa$ and $A^* \subseteq^* B$ for every $B \in \mathcal{D}$.

(3) For $\alpha < \Upsilon$ and $a \in \kappa^+$, the set

$$\mathcal{K}_{a,\alpha} \stackrel{\text{def}}{=} \{p \in Q : \alpha \in u^p \ \&\ a \in \mathrm{Dom}(f^p_\alpha)\}$$

is dense open in $Q$.

**Proof of the Claim.** (1) Routine checking.

(2) For $\alpha < \kappa$, the set

$$\mathcal{I}_\alpha \stackrel{\text{def}}{=} \{p \in Q : (\exists \beta \geq \alpha)[\beta \in A^p]\}$$

is dense open in $V'$, hence $A^*$ is an unbounded subset of $\kappa$ in $V'[G]$. For $B \in \mathcal{D}$ the set

$$\mathcal{J}_B \stackrel{\text{def}}{=} \{p \in Q : B^p \subseteq B\}$$

is dense open. If $p \in \mathcal{J}_B \cap G$, then for any $q \in G$ with $q \geq p$ we have $A^q \setminus A^p \subseteq B^p$. Hence $A^* \setminus B \subseteq A^p$.

(3) Given $p \in Q$, clearly there is $q \geq p$ with $\alpha \in u^q$. Namely, we may let for $p \in Q$ such that $\alpha \notin u^p$ an extension $q$ be defined by $A^q = A^p$, $u^q = u^p \cup \{\alpha\}$, $f^q_\alpha = \emptyset$ and $f^q_\beta = f^p_\beta$ for $\beta \in u^p$. So without loss of generality $\alpha \in u^p$ and $a \notin \mathrm{Dom}(f^p_\alpha)$. Applying the Prikry Lemma, for every $b \in \mathrm{Dom}(f^p_\alpha)$ and $w \in [A^p]^{<\aleph_0}$, there is $B_{w,b} \subseteq B^p$ with $B_{w,b} \in \mathcal{D}$ and such that

$$(w, B_{w,b})||_{\mathrm{Pr}(\mathcal{D})} \text{“}\underset{\sim}{M}_\alpha \models b\underset{\sim}{R}_\alpha a\text{”}.$$

Choose $\gamma < \kappa$ such that $(\eta_\alpha \restriction a, \gamma) \notin \bigcup_{\beta \in u^p} \mathrm{Rang}(f^p_\beta)$, which is possible as for every relevant $\beta$ we have $|\mathrm{Dom}(f^p_\beta)| < \kappa$ and $|u^p| < \kappa$. Now we define $q$ by letting $A^q \stackrel{\mathrm{def}}{=} A^p$, $B^q \stackrel{\mathrm{def}}{=} \bigcap\{B_{w,b} : w \in [A]^{<\aleph_0} \ \& \ b \in \mathrm{Dom}(f^p_\alpha)\} \cap B^p$, $u^q \stackrel{\mathrm{def}}{=} u^p$ and

$$f^q_\beta \stackrel{\mathrm{def}}{=} \begin{cases} f^p_\beta & \text{if } \beta \neq \alpha \\ f^p_\alpha \cup \{(a, (\eta_\alpha \restriction a, \gamma))\} & \text{otherwise.} \end{cases}$$

To verify that $q$ is a condition we discuss 2.5(v). If $\beta \neq \alpha$ and $x', x'' \in \mathrm{Dom}(f^q_\alpha)$ then $\langle w, B^q\rangle$ decides in $\mathrm{Pr}(\mathcal{D})$ if $\underset{\sim}{M}_\beta \models \underset{\sim}{R}_\beta(x', x'')$ because already $\langle w, B^p\rangle$ does that. If $x', x'' \in \mathrm{Dom}(f^p_\alpha)$ the conclusion follows similarly. If $\{x', x''\} \supseteq \{a\}$ then the conclusion follows by the choice of $B^q$.

Suppose that $f^q_\alpha(x') = f^q_\beta(x') \neq f^q_\alpha(x'') = f^q_\beta(x'')$ for some $x' \neq x''$ and $\alpha \neq \beta$. If $x', x'' \in \mathrm{Dom}(f^p_\alpha)$ then

$$\langle w, B^q\rangle \Vdash_{\mathrm{Pr}(\mathcal{D})} \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(x', x'') \iff \underset{\sim}{M}_\beta \models \underset{\sim}{R}_\beta\text{”}$$

because this is already true of $\langle w, B^p\rangle$. So suppose without loss of generality that $x' = a$. But $\gamma$ was chosen so that $f^q_\alpha(a) = (\eta_\alpha \restriction a, \gamma)$ is not in $\mathrm{Rang}(f^q_\beta)$, hence the condition 2.5(v) is satisfied.

$\bigstar_{2.6}$

**Definition 2.7.** (Shelah, [Sh 80]) Let $\lambda \geq \aleph_0$ be a cardinal. A forcing notion $P$ is said to be *stationary* $\lambda^+$-*cc* iff for every $\langle p_\alpha : \alpha < \lambda^+\rangle$ in $P$, there is a club $C \subseteq \lambda^+$ and a regressive $h : \lambda^+ \to \lambda^+$ such that for all $\alpha, \beta \in C$,

$$[\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \lambda \ \& \ h(\alpha) = h(\beta)] \implies p_\alpha, p_\beta \text{ are compatible.}$$

**Theorem 2.8.** (Shelah, [Sh 80], [Sh 546]) Suppose that $\lambda^{<\lambda} = \lambda \geq \aleph_0$. Iterations with $(<\lambda)$-support of $(<\lambda)$-directed-closed stationary $\lambda^+$-cc forcing, are $(<\lambda)$-directed-closed and satisfy stationary $\lambda^+$-cc.

**Claim 2.9.** Suppose that $Q$ is as in Claim 2.6. Then $Q$ is $(<\kappa)$-directed-closed and satisfies stationary $\kappa^+$-cc.

**Proof of the Claim.** First suppose that $i^* < \kappa$ and $\{p_i : i < i^*\}$ is directed. For $i < i^*$ let $p_i \stackrel{\text{def}}{=} \langle A^i, B^i, u^i, \bar{f}^i \rangle$. We define $A \stackrel{\text{def}}{=} \bigcup_{i<i^*} A^i$, $B \stackrel{\text{def}}{=} \bigcap_{i<i^*} B^i$, $u \stackrel{\text{def}}{=} \cup_{i<i^*} u^i$, and for $\alpha \in u$ we let $f_\alpha \stackrel{\text{def}}{=} \cup_{i<i^*} f^i_\alpha$. It is easily verified that this defines a common upper bound of all $p_i$.

Hence $Q$ is $(<\kappa)$-directed-closed. Now we shall prove that it is $\kappa^+$-stationary-cc. Let $\langle p_i : i < \kappa^+ \rangle$ be given, where each $p_i = \langle A_i, B_i, u_i, \bar{f}^i \rangle$ and $\bar{f}_i = \langle f^i_\alpha : \alpha \in u_i \rangle$. Let $U \stackrel{\text{def}}{=} \bigcup \{u_i : i < \kappa^+\}$, hence $U \subseteq \Upsilon$ and $|U| \leq \kappa^+$. Let us fix a one-to-one enumeration of $U$ in an order type $\leq \kappa^+$, so $U = \{\alpha_s : s < s^* \leq \kappa^+\}$.

For $i < \kappa^+$ let $S_i = \{s : \alpha_s \in u_i\}$ be an increasing enumeration and let $\sigma_i \stackrel{\text{def}}{=} \text{otp}(S_i)$, hence $\sigma_i < \kappa$. For $s \in S_i$ let $d^i_s \stackrel{\text{def}}{=} \text{Dom}(f^i_{\alpha_s})$ and for $k < \sigma_i$ let $\alpha^i_k = \alpha$ iff $\alpha = \alpha_s$ for the $k$-the element $s$ of $S_i$. Let $\gamma_i < \kappa$ be given by

$$\gamma_i \stackrel{\text{def}}{=} \sup\{\gamma + 1 : (\exists \alpha \in u_i)(\exists \zeta \in \text{Dom}(f^i_\alpha))\,(f^i_\alpha(\zeta) = (\eta_\alpha \restriction \zeta, \gamma))\}.$$

For $A \in [\kappa]^{<\kappa}$ and $\sigma < \kappa$ define a language

$$\mathcal{L}_{A,\sigma} = \{\mathbf{R}_{w,k} : w \in [A]^{<\omega}, k < \sigma\} \cup \{<\} \cup \{g_k : k < \sigma\} \cup \{\mathbf{P}, \mathbf{Q}\},$$

where each $\mathbf{R}_{w,k}$ is a 2-place relation symbol, as is $<$, each $g_k$ is a 1-place function symbol and $\mathbf{P}, \mathbf{Q}$ are unary predicates. Note that the size of this language is $<\kappa$.

For $i < \kappa^+$ define a model $N_i$ of $\mathcal{L}_{A_i,\sigma_i}$ with the universe

$$\gamma_i \times \{0\} \cup \bigcup_{s \in S_i} d^i_s \times \{1\}$$

and the interpretation given by:

- $\mathbf{P}((a,b))$ iff $b = 0$,

- $\mathbf{Q}((a,b))$ iff $b=1$,
- $<$ is the partial ordering given by letting $(\alpha,a)<(\beta,b)$ iff $a=b$ and $\alpha<\beta$ as ordinals,
- $(\zeta,a),(\xi,b)\in\mathbf{R}_{w,k}$ iff $a=b=1$ and $\zeta,\xi\in d^i_{\alpha^i_k}$, while

$$(w,B_i)\Vdash_{\Pr(\mathcal{D})}\text{“}\zeta\underset{\sim}{R}_{\alpha^i_k}\xi\text{”},$$

- $g_k((\zeta,a))=(\gamma,b)$ iff $a=0=b,\zeta=\gamma$ or $a=1,b=0$ and

$$f^i_{\alpha^i_k}(\zeta)=(\eta_{\alpha^i_k}\restriction\zeta,\gamma).$$

For each relevant $A,\sigma$ consider the isomorphism types of models of $\mathcal{L}_{A,\sigma}$ whose universe is a disjoint union of two sets each of size $<\kappa$. There are $\leq\kappa$ such types (because $\kappa^{<\kappa}=\kappa$), let us enumerate them as

$$\{t^{A,\sigma}_\beta:\,\beta<\beta(A,\sigma)\leq\kappa\}.$$

For $i<\kappa^+$ let $\beta_i$ be such that the isomorphism type of $N_i$ as a model of $\mathcal{L}_{A_i,\sigma_i}$ is $t^{A_i,\sigma_i}_{\beta_i}$.

Let $F$ from $\kappa\times[\kappa]^{<\kappa}\times{}^{\kappa>}([\kappa^+]^{<\kappa})\times\kappa\times\kappa\times[\kappa^+]^{<\kappa}$ be a bijection onto $\kappa^+$. Let $C$ be a club of $j<\kappa^+$ such that for $j\in C$ with $\mathrm{cf}(j)=\kappa$ we have

$$F(\sigma,A,\langle d_k:\,k<\sigma\rangle,\beta,\gamma,S)<j\iff\sup_{k<\sigma}d_k,\sup(S)<j,$$

and such that for all $i<j$ we have $\sup\{s:\,\alpha_s\in u_i\}<j$ and

$$\sup\bigcup\{\mathrm{Dom}(f^i_\alpha):\,\alpha\in u_i\}<j.$$

Such a club exists because $\kappa^{<\kappa}=\kappa$.

We define $h:\,\kappa^+\to\kappa^+$ by letting $h(i)=0$ unless $i\in C$ and $\mathrm{cf}(i)=\kappa$, when $h(i)=F(\sigma_i,A_i,\langle d^i_s\cap i:\,s<i\rangle,\beta_i,\gamma_i,S_i\cap i)$. Hence $h$ is regressive.

Suppose that $i<j\in C$ and $\mathrm{cf}(i)=\mathrm{cf}(j)=\kappa$ are such that $h(i)=h(j)$, we claim that $p_i$ and $p_j$ are compatible. In order to prove this we proceed with several subclaims.

**Subclaim 2.10.** $A_i=A_j$ and $\sigma_i=\sigma_j$.

**Proof of the Subclaim.** This follows from the choice of $F$ and $h$. $\bigstar_{2.10}$

Let $A \stackrel{\text{def}}{=} A_i = A_j$, $\sigma \stackrel{\text{def}}{=} \sigma_i = \sigma_j$.

**Subclaim 2.11.** $N_i$ and $N_j$ are isomorphic as models of $\mathcal{L}_{A,\sigma}$.

**Proof of the Subclaim.** This follows from Subclaim 2.10 and the fact that $\beta_i = \beta_j$. $\bigstar_{2.11}$

**Subclaim 2.12.** If $\alpha \in u_i \cap u_j$ <u>then</u> $\alpha = \alpha^i_k = \alpha^j_k$ for the same $k$ and $\alpha = \alpha_s$ for some $s < i$.

**Proof of the Subclaim.** Since $\alpha \in u_i$ and $j \in C$ is of cofinality $\kappa$, we have that $\alpha = \alpha_s$ for some $s < j$. Hence $s < i$ by the choice of $h$ and so $\alpha = \alpha^i_k$ for some $k$. Since $S_i \cap i = S_j \cap j$ we have that $\alpha = \alpha^j_k$ as well. $\bigstar_{2.12}$

**Subclaim 2.13.** If $\alpha \in u_i \cap u_j$ and $\zeta \in d^i_\alpha \cap d^j_\alpha$, <u>then</u> $f^i_\alpha(\zeta) = f^j_\alpha(\zeta)$.

**Proof of the Subclaim.** By Subclaim 2.12 there is $k$ such that $\alpha = \alpha^i_k = \alpha^j_k$, which is $\alpha_s$ for some $s < i$. By the choice of $j$ we have $\sup(d^i_\alpha) < j$, so by the choice of $h$ we have $\zeta < i$. Since $N_i$ and $N_j$ are isomorphic, by the definition of $<$ in these models we have that $(\zeta, 1)$ is a fixed point of the isomorphism. Hence $g_k((\zeta, 1))$ is as well, so there is a unique $\gamma$ such that

$$f^i_\alpha(\zeta) = (\eta_\alpha \restriction \zeta, \gamma) = f^j_\alpha(\zeta).$$

$\bigstar_{2.13}$

For every $w \in [A]^{<\omega}$ and for every $\alpha \in u_i \cup u_j$ and $\zeta, \zeta' \in \bigcup_{l \in \{i,j\}, \alpha \in u_l} d^l_\alpha$ we can find $B^{\alpha,\zeta,\zeta'}$ such that

$$\langle w, B^{\alpha,\zeta,\zeta'} \rangle_{\Pr(\mathcal{D})} || \text{“} \underset{\sim}{M}_\alpha \vDash \underset{\sim}{R}_\alpha(\zeta, \zeta') \text{”}.$$

Let $B \stackrel{\text{def}}{=} B_i \cap B_j \cap \bigcap_{\alpha \in u_i \cup u_j, \zeta, \zeta' \in d^i_\alpha \cup d^j_\alpha} B^{\alpha,\zeta,\zeta'}$. We claim that a common upper bound of $p_i$ and $p_j$ is given by $q = \langle A, B, u = u_i \cup u_j, \bar{f} \rangle$ where $\bar{f} = \langle f_\alpha : \alpha \in u \rangle$ and $f_\alpha = \bigcup_{l \in \{i,j\}} f^l_\alpha$. To prove this it suffices to prove the following two claims:

**Subclaim 2.14.** Suppose that $\alpha, \beta \in u$ and $\zeta < \zeta'$ are such that

$$f_\alpha(\zeta) = f_\beta(\zeta) \neq f_\alpha(\zeta') = f_\beta(\zeta').$$

Then for every $w \in [A]^{<\omega}$ we have

$$\langle w, B\rangle_{\Pr(\mathcal{D})} \Vdash \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(\zeta, \zeta') \iff \underset{\sim}{M}_\beta \models \underset{\sim}{R}_\beta(\zeta, \zeta')\text{”}.$$

**Proof of the Subclaim.** We have to do a case analysis.

Case 1. For some $l \in \{i, j\}$ we have that $\alpha, \beta \in u_l$ and $\zeta, \zeta' \in d^l_\alpha \cap d^l_\beta$.

The conclusion follows by the analogous properties of $p_l$.

Case 2. $\alpha \in u_i \cap u_j$, $\zeta \in d^i_\alpha \setminus d^j_\alpha$ and $\zeta' \in d^j_\alpha \setminus d^i_\alpha$.

We have $\zeta' \notin d^i_\alpha$, hence $\zeta' \geq j$ by the choice of $h$. Hence $\zeta' \notin d^i_\beta$ and so $\zeta' \in d^j_\beta$. In particular

$$f^j_\alpha(\zeta') = f_\alpha(\zeta') = f_\beta(\zeta') = f^j_\beta(\zeta').$$

Since $\zeta \in d^i_\alpha \setminus d^j_\alpha$ we have $\zeta \in [i, j)$ and so $\zeta \in d^i_\beta \setminus d^j_\beta$ and

$$f^i_\alpha(\zeta) = f_\alpha(\zeta) = f_\beta(\zeta) = f^i_\beta(\zeta).$$

In particular $\beta \in u_i \cap u_j$. Let $\zeta''$ be such that $(\zeta'', 1) \in N_i$ is the isomorphic image of $(\zeta', 1)$ under an isomorphism between $N_j$ and $N_i$. Then

$$f^i_\alpha(\zeta'') = f^j_\alpha(\zeta') = f^j_\beta(\zeta') = f^i_\beta(\zeta'')$$

and so

$$f^i_\alpha(\zeta) = f^i_\beta(\zeta) \neq f^i_\alpha(\zeta'') = f^i_\beta(\zeta'').$$

So for every $w \in [A]^{<\omega}$ we have

$$\langle w, B_i\rangle_{\Pr(\mathcal{D})} \Vdash \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(\zeta, \zeta'') \iff \underset{\sim}{M}_\beta \models \underset{\sim}{R}_\beta(\zeta, \zeta'')\text{”}.$$

Let $w \in [A]^{<\omega}$ be given. By the choice of the function $\mathbf{R}_{w,k}$ for $k$ such that $\alpha = \alpha^i_k$ we have

$$\langle w, B_i\rangle_{\Pr(\mathcal{D})} \Vdash \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(\zeta, \zeta'')\text{” iff } \langle w, B_j\rangle_{\Pr(\mathcal{D})} \Vdash \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(\zeta, \zeta')\text{”}$$

and similarly for $\beta$ in place of $\alpha$. Hence

$$\langle w, B\rangle_{\Pr(\mathcal{D})} \Vdash \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(\zeta, \zeta') \iff \underset{\sim}{M}_\beta \models \underset{\sim}{R}_\beta(\zeta, \zeta')\text{”}$$

as required.

Case 3. $\beta \in u_i \cap u_j$, $\zeta \in d^i_\beta \setminus d^j_\beta$ and $\zeta' \in d^j_\beta \setminus d^i_\beta$.

Symmetric to Case 2 with $\alpha$ replaced by $\beta$.

Case 4. $\alpha \in u_i \cap u_j$, $\zeta \in d^i_\alpha \cap d^j_\alpha$ and $\zeta' \in d^j_\alpha \setminus d^i_\alpha$.

As in Case 2, $\zeta' \notin d^i_\alpha$ so $\zeta' \geq j$ and so $\zeta' \notin d^i_\beta$. Hence $\zeta' \in d^j_\beta \cap d^j_\alpha$ and so $f_\alpha(\zeta') = f^j_\alpha(\zeta')$ and $f_\beta(\zeta') = f^j_\beta(\zeta')$. Since $\zeta \in d^j_\alpha$ we have $f_\alpha(\zeta) = f^j_\alpha(\zeta)$. Since $\zeta \in d^i_\alpha$ we have $\zeta < j$ so $\zeta < i$. If $\zeta \in d^j_\beta$ we obtain the desired conclusion because $p_j \in Q$. But if not, then $\zeta \in d^i_\beta$, hence $\beta \in u_i \cap u_j$ and so $\zeta \in d^i_\beta \cap i = d^j_\beta \cap j$, a contradiction.

Case 5. $\beta \in u_i \cap u_j$, $\zeta \in d^i_\beta \cap d^j_\beta$ and $\zeta' \in d^j_\beta \setminus d^i_\beta$.

Symmetric to Case 4 with $\alpha$ replaced by $\beta$.

Case 6. $\alpha \in u_i \cap u_j$ and $\zeta' \in d^i_\alpha \setminus d^j_\alpha$ while $\zeta \in d^j_\alpha \setminus d^i_\alpha$.

This case cannot happen because $\zeta < \zeta'$.

Case 7. Symmetric to Case 6 with $\alpha$ replaced by $\beta$.

Cannot happen for the same reason as Case 6.

Case 8. $\alpha \in u_i \cap u_j$ and $\zeta, \zeta' \in d^i_\alpha \cap d^j_\alpha$.

If $\beta \in u_i \cap u_j$ then we are in Case 1. If $\beta \in u_i$ then since $\zeta, \zeta' \in d^i_\alpha$ we have $\zeta, \zeta' < j$ and so since $\alpha \in u_j$ we have $\zeta, \zeta' < i$. Hence $\zeta, \zeta' \in d^i_\beta$ and the conclusion follows because $p_i \in Q$. If $\beta \notin u_i$ then $\beta \in u_j$ and $\zeta, \zeta' \in d^j_\alpha \cap d^j_\beta$, hence the conclusion follows as $p_j \in Q$.

Case 9. $\beta \in u_i \cap u_j$ and $\zeta, \zeta' \in d^i_\beta \cap d^j_\beta$.

Symmetric to Case 8.

Case 10. $\alpha \in u_i \setminus u_j$ and $\beta \in u_j \setminus u_i$.

Hence $\zeta, \zeta' \in d^i_\alpha$ and so $\zeta, \zeta' < j$. Let $k$ be such that $\beta = \alpha^j_k$ and let $\beta' = \alpha^i_k$. By the choice of $h$ and the fact that $N_i$ and $N_j$ are isomorphic, we have that $\zeta, \zeta' < i$ and $\zeta, \zeta' \in d^i_{\beta'}$, while

$$\langle w, B_i \rangle \Vdash_{\mathrm{Pr}(\mathcal{D})} \text{“}\underset{\sim}{M}_{\beta'} \models \zeta \underset{\sim}{R}_{\beta'} \zeta'\text{” iff } \langle w, B_j \rangle \Vdash_{\mathrm{Pr}(\mathcal{D})} \text{“}\underset{\sim}{M}_\beta \models \zeta \underset{\sim}{R}_\beta \zeta'\text{”}.$$

Moreover $f^i_{\beta'}(\zeta) = f^j_\beta(\zeta)$ and similarly for $\zeta'$. We get the desired conclusion by applying this and the fact that $p_i \in Q$.

Case 11. $\alpha \in u_j \setminus u_i$ and $\beta \in u_i \setminus u_j$.

Symmetric to Case 10. $\bigstar_{2.14}$

**Subclaim 2.15.** Suppose that $\alpha \in u$ and $\zeta, \zeta' \in \mathrm{Dom}(f_\alpha)$.

Then $\langle w, B \rangle ||_{\mathrm{Pr}(\mathcal{D})}$“$\underset{\sim}{M}_\alpha \models \zeta \underset{\sim}{R}_\alpha \zeta'$”.

**Proof of the Subclaim.** Follows by the choice of $B$. $\bigstar_{2.15}$

This finishes the proof of the chain condition. $\bigstar_{2.9}$

**Observation 2.16.** Suppose that $\mathcal{D}$ is a normal ultrafilter over $\kappa$ and $Q$ is a forcing notion such that

$$\Vdash_Q \text{“}\mathcal{D} \subseteq \underset{\sim}{\mathcal{D}}' \text{ and } \underset{\sim}{\mathcal{D}}' \text{ is a normal ultrafilter over } \kappa\text{”}.$$

Then $\Pr(\mathcal{D}) \lessdot_e Q * \Pr(\underset{\sim}{\mathcal{D}}')$, where $e$ is the embedding given by

$$e((a, A)) \stackrel{\text{def}}{=} (\emptyset_Q, (a, A)).$$

**Definition 2.17.** Suppose that $Q$ is as in Claim 2.6, while $Q <\circ P$, and $\underset{\sim}{\mathcal{D}}'$ is a $P$-name of a normal ultrafilter over $\kappa$, extending $\mathcal{D} \cup \{\underset{\sim}{A}^*\}$. For $\alpha < \Upsilon$ we define $\underset{\sim}{Gr}_\alpha^{\mathcal{D}'}$, intended to be a $P * \Pr(\mathcal{D})$-name for a graph on $\{\eta_\alpha \restriction \zeta : \zeta < \kappa^+\} \times \kappa$ (see Claim 2.19 below), defined by letting for $y', y'' \in \{\eta_\alpha \restriction \zeta : \zeta < \kappa^+\} \times \kappa$,

$$\begin{aligned} y' \underset{\sim}{R} y'' \quad & \text{iff for some } \langle p, \langle w, B^p \rangle\rangle \in \underset{\sim}{G} \text{ with } \alpha \in u^p, p \in Q \text{ and } [w] \in [A^p]^{<\aleph_0} \\ & \text{and some } x', x'' \in \mathrm{Dom}(f_\alpha^p) \\ & \text{we have } f_\alpha^p(x') = y' \text{ and } f_\alpha^p(x'') = y'', \\ & \text{AND } \langle w, B^p \rangle \Vdash_{\Pr(\mathcal{D})} \text{“}\underset{\sim}{M}_\alpha \models \underset{\sim}{R}_\alpha(x', x'')\text{”}. \end{aligned}$$

**Notation 2.18.** Suppose that $Q$ is as in Claim 2.6. For $\alpha < \Upsilon$ let

$$\underset{\sim}{f}_\alpha \stackrel{\text{def}}{=} \cup\{f_\alpha^p : \alpha \in u^p \ \&\ p \in \underset{\sim}{G}_Q\}.$$

**Claim 2.19.** Suppose $Q$ is as in Claim 2.6, while $Q \lessdot P$, and $\underset{\sim}{\mathcal{D}}'$ is a $P$-name of a normal ultrafilter over $\kappa$, extending $\mathcal{D} \cup \{\underset{\sim}{A}^*\}$ (equivalently, $\underset{\sim}{A}^* \in \underset{\sim}{\mathcal{D}}'$).

Then

$$\langle \emptyset, \langle \emptyset, \underset{\sim}{A}^* \rangle\rangle \Vdash_{P * \Pr(\underset{\sim}{\mathcal{D}}')} \text{“}\underset{\sim}{f}_\alpha \text{ is an embedding of } \underset{\sim}{M}_\alpha \text{ into } \underset{\sim}{Gr}_\alpha^{\mathcal{D}'}.$$

**Proof of the Claim.** Let $G$ be $P * \Pr(\underset{\sim}{\mathcal{D}}')$-generic with $\langle \emptyset, \langle \emptyset, \underset{\sim}{A}^* \rangle\rangle \in G$ and suppose that $x', x''$ are such that $M_\alpha \models R_\alpha(x', x'')$ in $V[G]$. Let $\langle p^+, \langle w, \underset{\sim}{A}' \rangle\rangle$ be a condition in $G$ that forces this. Without loss of generality, we have

$$\langle p^+, \langle w, \underset{\sim}{A}' \rangle\rangle \geq \langle \emptyset, \langle \emptyset, \underset{\sim}{A}^* \rangle\rangle.$$

In particular, $p^+ \Vdash_P$ "$w \in [\underset{\sim}{A}^*]^{<\aleph_0}$". Considering $P$ as $Q * \underset{\sim}{P}/Q$, let us write $\langle p^+, \langle w, \underset{\sim}{A}'\rangle\rangle$ as $\langle p, p', \langle w, \underset{\sim}{A}'\rangle\rangle$. By extending $p^+$ if necessary, we may assume that $A^p \supseteq w$, and then using the density of $\mathcal{K}_{x',\alpha}$ and $\mathcal{K}_{x'',\alpha}$, we may also assume that $\alpha \in u^p$ and $x', x'' \in \mathrm{Dom}(f^p_\alpha)$. By extending further, we may assume that $p^+ \Vdash$ "$\underset{\sim}{A}' \subseteq B^p$". Then $\langle p^+, \langle w, \underset{\sim}{A}'\rangle\rangle$ extends $\langle p, \langle w, B^p\rangle\rangle$, hence the latter is in $G$. Since $p \Vdash_P$ "$\langle w, B^p\rangle \|_{\mathrm{Pr}(\mathcal{D})} \underset{\sim}{R}_\alpha(x', x'')$", it must be that $\langle w, B^p\rangle \Vdash_{\mathrm{Pr}(\mathcal{D})}$ "$M_\alpha \models R_\alpha(x', x'')$". Hence in $V[G]$ we have that

$$y' = f_\alpha(x') R y'' = f_\alpha(x'').$$

On the other hand, suppose that in $V[G]$ we have $y' = f_\alpha(x') R y'' = f_\alpha(x'')$ and let $\langle p, \langle w, B^p\rangle\rangle$ exemplify this. In particular, $\langle w, B^p\rangle$ forces in $\mathrm{Pr}(\mathcal{D})$ that "$M_\alpha \models R_\alpha(x', x'')$", and since $\langle p, \langle w, B^p\rangle\rangle \in G$, we have that $R_\alpha(x', x'')$ holds in $V[G]$.

As it is easily seen that each $f_\alpha$ is forced to be 1-1 and total, this finishes the proof. $\bigstar_{2.19}$

**Claim 2.20.** Suppose that $Q$ and $\underset{\sim}{\mathcal{D}}'$ are as in Claim 2.19, while $G$ is $Q$-generic over $V'$. Further suppose that $H$ is a $\mathrm{Pr}(\mathcal{D}')$-generic filter over $V'[G]$ with $\langle \emptyset, A^*\rangle \in H$.

Then in $V'[G][H]$, there is a graph $Gr^*$ of size $\kappa^+$ such that for every filter $J$ in $\mathrm{Pr}(\mathcal{D})$ satisfying

$$\{(\emptyset_Q, p) : p \in J\} \subseteq G * \underset{\sim}{H} \stackrel{\mathrm{def}}{=} \{(q, \underset{\sim}{s}) : q \in G \ \&\ q \Vdash \text{"}\underset{\sim}{s} \in \underset{\sim}{H}\text{"}\}$$

which is $\mathrm{Pr}(\mathcal{D})$-generic over $V'$, every graph of size $\kappa^+$ in $V'[J]$ is embedded into $Gr^*$.

**Proof of the Claim.** Define $Gr^*$ on $\cup_{\alpha<\Upsilon}\{\eta_\alpha \restriction \zeta : \zeta < \kappa^+\} \times \kappa$, hence $|Gr^*| = \kappa^+$, by our assumptions on $\bar{\eta}$. We let

$$Gr^* \models \text{"}(\eta_\alpha \restriction \zeta, i) R (\eta_\alpha \restriction \xi, j)\text{" iff } Gr^{\mathcal{D}'}_\alpha \models \text{"}(\eta_\alpha \restriction \zeta, i) R (\eta_\alpha \restriction \xi, j)\text{"}.$$

Then $Gr^*$ is a well defined graph, as follows by the definition of $Q$.

Given $M$ a graph on $\kappa^+$ in $V'[J]$. Let $\langle w, A\rangle \in J$ force in $\mathrm{Pr}(\mathcal{D})$ that $M$ is a graph on $\kappa^+$. By Observation 2.16, $(\emptyset_Q, \langle w, A\rangle)$ forces in $Q * \mathrm{Pr}(\mathcal{D}')$ that

$M$ is a graph on $\kappa^+$, so since $(\emptyset_Q, \langle w, A\rangle) \in G * \underset{\sim}{H}$ we have that for some $\alpha$ it is true that $M = \underset{\sim}{M}_\alpha[G][H]$. Since $(\emptyset_Q, \langle \emptyset, A^*\rangle) \in G * \underset{\sim}{H}$ we have by Claim 2.19 that $M$ embeds into $Gr^{\mathcal{D}'}_\alpha$ in $V'[G][H]$, but $Gr^{\mathcal{D}'}_\alpha$ embeds into $Gr^*$ by the definition of $Gr^*$. $\bigstar_{2.20}$

**Claim 2.21.** Let $\mathcal{D}$ be a normal ultrafilter over $\kappa$ and $A \in \mathcal{D}$. Suppose that $G$ is a $\Pr(\mathcal{D})$-generic filter over $V$. Then there is some $G'$ which is $\Pr(\mathcal{D})$-generic over $V$ and such that $(\emptyset, A) \in G'$ while $V[G] = V[G']$.

**Proof of the Claim.** Let $x = x_G = \cup\{s : (\exists B \in \mathcal{D})(s, B) \in G\}$, so

$$G = G_x = \{(s, B) \in \Pr(\mathcal{D}) : s \subseteq x_G \subseteq s \cup B\}.$$

Now we use the Mathias characterization of Prikry forcing, which says that for an infinite subset $x$ of $\kappa$ we have that $G_x$ is $\Pr(\mathcal{D})$-generic over $V$ iff $x_G \setminus B$ is finite for all $B \in \mathcal{D}$. Hence $x \setminus A$ is finite. Let $y = x_G \cap A$, so an infinite subset of $\kappa$ which clearly satisfies that $y \setminus B$ is finite for all $B \in \mathcal{D}$. Let $G' = G_y$, so $G'$ is $\Pr(\mathcal{D})$-generic over $V$ and $(\emptyset, A) \in G'$. We have $V[G'] \subseteq V[G]$ because $y \in G$ and $V[G] \subseteq V[G']$ because $x \setminus y$ is finite. $\bigstar_{2.21}$

**Conclusion 2.22.** Suppose that $Q$, $\underset{\sim}{\mathcal{D}}'$, $G$ and $V'$ are as in Claim 2.20 and $H$ is a $\Pr(\mathcal{D}')$-generic filter over $V'[G]$. Then the conclusion of Claim 2.20 holds in $V'[G][H]$.

**Proof.** The conclusion follows by Claim 2.20 and Claim 2.21. $\bigstar_{2.22}$

**Claim 2.23.** Suppose that $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i, \underset{\sim}{A}_i : i < \chi\rangle \in \mathcal{K}^*_{\kappa^+}$ is given by determining $Q_0$ as in Definition 2.2 and defining $\underset{\sim}{Q}^i_{\underset{\sim}{\mathcal{D}}} = \underset{\sim}{Q}^{V[G_{P_i}]}_{\mathcal{D}}$ (as defined in Definition 2.5) and $\underset{\sim}{A}_i = \underset{\sim}{A}^*_i$, where $\underset{\sim}{A}^*_i$ was defined in Claim 2.6(2).

Then $\bar{Q}$ is fitted.

**Proof of the Claim.** We shall take $\mathbf{R}[\mathbb{P}, \underset{\sim}{\mathcal{D}}] = Q^{V^{\mathbb{P}}}_{\underset{\sim}{\mathcal{D}},\kappa,\bar{\eta}}$ if this is well defined (i.e. $V^{\mathbb{P}}$ satisfies the conditions on $V'$ in Definition 2.5) and $\mathbf{R}[\mathbb{P}, \underset{\sim}{\mathcal{D}}] = \{\emptyset\}$ otherwise. By Claim 1.21, it suffices to give a definition of $\mathbf{h}$ satisfying the requirements of that Claim. Suppose that $\mathbb{P}, \underset{\sim}{\mathcal{D}}$ are such that $\mathbf{R}[\mathbb{P}, \underset{\sim}{\mathcal{D}}]$ is non-trivial, working in $V^{\mathbb{P}}$ we define

$$\mathbf{h} = \mathbf{h}_{[\mathbb{P},\mathcal{D}]} : Q_{\mathcal{D}} = \mathbf{R}_{[\mathbb{P},\mathcal{D}]} \to \mathcal{D}$$

by letting $\mathbf{h}(p) \stackrel{\text{def}}{=} B^p$ for $p = (A^p, B^p, u^p, \bar{f}^p)$. We check that this definition is as required. So suppose that $\kappa' < \kappa$ is inaccessible and $\mathbf{g}$ is a $(< \kappa')$-directed family of conditions in $Q_{\mathcal{D}}$ with the property that for all $p \in \mathbf{g}$ we have $\kappa' \in B^p$. We define $r$ by letting

$$A^r \stackrel{\text{def}}{=} \bigcup_{p\in\mathbf{g}} A^p \cup \{\kappa'\}, B^r \stackrel{\text{def}}{=} \bigcap_{p\in\mathbf{g}} B^p \setminus \{\kappa'\}, u^r \stackrel{\text{def}}{=} \cup_{p\in\mathbf{g}} u^p,$$

and for $\alpha \in u^r$, we let $f^r_\alpha \stackrel{\text{def}}{=} \cup_{p\in\mathbf{g}\ \&\ \alpha\in u^p} f^p_\alpha$. It is easy to check that this condition is as desired. $\bigstar_{2.23}$

**Remark 2.24.** The inaccessibility of $\kappa'$ was not used in the Proof of Claim 2.23.

*Proof of the Theorem finished.*

To finish the proof of the Theorem, in $V_0$ let $\bar{Q}$ be as in Claim 2.23. By Claim 2.23 and the definition of fittedness, we can find sequences $\langle p^*_i : i < \chi\rangle$ and $\langle \alpha_i : i < \chi\rangle$ witnessing that $\bar{Q}$ is fitted. Let $\underset{\sim}{\mathcal{D}}_i \stackrel{\text{def}}{=} p^*_i(\alpha_i)$ for $i < \chi$. If we force in $V_0$ by

$$P^* \stackrel{\text{def}}{=} \lim\langle P_{\alpha_i}/(p^*_i \restriction \alpha_i), \underset{\sim}{Q}_{\mathcal{D}_i} : i < \chi\rangle,$$

we obtain a universe $V^*$ in which $\langle \mathcal{D}_i : \text{cf}(i) = \kappa^+\rangle$ is an increasing sequence of normal filters over $\kappa$, and $\mathcal{D} \stackrel{\text{def}}{=} \bigcup_{i\in S^\chi_{\kappa^+}} \mathcal{D}_i$ is a normal ultrafilter over $\kappa$. For, in $V^{P_{\alpha_i}/(p^*_i\restriction\alpha_i)}$, we have that $\mathcal{D}_i$ is an ultrafilter over $\kappa$, and $\text{cf}(\chi) > \kappa$, while the iteration is with $(< \kappa)$-supports and $\kappa^{<\kappa} = \kappa$. Hence every subset of $\kappa$ in $V^*$ appears as an element of $V^{P_{\alpha_i}/(p^*_i\restriction\alpha_i)}$ for some $i$, and so $\mathcal{D}$ is an ultrafilter.

Also, for every $i \in S^{\chi}_{\kappa^+}$ we have that $A^*_i \in \mathcal{D}$. Let $\underset{\sim}{\mathcal{D}}$ be a $P^*$-name for $\mathcal{D}$ of $V^*$. Let

$$E \stackrel{\text{def}}{=} \left\{ \delta < \chi : \begin{array}{c} (\forall \alpha < \delta)(\exists \beta \in (\alpha, \delta))[\alpha_\beta = \beta \text{ and} \\ \underset{\sim}{\mathcal{D}} \cap \mathcal{P}(\kappa)^{V_0^{P_\beta}} \text{ is a } P_\beta/(p_\beta \restriction \beta)\text{-name} \\ \text{and } p_{\beta+1}(\beta) = \mathcal{D} \cap \mathcal{P}(\kappa)^{V_0^{P_\beta}}] \end{array} \right\}.$$

Hence $E$ is a club of $\chi$. Let $\delta \in E \cap S^{\chi}_{\kappa^{++}}$ be larger than $\kappa^{+++}$. Force with $P^* \restriction \delta$, so obtaining $V_1$ in which $2^{\kappa^+} \geq 2^\kappa \geq \kappa^{+++}$, as each coordinate of $P^* \restriction \delta$ adds a subset of $\kappa$, and cardinals are preserved. In $V_1$ force with the Prikry forcing for $\mathcal{D}_\delta \stackrel{\text{def}}{=} \bigcup_{i \in S^{\delta}_{\kappa^+}} \mathcal{D}_i$. Let $W \stackrel{\text{def}}{=} V_1[\Pr(\mathcal{D}_\delta)]$. For $i \in S^{\delta}_{\kappa^+}$, let $Gr^*_i$ be a graph obtained in $W$ satisfying the conditions of Conclusion 2.22 with $\mathcal{D}_\delta$ in place of $\mathcal{D}'$ and $\mathcal{D}_i$ in place of $\mathcal{D}$. Let $C$ be a club of $\delta$ of order type $\kappa^{++}$, and let $g$ be its increasing enumeration.

We claim that $W$ is as required, and that

$$\{Gr^*_{g(i)} : i < \kappa^{++} \ \& \ \text{cf}(g(i)) = \kappa^+\}$$

are universal for graphs of size $\kappa^+$. Clearly the cofinality of $\kappa$ in $W$ is $\aleph_0$ and $\kappa$ is a strong limit. Suppose that $Gr$ is a graph on $\kappa^+$ in $W$ and let $\underset{\sim}{Gr}$ be a $\Pr(\mathcal{D}_\delta)$-name for it. Hence, there is a $i < \kappa^{++}$ with $\text{cf}(g(i)) = \kappa^+$ such that $\underset{\sim}{Gr}$ is a $\Pr(\mathcal{D}_{g(i)})$-name for a graph on $\kappa^+$. The conclusion follows by the choice of $Gr^*_i$. $\bigstar_{2.1}$

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